Subject – Math(Higher Level)
Topic - Algebra
Year - Nov 2011 – Nov 2019

Question -1

[Maximum mark: 6]

Find the cube roots of i in the form $a+b\text{i}$, where $a, b \in \mathbb{R}$.

Question -2

[Maximum mark: 7]

Given that $y=\frac{1}{1-x}$, use mathematical induction to prove that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}=\frac{n!}{(1-x)^{n+1}}$, $n \in \mathbb{Z}^+$.

Question -3

[Maximum mark: 4]

Find the value of k if $\sum_{r=1}^{\infty} k\left(\frac{1}{3}\right)^r = 7$.

Question -4

[Maximum mark: 7]

If $z_1 = a + a\sqrt{3}\,\text{i}$ and $z_2 = 1-\text{i}$, where a is a real constant, express z_1 and z_2 in the form $r\,\text{cis}\,\theta$, and hence find an expression for $\left(\frac{z_1}{z_2}\right)^6$ in terms of a and i.

Question -5

[Maximum mark: 6]

Given that z is the complex number $x+\text{i}y$ and that $|z|+z=6-2\text{i}$, find the value of x and the value of y.

Question -6

[Maximum mark: 5]

Solve the equation $2-\log_3(x+7)=\log_{\frac{1}{3}} 2x$.

Question -7

[Maximum mark: 5]

(a) Expand and simplify $\left(x-\frac{2}{x}\right)^4$. *[3 marks]*

(b) Hence determine the constant term in the expansion $(2x^2+1)\left(x-\frac{2}{x}\right)^4$. *[2 marks]*

Question – 8

[Maximum mark: 7]

Given that $(4-5\mathrm{i})m+4n=16+15\mathrm{i}$, where $\mathrm{i}^2=-1$, find m and n if

(a) m and n are real numbers; *[3 marks]*

(b) m and n are conjugate complex numbers. *[4 marks]*

Question -9

[Total mark: 29]

Part A *[Maximum mark: 12]*

(a) Given that $(x+\mathrm{i}y)^2=-5+12\mathrm{i}$, x, $y\in\mathbb{R}$. Show that

(i) $x^2-y^2=-5$;

(ii) $xy=6$. *[2 marks]*

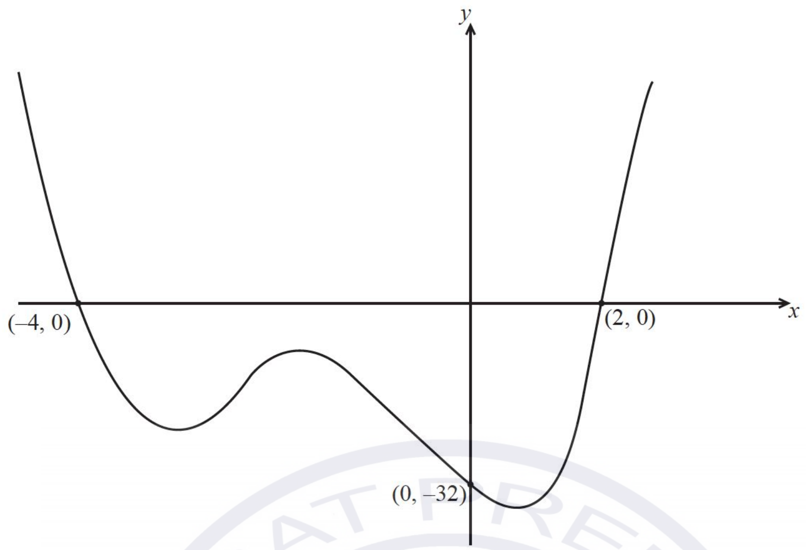
(b) Hence find the two square roots of $-5+12\mathrm{i}$. *[5 marks]*

(c) For any complex number z, show that $(z^*)^2=(z^2)^*$. *[3 marks]*

(d) Hence write down the two square roots of $-5-12\mathrm{i}$. *[2 marks]*

Part B *[Maximum mark: 17]*

The graph of a polynomial function f of degree 4 is shown below.



(a) Explain why, of the four roots of the equation $f(x) = 0$, two are real and two are complex. *[2 marks]*

(b) The curve passes through the point $(-1, -18)$. Find $f(x)$ in the form $f(x) = (x-a)(x-b)(x^2+cx+d)$, where $a, b, c, d \in \mathbb{Z}$. *[5 marks]*

(c) Find the two complex roots of the equation $f(x) = 0$ in Cartesian form. *[2 marks]*

(d) Draw the four roots on the complex plane (the Argand diagram). *[2 marks]*

(e) Express each of the four roots of the equation in the form $re^{i\theta}$. *[6 marks]*

Question -10

[Maximum mark: 4]

Expand and simplify $\left(\frac{x}{y} - \frac{y}{x}\right)^4$.

Question -11

[Maximum mark: 19]

Consider the complex numbers

$$z_1 = 2\sqrt{3}\,\text{cis}\,\frac{3\pi}{2} \text{ and } z_2 = -1 + \sqrt{3}\text{i}.$$

(a) (i) Write down z_1 in Cartesian form.

(ii) Hence determine $(z_1 + z_2)^*$ in Cartesian form. *[3 marks]*

(b) (i) Write z_2 in modulus-argument form.

(ii) Hence solve the equation $z^3 = z_2$. *[6 marks]*

(c) Let $z = r\,\text{cis}\,\theta$, where $r \in \mathbb{R}^+$ and $0 \le \theta < 2\pi$. Find all possible values of r and θ,

(i) if $z^2 = (1 + z_2)^2$;

(ii) if $z = -\dfrac{1}{z_2}$. *[6 marks]*

(d) Find the smallest positive value of n for which $\left(\dfrac{z_1}{z_2}\right)^n \in \mathbb{R}^+$. *[4 marks]*

Question -12

[Maximum mark: 6]

(a) If $w = 2 + 2\text{i}$, find the modulus and argument of w. *[2 marks]*

(b) Given $z = \cos\left(\dfrac{5\pi}{6}\right) + \text{i}\sin\left(\dfrac{5\pi}{6}\right)$, find in its simplest form $w^4 z^6$. *[4 marks]*

Question 13

[Maximum mark: 6]

The first terms of an arithmetic sequence are $\dfrac{1}{\log_2 x}, \dfrac{1}{\log_8 x}, \dfrac{1}{\log_{32} x}, \dfrac{1}{\log_{128} x}, \ldots$

Find x if the sum of the first 20 terms of the sequence is equal to 100.

Question 14

[Maximum mark: 4]

Expand $(2-3x)^5$ in ascending powers of x, simplifying coefficients.

Question 15

[Maximum mark: 7]

A geometric sequence has first term a, common ratio r and sum to infinity 76. A second geometric sequence has first term a, common ratio r^3 and sum to infinity 36.

Find r.

Question 16

[Maximum mark: 7]

Given the complex numbers $z_1 = 1+3\text{i}$ and $z_2 = -1-\text{i}$.

(a) Write down the exact values of $|z_1|$ and $\arg(z_2)$. *[2 marks]*

(b) Find the minimum value of $|z_1 + \alpha z_2|$, where $\alpha \in \mathbb{R}$. *[5 marks]*

Question 17

[Maximum mark: 18]

(a) (i) Express each of the complex numbers $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = -2\mathrm{i}$ in modulus-argument form.

(ii) Hence show that the points in the complex plane representing z_1, z_2 and z_3 form the vertices of an equilateral triangle.

(iii) Show that $z_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where $n \in \mathbb{N}$. *[9 marks]*

(b) (i) State the solutions of the equation $z^7 = 1$ for $z \in \mathbb{C}$, giving them in modulus-argument form.

(ii) If w is the solution to $z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of π.

(iii) Show that $z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial $z^7 - 1$. State the two other quadratic factors with real coefficients. *[9 marks]*

Question 18

[Maximum mark: 7]

Prove by mathematical induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

Question 19

[Maximum mark: 7]

The sum of the first two terms of a geometric series is 10 and the sum of the first four terms is 30.

(a) Show that the common ratio r satisfies $r^2 = 2$. *[4]*

(b) Given $r = \sqrt{2}$

(i) find the first term;

(ii) find the sum of the first ten terms. *[3]*

Question 20

[Maximum mark: 8]

Solve the following equations:

(a) $\log_2 (x-2) = \log_4 \left(x^2 - 6x + 12\right)$; *[3]*

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$. *[5]*

Question 21

[Maximum mark: 20]

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$. *[2]*

(b) Expand $\left(z + z^{-1}\right)^4$. *[1]*

(c) Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where p, q and r are constants to be determined. *[4]*

(d) Show that $\cos^6\theta = \dfrac{1}{32}\cos 6\theta + \dfrac{3}{16}\cos 4\theta + \dfrac{15}{32}\cos 2\theta + \dfrac{5}{16}$. *[3]*

(e) Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \, \mathrm{d}\theta$. *[3]*

The region S is bounded by the curve $y = \sin x \cos^2 x$ and the x-axis between $x = 0$ and $x = \dfrac{\pi}{2}$.

(f) S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4]*

(g) (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$.

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, \mathrm{d}\theta$ in terms of k. *[3]*

Question 22

[Maximum mark: 5]

Consider $a = \log_2 3 \times \log_3 4 \times \log_4 5 \times \ldots \times \log_{31} 32$. Given that $a \in \mathbb{Z}$, find the value of a.

Question 23

[Maximum mark: 17]

A geometric sequence $\{u_n\}$, with complex terms, is defined by $u_{n+1} = (1+\mathrm{i})u_n$ and $u_1 = 3$.

(a) Find the fourth term of the sequence, giving your answer in the form $x + y\mathrm{i}$, $x, y \in \mathbb{R}$. *[3]*

(b) Find the sum of the first 20 terms of $\{u_n\}$, giving your answer in the form $a \times (1 + 2^m)$ where $a \in \mathbb{C}$ and $m \in \mathbb{Z}$ are to be determined. *[4]*

A second sequence $\{v_n\}$ is defined by $v_n = u_n u_{n+k}$, $k \in \mathbb{N}$.

(c) (i) Show that $\{v_n\}$ is a geometric sequence.

(ii) State the first term.

(iii) Show that the common ratio is independent of k. *[5]*

A third sequence $\{w_n\}$ is defined by $w_n = |u_n - u_{n+1}|$.

(d) (i) Show that $\{w_n\}$ is a geometric sequence.

(ii) State the geometrical significance of this result with reference to points on the complex plane. *[5]*

Question 24

[Maximum mark: 5]

Solve the equation $8^{x-1} = 6^{3x}$. Express your answer in terms of $\ln 2$ and $\ln 3$.

Question 25

[Maximum mark: 7]

Consider the complex numbers $u = 2 + 3\mathrm{i}$ and $v = 3 + 2\mathrm{i}$.

(a) Given that $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{10}{w}$, express w in the form $a + b\mathrm{i}$, $a, b \in \mathbb{R}$. *[4]*

(b) Find w^* and express it in the form $re^{\mathrm{i}\theta}$. *[3]*

Question 26

[Maximum mark: 7]

The first three terms of a geometric sequence are $\sin x$, $\sin 2x$ and $4\sin x\cos^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio r. *[1]*

(b) Find the set of values of x for which the geometric series $\sin x + \sin 2x + 4\sin x\cos^2 x + \ldots$ converges. *[3]*

Consider $x = \arccos\left(\frac{1}{4}\right)$, $x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$. *[3]*

Question 27

[Maximum mark: 7]

Use mathematical induction to prove that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$.

Question 28

[Maximum mark: 18]

Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 0$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

(a) (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant.

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d and n. [4]

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

(b) (i) Find S_n, in terms of a, d and n.

(ii) Find the values of d for which $\sum_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d.

(iv) Given that $S_\infty = 2^{a+1}$ find the value of d. [8]

Let $\{w_n\}$, $n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$.

(c) Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q. [6]

Question 29

[Maximum mark: 4]

Expand $(3-x)^4$ in ascending powers of x and simplify your answer.

Question 30

[Maximum mark: 9]

(a) Find three distinct roots of the equation $8z^3 + 27 = 0$, $z \in \mathbb{C}$ giving your answers in modulus-argument form. [6]

The roots are represented by the vertices of a triangle in an Argand diagram.

(b) Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$. [3]

Question 31

[Maximum mark: 8]

(a) State the set of values of a for which the function $x \mapsto \log_a x$ exists, for all $x \in \mathbb{R}^+$. [2]

(b) Given that $\log_x y = 4\log_y x$, find all the possible expressions of y as a function of x. [6]

Question 32

[Maximum mark: 13]

(a) Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0$, $n \in \mathbb{Z}$. [2]

(b) Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. [2]

(c) Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2$, $n \in \mathbb{Z}$. [9]

Question 33

[Maximum mark: 6]

The fifth term of an arithmetic sequence is equal to 6 and the sum of the first 12 terms is 45. Find the first term and the common difference.

Question 34

[Maximum mark: 4]

Find integer values of m and n for which

$$m - n\log_3 2 = 10\log_9 6$$

Question 35

[Maximum mark: 21]

(a) Use de Moivre's theorem to find the value of $\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3$. [2]

(b) Use mathematical induction to prove that

$$(\cos\theta - \mathrm{i}\sin\theta)^n = \cos n\theta - \mathrm{i}\sin n\theta \text{ for } n \in \mathbb{Z}^+.$$ [6]

Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(c) Find an expression in terms of θ for $(z)^n + (z^*)^n$, $n \in \mathbb{Z}^+$ where z^* is the complex conjugate of z. [2]

(d) (i) Show that $zz^* = 1$.

(ii) Write down the binomial expansion of $(z + z^*)^3$ in terms of z and z^*.

(iii) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [5]

(e) Hence solve $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \le \theta < \pi$. [6]

Question 36

[Maximum mark: 8]

Consider the expansion of $(1+x)^n$ in ascending powers of x, where $n \geq 3$.

(a) Write down the first four terms of the expansion. [2]

The coefficients of the second, third and fourth terms of the expansion are consecutive terms of an arithmetic sequence.

(b) (i) Show that $n^3 - 9n^2 + 14n = 0$.

(ii) Hence find the value of n. [6]

Question 37

[Maximum mark: 8]

Use mathematical induction to prove that $n\left(n^2+5\right)$ is divisible by 6 for $n \in \mathbb{Z}^+$.

Question 38

[Maximum mark: 23]

Let $w = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$.

(a) Verify that w is a root of the equation $z^7 - 1 = 0,\ z \in \mathbb{C}$. [3]

(b) (i) Expand $(w-1)\left(1+w+w^2+w^3+w^4+w^5+w^6\right)$.

(ii) Hence deduce that $1+w+w^2+w^3+w^4+w^5+w^6 = 0$. [3]

(c) Write down the roots of the equation $z^7 - 1 = 0,\ z \in \mathbb{C}$ in terms of w and plot these roots on an Argand diagram. [3]

Consider the quadratic equation $z^2 + bz + c = 0$ where $b, c \in \mathbb{R}, z \in \mathbb{C}$. The roots of this equation are α and α^* where α^* is the complex conjugate of α.

(d) (i) Given that $\alpha = w + w^2 + w^4$, show that $\alpha^* = w^6 + w^5 + w^3$.

(ii) Find the value of b and the value of c. [10]

(e) Using the values for b and c obtained in part (d)(ii), find the imaginary part of α, giving your answer in surd form. [4]

Question 39

[Maximum mark: 7]

The sum of the first n terms of a sequence $\{u_n\}$ is given by $S_n = 3n^2 - 2n$, where $n \in \mathbb{Z}^+$.

(a) Write down the value of u_1. [1]

(b) Find the value of u_6. [2]

(c) Prove that $\{u_n\}$ is an arithmetic sequence, stating clearly its common difference. [4]

Question 40

[Maximum mark: 5]

Solve the equation $4^x + 2^{x+2} = 3$.

Question 41

[Maximum mark: 19]

Let ω be one of the non-real solutions of the equation $z^3 = 1$.

(a) Determine the value of

(i) $1 + \omega + \omega^2$;

(ii) $1 + \omega^* + (\omega^*)^2$. [4]

(b) Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$. [4]

Consider the complex numbers $p = 1 - 3\mathrm{i}$ and $q = x + (2x + 1)\mathrm{i}$, where $x \in \mathbb{R}$.

(c) Find the values of x that satisfy the equation $|p| = |q|$. [5]

(d) Solve the inequality $\mathrm{Re}(pq) + 8 < (\mathrm{Im}(pq))^2$. [6]

Question 42

[Maximum mark: 4]

Find the solution of $\log_2 x - \log_2 5 = 2 + \log_2 3$.

Question 43

[Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + \sqrt{3}\,\mathrm{i}$, $z_2 = 1 + \mathrm{i}$ and $w = \frac{z_1}{z_2}$.

(a) By expressing z_1 and z_2 in modulus-argument form write down

(i) the modulus of w;

(ii) the argument of w. [4]

(b) Find the smallest positive integer value of n, such that w^n is a real number. [2]

Question 44

[Maximum mark: 7]

An arithmetic sequence $u_1, u_2, u_3 \ldots$ has $u_1 = 1$ and common difference $d \neq 0$. Given that u_2, u_3 and u_6 are the first three terms of a geometric sequence

(a) find the value of d. [4]

Given that $u_N = -15$

(b) determine the value of $\sum_{r=1}^{N} u_r$. [3]

Question 45

[Maximum mark: 6]

Use the method of mathematical induction to prove that $4^n + 15n - 1$ is divisible by 9 for $n \in \mathbb{Z}^+$.

Question 46

[Maximum mark: 5]

Find the term independent of x in the binomial expansion of $\left(2x^2 + \frac{1}{2x^3}\right)^{10}$.

Question 47

[Maximum mark: 5]

The 1st, 4th and 8th terms of an arithmetic sequence, with common difference d, $d \neq 0$, are the first three terms of a geometric sequence, with common ratio r. Given that the 1st term of both sequences is 9 find

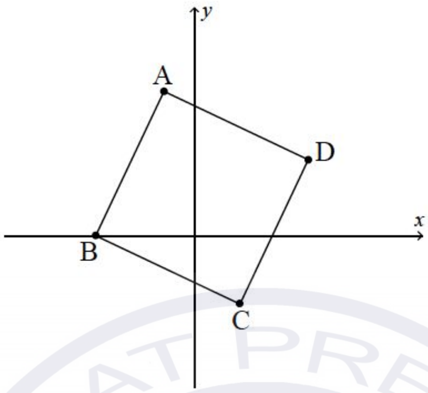
(a) the value of d; [4]

(b) the value of r. [1]

Question 48

[Maximum mark: 4]

In the following Argand diagram the point A represents the complex number $-1 + 4\text{i}$ and the point B represents the complex number $-3 + 0\text{i}$. The shape of ABCD is a square. Determine the complex numbers represented by the points C and D.



Question 49

[Maximum mark: 9]

Prove by mathematical induction that $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}$, where $n \in \mathbb{Z}, n \geq 3$.

Question 50

(c) Let $z = 1 - \cos 2\theta - \text{i} \sin 2\theta, z \in \mathbb{C}, 0 \leq \theta \leq \pi$.

(i) Find the modulus and argument of z in terms of θ. Express each answer in its simplest form.

(ii) Hence find the cube roots of z in modulus-argument form. [14]

Question 51

[Maximum mark: 5]

Solve the equation $\log_2(x + 3) + \log_2(x - 3) = 4$.

Question 52

[Maximum mark: 4]

Find the coefficient of x^8 in the expansion of $\left(x^2 - \frac{2}{x}\right)^7$.

Question 53

[Maximum mark: 7]

Determine the roots of the equation $(z+2\mathrm{i})^3 = 216\mathrm{i}$, $z \in \mathbb{C}$, giving the answers in the form $z = a\sqrt{3} + b\mathrm{i}$ where a, $b \in \mathbb{Z}$.

Question 54

[Maximum mark: 7]

(a) Show that $\log_{r^2} x = \frac{1}{2}\log_r x$ where $r, x \in \mathbb{R}^+$. [2]

It is given that $\log_2 y + \log_4 x + \log_4 2x = 0$.

(b) Express y in terms of x. Give your answer in the form $y = px^q$, where p, q are constants. [5]

Question 55

[Maximum mark: 6]

Consider the distinct complex numbers $z = a + \mathrm{i}b$, $w = c + \mathrm{i}d$, where a, b, c, $d \in \mathbb{R}$.

(a) Find the real part of $\frac{z+w}{z-w}$. [4]

(b) Find the value of the real part of $\frac{z+w}{z-w}$ when $|z| = |w|$. [2]

Question 56

[Maximum mark: 7]

The geometric sequence $u_1, u_2, u_3, \ldots$ has common ratio r.
Consider the sequence $A = \{a_n = \log_2 |u_n| : n \in \mathbb{Z}^+\}$.

(a) Show that A is an arithmetic sequence, stating its common difference d in terms of r. [4]

A particular geometric sequence has $u_1 = 3$ and a sum to infinity of 4.

(b) Find the value of d. [3]

Question 57

[Maximum mark: 14]

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$.

(a) (i) Express w^2 and w^3 in modulus-argument form.

(ii) Sketch on an Argand diagram the points represented by w^0, w^1, w^2 and w^3. [5]

These four points form the vertices of a quadrilateral, Q.

(b) Show that the area of the quadrilateral Q is $\frac{21\sqrt{3}}{2}$. [3]

Let $z = 2\left(\cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}\right)$, $n \in \mathbb{Z}^+$. The points represented on an Argand diagram by $z^0, z^1, z^2, \ldots, z^n$ form the vertices of a polygon P_n.

(c) Show that the area of the polygon P_n can be expressed in the form $a\left(b^n - 1\right)\sin\frac{\pi}{n}$, where $a, b \in \mathbb{R}$. [6]

Question 58

[Maximum mark: 7]

Use the principle of mathematical induction to prove that

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}, \text{ where } n \in \mathbb{Z}^+.$$

Question 59

[Maximum mark: 6]

Solve $(\ln x)^2 - (\ln 2)(\ln x) < 2(\ln 2)^2$.

Question 60

[Maximum mark: 5]

Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants.
The remainder when $f(x)$ is divided by $(x+1)$ is 7, and the remainder when $f(x)$ is divided by $(x-2)$ is 1. Find the value of p and the value of q.

Question 61

[Maximum mark: 16]

(a) Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answers in the form $re^{i\theta}$, where $r, \theta \in \mathbb{R}^+$. [5]

(b) Let S be the sum of the roots found in part (a).

(i) Show that $\operatorname{Re} S = \operatorname{Im} S$.

(ii) By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of $\cos\frac{\pi}{12}$ in the form $\frac{\sqrt{a} + \sqrt{b}}{c}$, where a, b and c are integers to be determined.

(iii) Hence, or otherwise, show that $S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + i)$. [11]

Question 62

[Maximum mark: 7]

Consider the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$.
Two of the roots of the equation are $\log_2 6$ and $i\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$.
Show that $6a + d + 12 = 0$.

Question 63

[Maximum mark: 6]

Use mathematical induction to prove that $\sum_{r=1}^{n} r(r!) = (n + 1)! - 1$, for $n \in \mathbb{Z}^+$.

Question 64

[Maximum mark: 7]

Consider the following system of equations where $a \in \mathbb{R}$.

$$\begin{aligned} 2x + 4y - z &= 10 \\ x + 2y + az &= 5 \\ 5x + 12y &= 2a. \end{aligned}$$

(a) Find the value of a for which the system of equations does not have a unique solution. [2]

(b) Find the solution of the system of equations when $a = 2$. [5]

Question 65

[Maximum mark: 5]

A team of four is to be chosen from a group of four boys and four girls.

(a) Find the number of different possible teams that could be chosen. [3]

(b) Find the number of different possible teams that could be chosen, given that the team must include at least one girl and at least one boy. [2]

Question 66

[Maximum mark: 7]

Solve the simultaneous equations

$$\log_2 6x = 1 + 2\log_2 y$$
$$1 + \log_6 x = \log_6(15y - 25).$$

Question 67

[Maximum mark: 5]

Consider the function $f(x) = x^4 - 6x^2 - 2x + 4$, $x \in \mathbb{R}$.
The graph of f is translated two units to the left to form the function $g(x)$.
Express $g(x)$ in the form $ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d, e \in \mathbb{Z}$.

Question 68

[Maximum mark: 4]

In an arithmetic sequence, the sum of the 3rd and 8th terms is 1.
Given that the sum of the first seven terms is 35, determine the first term and the common difference.

Question 69

[Maximum mark: 9]

The function $p(x)$ is defined by $p(x) = x^3 - 3x^2 + 8x - 24$ where $x \in \mathbb{R}$.

(a) Find the remainder when $p(x)$ is divided by

(i) $(x - 2)$

(ii) $(x - 3)$. [3]

(b) Prove that $p(x)$ has only one real zero. [4]

(c) Write down the transformation that will transform the graph of $y = p(x)$ onto the graph of $y = 8x^3 - 12x^2 + 16x - 24$. [2]

Question 70

[Maximum mark: 4]

Determine the first three terms of $(1 - 2x)^{11}$ in ascending powers of x, giving each term in its simplest form.

Question 71

[Maximum mark: 7]

Consider the equation $z^4 = -4$, where $z \in \mathbb{C}$.

(a) Solve the equation, giving the solutions in the form $a + ib$, where $a, b \in \mathbb{R}$. [5]

(b) The solutions form the vertices of a polygon in the complex plane. Find the area of the polygon. [2]

Question 72

[Maximum mark: 5]

Three planes have equations:

$$\begin{aligned} 2x - y + z &= 5 \\ x + 3y - z &= 4 \\ 3x - 5y + az &= b \end{aligned} \quad \text{, where } a, b \in \mathbb{R}.$$

Find the set of values of a and b such that the three planes have no points of intersection.