Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(c\,f)' = c\,f'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(f\,g)' = f'\,g + f\,g'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(a^x\right) = a^x \ln(a)$

$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$

$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}, \quad x > 0$

$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}, \quad x \neq 0$

$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln a}, \quad x > 0$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right)=n\left[f(x)\right]^{n-1}f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right)=f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right)=\frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right)=f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right)=-f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right)=f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right)=f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right)=\frac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The Second Derivative is denoted as

$f''(x)=f^{(2)}(x)=\frac{d^2 f}{dx^2}$ and is defined as

$f''(x)=\left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n$^{\text{th}}$ Derivative is denoted as

$f^{(n)}(x)=\frac{d^n f}{dx^n}$ and is defined as

$f^{(n)}(x)=\left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the $(n$-$1)^{\text{st}}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y}+x^3y^2=\sin(y)+11x$. Remember $y=y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$\mathbf{e}^{2x-9y}(2-9y')+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11$$
$$2\mathbf{e}^{2x-9y}-9y'\mathbf{e}^{2x-9y}+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11 \qquad \Rightarrow \qquad y'=\frac{11-2\mathbf{e}^{2x-9y}-3x^2y^2}{2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)}$$
$$\left(2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)\right)y'=11-2\mathbf{e}^{2x-9y}-3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x=c$ is a critical point of $f(x)$ provided either **1.** $f'(c)=0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x)>0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x)<0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x)=0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x)>0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x)<0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x=c$ is a inflection point of $f(x)$ if the concavity changes at $x=c$.