Subject – Math(Higher Level)
Topic - Statistics and Probability
Year - Nov 2011 – Nov 2019

Question 1

[Maximum mark: 6]

In a particular city 20 % of the inhabitants have been immunized against a certain disease. The probability of infection from the disease among those immunized is $\frac{1}{10}$, and among those not immunized the probability is $\frac{3}{4}$. If a person is chosen at random and found to be infected, find the probability that this person has been immunized.

Question 2

[Maximum mark: 6]

A target consists of three concentric circles of radii 1 m, 3 m and 5 m respectively, as shown in the diagram.

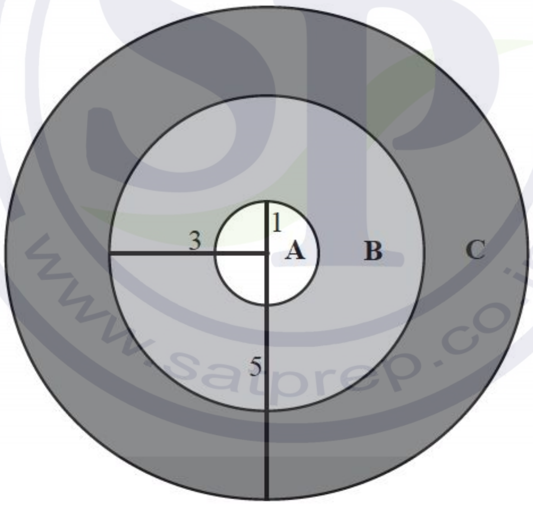


diagram not to scale

Nina shoots an arrow at the target. She has a probability of $\frac{1}{2}$ of hitting the target. If the arrow hits the target it does so at a random point on the target. Ten points are scored for hitting region A, six points for hitting region B, and three points for hitting region C. Find the expected number of points Nina scores each time she shoots an arrow at the target.

Question 3

[Maximum mark: 10]

A continuous random variable X has the probability density function

$$f(x) = \begin{cases} k \sin x, & 0 \le x \le \dfrac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. *[2 marks]*

(b) Find $\mathrm{E}(X)$. *[5 marks]*

(c) Find the median of X. *[3 marks]*

Question 4

[Maximum mark: 5]

On a particular day, the probability that it rains is $\frac{2}{5}$. The probability that the "Tigers" soccer team wins on a day when it rains is $\frac{2}{7}$ and the probability that they win on a day when it does not rain is $\frac{4}{7}$.

(a) Draw a tree diagram to represent these events and their outcomes. *[1 mark]*

(b) What is the probability that the "Tigers" soccer team wins? *[2 marks]*

(c) Given that the "Tigers" soccer team won, what is the probability that it rained on that day? *[2 marks]*

Question 5

Consider the following functions:

$$f(x) = \frac{2x^2 + 3}{75},\ x \ge 0$$

$$g(x) = \frac{|3x - 4|}{10},\ x \in \mathbb{R}.$$

The domains of f and g are now restricted to $\{0, 1, 2, 3, 4\}$.

(d) By considering the values of f and g on this new domain, determine which of f and g could be used to find a probability distribution for a discrete random variable X, stating your reasons clearly. *[6 marks]*

(e) Using this probability distribution, calculate the mean of X. *[2 marks]*

Question 6

[Maximum mark: 8]

The continuous random variable X has probability density function given by

$$f(x)=\begin{cases} ae^{-x}, & 0\le x\le 1 \\ 0, & \text{otherwise.}\end{cases}$$

(a) State the mode of X. *[1 mark]*

(b) Determine the value of a. *[3 marks]*

(c) Find $\mathrm{E}(X)$. *[4 marks]*

Question 7

[Maximum mark: 5]

The probability density function of the random variable X is defined as

$$f(x)=\begin{cases} \sin x, & 0\le x\le \dfrac{\pi}{2} \\ 0, & \text{otherwise.}\end{cases}$$

Find $\mathrm{E}(X)$.

Question 8

[Maximum mark: 6]

Two events A and B are such that $\mathrm{P}(A\cup B)=0.7$ and $\mathrm{P}(A\,|\,B')=0.6$.

Find $\mathrm{P}(B)$.

Question 9

[Maximum mark: 22]

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is $\frac{2}{3}$.

(a) Show that the probability that Alfred wins exactly 4 of the games is $\frac{80}{243}$. *[3 marks]*

(b) (i) Explain why the total number of possible outcomes for the results of the 6 games is 64.

(ii) By expanding $(1+x)^6$ and choosing a suitable value for x, prove

$$64 = \binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}.$$

(iii) State the meaning of this equality in the context of the 6 games played. *[4 marks]*

(c) The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still $\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2\left(\frac{2}{3}\right)^s\left(\frac{1}{3}\right)^t$ where the values of r, s and t are to be found.

(ii) Using your answer to (c)(i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that $\binom{12}{6} = \binom{6}{0}^2+\binom{6}{1}^2+\binom{6}{2}^2+\binom{6}{3}^2+\binom{6}{4}^2+\binom{6}{5}^2+\binom{6}{6}^2$. *[9 marks]*

(d) Alfred and Beatrice play n games. Let A denote the number of games Alfred wins. The expected value of A can be written as $\mathrm{E}(A) = \sum_{r=0}^{n} r\binom{n}{r}\frac{a^r}{b^n}$.

(i) Find the values of a and b.

(ii) By differentiating the expansion of $(1+x)^n$, prove that the expected number of games Alfred wins is $\frac{2n}{3}$. *[6 marks]*

Question 10

[Maximum mark: 5]

Tim and Caz buy a box of 16 chocolates of which 10 are milk and 6 are dark. Caz randomly takes a chocolate and eats it. Then Tim randomly takes a chocolate and eats it.

(a) Draw a tree diagram representing the possible outcomes, clearly labelling each branch with the correct probability. *[3 marks]*

(b) Find the probability that Tim and Caz eat the same type of chocolate. *[2 marks]*

Question 11

[Maximum mark: 6]

The discrete random variable X has probability distribution:

x	0	1	2	3
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	a

(a) Find the value of a. *[1]*

(b) Find $E(X)$. *[2]*

(c) Find $\text{Var}(X)$. *[3]*

Question 12

[Maximum mark: 4]

Four numbers are such that their mean is 13, their median is 14 and their mode is 15. Find the four numbers.

Question 13

[Maximum mark: 6]

Events A and B are such that $P(A) = \frac{2}{5}$, $P(B) = \frac{11}{20}$ and $P(A|B) = \frac{2}{11}$.

(a) Find $P(A \cap B)$. *[2]*

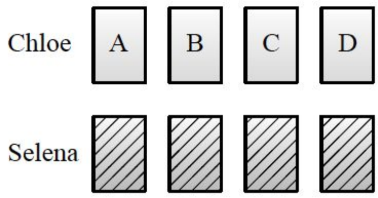
(b) Find $P(A \cup B)$. *[2]*

(c) State with a reason whether or not events A and B are independent. *[2]*

Question 14

[Maximum mark: 11]

Chloe and Selena play a game where each have four cards showing capital letters A, B, C and D.
Chloe lays her cards face up on the table in order $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ as shown in the following diagram.



Selena shuffles her cards and lays them face down on the table. She then turns them over one by one to see if her card matches with Chloe's card directly above.
Chloe wins if **no** matches occur; otherwise Selena wins.

(a) Show that the probability that Chloe wins the game is $\frac{3}{8}$. [6]

Chloe and Selena repeat their game so that they play a total of 50 times.
Suppose the discrete random variable X represents the number of times Chloe wins.

(b) Determine

(i) the mean of X;

(ii) the variance of X. [5]

Question 15

[Maximum mark: 7]

Events A and B are such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.5$.

(a) Determine the value of $\mathrm{P}(A \cup B)$ when

(i) A and B are mutually exclusive;

(ii) A and B are independent. *[4]*

(b) Determine the range of possible values of $\mathrm{P}(A|B)$. *[3]*

Question 16

[Maximum mark: 6]

A continuous random variable T has probability density function f defined by

$$f(t)=\begin{cases} |2-t|, & 1\le t\le 3 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Sketch the graph of $y=f(t)$. *[2]*

(b) Find the interquartile range of T. *[4]*

Question 17

[Maximum mark: 7]

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

(a) Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7. *[3]*

(b) Find the number of selections Grace could make if at least two of the four integers drawn are even. *[4]*

Question 18

[Maximum mark: 5]

A mathematics test is given to a class of 20 students. One student scores 0, but all the other students score 10.

(a) Find the mean score for the class. [2]

(b) Write down the median score. [1]

(c) Write down the number of students who scored

(i) above the mean score;

(ii) below the median score. [2]

Question 19

[Maximum mark: 6]

A football team, Melchester Rovers are playing a tournament of five matches.
The probabilities that they win, draw or lose a match are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively.
These probabilities remain constant; the result of a match is independent of the results of other matches. At the end of the tournament their coach Roy loses his job if they lose three **consecutive** matches, otherwise he does not lose his job. Find the probability that Roy loses his job.

Question 20

[Maximum mark: 4]

A and B are two events such that $\mathrm{P}(A) = 0.25$, $\mathrm{P}(B) = 0.6$ and $\mathrm{P}(A \cup B) = 0.7$.

(a) Find $\mathrm{P}(A \cap B)$. [2]

(b) Determine whether events A and B are independent. [2]

Question 21

[Maximum mark: 7]

A box contains four red balls and two white balls. Darren and Marty play a game by each taking it in turn to take a ball from the box, without replacement. The first player to take a white ball is the winner.

(a) Darren plays first, find the probability that he wins. [4]

The game is now changed so that the ball chosen is replaced after each turn.
Darren still plays first.

(b) Show that the probability of Darren winning has not changed. [3]

Question 22

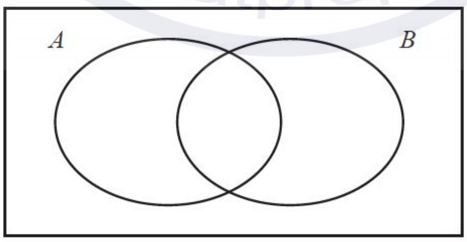
[Maximum mark: 4]

At a skiing competition the mean time of the first three skiers is 34.1 seconds. The time for the fourth skier is then recorded and the mean time of the first four skiers is 35.0 seconds. Find the time achieved by the fourth skier.

Question 23

[Maximum mark: 5]

(a) On the Venn diagram shade the region $A' \cap B'$. [1]



Two events A and B are such that $\mathrm{P}(A \cap B') = 0.2$ and $\mathrm{P}(A \cup B) = 0.9$.

(b) Find $\mathrm{P}(A' | B')$. [4]

Question 24

[Maximum mark: 8]

A biased coin is tossed five times. The probability of obtaining a head in any one throw is p.

Let X be the number of heads obtained.

(a) Find, in terms of p, an expression for $\mathrm{P}(X = 4)$. [2]

(b) (i) Determine the value of p for which $\mathrm{P}(X = 4)$ is a maximum.

(ii) For this value of p, determine the expected number of heads. [6]

Question 25

[Maximum mark: 6]

A and B are independent events such that $\mathrm{P}(A) = \mathrm{P}(B) = p$, $p \neq 0$.

(a) Show that $\mathrm{P}(A \cup B) = 2p - p^2$. [2]

(b) Find $\mathrm{P}(A \mid A \cup B)$ in simplest form. [4]

Question 26

[Maximum mark: 4]

The faces of a fair six-sided die are numbered $1, 2, 2, 4, 4, 6$. Let X be the discrete random variable that models the score obtained when this die is rolled.

(a) Complete the probability distribution table for X. [2]

x				
$\mathrm{P}(X = x)$				

(b) Find the expected value of X. [2]

Question 27

[Maximum mark: 9]

Consider two events A and B defined in the same sample space.

(a) Show that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A' \cap B)$. [3]

(b) Given that $\mathrm{P}(A \cup B) = \frac{4}{9}$, $\mathrm{P}(B \mid A) = \frac{1}{3}$ and $\mathrm{P}(B \mid A') = \frac{1}{6}$,

(i) show that $\mathrm{P}(A) = \frac{1}{3}$;

(ii) hence find $\mathrm{P}(B)$. [6]

Question 28

[Maximum mark: 15]

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} k\sin\left(\frac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. [4]

(b) By considering the graph of f write down

(i) the mean of X;

(ii) the median of X;

(iii) the mode of X. [3]

(c) (i) Show that $P(0 \le X \le 2) = \frac{1}{4}$.

(ii) Hence state the interquartile range of X. [6]

(d) Calculate $P(X \le 4 \mid X \ge 3)$. [2]

Question 29

[Maximum mark: 7]

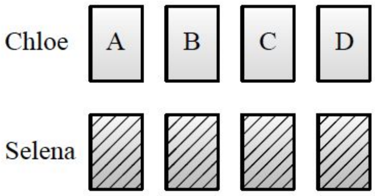
(a) The random variable X has the Poisson distribution $\mathrm{Po}(m)$. Given that $\mathrm{P}(X > 0) = \frac{3}{4}$, find the value of m in the form $\ln a$ where a is an integer. [3]

(b) The random variable Y has the Poisson distribution $\mathrm{Po}(2m)$. Find $\mathrm{P}(Y > 1)$ in the form $\frac{b - \ln c}{c}$ where b and c are integers. [4]

Question 30

[Maximum mark: 11]

Chloe and Selena play a game where each have four cards showing capital letters A, B, C and D.
Chloe lays her cards face up on the table in order A, B, C, D as shown in the following diagram.



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Chloe and Selena repeat their game so that they play a total of 50 times.
Suppose the discrete random variable X represents the number of times Chloe wins.

(b) Determine

(i) the mean of X;

(ii) the variance of X. [5]

Question 31

[Maximum mark: 6]

The discrete random variable X has the following probability distribution, where p is a constant.

x	0	1	2	3	4
$\mathrm{P}(X=x)$	p	$0.5-p$	0.25	0.125	p^3

(a) Find the value of p. [2]

(b) (i) Find μ, the expected value of X.

(ii) Find $\mathrm{P}(X>\mu)$. [4]

Question 32

[Maximum mark: 5]

Two unbiased tetrahedral (four-sided) dice with faces labelled $1, 2, 3, 4$ are thrown and the scores recorded. Let the random variable T be the maximum of these two scores. The probability distribution of T is given in the following table.

t	1	2	3	4
$P(T=t)$	$\frac{1}{16}$	a	b	$\frac{7}{16}$

(a) Find the value of a and the value of b. [3]

(b) Find the expected value of T. [2]

Question 33

[Maximum mark: 6]

Consider two events, A and B, such that $P(A) = P(A' \cap B) = 0.4$ and $P(A \cap B) = 0.1$.

(a) By drawing a Venn diagram, or otherwise, find $P(A \cup B)$. [3]

(b) Show that the events A and B are not independent. [3]

Question 34

[Maximum mark: 16]

The random variable X has probability density function f given by

$$f(x) = \begin{cases} k(\pi - \arcsin x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$, where k is a positive constant.

(a) State the mode of X. [1]

(b) (i) Find $\int \arcsin x \, dx$.

(ii) Hence show that $k = \frac{2}{2+\pi}$. [6]

(c) Given that $y = \left(\frac{x^2}{2}\right)\arcsin x - \left(\frac{1}{4}\right)\arcsin x + \left(\frac{x}{4}\right)\sqrt{1-x^2}$, show that

(i) $\frac{dy}{dx} = x \arcsin x$;

(ii) $E(X) = \frac{3\pi}{4(\pi+2)}$. [9]

Question 35

[Maximum mark: 7]

Let X be a random variable which follows a normal distribution with mean μ. Given that $\mathrm{P}(X < \mu - 5) = 0.2$, find

(a) $\mathrm{P}(X > \mu + 5)$; [2]

(b) $\mathrm{P}(X < \mu + 5 \mid X > \mu - 5)$. [5]

Question 36

[Maximum mark: 5]

The probability distribution of a discrete random variable, X, is given by the following table, where N and p are constants.

x	1	5	10	N
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	p

(a) Find the value of p. [2]

(b) Given that $\mathrm{E}(X) = 10$, find the value of N. [3]