SATPREP

Assignment: Applications of Differentiation

1. The function $f(x)$ is given by $f(x) = x^3 - 3x^2 + 3x$, for $-1 \leq x \leq 3$.

(a) Differentiate $f(x)$ with respect to x.

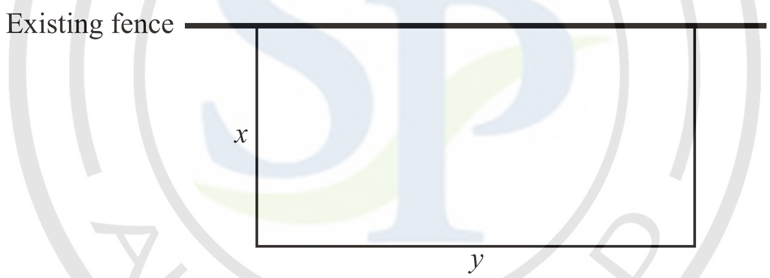
(b) Copy and complete the table below.

x	-1	0	1	2	3
$f(x)$		0	1	2	9
$f'(x)$	12		0		12

(c) Use the information in your table to sketch the graph of $f(x)$.

(d) Write down the gradient of the tangent to the curve at the point (3, 9).

2. A farmer wishes to enclose a rectangular field using an existing fence for one of the four sides.



(a) Write an expression in terms of x and y that shows the total length of the new fence.

(b) The farmer has enough materials for 2500 metres of new fence. Show that

$$y = 2500 - 2x$$

(c) $A(x)$ represents the area of the field in terms of x.

(i) Show that

$$A(x) = 2500x - 2x^2$$

(ii) Find $A'(x)$.

(iii) Hence or otherwise find the value of x that produces the maximum area of the field.

(iv) Find the maximum area of the field.

3. The function g is defined as follows

$$g : x \mapsto px^2 + qx + c, \qquad p, q, c \in \mathbb{R}$$

(a) Find $g'(x)$

(b) If $g'(x) = 2x + 6$, find the values of p and q.

(c) $g(x)$ has a minimum value of -12 at the point A. Find

(i) the x-coordinate of A;

(ii) the value of c.

4. The function $f(x)$ is given by the formula

$$f(x) = 2x^3 - 5x^2 + 7x - 1$$

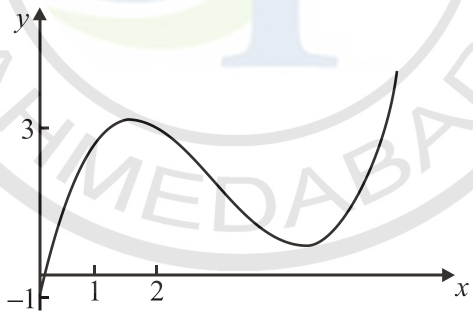
(a) Evaluate $f(1)$.

(b) Calculate $f'(x)$.

(c) Evaluate $f'(2)$.

(d) State whether the function $f(x)$ is increasing or decreasing at $x = 2$.

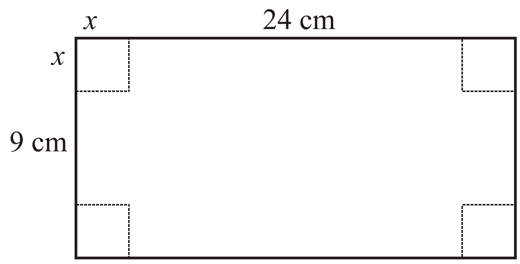
(e) The sketch graph shown below is the graph of a cubic function.



(i) Is it possible that this is the graph of the function $f(x)$ above?

(ii) State one reason for your decision.

5. A rectangular piece of card measures 24 cm by 9 cm. Equal squares of length x cm are cut from each corner of the card as shown in the diagram below. What is left is then folded to make an **open** box, of length l cm and width w cm.



(a) Write expressions, in terms of x, for

(i) the length, l;

(ii) the width, w.

(b) Show that the volume (B m^3) of the box is given by $B = 4x^3 - 66x^2 + 216x$.

(c) Find $\frac{dB}{dx}$.

(d) (i) Find the value of x which gives the maximum volume of the box.

(ii) Calculate the maximum volume of the box.

6. A closed box has a square base of side x and height h.

(a) Write down an expression for the volume, V, of the box.

(b) Write down an expression for the total surface area, A, of the box.

The volume of the box is 1000 cm^3

(c) Express h in terms of x.

(d) Hence show that $A = 4000x^{-1} + 2x^2$.

(e) Find $\frac{dA}{dx}$.

(f) Calculate the value of x that gives a minimum surface area.

(g) Find the surface area for this value of x.

7. Consider the function $f(x) = 2x^3 - 3x^2 - 12x + 5$.

(a) (i) Find $f'(x)$.

(ii) Find the gradient of the curve $f(x)$ when $x = 3$.

(b) Find the x-coordinates of the points on the curve where the gradient is equal to -12.

(c) (i) Calculate the x-coordinates of the local maximum and minimum points.

(ii) Hence find the coordinates of the local minimum.

(d) For what values of x is the value of $f(x)$ increasing?

8. The cost of producing a mathematics textbook is \$ 15 (US dollars) and it is then sold for \$ x.

(a) Find an expression for the profit made on each book sold.

A total of $(100\,000 - 4000x)$ books is sold.

(b) Show that the profit made on all the books sold is

$$P = 160\,000x - 4000x^2 - 1500\,000.$$

(c) (i) Find $\frac{\mathrm{d}P}{\mathrm{d}x}$.

(ii) Hence calculate the value of x to make a maximum profit

(d) Calculate the number of books sold to make this maximum profit.

9. A function $g(x) = x^3 + 6x^2 + 12x + 18$

(a) Find $g'(x)$.

(b) Solve $g'(x) = 0$.

(c) (i) Calculate the values of $g'(x)$ when

(a) $x = -3$;

(b) $x = 0$.

(ii) Hence state whether the function is increasing or decreasing at

(a) $x = -3$;

(b) $x = 0$.

10. A function is given as $y = ax^2 + bx + 6$.

(a) Find $\frac{dy}{dx}$.

(b) If the gradient of this function is 2 when x is 6 write an equation in terms of a and b.

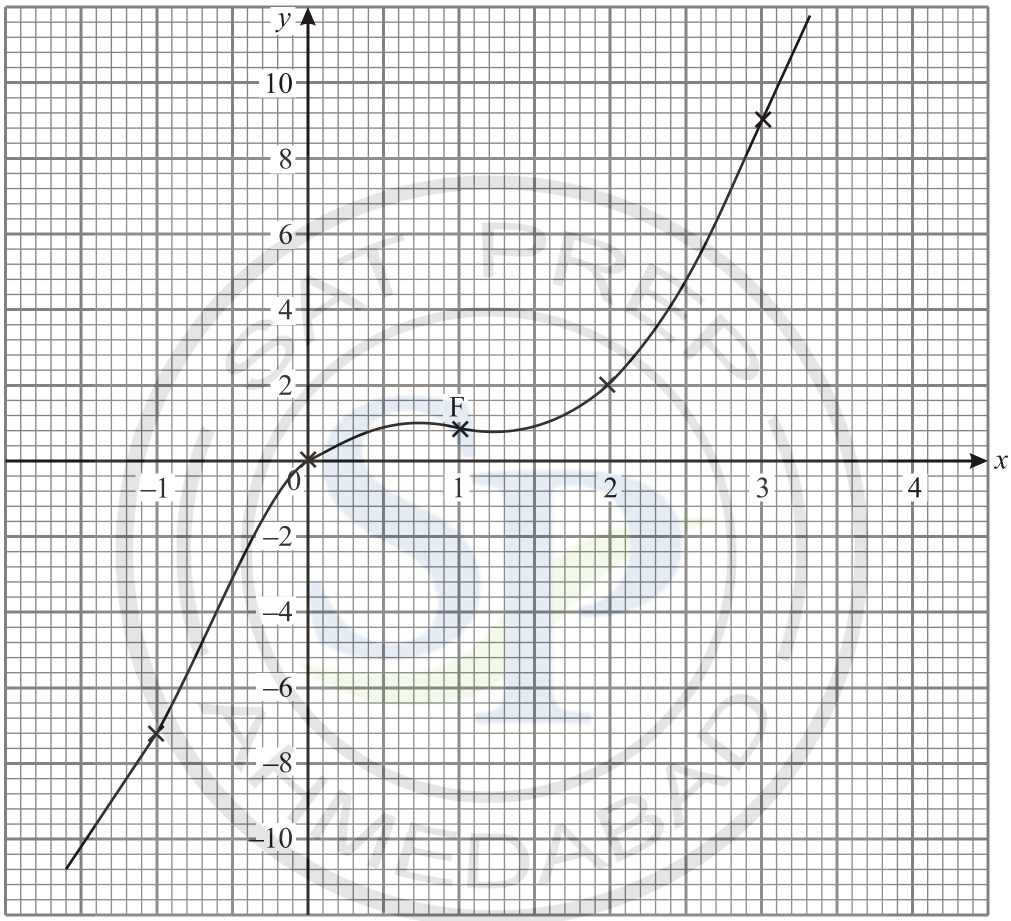
(c) If the point (3, –15) lies on the graph of the function find a second equation in terms of a and b.

1. (a) $f'(x) = 3x^2 - 6x + 3$

(b)

x	-1	0	1	2	3
$f(x)$	-7	0	1	2	9
$f'(x)$	12	**3**	0	**3**	12

(c)



(d) 12

2. (a) $2x + y$

(b) $2500 = 2x + y$
$2500 - 2x = y$

(c) (i) Area $A(x) = xy$

$= x(2500 - 2x)$

$= 2500x - 2x^2$

(ii) $A'(x) = 2500 - 4x$

(iii) $A'(x) = 0$

$0 = 2500 - 4x$

$4x = 2500$

$x = 625$

(iv) $A(x) = 2500x - 2x^2$

$A(625) = 2500 \times 625 - 2(625)^2$

$= 781250$

$= 781000 \text{ m}^2$

3. (a) $g'(x) = 2px + q$

(b) $2px + q = 2x + 6$

(c) (i) $g'(x) = 0$

$\Rightarrow 2x + 6 = 0$

$\Rightarrow x = -3$

(ii) $-12 = (-3)^2 + 6(-3) + c$

$-12 = 9 - 18 + c$

$\Rightarrow c = -3$

4. (a) Substitute $x = 1$ into $f(x)$, $f(1) = 3$.

(b) $f'(x) = 6x^2 - 10x + 7$

(c) Substitute $x = 2$ into (b) $f'(2) = 11$.

(d) Increasing.

(e) (i) No.

(ii) Because the gradient at $x = 2$ is wrong (or wrong sign) or **any other valid reason** (*e.g.* $f(x)$ has an inflection not a max/min), (but note that $f(1)$ and $f(0)$ both agree, and both the formula and the graph have a single real root near to 0, so none of these are valid reasons).

A sketch of the graph from the GDC with no detailed reason can be awarded ***(G1)*** if it is reasonable.

5. (a) (i) $l = 24 - 2x$

(ii) $w = 9 - 2x$

(b) $B = x(24 - 2x)(9 - 2x)$
$= 4x^3 - 66x^2 + 216x$

(c) $\frac{dB}{dx} = 12x^2 - 132x + 216$

(d) (i) $\frac{dB}{dx} = 0 \Rightarrow x^2 - 11x + 18 = 0$
$(x - 2)(x - 9) = 0$
$\Rightarrow x = 2$ or $x = 9$ (not possible)
Therefore, $x = 2$ cm.

(ii) $B = 4(2)^3 - 66(2)^2 + 216(2)$ (or $2 \times 20 \times 5$)
$= 200 \text{ cm}^3$

6. (a) $V = x^2h$

(b) $A = 2x^2 + 4xh$

(c) $1000 = x^2h$
$h = \frac{1000}{x^2}$

(d) $A = 2x^2 + 4x\left(\frac{1000}{x^2}\right)$
$A = 2x^2 + \frac{4000}{x}$
$= 2x^2 + 4000x^{-1}$

(e) $\frac{dA}{dx} = 4x - 4000x^{-2}$

(f) $4x - 4000x^{-2} = 0$
$4x^3 - 4000 = 0$
$4x^3 = 4000$
$x^3 = 1000$
$x = 10$

OR

$x = 10$

(g) $h = \frac{1000}{100} = 10$

$A = 2(100) + 4(10)(10)$
$= 200 + 400 = 600$

OR

$A = 600$

7. (a) (i) $f'(x) = 6x^2 - 6x - 12\ (+0) = 6x^2 - 6x - 12$

(ii) $f'(3) = 6(3)^2 - 6(3) - 12 = 24$

(b) $6x^2 - 6x\ 12 = -12$
$\Rightarrow 6x^2 - 6x = 0$
$\Rightarrow 6x\,(x - 1) = 0$
$\Rightarrow x = 0$ or $x = 1$

(c) (i) $f'(x) = 0 \Rightarrow 6x - 12 = 0$
$\Rightarrow 6\,(x^2 - x - 2) = 0$
$\Rightarrow 6(x - 2)\,(x + 1) = 0$
$\Rightarrow x = 2$ or $x = -1$

(ii) $x = 2,\ y = -15$
Therefore, minimum is $(2, -15)$

(d) $x < -1$ and $x > 2$

8. (a) $x - 15$

(b) Profit $= (x - 15)\,(100\,000 - 4000x)$
$= 100000x - 4000x^2 - 1500\,000 + 60\,000x$

$= 160\,000x - 4000x^2 - 1500\,000$

(c) (i) $\frac{\mathrm{d}P}{\mathrm{d}x} = 160000 - 8000x$

(ii) $0 = 160000 - 8000x$
$x = \frac{160000}{8000}$
$x = 20$

(d) Books sold $= 100\,000 - 4000 \times 20$
$= 20000$

9. (a) $g'(x) = 3x^2 + 12x + 12$

(b) $3x^2 + 12x + 12 = 0$
$x^2 + 4x + 4 = 0$
$(x + 2)^2 = 0$
$x = -2$

(c) (i) $x = -3 \Rightarrow \frac{dy}{dx} = 3$

(ii) $x = 0 \Rightarrow \frac{dy}{dx} = 12$

(iii) (a) Increasing

(b) Increasing

10. (a) $y = ax^2 + bx + 6$
$\frac{dy}{dx} = 2ax + b$

(b) Gradient = 2 when $x = 6$.
Therefore, $2 = 2a \times 6 + b$
$2 = 12= + b$

(c) $y = -15$ when $x = 3$.
Therefore, $-15 = 9a + 3b + 6$
or $-21 = 9a + 3b$ or $-7 = 3a + b$