SATPREP

Name____________________________________

Assignment:Induction (Series)

Date________________

Use mathematical induction to prove that each statement is true for all positive integers.

1) $1 + 9 + 25 + \cdots + (2n - 1)^2 = \dfrac{n(4n^2 - 1)}{3}$

2) $45 + 207 + 477 + \cdots + 54n^2 - 9 = 9n^2(2n + 3)$

3) $-1 + 2 + 5 + \cdots + 3n - 4 = \dfrac{n(3n - 5)}{2}$

4) $1 + 25 + 625 + \cdots + 5^{2n - 2} = \dfrac{5^{2n} - 1}{5^2 - 1}$

5) $4 + 16 + 64 + \cdots + 2^{2n} = \dfrac{2^2(2^{2n} - 1)}{2^2 - 1}$

6) $8 + 10 + 12 + \cdots + 2n + 6 = n(n + 7)$

7) $10 + 12 + 14 + \cdots + 2n + 8 = n(n + 9)$

8) $15 + 23 + 31 + \cdots + 8n + 7 = n(4n + 11)$

9) $40 + 184 + 424 + \cdots + 48n^2 - 8 = 8n^2(2n + 3)$

10) $35 + 161 + 371 + \cdots + 42n^2 - 7 = 7n^2(2n + 3)$

Answers to Assignment:Series (ID: 1)

1) Let P_n be the statement $1 + 9 + 25 + \cdots + (2n - 1)^2 = \dfrac{n(4n^2 - 1)}{3}$

Anchor Step

P_1 is true since $(2 - 1)^2 = \dfrac{4 \cdot 1^2 - 1}{3}$

Inductive Hypothesis

Assume that P_k is true: $1 + 9 + 25 + \cdots + (2k - 1)^2 = \dfrac{k(4k^2 - 1)}{3}$

Inductive Step

We now show that P_{k+1} is true:

$$1 + 9 + 25 + \cdots + (2k - 1)^2 + (2(k + 1) - 1)^2 = \frac{k(4k^2 - 1)}{3} + (2(k + 1) - 1)^2$$

$$\frac{k(4k^2 - 1)}{3} + (2k + 1)^2$$

$$\frac{4k^3 - k}{3} + \frac{3(4k^2 + 4k + 1)}{3}$$

$$\frac{4k^3 - k}{3} + \frac{12k^2 + 12k + 3}{3}$$

$$\frac{4k^3 + 12k^2 + 11k + 3}{3}$$

$$\frac{4k^3 + 12k^2 + 12k + 4 - k - 1}{3}$$

$$\frac{4(k^3 + 3k^2 + 3k + 1) - (k + 1)}{3}$$

$$\frac{4(k + 1)^3 - (k + 1)}{3}$$

$$\frac{(k + 1)(4(k + 1)^2 - 1)}{3}$$

Conclusion

By induction P_n is true for all $n \geq 1$.

2) Let P_n be the statement $45 + 207 + 477 + \cdots + 54n^2 - 9 = 9n^2(2n + 3)$

Anchor Step

P_1 is true since $54 \cdot 1^2 - 9 = 9 \cdot 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $45 + 207 + 477 + \cdots + 54k^2 - 9 = 9k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
45 + 207 + 477 + \cdots + 54k^2 - 9 + 54(k+1)^2 - 9 &= 9k^2(2k+3) + 54(k+1)^2 - 9 \\
&18k^3 + 27k^2 + 54(k^2 + 2k + 1) - 9 \\
&18k^3 + 27k^2 + 54k^2 + 108k + 54 - 9 \\
&18k^3 + 81k^2 + 108k + 45 \\
&9(2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3) \\
&9(2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1)) \\
&9(2(k+1)^3 + 3(k+1)^2) \\
&9(k+1)^2(2(k+1) + 3)
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

3) Let P_n be the statement $-1 + 2 + 5 + \cdots + 3n - 4 = \frac{n(3n-5)}{2}$

Anchor Step

P_1 is true since $3 - 4 = \frac{3-5}{2}$

Inductive Hypothesis

Assume that P_k is true: $-1 + 2 + 5 + \cdots + 3k - 4 = \frac{k(3k-5)}{2}$

Inductive Step

We now show that P_{k+1} is true:

$$\begin{aligned}
-1 + 2 + 5 + \cdots + 3k - 4 + 3(k+1) - 4 &= \frac{k(3k-5)}{2} + 3(k+1) - 4 \\
&\frac{3k^2-5k}{2} + 3k + 3 - 4 \\
&\frac{3k^2-5k}{2} + \frac{6k-2}{2} \\
&\frac{3k^2+k-2}{2} \\
&\frac{3k^2+6k+3-5k-5}{2} \\
&\frac{3(k^2+2k+1)-5(k+1)}{2} \\
&\frac{3(k+1)^2-5(k+1)}{2} \\
&\frac{(k+1)(3(k+1)-5)}{2}
\end{aligned}$$

Conclusion

By induction P_n is true for all $n \geq 1$.

4) Let P_n be the statement $1 + 25 + 625 + \cdots + 5^{2n-2} = \dfrac{5^{2n} - 1}{5^2 - 1}$

Anchor Step

P_1 is true since $5^{2-2} = \dfrac{5^2 - 1}{5^2 - 1}$

Inductive Hypothesis

Assume that P_k is true: $1 + 25 + 625 + \cdots + 5^{2k-2} = \dfrac{5^{2k} - 1}{5^2 - 1}$

Inductive Step

We now show that P_{k+1} is true:

$$1 + 25 + 625 + \cdots + 5^{2k-2} + 5^{2(k+1)-2} = \frac{5^{2k} - 1}{24} + 5^{2(k+1)-2}$$

$$\frac{5^0\left(5^{2k} - 1\right)}{24} + 5^{2k+2-2}$$

$$\frac{5^0\left(5^{2k} - 1\right)}{24} + \frac{5^{2k} \cdot 5^0\left(5^2 - 1\right)}{24}$$

$$\frac{5^0\left(5^{2k} - 1 + 5^{2k+2} - 5^{2k}\right)}{24}$$

$$\frac{5^{2(k+1)} - 1}{24}$$

Conclusion

By induction P_n is true for all $n \geq 1$.

5) Let P_n be the statement $4 + 16 + 64 + \cdots + 2^{2n} = \frac{2^2(2^{2n} - 1)}{2^2 - 1}$

Anchor Step

P_1 is true since $2^2 = \frac{2^2(2^2 - 1)}{2^2 - 1}$

Inductive Hypothesis

Assume that P_k is true: $4 + 16 + 64 + \cdots + 2^{2k} = \frac{2^2(2^{2k} - 1)}{2^2 - 1}$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
4 + 16 + 64 + \cdots + 2^{2k} + 2^{2(k+1)} &= \frac{4(2^{2k} - 1)}{3} + 2^{2(k+1)} \\
&\frac{2^2(2^{2k} - 1)}{3} + 2^{2k+2} \\
&\frac{2^2(2^{2k} - 1)}{3} + \frac{2^{2k} \cdot 2^2(2^2 - 1)}{3} \\
&\frac{2^2(2^{2k} - 1 + 2^{2k+2} - 2^{2k})}{3} \\
&\frac{4(2^{2(k+1)} - 1)}{3}
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

6) Let P_n be the statement $8 + 10 + 12 + \cdots + 2n + 6 = n(n + 7)$

Anchor Step

P_1 is true since $2 + 6 = 1 + 7$

Inductive Hypothesis

Assume that P_k is true: $8 + 10 + 12 + \cdots + 2k + 6 = k(k + 7)$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
8 + 10 + 12 + \cdots + 2k + 6 + 2(k + 1) + 6 &= k(k + 7) + 2(k + 1) + 6 \\
&k^2 + 7k + 2k + 2 + 6 \\
&k^2 + 9k + 8 \\
&k^2 + 2k + 1 + 7k + 7 \\
&k^2 + 2k + 1 + 7(k + 1) \\
&(k + 1)^2 + 7(k + 1) \\
&(k + 1)(k + 1 + 7)
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

7) Let P_n be the statement $10 + 12 + 14 + \cdots + 2n + 8 = n(n + 9)$

Anchor Step

P_1 is true since $2 + 8 = 1 + 9$

Inductive Hypothesis

Assume that P_k is true: $10 + 12 + 14 + \cdots + 2k + 8 = k(k + 9)$

Inductive Step

We now show that P_{k+1} is true:

$$\begin{aligned}10 + 12 + 14 + \cdots + 2k + 8 + 2(k + 1) + 8 &= k(k + 9) + 2(k + 1) + 8\\ &k^2 + 9k + 2k + 2 + 8\\ &k^2 + 11k + 10\\ &k^2 + 2k + 1 + 9k + 9\\ &k^2 + 2k + 1 + 9(k + 1)\\ &(k + 1)^2 + 9(k + 1)\\ &(k + 1)(k + 1 + 9)\end{aligned}$$

Conclusion

By induction P_n is true for all $n \geq 1$.

8) Let P_n be the statement $15 + 23 + 31 + \cdots + 8n + 7 = n(4n + 11)$

Anchor Step

P_1 is true since $8 + 7 = 4 + 11$

Inductive Hypothesis

Assume that P_k is true: $15 + 23 + 31 + \cdots + 8k + 7 = k(4k + 11)$

Inductive Step

We now show that P_{k+1} is true:

$$\begin{aligned}15 + 23 + 31 + \cdots + 8k + 7 + 8(k + 1) + 7 &= k(4k + 11) + 8(k + 1) + 7\\ &4k^2 + 11k + 8k + 8 + 7\\ &4k^2 + 19k + 15\\ &4k^2 + 8k + 4 + 11k + 11\\ &4(k^2 + 2k + 1) + 11(k + 1)\\ &4(k + 1)^2 + 11(k + 1)\\ &(k + 1)(4(k + 1) + 11)\end{aligned}$$

Conclusion

By induction P_n is true for all $n \geq 1$.

9) Let P_n be the statement $40 + 184 + 424 + \cdots + 48n^2 - 8 = 8n^2(2n + 3)$

Anchor Step

P_1 is true since $48 \cdot 1^2 - 8 = 8 \cdot 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $40 + 184 + 424 + \cdots + 48k^2 - 8 = 8k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$\begin{aligned}
40 + 184 + 424 + \cdots + 48k^2 - 8 + 48(k+1)^2 - 8 = {} & 8k^2(2k + 3) + 48(k+1)^2 - 8 \\
& 16k^3 + 24k^2 + 48(k^2 + 2k + 1) - 8 \\
& 16k^3 + 24k^2 + 48k^2 + 96k + 48 - 8 \\
& 16k^3 + 72k^2 + 96k + 40 \\
& 8(2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3) \\
& 8(2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1)) \\
& 8(2(k+1)^3 + 3(k+1)^2) \\
& 8(k+1)^2(2(k+1) + 3)
\end{aligned}$$

Conclusion

By induction P_n is true for all $n \geq 1$.

10) Let P_n be the statement $35 + 161 + 371 + \cdots + 42n^2 - 7 = 7n^2(2n + 3)$

Anchor Step

P_1 is true since $42 \cdot 1^2 - 7 = 7 \cdot 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $35 + 161 + 371 + \cdots + 42k^2 - 7 = 7k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$\begin{aligned}
35 + 161 + 371 + \cdots + 42k^2 - 7 + 42(k+1)^2 - 7 = {} & 7k^2(2k + 3) + 42(k+1)^2 - 7 \\
& 14k^3 + 21k^2 + 42(k^2 + 2k + 1) - 7 \\
& 14k^3 + 21k^2 + 42k^2 + 84k + 42 - 7 \\
& 14k^3 + 63k^2 + 84k + 35 \\
& 7(2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3) \\
& 7(2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1)) \\
& 7(2(k+1)^3 + 3(k+1)^2) \\
& 7(k+1)^2(2(k+1) + 3)
\end{aligned}$$

Conclusion

By induction P_n is true for all $n \geq 1$.