Name___________________________________

SATPREP

Date ____________________

Parametric and Polar Equations

1997 CALCULUS BC *(a graphing calculator maybe used)*

1. During the time period $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

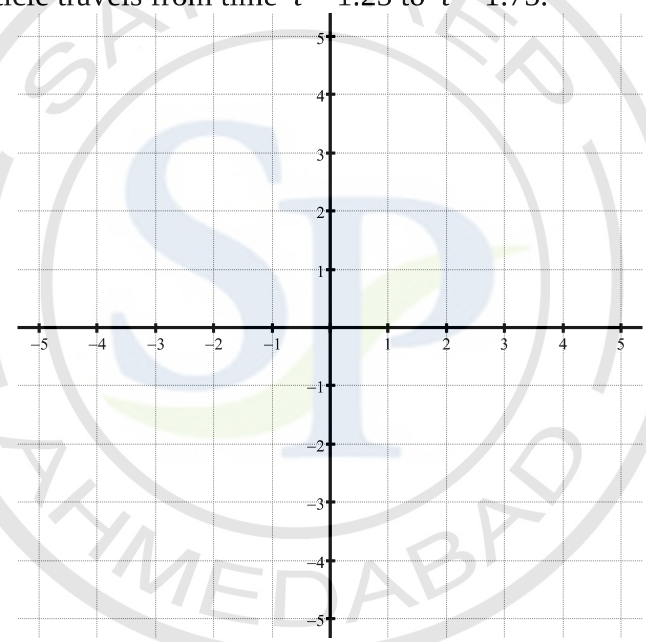
   a) Find the position of the particle when $t = \frac{5}{2}$.

   b) On the axes provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along its path.

   c) How many times does the particle pass through the point found in part (a)?

   d) Find the velocity vector for the particle at any time t.

   e) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time $t = 1.25$ to $t = 1.75$.



1998 CALCULUS BC *(a graphing calculator maybe used)*

2. A particle moves along the curve defined by the equation $y = x^3 - 3x$. The x-coordinate of the particle, $x(t)$, satisfies the equation $\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$, for $t \geq 0$ with initial condition $x(0) = -4$.

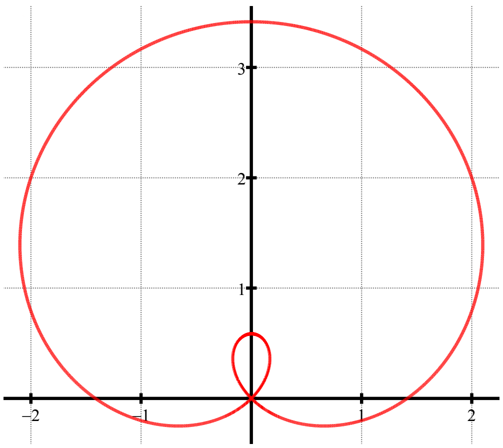
   a) Find $x(t)$ in term of t.

   b) Find $\frac{dy}{dt}$ in terms of t.

   c) Find the location and speed of the particle at time $t = 4$.

SEE OTHER SIDE

3. The figure to the left shows the graph of $r = \sqrt{2} + 2\sin\theta$ for $0 \le \theta \le 2\pi$.



a) For $0 < \theta < \frac{\pi}{2}$, $\frac{dr}{d\theta}$ is positive. What does this fact say about r? What does this fact say about the curve?

b) For $0 \le \theta \le 2\pi$, find the angle(s) θ that corresponds to the point on the curve with y-coordinate 3.

c) Set up an integral to find the area inside the inner loop for the curve $r = \sqrt{2} + 2\sin\theta$. Use your calculator to evaluate your integral.

d) Set up an integral to find the arc length of the inner loop. Use your calculator to evaluate your integral.

e) Set up an integral to find the area inside the outer loop for the curve $r = \sqrt{2} + 2\sin\theta$. Use your calculator to evaluate your integral.

f) Set up an integral to find the arc length of the outer loop. Use your calculator to evaluate your integral.

g) What is the area between the two loops?

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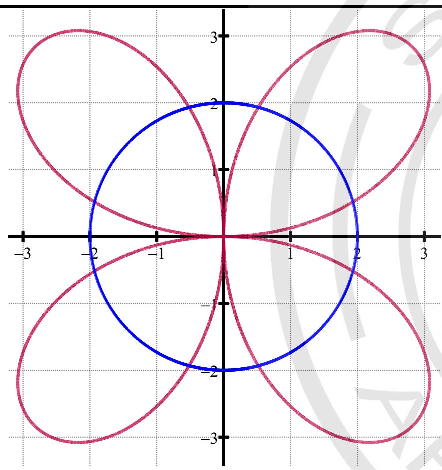
4. The figure to the left shows the graphs of $r = 4\sin 2\theta$ and $r = 2$ for $0 \le \theta \le 2\pi$.

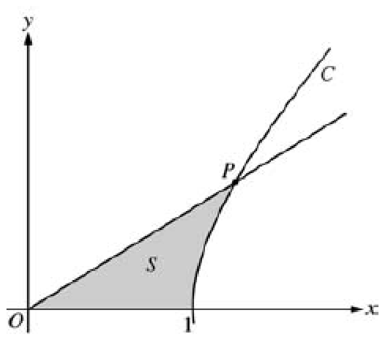


a) Set up two or more integrals to find the area common to both curves in the first quadrant. Use your calculator to evaluate the integrals and find such area.

b) Find the total area common to both curves for $0 \le \theta \le 2\pi$.

c) Set up two or more integrals to find the perimeter of the region common to both curves in the first quadrant. Use your calculator to evaluate the integrals and find the perimeter.

2003 CALCULUS BC *(a graphing calculator maybe used)*



5. The figure to the left shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

b) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

c) Use the polar equation given in part (b) to set up and integral expression with respect to the polar angle θ that represents the area of S.

Answer

1997 CALCULUS BC

1.

a) $x\left(\frac{5}{2}\right)=0$; $y\left(\frac{5}{2}\right)=5 \Rightarrow (0,5)$ ***(1 point)***

b) ***(2 points: sketch, direction)***

c) 3 times ***(1 point)***

d) $\vec{v}=\langle -3\pi\sin(\pi t),\ 5\pi\cos(\pi t)\rangle$ ***(2 points)***

e) $s=\int_{1.25}^{1.75}\sqrt{(-3\pi\sin(\pi t))^2+(5\pi\cos(\pi t))^2}\,dt \approx 5.392$ ***(3 points: limits, integrand, answer)***

1998 CALCULUS BC

2.

a) $x(t)=\int \frac{1}{\sqrt{2t+1}}\,dt=\sqrt{2t+1}+C$ (using u-substitution.)

(2 points: set up, antiderivative with +C)

Since $x(0)=-4 \Rightarrow \sqrt{1}+C=-4 \Rightarrow C=-5$.

So $x(t)=\sqrt{2t+1}-5$ ***(1 point)***

b) *Method #1*: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=\left(3x^2-3\right)\cdot\frac{1}{\sqrt{2t+1}}$

Since $x(t)=\sqrt{2t+1}-5 \Rightarrow \frac{dy}{dt}=\frac{3\cdot\left(\sqrt{2t+1}-5\right)^2-3}{\sqrt{2t+1}}$ ***(2 points)***

Method #2: Implicit differentiation: $\frac{dy}{dt}=3x^2\cdot\frac{dx}{dt}-3\cdot\frac{dx}{dt}$

Since $\left.\begin{array}{l} x(t)=\sqrt{2t+1}-5 \\ \frac{dx}{dt}=\frac{1}{\sqrt{2t+1}} \end{array}\right\} \Rightarrow \frac{dy}{dt}=\frac{3\cdot\left(\sqrt{2t+1}-5\right)^2-3}{\sqrt{2t+1}}$

Method #3: Since $x(t)=\sqrt{2t+1}-5 \Rightarrow y=\left(\sqrt{2t+1}-5\right)^3-3\left(\sqrt{2t+1}-5\right)$

Taking derivatives: $\frac{dy}{dt}=3\cdot\left(\sqrt{2t+1}-5\right)^2\cdot\frac{1}{\sqrt{2t+1}}-3\cdot\frac{1}{\sqrt{2t+1}}=\frac{3\cdot\left(\sqrt{2t+1}-5\right)^2-3}{\sqrt{2t+1}}$

c) $x(4)=\sqrt{2(4)+1}-5=-2$

So: $y=(-2)^3-3(-2)=-2 \Rightarrow$ At $t=4$: $(-2,-2)$ ***(1 point)***

At $t=4$: $\frac{dx}{dt}=\frac{1}{3}$ and $\frac{dy}{dt}=\frac{3\cdot(3-5)^2-3}{3}=3$ ***(1 point)***

So: $speed=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{\left(\frac{1}{3}\right)^2+3^2}=\frac{\sqrt{82}}{3}\approx 3.018$

(2 points: speed formula, answer)

3. $r=\sqrt{2}+2\sin\theta=0 \Rightarrow \theta=\frac{5\pi}{4};\frac{7\pi}{4}$

a) Since $\frac{dr}{d\theta}>0$, r is increasing. This means that the curve is getting away from the origin.

b) $y=r\sin\theta \Rightarrow 3=\left(\sqrt{2}+2\sin\theta\right)\cdot\sin\theta$. Using the calculator: $\theta\approx 1.171$ or 1.970

c) $\frac{1}{2}\int_{5\pi/4}^{7\pi/4}\left(\sqrt{2}+2\sin\theta\right)^2 d\theta \approx 0.142$

d) $\frac{dr}{d\theta}=2\cos\theta \Rightarrow \int_{5\pi/4}^{7\pi/4}\sqrt{\left(\sqrt{2}+2\sin\theta\right)^2+\left(2\cos\theta\right)^2}\,d\theta \approx 1.440$

e) $\frac{1}{2}\int_{7\pi/4}^{13\pi/4}\left(\sqrt{2}+2\sin\theta\right)^2 d\theta \approx 12.425$ or $\frac{1}{2}\int_{0}^{5\pi/4}\left(\sqrt{2}+2\sin\theta\right)^2 d\theta+\frac{1}{2}\int_{7\pi/4}^{2\pi}\left(\sqrt{2}+2\sin\theta\right)^2 d\theta \approx 12.425$ or

$2\cdot\left[\frac{1}{2}\int_{\pi/2}^{5\pi/4}\left(\sqrt{2}+2\sin\theta\right)^2 d\theta\right]\approx 12.425$

f) $\int_{7\pi/4}^{13\pi/4}\sqrt{\left(\sqrt{2}+2\sin\theta\right)^2+\left(2\cos\theta\right)^2}\,d\theta \approx 12.754$ or

$\int_{0}^{5\pi/4}\sqrt{\left(\sqrt{2}+2\sin\theta\right)^2+\left(2\cos\theta\right)^2}\,d\theta+\int_{7\pi/4}^{2\pi}\sqrt{\left(\sqrt{2}+2\sin\theta\right)^2+\left(2\cos\theta\right)^2}\,d\theta \approx 12.754$ or

$2\cdot\left[\int_{\pi/2}^{5\pi/4}\sqrt{\left(\sqrt{2}+2\sin\theta\right)^2+\left(2\cos\theta\right)^2}\,d\theta\right]\approx 12.754$

g) 12.283

4. $r=4\sin 2\theta=2 \Rightarrow \theta=\frac{\pi}{12};\frac{5\pi}{12}$

a) $\frac{1}{2}\int_{0}^{\pi/12}\left(4\sin 2\theta\right)^2 d\theta+\frac{1}{2}\int_{\pi/12}^{5\pi/12}\left(2\right)^2 d\theta+\frac{1}{2}\int_{5\pi/12}^{\pi/2}\left(4\sin 2\theta\right)^2 d\theta \approx 2.457$ or

$2\cdot\left[\frac{1}{2}\int_{0}^{\pi/12}\left(4\sin 2\theta\right)^2 d\theta\right]+\frac{1}{2}\int_{\pi/12}^{5\pi/12}\left(2\right)^2 d\theta \approx 2.457$

b) 2.457 × 4 =9.827

c) $\frac{dr}{d\theta}=8\cos 2\theta$ and $\frac{dr}{d\theta}=0$

$\int_{0}^{\pi/12}\sqrt{\left(4\sin 2\theta\right)^2+\left(8\cos 2\theta\right)^2}\,d\theta+\int_{\pi/12}^{5\pi/12}\sqrt{\left(2\right)^2+\left(0\right)^2}\,d\theta+\int_{5\pi/12}^{\pi/2}\sqrt{\left(4\sin 2\theta\right)^2+\left(8\cos 2\theta\right)^2}\,d\theta \approx 6.143$ or

$2\cdot\left[\int_{0}^{\pi/12}\sqrt{\left(4\sin 2\theta\right)^2+\left(8\cos 2\theta\right)^2}\,d\theta\right]+\int_{\pi/12}^{5\pi/12}\sqrt{\left(2\right)^2+\left(0\right)^2}\,d\theta \approx 6.143$

5. a)

At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$.

Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}$. At P, $\frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$2 : \begin{cases} 1 : \text{coordinates of } P \\ 1 : \dfrac{dx}{dy} \text{ at } P \end{cases}$$

b)

$x = r\cos\theta$; $y = r\sin\theta$

$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$

$$2 : \begin{cases} 1 : \text{substitutes } x = r\cos\theta \text{ and} \\ \qquad y = r\sin\theta \text{ into } x^2 - y^2 = 1 \\ 1 : \text{isolates } r^2 \end{cases}$$

c)

Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$\text{Area} = \int_0^{\tan^{-1}(3/5)} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta} \, d\theta$$

$$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand and constant} \end{cases}$$