Subject – Math(Higher Level)
Topic - Circular trigonometry
Year - Nov 2011 – Nov 2019

Question -1

[Maximum mark: 6]

From a vertex of an equilateral triangle of side $2x$, a circular arc is drawn to divide the triangle into two regions, as shown in the diagram below.

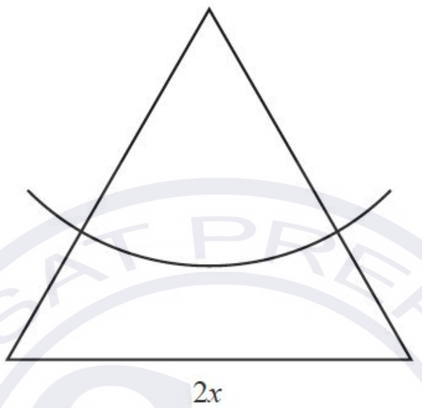


diagram not to scale

Given that the areas of the two regions are equal, find the radius of the arc in terms of x.

Question -2

[Maximum mark: 7]

Let $f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

(a) For what values of x does $f(x)$ not exist? *[2 marks]*

(b) Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$. *[5 marks]*

Question -3

[Maximum mark: 21]

In the triangle ABC, $A\hat{B}C = 90^\circ$, $AC = \sqrt{2}$ and $AB = BC + 1$.

(a) Show that $\cos \hat{A} - \sin \hat{A} = \frac{1}{\sqrt{2}}$. *[3 marks]*

(b) By squaring both sides of the equation in part (a), solve the equation to find the angles in the triangle. *[8 marks]*

(c) Apply Pythagoras' theorem in the triangle ABC to find BC, and hence show that $\sin \hat{A} = \frac{\sqrt{6} - \sqrt{2}}{4}$. *[6 marks]*

(d) Hence, or otherwise, calculate the length of the perpendicular from B to [AC]. *[4 marks]*

Question -4

[Maximum mark: 6]

Show that $\frac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$.

Question -5

[Maximum mark: 4]

Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos \alpha = -\frac{3}{4}$, find the value of $\sin 2\alpha$.

Question -6

[Maximum mark: 7]

In the triangle PQR, $PQ = 6$, $PR = k$ and $P\hat{Q}R = 30^\circ$.

(a) For the case $k = 4$, find the two possible values of QR. *[4 marks]*

(b) Determine the values of k for which the conditions above define a unique triangle. *[3 marks]*

Question -7

[Maximum mark: 19]

(a) (i) Express $\cos\left(\frac{\pi}{6}+x\right)$ in the form $a\cos x - b\sin x$ where $a, b \in \mathbb{R}$.

(ii) Hence solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le 2\pi$. *[7 marks]*

(b) Let $p(x) = 2x^3 - x^2 - 2x + 1$.

(i) Show that $x = 1$ is a zero of p.

(ii) Hence find all the solutions of $2x^3 - x^2 - 2x + 1 = 0$.

(iii) Express $\sin 2\theta \cos\theta + \sin^2\theta$ in terms of $\sin\theta$.

(iv) Hence solve $\sin 2\theta \cos\theta + \sin^2\theta = 1$ for $0 \le \theta \le 2\pi$. *[12 marks]*

Question -8

[Maximum mark: 6]

(a) Given that $\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find p. *[3 marks]*

(b) Hence find the value of $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$. *[3 marks]*

Question 9

[Maximum mark: 8]

(a) Prove the trigonometric identity $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$. *[4]*

(b) Given $f(x) = \sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)$, $x \in [0, \pi]$, find the range of f. *[2]*

(c) Given $g(x) = \csc\left(x+\frac{\pi}{6}\right)\csc\left(x-\frac{\pi}{6}\right)$, $x \in [0, \pi]$, $x \ne \frac{\pi}{6}$, $x \ne \frac{5\pi}{6}$, find the range of g. *[2]*

Question 10

[Maximum mark: 6]

Given that $\sin x + \cos x = \frac{2}{3}$, find $\cos 4x$.

Question 11

[Maximum mark: 5]

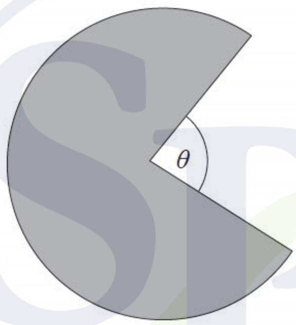
The triangle ABC is equilateral of side $3\,\text{cm}$. The point D lies on [BC] such that $\text{BD} = 1\,\text{cm}$. Find $\cos \text{D}\hat{\text{A}}\text{C}$.

Question 12

[Maximum mark: 5]

The logo, for a company that makes chocolate, is a sector of a circle of radius $2\,\text{cm}$, shown as shaded in the diagram. The area of the logo is $3\pi\,\text{cm}^2$.

diagram not to scale



(a) Find, in radians, the value of the angle θ, as indicated on the diagram. [3]

(b) Find the total length of the perimeter of the logo. [2]

Question 13

[Maximum mark: 6]

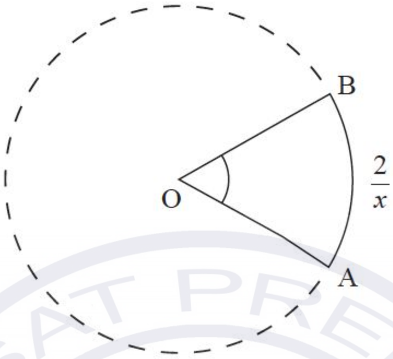
Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0^\circ \le x < 360^\circ$.

Question 14

[Maximum mark: 4]

The following diagram shows a sector of a circle where $A\hat{O}B = x$ radians and the length of the arc $AB = \frac{2}{x}\text{cm}$.

Given that the area of the sector is $16\,\text{cm}^2$, find the length of the arc AB.



Question 15

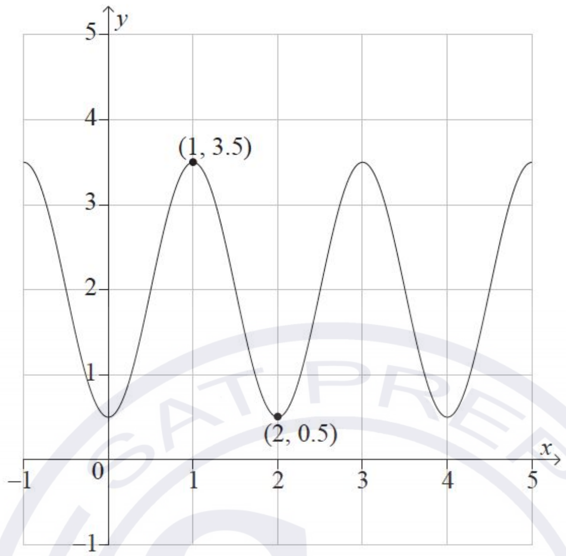
[Maximum mark: 7]

Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi, \pi]$.

Question 16

[Maximum mark: 6]

The following diagram shows the curve $y = a\sin\left(b(x+c)\right) + d$, where a, b, c and d are all positive constants. The curve has a maximum point at $(1, 3.5)$ and a minimum point at $(2, 0.5)$.



(a) Write down the value of a and the value of d. [2]

(b) Find the value of b. [2]

(c) Find the smallest possible value of c, given $c > 0$. [2]

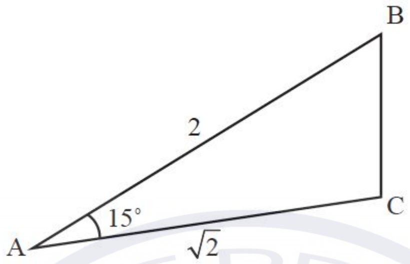
Question 17

[Maximum mark: 8]

(a) Expand and simplify $(1-\sqrt{3})^2$. [1]

(b) By writing $15°$ as $60° - 45°$ find the value of $\cos(15°)$. [3]

The following diagram shows the triangle ABC where $AB = 2$, $AC = \sqrt{2}$ and $B\hat{A}C = 15°$.



(c) Find BC in the form $a + \sqrt{b}$ where $a, b \in \mathbb{Z}$. [4]

Question 18

[Maximum mark: 8]

Consider the equation $\dfrac{\sqrt{3}-1}{\sin x} + \dfrac{\sqrt{3}+1}{\cos x} = 4\sqrt{2},\ 0 < x < \dfrac{\pi}{2}$. Given that $\sin\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

(a) verify that $x = \dfrac{\pi}{12}$ is a solution to the equation; [3]

(b) hence find the other solution to the equation for $0 < x < \dfrac{\pi}{2}$. [5]

Question 19

[Maximum mark: 19]

(a) Find the value of $\sin\dfrac{\pi}{4} + \sin\dfrac{3\pi}{4} + \sin\dfrac{5\pi}{4} + \sin\dfrac{7\pi}{4} + \sin\dfrac{9\pi}{4}$. [2]

(b) Show that $\dfrac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in \mathbb{Z}$. [2]

(c) Use the principle of mathematical induction to prove that
$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \dfrac{1-\cos 2nx}{2\sin x},\ n \in \mathbb{Z}^+,\ x \neq k\pi$ where $k \in \mathbb{Z}$. [9]

(d) Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. [6]

Question 20

[Maximum mark: 5]

Solve the equation $\sec^2 x + 2\tan x = 0$, $0 \le x \le 2\pi$.

Question 21

[Maximum mark: 5]

Let $a = \sin b$, $0 < b < \frac{\pi}{2}$.

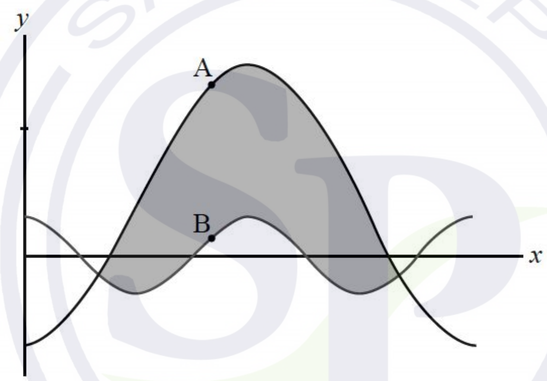
Find, in terms of b, the solutions of $\sin 2x = -a$, $0 \le x \le \pi$.

Question 22

[Maximum mark: 17]

Consider the functions f and g defined on the domain $0 < x < 2\pi$ by $f(x) = 3\cos 2x$ and $g(x) = 4 - 11\cos x$.

The following diagram shows the graphs of $y = f(x)$ and $y = g(x)$.



(a) Find the x-coordinates of the points of intersection of the two graphs. [6]

(b) Find the exact area of the shaded region, giving your answer in the form $p\pi + q\sqrt{3}$, where $p, q \in \mathbb{Q}$. [5]

At the points A and B on the diagram, the gradients of the two graphs are equal.

(c) Determine the y-coordinate of A on the graph of g. [6]

Question 23

[Maximum mark: 7]

The lengths of two of the sides in a triangle are $4\,\text{cm}$ and $5\,\text{cm}$. Let θ be the angle between the two given sides. The triangle has an area of $\frac{5\sqrt{15}}{2}\,\text{cm}^2$.

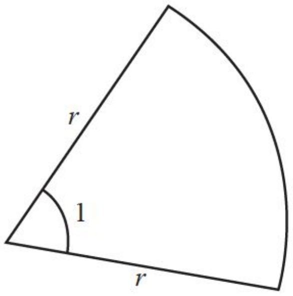
(a) Show that $\sin\theta = \frac{\sqrt{15}}{4}$. [1]

(b) Find the two possible values for the length of the third side. [6]

Question 24

[Maximum mark: 4]

A sector of a circle with radius r cm, where $r > 0$, is shown on the following diagram.
The sector has an angle of 1 radian at the centre.



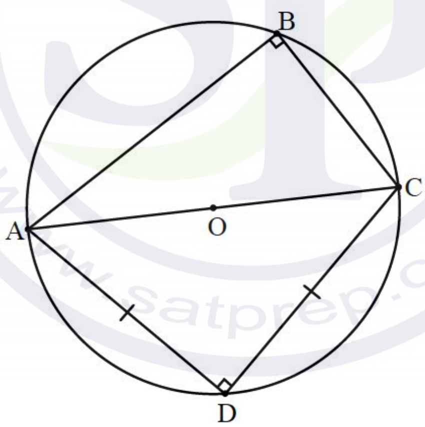
Let the area of the sector be $A\,\text{cm}^2$ and the perimeter be P cm. Given that $A = P$, find the value of r.

Question 25

[Maximum mark: 14]

(a) Given that $\cos 75^\circ = q$, show that $\cos 105^\circ = -q$. [1]

In the following diagram, the points A, B, C and D are on the circumference of a circle with centre O and radius r. [AC] is a diameter of the circle. $\text{BC} = r$, $\text{AD} = \text{CD}$ and $\text{A\hat{B}C} = \text{A\hat{D}C} = 90^\circ$.



(b) Show that $\text{B\hat{A}D} = 75^\circ$. [3]

(c) (i) By considering triangle ABD, show that $\text{BD}^2 = 5r^2 - 2r^2q\sqrt{6}$.

(ii) By considering triangle CBD, find another expression for BD^2 in terms of r and q. [7]

(d) Use your answers to part (c) to show that $\cos 75^\circ = \dfrac{1}{\sqrt{6}+\sqrt{2}}$. [3]

Question 26

[Maximum mark: 7]

A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$.

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$.