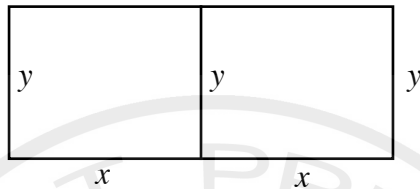
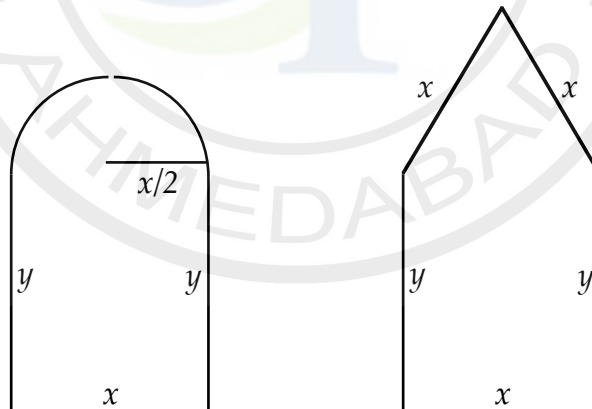


Assignment :Optimization

1. A rectangle has a perimeter of 200 inches. Determine the dimensions that maximize the area, and give the maximum possible area.
2. A rectangle has area is 150 sq. in. Determine the dimensions that minimize the perimeter, and give the minimum possible perimeter.
3. A pen is to be constructed with 200 ft of fence. It must consist of two congruent rectangles with one common side. Determine the dimensions that maximize the area and give the maximum area.



4. An open-topped box is constructed as follows: A 24 in. \times 24 in. square bottom has four identical squares cut off at each corner. The sides are then folded up. What sized corners should be cut out to maximize the volume of the box. Give this maximum possible volume.
5. An arched window is to have a rectangular base with either
 - (i) a semicircular top or
 - (ii) an equilateral triangle top



In each case, the total perimeter must be 30 ft. In each case, determine the dimensions that maximize the area and give the maximum area.

Solutions

1. Let x and y be the sides of the rectangle so that $2x + 2y = 200$, or $x + y = 100$. Then $y = 100 - x$. Then the area is $A(x) = xy = x(100 - x) = 100x - x^2$ for $0 < x < 100$. The parabola $A(x)$ obtains a maximum value when $0 = A'(x) = 100 - 2x$, which gives dimensions of $x = 50$ in, $y = 50$ in, and a maximum area of 2500 sq. in.

2. Let x and y be the sides so that $xy = 150$ and $y = 150/x$. Then the perimeter is $P(x) = 2x + 2y = 2x + \frac{300}{x}$ for $x > 0$. The function $P(x)$ obtains a minimum value when $0 = P'(x) = 2 - \frac{300}{x^2}$, which gives $x^2 = 150$. So, $x = \sqrt{150}$ in., $y = 150/\sqrt{150} = \sqrt{150}$ in., and the minimum perimeter is $2\sqrt{150} + 2\sqrt{150} = 4\sqrt{150}$ in.

3. We must have $4x + 3y = 200$ so that $y = \frac{200 - 4x}{3}$. The area is

$$A(x) = 2xy = 2x \times \frac{200 - 4x}{3} = \frac{400x - 8x^2}{3}.$$

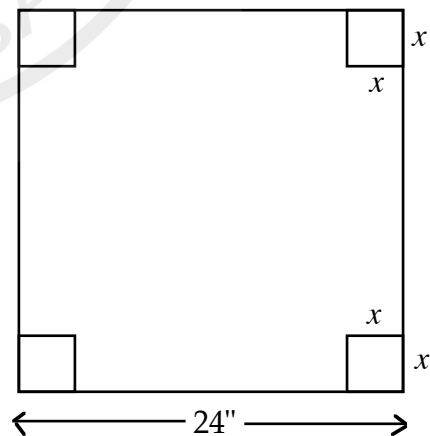
We must have $x > 0$, and as y decreases to 0, we have $4x$ increasing to 200. So we must have $0 < x < 50$. Now $A(x)$ is maximized when $0 = A'(x) = \frac{400 - 16x}{3}$, which gives $16x = 400$ and $x = 25$ ft. Then $y = 100/3$ ft, or 33 ft, 4 in. The maximum area is then $A = 2 \times 25 \times \frac{100}{3} = 1666\frac{2}{3}$ sq. ft.

4. After cutting out the corners, each side of the box base has length $24 - 2x$, while the height of the box becomes x . The volume of the box is then

$$\begin{aligned} V(x) &= x(24 - 2x)^2 = x(576 - 96x + 4x^2) \\ &= 576x - 96x^2 + 4x^3 \quad \text{for } 0 < x < 12 \end{aligned}$$

The max of $V(x)$ occurs at critical point where

$$\begin{aligned} 0 &= V'(x) = 576 - 192x + 12x^2 \\ &= 12(x^2 - 16x + 48) = 12(x - 12)(x - 4) \end{aligned}$$

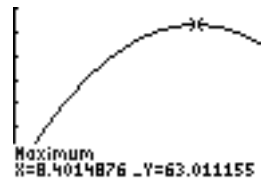


Thus, $x = 12$ or $x = 4$. Due to the constraint, we must have $x = 4$ in. which leaves a 16 in. \times 16 in. base and gives a maximum volume of $4 \times 16^2 = 1024$ in³.

5. (i) The perimeter of the semicircle is $\frac{1}{2}\pi \times \text{diameter} = \frac{\pi x}{2}$, and the total perimeter must be 30 ft. So we must have $x + 2y + \frac{\pi x}{2} = 30$, or $y = 15 - \frac{x}{2} - \frac{\pi x}{4}$. The area of the window is then

$$\begin{aligned} A(x) &= xy + \frac{1}{2}\pi(x/2)^2 = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{\pi}{8}x^2 \\ &= 15x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi}{8}x^2 = 15x - \frac{x^2}{2} - \frac{\pi x^2}{8}. \end{aligned}$$

As y decreases to 0, then $30 \approx x + \frac{\pi x}{2} = \frac{2x + \pi x}{2} = \frac{(2 + \pi)x}{2}$. So $x \approx \frac{60}{2 + \pi} \approx 11.67$ ft. Thus, the domain of $A(x)$ is $0 < x < \frac{60}{2 + \pi}$, and $A(x)$ has a maximum value at a critical point within this domain.



We then have

$$0 = A'(x) = 15 - x - \frac{\pi x}{4} = 15 - x\left(1 + \frac{\pi}{4}\right) = 15 - x\left(\frac{4 + \pi}{4}\right),$$

$$\text{which gives } x = \frac{60}{4 + \pi} \approx 8.4 \text{ ft.}$$

Then,

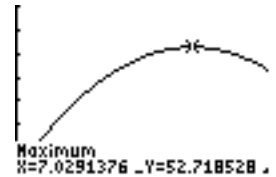
$$\begin{aligned} y &= 15 - \frac{1}{2} \times \frac{60}{4 + \pi} - \frac{\pi}{4} \times \frac{60}{4 + \pi} = 15 - \frac{30}{4 + \pi} - \frac{15\pi}{4 + \pi} \\ &= \frac{60 + 15\pi - 30 - 15\pi}{4 + \pi} = \frac{30}{4 + \pi} = \frac{1}{2}x \approx 4.2 \text{ ft.} \end{aligned}$$

The maximum area is approximately 63 sq. ft.

(ii) Now we have $3x + 2y = 30$ so that $y = \frac{30 - 3x}{2}$. The height of the equilateral triangle is given by $x \times \sin 60^\circ = \frac{\sqrt{3}x}{2}$, so its area is $\frac{1}{2} \text{base} \times \text{height} = \frac{\sqrt{3}x^2}{4}$. The total area of the window is then

$$A(x) = xy + \frac{\sqrt{3}x^2}{4} = \frac{30x - 3x^2}{2} + \frac{\sqrt{3}x^2}{4} = 15x + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2.$$

We know that $x > 0$, and as y decreases to 0 then $3x \approx 30$. So the domain of $A(x)$ is $0 < x < 10$. The maximum value of the parabola $A(x)$ does occur at a critical point within the domain. So we solve $A'(x) = 0$:



$$0 = A'(x) = 15 + \left(\frac{\sqrt{3}}{2} - 3\right)x = 15 + \left(\frac{\sqrt{3} - 6}{2}\right)x = 15 - \left(\frac{6 - \sqrt{3}}{2}\right)x.$$

Thus, $x = \frac{30}{6 - \sqrt{3}} \approx 7.029$ ft. Then

$$y = 15 - \frac{3}{2} \times \frac{30}{6 - \sqrt{3}} = 15 - \frac{45}{6 - \sqrt{3}} = \frac{45 - 15\sqrt{3}}{6 - \sqrt{3}} \approx 4.4563 \text{ ft}$$

and the maximum area is approximately 52.7185 sq. ft.