SATPREP

Assignment : Properties of curve : Implicit differentiation

1. Show that the points (0, 0) and $(\sqrt{2\pi}, -\sqrt{2\pi})$ on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent.

2. The function f is defined by $f(x) = e^{x^2-2x-1.5}$.

(a) Find $f'(x)$.

(b) You are given that $y = \dfrac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the value of a.

3. Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point (1, –2).

4. The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

(a) Find the value of p and the value of q.

(b) The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph.

5. Find the equation of the normal to the curve $5xy^2 - 2x^2 = 18$ at the point (1, 2).

6. A curve C is defined implicitly by $xe^y = x^2 + y^2$. Find the equation of the tangent to C at the point (1, 0).

7. Consider the curve with equation $f(x) = e^{-2x^2}$ for $x < 0$.

Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

8. The function f is defined on the domain $x \geq 0$ by $f(x) = \dfrac{x^2}{e^x}$.

(a) Find the maximum value of $f(x)$, and justify that it is a maximum.

(b) Find the x coordinates of the points of inflexion on the graph of f.

9. The function f is defined by $f(x) = (\ln(x-2))^2$. Find the coordinates of the point of inflexion of f.

10. The normal to the curve $xe^{-y} + e^{y} = 1 + x$, at the point $(c, \ln c)$, has a y-intercept $c^2 + 1$.

Determine the value of c.

11. Find the equation of the normal to the curve $x^3y^3 - xy = 0$ at the point (1, 1).

12. Find the gradient of the curve $e^{xy} + \ln(y^2) + e^{y} = 1 + e$ at the point (0, 1).

Solutions

1. Attempt at implicit differentiation

$$e^{(x+y)}\left(1+\frac{dy}{dx}\right) = -\sin(xy)\left(x\frac{dy}{dx}+y\right)$$

let $x = 0$, $y = 0$

$$e^0\left(1+\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -1$$

let $x = \sqrt{2\pi}$, $y = -\sqrt{2\pi}$

$$e^0\left(1+\frac{dy}{dx}\right) = -\sin(-2\pi)\left(x\frac{dy}{dx}+y\right) = 0$$

so $\frac{dy}{dx} = -1$

since both points lie on the line $y = -x$ this is a common tangent

Note: $y = -x$ must be seen for the final R1. It is not sufficient to note that the gradients are equal.

2. (a) $\left(u = x^2 - 2x - 1.5; \frac{du}{dx} = 2x - 2\right)$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = e^x(2x-2)$$

$$= 2(x-1)\,e^{x^2-2x-1.5}$$

(b) $$\frac{dy}{dx} = \frac{(x-1)\times 2(x-1)e^{x^2-2x-1.5} - 1\times e^{x^2-2x-1.5}}{(x-1)^2}$$

$$= \frac{2x^2-4x+1}{(x+1)^2}e^{x^2-2x-1.5}$$

minimum occurs when $\frac{dy}{dx} = 0$

$x = 1 \pm\sqrt{\frac{1}{2}}$ $\left(\text{accept } x = \frac{4\pm\sqrt{8}}{4}\right)$

$a = 1 + \sqrt{\frac{1}{2}}$ $\left(\text{accept } a = \frac{4+\sqrt{8}}{4}\right)$

3. Attempting to differentiate implicitly

$3x^2y + 2xy^2 = 2 \Rightarrow 6xy + 3x^2\dfrac{\text{d}y}{\text{d}x} + 2y^2 + 4xy\dfrac{\text{d}y}{\text{d}y} = 0$

Substituting $x = 1$ and $y = -2$

$-12 + 3\dfrac{\text{d}y}{\text{d}x} + 8 - 8\dfrac{\text{d}y}{\text{d}x} = 0$

$\Rightarrow -5\dfrac{\text{d}y}{\text{d}x} = 4 \Rightarrow \dfrac{\text{d}y}{\text{d}x} = -\dfrac{4}{5}$

Gradient of normal is $\dfrac{5}{4}$

4. (a) **METHOD 1**

$f'(x) = q - 2x = 0$
$f'(3) = q - 6 = 0$
$q = 6$
$f(3) = p + 18 - 9 = 5$

$p = -4$

METHOD 2

$f(x) = -(x - 3)^2 + 5$
$= -x^2 + 6x - 4$
$q = 6, p = -4$

(b) $g(x) = -4 + 6(x - 3) - (x - 3)^2$ $(= -31 + 12x - x^2)$

Note: Accept any alternative form that is correct.
Award M1A0 for a substitution of $(x + 3)$.

5. $5y^2 + 10xy\dfrac{\text{d}y}{\text{d}x} - 4x = 0$

Note: Award A1A1 for correct differentiation of $5xy^2$.
A1 for correct differentiation of $-2x^2$ and 18.

At the point (1, 2), $20 + 20\dfrac{\text{d}y}{\text{d}x} - 4 = 0$

$\Rightarrow \dfrac{\text{d}y}{\text{d}x} = -\dfrac{4}{5}$

Gradient of normal = $\dfrac{5}{4}$

Equation of normal $y - 2 = \dfrac{5}{4}(x - 1)$

$y = \dfrac{5}{4}x - \dfrac{5}{4} + \dfrac{8}{4}$

$y = \dfrac{5}{4}x + \dfrac{3}{4}$ $\quad (4y = 5x + 3)$

6. $xe^y = x^2 + y^2$

$e^y + xe^y \frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$

(1, 0) fits $\Rightarrow 1 + \frac{dy}{dx} = 2 + 0$

$\Rightarrow \frac{dy}{dx} = 1$

Equation of tangent is $y = x + c$

(1, 0) fits $\Rightarrow c = -1$

$\Rightarrow y = x - 1$

7. .

$f'(x) = -4x\,e^{-2x^2}$

$f''(x) = -4\,e^{-2x^2} + 16x^2\,e^{-2x^2} \quad \left(= \left(16x^2 - 4\right)e^{-2x^2}\right)$

Attempting to solve $f''(x) = 0$

$x = -\frac{1}{2}$

Note: Do not award this A1 for stating $x = \pm\frac{1}{2}$ as the final answer for x.

$f\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{e}}\ (= 0.607)$

Note: Do not award this A1 for also stating $\left(\frac{1}{2}, \frac{1}{\sqrt{e}}\right)$ as a coordinate.

EITHER

Correctly labelled graph of $f'(x)$ for $x < 0$ denoting the maximum $f'(x)$

(e.g. $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated)

OR

Correctly labelled graph of $f''(x)$ for $x < 0$ denoting the maximum $f'(x)$

(e.g. $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated)

OR

$f'(0.5) \approx 1.21$. $f'(x) < 1.21$ just to the left of $x = -\dfrac{1}{2}$

and $f'(x) < 1.21$ just to the right of $x = -\dfrac{1}{2}$

(e.g. $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated)

OR

$f''(x) > 0$ just to the left of $x = -\dfrac{1}{2}$ and $f''(x) < 0$ just to the right of $x = -\dfrac{1}{2}$

(e.g. $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated)

8. (a) $f'(x) = \dfrac{2xe^x - x^2e^x}{e^{2x}}$ $\left(= \dfrac{2x - x^2}{e^x}\right)$

For a maximum $f'(x) = 0$

$2x - x^2 = 0$

giving $x = 0$ or 2

$f''(x) = \dfrac{(2-2x)e^x - e^x(2x - x^2)}{e^{2x}}$ $\left(= \dfrac{x^2 - 4x + 2}{e^x}\right)$

$f''(0) = 2 > 0 \Rightarrow$ minimum

$f''(2) = -\dfrac{2}{e^2} < 0 \Rightarrow$ maximum

Maximum value $= \dfrac{4}{e^2}$

(b) For a point of inflexion,

$f''(x) = \dfrac{x^2 - 4x + 2}{e^x} = 0$

giving $x = \dfrac{4 \pm \sqrt{16 - 8}}{2}$

$= 2 \pm \sqrt{2}$

(c) $\int_0^1 x^2 e^{-x} dx = \left[-x^2 e^{-x}\right]_0^1 + 2\int_0^1 xe^{-x} dx$

$= -e^{-1} - 2\left[xe^{-x}\right]_0^1 + 2\int_0^1 e^{-x} dx$

$= -e^{-1} - 2e^{-1} - 2\left[e^{-x}\right]_0^1$

$= -3e^{-1} - 2e^{-1} + 2$ $(= 2 - 5e^{-1})$

9. $f'(x) = \dfrac{2(\ln(x-2))}{x-2}$

$f''(x) = \dfrac{(x-2)\left(\dfrac{1}{x-2}\right) - 2\ln(x-2)\times 1}{(x-2)^2}$

$= \dfrac{2 - 2\ln(x-2)}{(x-2)^2}$

$f''(x) = 0$ for point of inflexion

$\Rightarrow 2 - 2\ln(x-2) = 0$

$\ln(x-2) = 1$

$x - 2 = e$

$x = e + 2$

$\Rightarrow f(x) = (\ln(e + 2 - 2))^2 = (\ln e)^2 = 1$

($\Rightarrow$ coordinates are $(e + 2, 1)$)

10. EITHER

differentiating implicitly:

$1 \times e^{-y} - xe^{-y}\dfrac{dy}{dx} + e^{y}\dfrac{dy}{dx} = 1$

at the point $(c, \ln c)$

$\dfrac{1}{c} - c \times \dfrac{1}{c}\dfrac{dy}{dx} + c\dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{c}$ $(c \neq 1)$

OR

reasonable attempt to make expression explicit

$xe^{-y} + e^{y} = 1 + x$

$x + e^{2y} = e^{y}(1 + x)$

$e^{2y} - e^{y}(1 + x) + x = 0$

$(e^{y} - 1)(e^{y} - x) = 0$

$e^{y} = 1,\ e^{y} = x$

$y = 0,\ y = \ln x$

Note: Do not penalize if $y = 0$ not stated.

$\frac{dy}{dx} = \frac{1}{2}$

gradient of tangent = $\frac{1}{c}$

Note: If candidate starts with $y = \ln x$ with no justification, award (M0)(A0)A1A1.

THEN

the equation of the normal is

$y - \ln c = -c(x - c)$

$x = 0,\ y = c^2 + 1$

$c^2 + 1 - \ln c = c^2$

$\ln c = 1$

$c = e$

11. $x^3y^3 - xy = 0$

$3x^2y^3 + 3x^3y^2y' - y - xy' = 0$

Note: Award A1 for correctly differentiating each term.

$x = 1,\ y = 1 \quad 3 + 3y' - 1 - y' = 0$

$2y' = -2$

$y' = -1$

gradient of normal = 1

equation of the normal $y - 1 = x - 1$

$y = x$

12. $e^{xy} + \ln(y^2) + e^y = 1 + e$

$$e^{xy}\left(y + x\frac{dy}{dx}\right) + \frac{2}{y}\frac{dy}{dx} + e^{y}\frac{dy}{dx} = 0, \text{ at } (0, 1)$$

$$1(1 + 0) + 2\frac{dy}{dx} + e\frac{dy}{dx} = 0$$

$$1 + 2\frac{dy}{dx} + e\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{2 + e} \ (= -0.212)$$