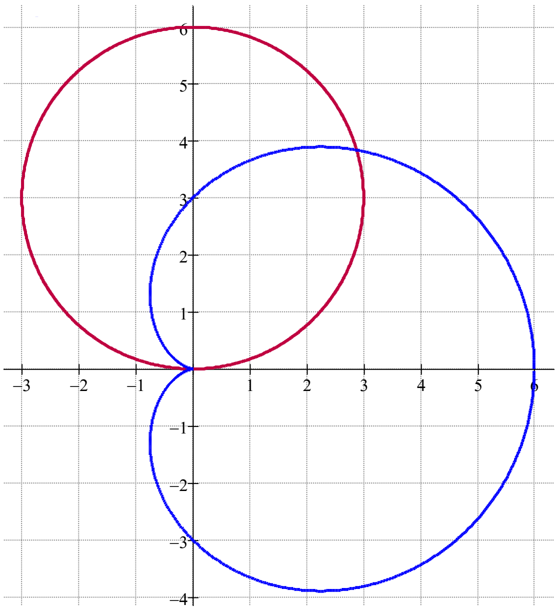
SATPREP

Name________________________________

Date ________________

Assignment : Polar Equation



1. The figure to the left shows the graphs of $r = 6\sin\theta$ and $r = 3 + 3\cos\theta$ for $0 \le \theta \le 2\pi$.

   a) Set up an equation to find the value of θ for the intersection(s) of both graphs. Use your calculator to solve your equation and find the polar coordinates of the point(s) of intersection.

   b) Set up an expression with two or more integrals to find the area common to both curves. Use your calculator to evaluate the integrals and find such area.

   c) Set up an expression with two or more integrals to find the perimeter of the region common to both curves. Use your calculator to evaluate the integrals and find such area.

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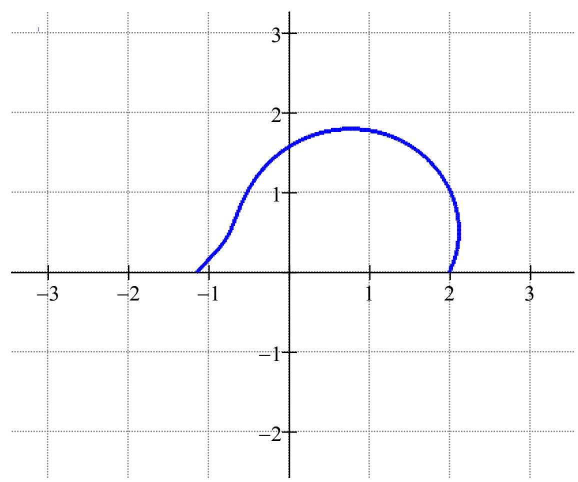
2. The figure to the right shows the graph of $r = \theta + 2\cos\theta$ for $0 \le \theta \le \pi$.



   a) Find the area bounded by the curve and the *x*-axis.

   b) Find the angle(s) θ that corresponds to the point(s) on the curve with y-coordinate 1.

   c) Find an expression for $\dfrac{dr}{d\theta}$. Evaluate your expression for $\dfrac{dr}{d\theta}$ at $\theta = \dfrac{\pi}{3}$. Write a sentence interpreting your result.

   d) Find the value of θ for $0 \le \theta \le \dfrac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance to the origin. Justify your answer.

   e) Find the slope of the curve at the point where $\theta = \dfrac{\pi}{3}$. Show all your work.

SEE OTHER SIDE

3. A particle moves in the *xy*-plane with position vector $\langle x(t), y(t)\rangle$ such that $x(t) = t^3 - 6t^2 + 9t + 1$ and $y(t) = -t^2 + 6t + 2$ in the time interval $0 \le t \le 5$.

a) Find the velocity vector of the particle at $t = 5$.

b) Is the particle moving to the left or to the right when $t = 5$? Is the particle moving up or down when $t = 5$? Justify each answer.

c) Find the equation of the tangent line to the path of the particle when $t = 5$.

d) At what time is the particle at rest? Justify your answer.

e) Find the acceleration vector at the time when the particle is at rest.

f) How fast is the particle moving when $t = 5$?

g) Is the speed of the particle increasing or decreasing when $t = 5$? Justify your answer.

h) Find the total distance traveled by the particle for the time interval $0 \le t \le 5$.

i) Find the average speed of the particle for the time interval $0 \le t \le 5$.

j) If the path followed by the particle was graphed, would the graph be concave up or down at $t = 5$? (*Note*: use your calculator wisely to evaluate derivatives for this question!)

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4. The asymptotes of the graph of the parametric equations $x = \frac{1}{t}$ and $y = \frac{t}{1+t}$ are

(*Hint: rewrite the curve in rectangular coordinates, then find its asymptotes.)*

(A) $x = 0,\ y = 0$
(B) $x = 0$ only
(C) $x = -1,\ y = 0$
(D) $x = -1$ only
(E) $x = 0,\ y = 1$

Answer

1.

a) $3+3\cos\theta = 6\sin\theta \Rightarrow \theta \approx 0.927$ and $\theta = \pi$.
Points of intersection: $(4.8,\ 0.927)$ and $(0,\ \pi)$
(*Note*: store the exact value of $\theta \approx 0.927$ in a memory of your calculator for the rest of the problem.)

b) $\frac{1}{2}\int_0^{0.927}(6\sin\theta)^2 \cdot d\theta + \frac{1}{2}\int_{0.927}^{\pi}(3+3\cos\theta)^2 \cdot d\theta \approx 4.026 + 6.666 \approx 10.692$

c) $\int_0^{0.927}\sqrt{(6\sin\theta)^2+(6\cos\theta)^2}\cdot d\theta + \int_{0.927}^{\pi}\sqrt{(3+3\cos\theta)^2+(-3\sin\theta)^2}\cdot d\theta \approx 5.564 + 6.633 \approx 12.197$

2.

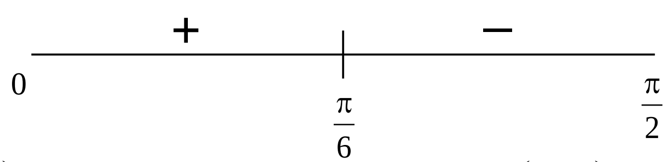
a) $\frac{1}{2}\int_0^{\pi}(\theta+2\cos\theta)^2 \cdot d\theta \approx 4.309$

b) $y = r\cdot\sin\theta = 1 \Rightarrow (\theta+2\cos\theta)\cdot\sin\theta = 1 \Rightarrow \theta \approx 0.460$ and $\theta \approx 2.051$

c) $\frac{dr}{d\theta} = 1-2\sin\theta \Rightarrow$ At $\theta = \frac{\pi}{3} \Rightarrow \frac{dr}{d\theta} = 1-\sqrt{3} \approx -0.732$. This means that as we trace the curve the distance to the pole decreases.

d) Absolute maximum for $r = \theta + 2\cos\theta$ in $0 \le \theta \le \frac{\pi}{2}$: $\frac{dr}{d\theta} = 1-2\sin\theta = 0 \Rightarrow \theta = \frac{\pi}{6} \approx 0.524$.

Making a sign analysis in $0 \le \theta \le \frac{\pi}{2}$:



r increases always in the interval $\left(0, \frac{\pi}{6}\right)$ and decreases always in the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, so the absolute maximum in $0 \le \theta \le \frac{\pi}{2}$ is at $\theta = \frac{\pi}{6} \approx 0.524$

e) $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\cdot\sin\theta + r\cdot\cos\theta}{\frac{dr}{d\theta}\cdot\cos\theta - r\cdot\sin\theta} = \frac{(1-2\sin\theta)\cdot\sin\theta + (\theta+2\cos\theta)\cdot\cos\theta}{(1-2\sin\theta)\cdot\cos\theta - (\theta+2\cos\theta)\cdot\sin\theta}$$

At $\theta = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} \approx -0.182$
(*Note*: the answer can also be found using the "CALC" menu of your graphing calculator.)

3\.

a) $\vec{v}(t) = \langle 3t^2 - 12t + 9, -2t + 6 \rangle \Rightarrow \vec{v}(5) = \langle 24, -4 \rangle$

b) The particle moves to the right and down when $t = 5$, because $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$.

c) At $t = 5 \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4}{24} = -\frac{1}{6}$ and since $x(5) = 21$ and $y(5) = 7 \Rightarrow y - 7 = -\frac{1}{6}(x - 21)$

d) $\frac{dx}{dt} = 3t^2 - 12t + 9 = 0 \Rightarrow t = 1$ or 3

$\frac{dy}{dx} = -2t + 6 = 0 \Rightarrow t = 3$

The particle is at rest at $t = 3$, when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

e) $\vec{a}(t) = \langle 6t - 12, -2 \rangle \Rightarrow \vec{a}(3) = \langle 6, -2 \rangle$

f) Speed = $\sqrt{24^2 + (-4)^2} = \sqrt{592} \approx 24.331$

g) Using the calculator: $\frac{d}{dt}\left[\sqrt{(3t^2 - 12t + 9)^2 + (-2t + 6)^2}\right]\Bigg|_{t=5} \approx 18.083 > 0$. The speed is increasing because its derivative is positive.

h) $\int_0^5 \sqrt{(3t^2 - 12t + 9)^2 + (-2t + 6)^2} \cdot dt \approx 33.043$

i) $\frac{1}{5-0}\int_0^5 \sqrt{(3t^2 - 12t + 9)^2 + (-2t + 6)^2} \cdot dt \approx 6.609$

j) We need $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{-2t + 6}{3t^2 - 12t + 9}\right]}{3t^2 - 12t + 9}$ at $t = 5$.

Using the calculator: $\frac{d}{dt}\left[\frac{-2t + 6}{3t^2 - 12t + 9}\right]\Bigg|_{t=5} \approx 0.042$

And therefore: $\left.\frac{d^2y}{dx^2}\right|_{t=5} \approx 0.002 > 0 \Rightarrow$ the graph is concave up.

4. (C)

Rewriting in Cartesian coordinates: $t = \frac{1}{x} \Rightarrow y = \frac{\frac{1}{x}}{1 + \frac{1}{x}}$ or what is the same $y = \frac{1}{x+1}$. This graph has a vertical asymptote at $x = -1$ because $y(-1) = \frac{1}{0}$. The graph also has a horizontal asymptote at $y = 0$, because $\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1}{x+1} = 0$