Subject – Math(Higher Level)
Topic - Algebra
Year - Nov 2011 – Nov 2019

Question -1

$\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$ **(A1)**

$z_1 = \mathrm{i}^{\frac{1}{3}} = \left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)^{\frac{1}{3}} = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} \quad \left(= \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$ **M1A1**

$z_2 = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} \quad \left(= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$ **(M1)A1**

$z_3 = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = -\mathrm{i}$ **A1**

Question-2

proposition is true for $n = 1$ since $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(1-x)^2}$ **M1**

$= \frac{1!}{(1-x)^2}$ **A1**

te: Must see the 1! for the ***A1***.

assume true for $n = k$, $k \in \mathbb{Z}^+$, *i.e.* $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \frac{k!}{(1-x)^{k+1}}$ **M1**

consider $\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)}{\mathrm{d}x}$ **(M1)**

$= (k+1)k!(1-x)^{-(k+1)-1}$ **A1**

$= \frac{(k+1)!}{(1-x)^{k+2}}$ **A1**

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers **R1**

Question 3

$u_1 = \frac{1}{3}k$, $r = \frac{1}{3}$ **(A1)(A1)**

$7 = \frac{\frac{1}{3}k}{1 - \frac{1}{3}}$ **M1**

$k = 14$ **A1**

[4 marks]

Question -4

$z_1 = 2a\,\text{cis}\left(\frac{\pi}{3}\right),\ z_2 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ ***M1A1A1***

EITHER

$\left(\frac{z_1}{z_2}\right)^6 = \frac{2^6 a^6 \text{cis}(0)}{\sqrt{2}^6 \text{cis}\left(\frac{\pi}{2}\right)} \quad \left(= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)\right)$ ***M1A1A1***

OR

$\left(\frac{z_1}{z_2}\right)^6 = \left(\frac{2a}{\sqrt{2}}\text{cis}\left(\frac{7\pi}{12}\right)\right)^6$ ***M1A1***

$= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)$ ***A1***

THEN

$= -8a^6 \text{i}$ ***A1***

Question -5

$\sqrt{x^2 + y^2} + x + y\text{i} = 6 - 2\text{i}$ ***(A1)***

equating real and imaginary parts ***M1***

$y = -2$ ***A1***

$\sqrt{x^2 + 4} + x = 6$ ***A1***

$x^2 + 4 = (6 - x)^2$ ***M1***

$-32 = -12x \Rightarrow x = \frac{8}{3}$ ***A1***

[6 marks]

Question – 6

$\log_3\left(\frac{9}{x+7}\right) = \log_3 \frac{1}{2x}$ ***M1M1A1***

te: Award ***M1*** for changing to single base, ***M1*** for incorporating the 2 into a log and ***A1*** for a correct equation with maximum one log expression each side.

$x + 7 = 18x$ ***M1***

$x = \frac{7}{17}$ ***A1***

[5 marks]

Question -7

(a) $\left(x-\frac{2}{x}\right)^4 = x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$ ***(A2)***

Note: Award ***(A1)*** for 3 or 4 correct terms.

Note: Accept combinatorial expressions, *e.g.* $\binom{4}{2}$ for 6.

$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ ***A1***

[3 marks]

(b) constant term from expansion of $(2x^2+1)\left(x-\frac{2}{x}\right)^4 = -64 + 24 = -40$ ***A2***

Note: Award ***A1*** for – 64 or 24 seen.

[2 marks]

Question -8

(a) attempt to equate real and imaginary parts ***M1***
equate real parts: $4m + 4n = 16$; equate imaginary parts: $-5m = 15$ ***A1***
$\Rightarrow m = -3,\ n = 7$ ***A1***

[3 marks]

(b) let $m = x + \mathrm{i}y,\ n = x - \mathrm{i}y$ ***M1***
$\Rightarrow (4-5\mathrm{i})(x+\mathrm{i}y) + 4(x-\mathrm{i}y) = 16 + 15\mathrm{i}$
$\Rightarrow 4x - 5\mathrm{i}x + 4\mathrm{i}y + 5y + 4x - 4\mathrm{i}y = 16 + 15\mathrm{i}$
attempt to equate real and imaginary parts ***M1***
$8x + 5y = 16,\ -5x = 15$ ***A1***
$\Rightarrow x = -3,\ y = 8$ ***A1***
$(\Rightarrow m = -3 + 8\mathrm{i},\ n = -3 - 8\mathrm{i})$

[4 marks]

Total [7 marks]

Question -9

Part A

(a) (i) $(x+\text{i}y)^2=-5+12\text{i}$

$x^2+2\text{i}xy+\text{i}^2y^2=-5+12\text{i}$ ***A1***

(ii) equating real and imaginary parts ***M1***

$x^2-y^2=-5$ ***AG***

$xy=6$ ***AG***

[2 marks]

(b) substituting ***M1***

EITHER

$x^2-\frac{36}{x^2}=-5$

$x^4+5x^2-36=0$ ***A1***

$x^2=4,-9$ ***A1***

$x=\pm2$ and $y=\pm3$ ***(A1)***

OR

$\frac{36}{y^2}-y^2=-5$

$y^4-5y^2-36=0$ ***A1***

$y^2=9,-4$ ***A1***

$y=\pm3$ **and** $x=\pm2$ ***(A1)***

Note: Accept solution by inspection if completely correct.

THEN

the square roots are $(2+3\text{i})$ and $(-2-3\text{i})$ ***A1***

[5 marks]

(c) **EITHER**

consider $z=x+\text{i}y$

$z^*=x-\text{i}y$

$(z^*)^2=x^2-y^2-2\text{i}xy$ ***A1***

$(z^2)=x^2-y^2+2\text{i}xy$ ***A1***

$(z^2)^*=x^2-y^2-2\text{i}xy$ ***A1***

$(z^*)^2=(z^2)^*$ ***AG***

OR

$z^*=r\text{e}^{-\text{i}\theta}$

$(z^*)^2=r^2\text{e}^{-2\text{i}\theta}$ ***A1***

$z^2=r^2\text{e}^{2\text{i}\theta}$ ***A1***

$(z^2)^* = r^2 e^{-2i\theta}$ **A1**

$(z^*)^2 = (z^2)^*$ **AG**

[3 marks]

(d) $(2-3i)$ and $(-2+3i)$ **A1A1**

[2 marks]

Part B

(a) the graph crosses the x-axis twice, indicating two real roots **R1**

since the quartic equation has four roots and only two are real, the other two roots must be complex **R1**

[2 marks]

(b) $f(x) = (x+4)(x-2)(x^2 + cx + d)$ **A1A1**

$f(0) = -32 \Rightarrow d = 4$ **A1**

Since the curve passes through $(-1, -18)$,

$-18 = 3 \times (-3)(5-c)$ **M1**

$c = 3$ **A1**

Hence $f(x) = (x+4)(x-2)(x^2 + 3x + 4)$

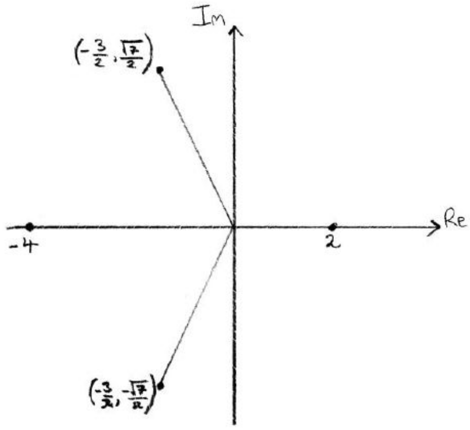
[5 marks]

(c) $x = \dfrac{-3 \pm \sqrt{9-16}}{2}$ **(M1)**

$\Rightarrow x = -\dfrac{3}{2} \pm i\dfrac{\sqrt{7}}{2}$ **A1**

[2 marks]

(d)



A1A1

Note:	Accept points or vectors on complex plane. Award ***A1*** for two real roots and ***A1*** for two complex roots.

[2 marks]

(e) real roots are $4e^{i\pi}$ and $2e^{i0}$ ***A1A1***

considering $-\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

$r = \sqrt{\frac{9}{4} + \frac{7}{4}} = 2$ ***A1***

finding θ using $\arctan\left(\frac{\sqrt{7}}{3}\right)$ ***M1***

$\theta = \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi$ or $\theta = \arctan\left(-\frac{\sqrt{7}}{3}\right) + \pi$ ***A1***

$\Rightarrow z = 2e^{i\left(\arctan\left(\frac{\sqrt{7}}{3}\right)+\pi\right)}$ or $\Rightarrow z = 2e^{i\left(\arctan\left(\frac{-\sqrt{7}}{3}\right)+\pi\right)}$ ***A1***

Note:	Accept arguments in the range $-\pi$ to π or 0 to 2π. Accept answers in degrees.

[6 marks]

Total [29 marks]

Question 10

$$\left(\frac{x}{y}-\frac{y}{x}\right)^4=\left(\frac{x}{y}\right)^4+4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right)+6\left(\frac{x}{y}\right)^2\left(-\frac{y}{x}\right)^2+4\left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3+\left(-\frac{y}{x}\right)^4$$ ***(M1)(A1)***

: Award ***M1*** for attempt to expand and ***A1*** for correct unsimplified expansion.

$$=\frac{x^4}{y^4}-4\frac{x^2}{y^2}+6-4\frac{y^2}{x^2}+\frac{y^4}{x^4}\quad\left(=\frac{x^8-4x^6y^2+6x^4y^4-4x^2y^6+y^8}{x^4y^4}\right)$$ ***A1A1***

: Award ***A1*** for powers, ***A1*** for coefficients and signs.

: Final two ***A*** marks are independent of first ***A*** mark.

[4 marks]

Question 11

(a) (i) $z_1 = 2\sqrt{3}\,\text{cis}\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\text{i}$ *A1*

(ii) $z_1 + z_2 = -2\sqrt{3}\text{i} - 1 + \sqrt{3}\text{i} = -1 - \sqrt{3}\text{i}$ *A1*

$(z_1 + z_2)^* = -1 + \sqrt{3}\text{i}$ *A1*

[3 marks]

(b) (i) $|z_2| = 2$

$\tan\theta = -\sqrt{3}$ *(M1)*

z_2 lies on the second quadrant

$\theta = \arg z_2 = \frac{2\pi}{3}$

$z_2 = 2\text{cis}\frac{2\pi}{3}$ *A1A1*

(ii) attempt to use De Moivre's theorem *M1*

$z = \sqrt[3]{2}\,\text{cis}\frac{\frac{2\pi}{3} + 2k\pi}{3},\ k = 0, 1 \text{ and } 2$

$z = \sqrt[3]{2}\,\text{cis}\frac{2\pi}{9},\ \sqrt[3]{2}\,\text{cis}\frac{8\pi}{9},\ \sqrt[3]{2}\,\text{cis}\frac{14\pi}{9}\left(= \sqrt[3]{2}\,\text{cis}\left(\frac{-4\pi}{9}\right)\right)$ *A1A1*

Note: Award ***A1*** for modulus, ***A1*** for arguments.

Note: Allow equivalent forms for z.

[6 marks]

(c) (i) **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3\left(\Rightarrow z = \pm\sqrt{3}\text{i}\right)$ *M1*

$z = \sqrt{3}\,\text{cis}\frac{\pi}{2}$ or $z_1 = \sqrt{3}\,\text{cis}\frac{3\pi}{2}\left(= \sqrt{3}\,\text{cis}\left(\frac{-\pi}{2}\right)\right)$ *A1A1*

so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}\left(= \frac{-\pi}{2}\right)$

Note: Accept $r\text{cis}(\theta)$ form.

METHOD 2

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3 \Rightarrow z^2 = 3\text{cis}\left((2n+1)\pi\right)$ *M1*

$r^2 = 3 \Rightarrow r = \sqrt{3}$ *A1*

$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (as $0 \le \theta < 2\pi$) *A1*

(ii) **METHOD 1**

$z = -\dfrac{1}{2cis\dfrac{2\pi}{3}} \Rightarrow z = \dfrac{\text{cis}\pi}{2\text{cis}\dfrac{2\pi}{3}}$ ***M1***

$\Rightarrow z = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$

so $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ ***A1A1***

METHOD 2

$z_1 = -\dfrac{1}{-1+\sqrt{3}\text{i}} \Rightarrow z_1 = -\dfrac{-1-\sqrt{3}\text{i}}{\left(-1+\sqrt{3}\text{i}\right)\left(-1-\sqrt{3}\text{i}\right)}$ ***M1***

$z = \dfrac{1+\sqrt{3}\text{i}}{4} \Rightarrow z = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$

so $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ ***A1A1***

[6 marks]

(d) $\dfrac{z_1}{z_2} = \sqrt{3}\,\text{cis}\dfrac{5\pi}{6}$ ***(A1)***

$\left(\dfrac{z_1}{z_2}\right)^n = \sqrt{3}^n\text{cis}\dfrac{5n\pi}{6}$ ***A1***

equating imaginary part to zero and attempting to solve ***M1***

obtain $n = 12$ ***A1***

Note: Working which only includes the argument is valid.

[4 marks]

Total [19 marks]

(a) (i) $z_1 = 2\sqrt{3}\,\text{cis}\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\text{i}$ *A1*

(ii) $z_1 + z_2 = -2\sqrt{3}\text{i} - 1 + \sqrt{3}\text{i} = -1 - \sqrt{3}\text{i}$ *A1*

$(z_1 + z_2)^* = -1 + \sqrt{3}\text{i}$ *A1*

[3 marks]

(b) (i) $|z_2| = 2$

$\tan\theta = -\sqrt{3}$ *(M1)*

z_2 lies on the second quadrant

$\theta = \arg z_2 = \frac{2\pi}{3}$

$z_2 = 2\text{cis}\frac{2\pi}{3}$ *A1A1*

(ii) attempt to use De Moivre's theorem *M1*

$z = \sqrt[3]{2}\,\text{cis}\frac{\frac{2\pi}{3} + 2k\pi}{3}$, $k = 0, 1$ and 2

$z = \sqrt[3]{2}\,\text{cis}\frac{2\pi}{9}, \sqrt[3]{2}\,\text{cis}\frac{8\pi}{9}, \sqrt[3]{2}\,\text{cis}\frac{14\pi}{9}\left(= \sqrt[3]{2}\,\text{cis}\left(\frac{-4\pi}{9}\right)\right)$ *A1A1*

Note: Award ***A1*** for modulus, ***A1*** for arguments.

Note: Allow equivalent forms for z.

[6 marks]

(c) (i) **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3\left(\Rightarrow z = \pm\sqrt{3}\text{i}\right)$ *M1*

$z = \sqrt{3}\,\text{cis}\frac{\pi}{2}$ or $z_1 = \sqrt{3}\,\text{cis}\frac{3\pi}{2}\left(= \sqrt{3}\,\text{cis}\left(\frac{-\pi}{2}\right)\right)$ *A1A1*

so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}\left(= \frac{-\pi}{2}\right)$

Note: Accept $r\text{cis}(\theta)$ form.

METHOD 2

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3 \Rightarrow z^2 = 3\text{cis}\left((2n+1)\pi\right)$ *M1*

$r^2 = 3 \Rightarrow r = \sqrt{3}$ *A1*

$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (as $0 \le \theta < 2\pi$) *A1*

(ii) **METHOD 1**

$z=-\frac{1}{2cis\frac{2\pi}{3}} \Rightarrow z=\frac{\text{cis}\pi}{2\text{cis}\frac{2\pi}{3}}$ ***M1***

$\Rightarrow z=\frac{1}{2}\text{cis}\frac{\pi}{3}$

so $r=\frac{1}{2}$ and $\theta=\frac{\pi}{3}$ ***A1A1***

METHOD 2

$z_1=-\frac{1}{-1+\sqrt{3}\text{i}} \Rightarrow z_1=-\frac{-1-\sqrt{3}\text{i}}{\left(-1+\sqrt{3}\text{i}\right)\left(-1-\sqrt{3}\text{i}\right)}$ ***M1***

$z=\frac{1+\sqrt{3}\text{i}}{4} \Rightarrow z=\frac{1}{2}\text{cis}\frac{\pi}{3}$

so $r=\frac{1}{2}$ and $\theta=\frac{\pi}{3}$ ***A1A1***

[6 marks]

(d) $\frac{z_1}{z_2}=\sqrt{3}\,\text{cis}\frac{5\pi}{6}$ ***(A1)***

$\left(\frac{z_1}{z_2}\right)^n=\sqrt{3}^n\text{cis}\frac{5n\pi}{6}$ ***A1***

equating imaginary part to zero and attempting to solve ***M1***

obtain $n=12$ ***A1***

Note: Working which only includes the argument is valid.

[4 marks]

Total [19 marks]

Question 12

(a) modulus $= \sqrt{8}$ *A1*

argument $= \dfrac{\pi}{4}$ (accept 45°) *A1*

Note: *A0* if extra values given. ***[2 marks]***

(b) **METHOD 1**

$w^4 z^6 = 64e^{\pi i} \times e^{5\pi i}$ ***(A1)(A1)***

Note: Allow alternative notation.

$= 64e^{6\pi i}$ ***(M1)***

$= 64$ ***A1***

METHOD 2

$w^4 = -64$ ***(M1)(A1)***

$z^6 = -1$ ***(A1)***

$w^4 z^6 = 64$ ***A1***

[4 marks]

Total [6 marks]

Question 13

METHOD 1

$$d=\frac{1}{\log_8 x}-\frac{1}{\log_2 x}$$ ***(M1)***

$$=\frac{\log_2 8}{\log_2 x}-\frac{1}{\log_2 x}$$ ***(M1)***

te: Award this ***M1*** for a correct change of base anywhere in the question.

$$=\frac{2}{\log_2 x}$$ ***(A1)***

$$\frac{20}{2}\left(2\times\frac{1}{\log_2 x}+19\times\frac{2}{\log_2 x}\right)$$ ***M1***

$$=\frac{400}{\log_2 x}$$ ***(A1)***

$$100=\frac{400}{\log_2 x}$$

$$\log_2 x=4\Rightarrow x=2^4=16$$ ***A1***

METHOD 2

$$20^{\text{th}}\text{ term }=\frac{1}{\log_{2^{39}} x}$$ ***A1***

$$100=\frac{20}{2}\left(\frac{1}{\log_2 x}+\frac{1}{\log_{2^{39}} x}\right)$$ ***M1***

$$100=\frac{20}{2}\left(\frac{1}{\log_2 x}+\frac{\log_2 2^{39}}{\log_2 x}\right)$$ ***M1(A1)***

te: Award this ***M1*** for a correct change of base anywhere in the question.

$$100=\frac{400}{\log_2 x}$$ ***(A1)***

$$\log_2 x=4\Rightarrow x=2^4=16$$ ***A1***

METHOD 3

$$\frac{1}{\log_2 x}+\frac{1}{\log_8 x}+\frac{1}{\log_{32} x}+\frac{1}{\log_{128} x}+\ldots$$

$$\frac{1}{\log_2 x}+\frac{\log_2 8}{\log_2 x}+\frac{\log_2 32}{\log_2 x}+\frac{\log_2 128}{\log_2 x}+\ldots$$ ***(M1)(A1)***

ote: Award this ***M1*** for a correct change of base anywhere in the question.

$$=\frac{1}{\log_2 x}(1+3+5+\ldots)$$ ***A1***

$$=\frac{1}{\log_2 x}\left(\frac{20}{2}(2+38)\right)$$ ***(M1)(A1)***

$$100=\frac{400}{\log_2 x}$$

$$\log_2 x=4\Rightarrow x=2^4=16$$ ***A1***

[6 marks]

Question 14

clear attempt at binomial expansion for exponent 5 ***M1***

$$2^5+5\times2^4\times(-3x)+\frac{5\times4}{2}\times2^3\times(-3x)^2+\frac{5\times4\times3}{6}\times2^2\times(-3x)^3$$

$$+\frac{5\times4\times3\times2}{24}\times2\times(-3x)^4+(-3x)^5$$ ***(A1)***

e: Only award ***M1*** if binomial coefficients are seen.

$$=32-240x+720x^2-1080x^3+810x^4-243x^5$$ ***A2***

e: Award ***A1*** for correct moduli of coefficients and powers. ***A1*** for correct signs.

Total [4 marks]

Question 15

for the first series $\frac{a}{1-r}=76$ ***A1***

for the second series $\frac{a}{1-r^3}=36$ ***A1***

attempt to eliminate a *e.g.* $\frac{76(1-r)}{1-r^3}=36$ ***M1***

simplify and obtain $9r^2+9r-10=0$ ***(M1)A1***

Note: Only award the ***M1*** if a quadratic is seen.

obtain $r=\frac{12}{18}$ and $-\frac{30}{18}$ ***(A1)***

$r=\frac{12}{18}\left(=\frac{2}{3}=0.666\ldots\right)$ ***A1***

Note: Award ***A0*** if the extra value of r is given in the final answer.

Total [7 marks]

Question 16

(a) $|z_1| = \sqrt{10}$; $\arg(z_2) = -\frac{3\pi}{4}$ $\left(\text{accept } \frac{5\pi}{4}\right)$ *A1A1*

[2 marks]

(b) $|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus *(M1)(A1)*

attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root *M1*

obtain $\alpha = 2$ at minimum *(A1)*

state $\sqrt{2}$ as final answer *A1*

[5 marks]

Total [7 marks]

Question 17

(a) (i) $z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right)$, $z_3 = 2\text{cis}\left(-\frac{\pi}{2}\right)$ or $2\text{cis}\left(\frac{3\pi}{2}\right)$ *A1A1A1*

Note:	Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note:	Accept arguments given in rational degrees, except where exponential form is used.

(ii) the points lie on a circle of radius 2 centre the origin *A1*

differences are all $\frac{2\pi}{3}$ (mod 2π) *A1*

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral *R1AG*

Note:	Accept an approach based on a clearly marked diagram.

(iii) $z_1^{3n} + z_2^{3n} = 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\text{cis}\left(\frac{5n\pi}{2}\right)$ *M1*

$= 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ *A1*

$2z_3^{3n} = 2 \times 2^{3n}\text{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ *A1AG*

[9 marks]

(b) (i) attempt to obtain **seven** solutions in modulus argument form *M1*

$z = \text{cis}\left(\frac{2k\pi}{7}\right), k = 0, 1 \ldots 6$ *A1*

(ii) w has argument $\frac{2\pi}{7}$ and $1+w$ has argument ϕ,

then $\tan(\phi) = \frac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}$ *M1*

$= \frac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}$ *A1*

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$ *A1*

(iii) since roots occur in conjugate pairs, *(R1)*

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$ ***A1***

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ ***AG***

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$ ***A1***

and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1$ ***A1***

[9 marks]

Total [18 marks]

Question 18

$n = 1:\ 1^3 + 11 = 12$

$= 3 \times 4$ **or** a multiple of 3 ***A1***

assume the proposition is true for $n = k$ $(ie\ k^3 + 11k = 3\text{m})$ ***M1***

:e: Do not award ***M1*** for statements with "Let $n = k$".

consider $n = k + 1$: $(k+1)^3 + 11(k+1)$ ***M1***

$= k^3 + 3k^2 + 3k + 1 + 11k + 11$ ***A1***

$= k^3 + 11k + (3k^2 + 3k + 12)$ ***M1***

$= 3(m + k^2 + k + 4)$ ***A1***

:e: Accept $k^3 + 11k + 3(k^2 + k + 4)$ or statement that $k^3 + 11k + (3k^2 + 3k + 12)$ is a multiple of 3.

true for $n = 1$, and $n = k$ true $\Rightarrow n = k + 1$ true

hence true for all $n \in \mathbb{Z}^+$ ***R1***

:e: Only award the final ***R1*** if at least 4 of the previous marks have been achieved.

Total [7 marks]

Question 19

(a) **METHOD 1**

$a + ar = 10$ ***A1***

$a + ar + ar^2 + ar^3 = 30$ ***A1***

$a + ar = 10 \Rightarrow ar^2 + ar^3 = 10r^2$ **or** $ar^2 + ar^3 = 20$ ***M1***

$10 + 10r^2 = 30$ **or** $r^2(a + ar) = 20$ ***A1***

$\Rightarrow r^2 = 2$ ***AG***

METHOD 2

$\dfrac{a(1-r^2)}{1-r} = 10$ and $\dfrac{a(1-r^4)}{1-r} = 30$ ***M1A1***

$\Rightarrow \dfrac{1-r^4}{1-r^2} = 3$ ***M1***

leading to either $1 + r^2 = 3$ (or $r^4 - 3r^2 + 2 = 0$) ***A1***

$\Rightarrow r^2 = 2$ ***AG***

[4 marks]

(b) (i) $a + a\sqrt{2} = 10$

$\Rightarrow a = \dfrac{10}{1+\sqrt{2}}$ **or** $a = 10(\sqrt{2} - 1)$ ***A1***

(ii) $S_{10} = \dfrac{10}{1+\sqrt{2}}\left(\dfrac{\sqrt{2}^{10} - 1}{\sqrt{2} - 1}\right)(= 10 \times 31)$ ***M1***

$= 310$ ***A1***

[3 marks]

Total [7 marks]

Question 20

(a) $\log_2(x-2) = \log_4\left(x^2 - 6x + 12\right)$

EITHER

$\log_2(x-2) = \dfrac{\log_2\left(x^2-6x+12\right)}{\log_2 4}$ ***M1***

$2\log_2(x-2) = \log_2\left(x^2-6x+12\right)$

OR

$\dfrac{\log_4(x-2)}{\log_4 2} = \log_4\left(x^2-6x+12\right)$ ***M1***

$2\log_4(x-2) = \log_4\left(x^2-6x+12\right)$

THEN

$(x-2)^2 = x^2 - 6x + 12$ ***A1***

$x^2 - 4x + 4 = x^2 - 6x + 12$

$x = 4$ ***A1*** ***N1***

[3 marks]

(b) $x^{\ln x} = e^{(\ln x)^3}$

taking ln of both sides or writing $x = e^{\ln x}$ ***M1***

$(\ln x)^2 = (\ln x)^3$ ***A1***

$(\ln x)^2(\ln x - 1) = 0$ ***(A1)***

$x = 1,\ x = e$ ***A1A1*** ***N2***

Note: Award second ***(A1)*** only if factorisation seen or if two correct solutions are seen.

[5 marks]

Total [8 marks]

Question 21

(a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ ***M1***

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$ ***A1***

$= 2\cos n\theta$ ***AG***

[2 marks]

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

(c) **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***M1***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$ ***M1***

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$ ***A1***

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$ ***A1***

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

[4 marks]

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$ ***M1***

$\left(z+z^{-1}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ ***AG***

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$ ***M1A1***

$= \frac{5\pi}{32}$ ***A1***

[3 marks]

(f) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$ ***M1***

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \pi\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$ ***M1***

$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3\pi}{16}$ ***A1***

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$ ***A1***

Note: Follow through from an incorrect r in (c) provided the final answer is positive.

[4 marks]

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, d\theta = \frac{(2k)!}{(k!)^2}\frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, d\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2} \left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$ ***A1***

[3 marks]

Total [20 marks]

Question 22

$\frac{\log 3}{\log 2}\times\frac{\log 4}{\log 3}\times\ldots\times\frac{\log 32}{\log 31}$ ***M1A1***

$=\frac{\log 32}{\log 2}$ ***A1***

$=\frac{5\log 2}{\log 2}$ ***(M1)***

$=5$ ***A1***

hence $a=5$

[5 marks]

Note: Accept the above if done in a specific base *eg* $\log_2 x$.

Question 23

(a) $r = 1 + \mathrm{i}$ *(A1)*

$u_4 = 3(1+\mathrm{i})^3$ ***M1***

$= -6 + 6\mathrm{i}$ ***A1***

[3 marks]

(b) $S_{20} = \dfrac{3\left((1+\mathrm{i})^{20} - 1\right)}{\mathrm{i}}$ ***(M1)***

$= \dfrac{3\left((2i)^{10} - 1\right)}{\mathrm{i}}$ ***(M1)***

Note: Only one of the two ***M1***s can be implied. Other algebraic methods may be seen.

$= \dfrac{3\left(-2^{10} - 1\right)}{\mathrm{i}}$ ***(A1)***

$= 3\mathrm{i}\left(2^{10} + 1\right)$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

$v_n = \left(3(1+\mathrm{i})^{n-1}\right)\left(3(1+\mathrm{i})^{n-1+k}\right)$ ***M1***

$9(1+\mathrm{i})^k(1+i)^{2n-2}$ ***A1***

$= 9(1+\mathrm{i})^k((1+i)^2)^{n-1}\left(= 9(1+\mathrm{i})^k(2\mathrm{i})^{n-1}\right)$

this is the general term of a geometrical sequence ***R1AG***

Notes: Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further. If the final expression for v_n is $9(1+\mathrm{i})^k(1+i)^{2n-2}$ award ***M1A1R0***.

METHOD 2

$\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1}u_{n+k+1}}{u_n u_{n+k}}$ ***M1***

$= (1+\mathrm{i})(1+\mathrm{i})$ ***A1***

this is a constant, hence sequence is geometric ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) $9(1+\mathrm{i})^k$ ***A1***

(iii) common ratio is $(1+i)^2 (= 2i)$ (which is independent of k) ***A1***

(d) (i) **METHOD 1**

$w_n = \left|3(1+i)^{n-1} - 3(1+\mathrm{i})^n\right|$ ***M1***

$= 3\left|1+i\right|^{n-1}\left|1-(1+\mathrm{i})\right|$ ***M1***

$= 3\left|1+i\right|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

METHOD 2

$w_n = \left|u_n - (1+\mathrm{i})u_n\right|$ ***M1***

$= \left|u_n\right|\left|-\mathrm{i}\right|$

$= \left|u_n\right|$ ***A1***

$= \left|3(1+\mathrm{i})^{n-1}\right|$

$= 3\left|(1+\mathrm{i})\right|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) distance between successive points representing u_n in the complex plane forms a geometric sequence ***R1***

Note: Various possibilities but must mention distance between successive points.

[5 marks]

Total [17 marks]

Question 24

METHOD 1

$2^{3(x-1)} = (2\times3)^{3x}$ ***M1***

ote: Award ***M1*** for writing in terms of 2 and 3.

$2^{3x}\times2^{-3} = 2^{3x}\times3^{3x}$

$2^{-3} = 3^{3x}$ ***A1***

$\ln\left(2^{-3}\right) = \ln\left(3^{3x}\right)$ ***(M1)***

$-3\ln2 = 3x\ln3$ ***A1***

$x = -\frac{\ln2}{\ln3}$ ***A1***

METHOD 2

$\ln8^{x-1} = \ln6^{3x}$ ***(M1)***

$(x-1)\ln2^3 = 3x\ln(2\times3)$ ***M1A1***

$3x\ln2 - 3\ln2 = 3x\ln2 + 3x\ln3$ ***A1***

$x = -\frac{\ln2}{\ln3}$ ***A1***

METHOD 3

$\ln8^{x-1} = \ln6^{3x}$ ***(M1)***

$(x-1)\ln8 = 3x\ln6$ ***A1***

$x = \frac{\ln8}{\ln8 - 3\ln6}$ ***A1***

$x = \frac{3\ln2}{\ln\left(\frac{2^3}{6^3}\right)}$ ***M1***

$x = -\frac{\ln2}{\ln3}$ ***A1***

Total [5 marks]

Question 25

(a) **METHOD 1**

$$\frac{1}{2+3\mathrm{i}}+\frac{1}{3+2\mathrm{i}}=\frac{2-3\mathrm{i}}{4+9}+\frac{3-2\mathrm{i}}{9+4}$$ *M1A1*

$$\frac{10}{w}=\frac{5-5\mathrm{i}}{13}$$ *A1*

$$w=\frac{130}{5-5\mathrm{i}}$$

$$=\frac{130\times5\times(1+\mathrm{i})}{50}$$

$$w=13+13\mathrm{i}$$ *A1*

[4 marks]

METHOD 2

$$\frac{1}{2+3\mathrm{i}}+\frac{1}{3+2\mathrm{i}}=\frac{3+2\mathrm{i}+2+3\mathrm{i}}{(2+3\mathrm{i})(3+2\mathrm{i})}$$ *M1A1*

$$\frac{10}{w}=\frac{5+5\mathrm{i}}{13\mathrm{i}}$$ *A1*

$$\frac{w}{10}=\frac{13\mathrm{i}}{5+5\mathrm{i}}$$

$$w=\frac{130\mathrm{i}}{(5+5\mathrm{i})}\times\frac{(5-5\mathrm{i})}{(5-5\mathrm{i})}$$

$$=\frac{650+650\mathrm{i}}{50}$$

$$=13+13\mathrm{i}$$ *A1*

[4 marks]

(b) $w^*=13-13\mathrm{i}$ *A1*

$$z=\sqrt{338}e^{-\frac{\pi}{4}\mathrm{i}}\left(=13\sqrt{2}\,e^{-\frac{\pi}{4}\mathrm{i}}\right)$$ *A1A1*

Note: Accept $\theta=\frac{7\pi}{4}$.
Do not accept answers for θ given in degrees.

Question 26

(a) $\sin x$, $\sin 2x$ and $4\sin x\cos^2 x$

$r = \dfrac{2\sin x\cos x}{\sin x} = 2\cos x$ ***A1***

Note: Accept $\dfrac{\sin 2x}{\sin x}$.

[1 mark]

(b) **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1$ ***M1***

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$ ***M1***

THEN

$0 < \cos x < \dfrac{1}{2}$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$-\dfrac{\pi}{2} < x < -\dfrac{\pi}{3}$ or $\dfrac{\pi}{3} < x < \dfrac{\pi}{2}$ ***A1A1***

[3 marks]

(c) $S_\infty = \dfrac{\sin x}{1 - 2\cos x}$ ***M1***

$S_\infty = \dfrac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1 - 2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$

$= \dfrac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \dfrac{\sqrt{15}}{2}$ ***AG***

[3 marks]

Total [7marks]

Question 27

let $\text{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$

consider $\text{P}(1)$:

$2! = 2$ and $2^1 (1!)^2 = 2$ so $\text{P}(1)$ is true **R1**

assume $\text{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2$, $k \in \mathbb{Z}^+$ **M1**

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\text{P}(k+1)$:

$(2(k+1))! = (2k+2)(2k+1)(2k)!$ **M1**

$(2(k+1))! \geq (2k+2)(2k+1)(k!)^2 2^k$ **A1**

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k+1 > k+1$ **R1**

$= 2^{k+1}((k+1)!)^2$ **A1**

$\text{P}(k+1)$ is true whenever $\text{P}(k)$ is true and $\text{P}(1)$ is true, so $\text{P}(n)$ is true for $n \in \mathbb{Z}^+$ **R1**

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

Total [7 marks]

Question 28

(a) (i) **METHOD 1**

$\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}}$ ***M1***

$= 2^{u_{n+1} - u_n} = 2^d$ ***A1***

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}}$ ***M1***

$= 2^d$ ***A1***

(ii) 2^a ***A1***

Note: Accept 2^{u_1}.

(iii) **EITHER**

v_n is a GP with first term 2^a and common ratio 2^d

$v_n = 2^a (2^d)^{(n-1)}$

OR

$u_n = a + (n-1)d$ as it is an AP

THEN

$v_n = 2^{a+(n-1)d}$ ***A1***

[4 marks]

(b) (i) $S_n = \frac{2^a\left((2^d)^n - 1\right)}{2^d - 1} = \frac{2^a\left(2^{dn} - 1\right)}{2^d - 1}$ ***M1A1***

Note: Accept either expression.

(ii) for sum to infinity to exist need $-1 < 2^d < 1$ ***R1***

$\Rightarrow \log 2^d < 0 \Rightarrow d \log 2 < 0 \Rightarrow d < 0$ ***(M1)A1***

Note: Also allow graph of 2^d.

(iii) $S_\infty = \frac{2^a}{1 - 2^d}$ ***A1***

(iv) $\dfrac{2^a}{1-2^d} = 2^{a+1} \Rightarrow \dfrac{1}{1-2^d} = 2$ **M1**

$\Rightarrow 1 = 2 - 2^{d+1} \Rightarrow 2^{d+1} = 1$

$\Rightarrow d = -1$ **A1**

[8 marks]

(c) **METHOD 1**

$w_n = pq^{n-1},\ z_n = \ln pq^{n-1}$ ***(A1)***

$z_n = \ln p + (n-1)\ln q$ ***M1A1***

$z_{n+1} - z_n = (\ln p + n\ln q) - (\ln p + (n-1)\ln q) = \ln q$

which is a constant so this is an AP

(with first term $\ln p$ and common difference $\ln q$)

$\sum_{i=1}^{n} z_i = \dfrac{n}{2}\left(2\ln p + (n-1)\ln q\right)$ ***M1***

$= n\left(\ln p + \ln q^{\left(\frac{n-1}{2}\right)}\right) = n\ln\left(pq^{\left(\frac{n-1}{2}\right)}\right)$ ***(M1)***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***A1***

METHOD 2

$\sum_{i=1}^{n} z_i = \ln p + \ln pq + \ln pq^2 + \ldots + \ln pq^{n-1}$ ***(M1)A1***

$= \ln\left(p^n q^{(1+2+3+\ldots+(n-1))}\right)$ ***(M1)A1***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***(M1)A1***

[6 marks]

Total [18 marks]

Question 29

$(3-x)^4 = 1.3^4 + 4.3^3(-x) + 6.3^2(-x)^2 + 4.3(-x)^3 + 1(-x)^4$ or equivalent ***(M1)(A1)***

$= 81 - 108x + 54x^2 - 12x^3 + x^4$ ***A1A1***

Note: ***A1*** for ascending powers, ***A1*** for correct coefficients including signs.

[4 marks]

Question 30

(a) **METHOD 1**

$z^3 = -\frac{27}{8} = \frac{27}{8}(\cos\pi + \mathrm{i}\sin\pi)$ ***M1(A1)***

$= \frac{27}{8}(\cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi))$ ***(A1)***

$z = \frac{3}{2}\left(\cos\left(\frac{\pi + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2n\pi}{3}\right)\right)$ ***M1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)$. ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $(re^{\mathrm{i}\theta})$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\dfrac{3}{2}$ so $(2z+3)$ is a factor

Attempt to use long division or factor theorem: ***M1***

$\Rightarrow 8z^3 + 27 \equiv (2z+3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$ ***A1***

Attempt to solve quadratic: ***M1***

$z = \dfrac{3 \pm 3\sqrt{3}\,\mathrm{i}}{4}$ ***A1***

$z_1 = \dfrac{3}{2}\left(\cos\dfrac{\pi}{3} + \mathrm{i}\sin\dfrac{\pi}{3}\right),$

$z_2 = \dfrac{3}{2}(\cos\pi + \mathrm{i}\sin\pi),$

$z_3 = \dfrac{3}{2}\left(\cos\dfrac{5\pi}{3} + \mathrm{i}\sin\dfrac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\dfrac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

METHOD 3

$8z^3 + 27 = 0$

Substitute $z = x + \mathrm{i}\,y$ ***M1***

$8\left(x^3 + 3\mathrm{i}\,x^2 y - 3xy^2 - \mathrm{i}\,y^3\right) + 27 = 0$

$\Rightarrow 8x^3 - 24xy^2 + 27 = 0$ and $24x^2 y - 8y^3 = 0$ ***A1***

Attempt to solve simultaneously: ***M1***

$8y\left(3x^2 - y^2\right) = 0$

$y = 0, y = x\sqrt{3}, y = -x\sqrt{3}$

$\Rightarrow \left(x = -\frac{3}{2}, y = 0\right), x = \frac{3}{4}, y = \pm\frac{3\sqrt{3}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2 = \frac{3}{2}\left(\cos\pi + \mathrm{i}\sin\pi\right),$

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

(b) **EITHER**

Valid attempt to use area $= 3\left(\frac{1}{2}ab\sin C\right)$ ***M1***

$= 3 \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$ ***A1A1***

Note: Award ***A1*** for correct sides, ***A1*** for correct $\sin C$.

OR

Valid attempt to use area $= \frac{1}{2}$ base $\times$ height ***M1***

area $= \frac{1}{2} \times \left(\frac{3}{4} + \frac{3}{2}\right) \times \frac{6\sqrt{3}}{4}$ ***A1A1***

Note: ***A1*** for correct height, ***A1*** for correct base.

THEN

$= \frac{27\sqrt{3}}{16}$ ***AG***

[3 marks]

Total [9 marks]

Question 31

(a) $a > 0$ **A1**

$a \neq 1$ **A1**

[2 marks]

(b) **METHOD 1**

$\log_x y = \frac{\ln y}{\ln x}$ and $\log_y x = \frac{\ln x}{\ln y}$ ***M1A1***

Note: Use of any base is permissible here, not just "e".

$\left(\frac{\ln y}{\ln x}\right)^2 = 4$ ***A1***

$\ln y = \pm 2 \ln x$ ***A1***

$y = x^2$ or $\frac{1}{x^2}$ ***A1A1***

METHOD 2

$\log_y x = \frac{\log_x x}{\log_x y} = \frac{1}{\log_x y}$ ***M1A1***

$(\log_x y)^2 = 4$ ***A1***

$\log_x y = \pm 2$ ***A1***

$y = x^2$ or $y = \frac{1}{x^2}$ ***A1A1***

Note: The final two ***A*** marks are independent of the one coming before.

[6 marks]

Total [8 marks]

Question 32

(a) $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ ***M1***

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ ***A1***

$= \sqrt{n+1}-\sqrt{n}$ ***AG***

[2 marks]

(b) $\sqrt{2}-1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ ***A2***

$< \frac{1}{\sqrt{2}}$ ***AG***

[2 marks]

(c) consider the case $n = 2$: required to prove that $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ ***M1***

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2}-1$

hence $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ ***A1***

now assume true for $n = k$: $\sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ ***M1***

$\frac{1}{\sqrt{1}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k+1$: $\frac{1}{\sqrt{1}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

from assumption, we have that $\frac{1}{\sqrt{1}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$ ***M1***

so attempt to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

EITHER

$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ ***A1***

$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}$, (from part a), which is true ***A1***

OR

$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}$. ***A1***

$> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \sqrt{k+1}$ ***A1***

THEN

so true for $n = 2$ and $n = k$ true $\Rightarrow n = k+1$ true. Hence true for all $n \geq 2$ ***R1***

Note: Award ***R1*** only if all previous ***M*** marks have been awarded.

[9 marks]

Total [13 marks]

Question 33

use of either $u_n = u_1 + (n-1)d$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ ***M1***

$u_1 + 4d = 6$ ***(A1)***

$\frac{12}{2}(2u_1 + 11d) = 45$ ***(A1)***

$\Rightarrow 4u_1 + 22d = 15$

attempt to solve simultaneous equations ***M1***

$4(6-4d) + 22d = 15$

$6d = -9 \Rightarrow d = -1.5$ ***A1***

$u_1 = 12$ ***A1***

[6 marks]

Question 34

METHOD 1

$m - n\log_3 2 = 10\log_9 6$

$m - n\log_3 2 = 5\log_3 6$ ***M1***

$m = \log_3\left(6^5 2^n\right)$ ***(M1)***

$3^m 2^{-n} = 6^5 = 3^5 \times 2^5$ ***(M1)***

$m = 5,\ n = -5$ ***A1***

ote: First ***M1*** is for any correct change of base, second ***M1*** for writing as a single logarithm, third ***M1*** is for writing 6 as 2×3.

METHOD 2

$m - n\log_3 2 = 10\log_9 6$

$m - n\log_3 2 = 5\log_3 6$ ***M1***

$m - n\log_3 2 = 5\log_3 3 + 5\log_3 2$ ***(M1)***

$m - n\log_3 2 = 5 + 5\log_3 2$ ***(M1)***

$m = 5,\ n = -5$ ***A1***

ote: First ***M1*** is for any correct change of base, second ***M1*** for writing 6 as 2×3 and third ***M1*** is for forming an expression without $\log_3 3$.

[4 marks]

Question 35

(a) $\left(\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3=\cos\pi+\mathrm{i}\sin\pi$ **M1**

$=-1$ **A1**

[2 marks]

(b) show the expression is true for $n=1$ **R1**

assume true for $n=k$, $(\cos\theta-\mathrm{i}\sin\theta)^k=\cos k\theta-\mathrm{i}\sin k\theta$ **M1**

Note: Do not accept "let $n=k$" or "assume $n=k$", assumption of truth must be present.

$(\cos\theta-\mathrm{i}\sin\theta)^{k+1}=(\cos\theta-\mathrm{i}\sin\theta)^k(\cos\theta-\mathrm{i}\sin\theta)$

$=(\cos k\theta-\mathrm{i}\sin k\theta)(\cos\theta-\mathrm{i}\sin\theta)$ **M1**

$=\cos k\theta\cos\theta-\sin k\theta\sin\theta-\mathrm{i}(\cos k\theta\sin\theta+\sin k\theta\cos\theta)$ **A1**

Note: Award ***A1*** for any correct expansion.

$=\cos((k+1)\theta)-\mathrm{i}\sin((k+1)\theta)$ **A1**

therefore if true for $n=k$ true for $n=k+1$, true for $n=1$, so true for all $n(\in\mathbb{Z}^+)$ **R1**

Note: To award the final ***R*** mark the first 4 marks must be awarded.

[6 marks]

(c) $(z)^n+(z^*)^n=(\cos\theta+\mathrm{i}\sin\theta)^n+(\cos\theta-\mathrm{i}\sin\theta)^n$

$=\cos n\theta+\mathrm{i}\sin n\theta+\cos n\theta-\mathrm{i}\sin n\theta=2\cos(n\theta)$ **(M1)A1**

[2 marks]

(d) (i) $zz^*=(\cos\theta+\mathrm{i}\sin\theta)(\cos\theta-\mathrm{i}\sin\theta)$

$=\cos^2\theta+\sin^2\theta$ **A1**

$=1$ **AG**

e: Allow justification starting with $|z|=1$.

(ii) $(z+z^*)^3=z^3+3z^2z^*+3z(z^*)^2+(z^*)^3\left(=z^3+3z+3z^*+(z^*)^3\right)$ **A1**

(iii) $(z+z^*)^3=(2\cos\theta)^3$ **A1**

$z^3+3z+3z^*+(z^*)^3=2\cos 3\theta+6\cos\theta$ **M1A1**

$\cos 3\theta=4\cos^3\theta-3\cos\theta$ **AG**

Note: ***M1*** is for using $zz^*=1$, this might be seen in d(ii).

[5 marks]

(e) $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

$\cos(3\theta) = \cos(2\theta)$ ***A1A1***

Note: ***A1*** for $\cos(3\theta)$ and ***A1*** for $\cos(2\theta)$.

$\theta = 0$ ***A1***

or $3\theta = 2\pi - 2\theta$ (or $3\theta = 4\pi - 2\theta$) ***M1***

$\theta = \frac{2\pi}{5}, \frac{4\pi}{5}$ ***A1A1***

Note: Do not accept solutions via factor theorem or other methods that do not follow "hence".

[6 marks]

Total [21 marks]

Question 36

(a) $1,\ nx,\ \frac{n(n-1)}{2}x^2,\ \frac{n(n-1)(n-2)}{6}x^3$ ***A1A1***

Note: Award ***A1*** for the first two terms and ***A1*** for the next two terms.

Note: Accept $\binom{n}{r}$ notation.

Note: Allow the terms seen in the context of an arithmetic sum.

Note: Allow unsimplified terms, *eg*, those including powers of 1 if seen.

[2 marks]

(b) (i) **EITHER**

using $u_3 - u_2 = u_4 - u_3$ ***(M1)***

$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}$ ***A1***

attempting to remove denominators and expanding (or vice versa) ***M1***

$3n^2 - 9n = n^3 - 6n^2 + 5n$ (or equivalent, *eg*, $6n^2 - 12n = n^3 - 3n^2 + 2n$) ***A1***

OR

using $u_2 + u_4 = 2u_3$ ***(M1)***

$n + \frac{n(n-1)(n-2)}{6} = n(n-1)$ ***(A1)***

attempting to remove denominators and expanding (or vice versa) ***M1***

$6n + n^3 - 3n^2 + 2n = 6n^2 - 6n$ (or equivalent) ***(A1)***

THEN

$n^3 - 9n^2 + 14n = 0$ ***AG***

(ii) $n(n-2)(n-7) = 0$ or $(n-2)(n-7) = 0$ ***(A1)***

$n = 7$ only (as $n \geq 3$) ***A1***

[6 marks]

Total [8 marks]

Question 37

let $\mathrm{P}(n)$ be the proposition that $n\left(n^2+5\right)$ is divisible by 6 for $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $n\left(n^2+5\right)=1\times\left(1^2+5\right)=6$ and so $\mathrm{P}(1)$ is true **R1**

assume $\mathrm{P}(k)$ is true *ie*, $k\left(k^2+5\right)=6m$ where $k, m \in \mathbb{Z}^+$ **M1**

te: Do not award ***M1*** for statements such as "let $n=k$".

consider $\mathrm{P}(k+1)$:

$(k+1)\left((k+1)^2+5\right)$ **M1**

$=(k+1)\left(k^2+2k+6\right)$

$=k^3+3k^2+8k+6$ ***(A1)***

$=\left(k^3+5k\right)+\left(3k^2+3k+6\right)$ **A1**

$=k\left(k^2+5\right)+3k(k+1)+6$ **A1**

$k(k+1)$ is even hence all three terms are divisible by 6 **R1**

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ **R1**

ote: To obtain the final ***R1***, four of the previous marks must have been awarded.

[8 marks]

Question 38

(a) **EITHER**

$w^7=\left(\cos\frac{2\pi}{7}+\mathrm{i}\sin\frac{2\pi}{7}\right)^7$ ***(M1)***

$=\cos 2\pi+\mathrm{i}\sin 2\pi$ **A1**

$=1$ **A1**

so w is a root **AG**

OR

$z^7=1=\cos(2\pi k)+i\sin(2\pi k)$ ***(M1)***

$z=\cos\left(\frac{2\pi k}{7}\right)+i\sin\left(\frac{2\pi k}{7}\right)$ **A1**

$k=1 \Rightarrow z=\cos\left(\frac{2\pi}{7}\right)+i\sin\left(\frac{2\pi}{7}\right)$ **A1**

so w is a root **AG**

[3 marks]

(b) (i) $(w-1)\left(1+w+w^2+w^3+w^4+w^5+w^6\right)$

$=w+w^2+w^3+w^4+w^5+w^6+w^7-1-w-w^2-w^3-w^4-w^5-w^6$ **M1**

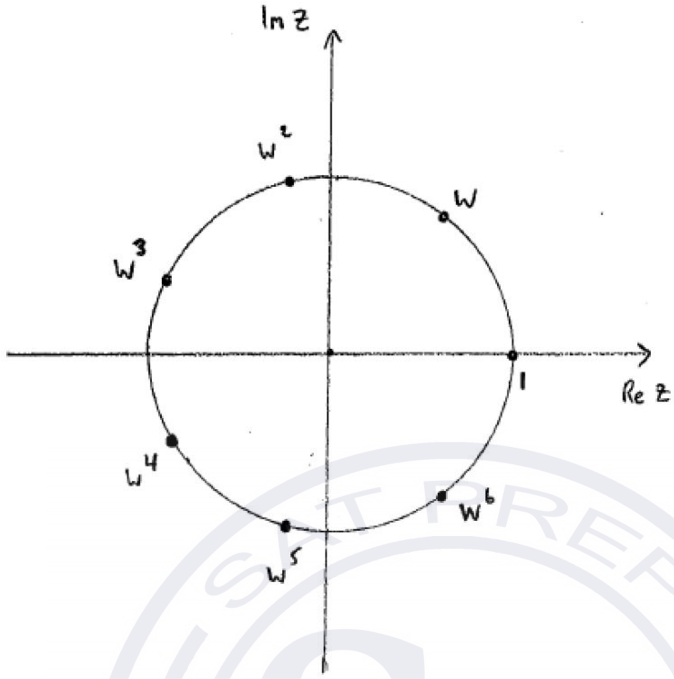
$=w^7-1\ (=0)$ **A1**

(ii) $w^7-1=0$ and $w-1\neq 0$ **R1**

so $1+w+w^2+w^3+w^4+w^5+w^6=0$ **AG**

[3 marks]

(c) the roots are 1, w, w^2, w^3, w^4, w^5 and w^6 ***A1***



7 points equidistant from the origin ***A1***

approximately correct angular positions for 1, w, w^2, w^3, w^4, w^5 and w^6 ***A1***

Note:	Condone use of *cis* notation for the final two ***A*** marks.

Note:	For the final ***A*** mark there should be one root in the first quadrant, two in the second, two in the third, one in the fourth, and one on the real axis.

[3 marks]

(d) (i) $\alpha^* = \left(w + w^2 + w^4\right)^*$

$= w^* + \left(w^2\right)^* + \left(w^4\right)^*$ ***A1***

since $w^* = w^6$, $\left(w^2\right)^* = w^5$ and $\left(w^4\right)^* = w^3$ ***R1***

$\Rightarrow \alpha^* = w^6 + w^5 + w^3$ ***AG***

(ii) $b = -\left(\alpha + \alpha^*\right)$ (using sum of roots (or otherwise)) ***(M1)***

$b = -\left(w + w^2 + w^3 + w^4 + w^5 + w^6\right)$ ***(A1)***

$= -(-1)$

$= 1$ ***A1***

$c = \alpha\alpha^*$ (using product of roots (or otherwise)) ***(M1)***

$c = \left(w + w^2 + w^4\right)\left(w^6 + w^5 + w^3\right)$

EITHER

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4$ ***A1***

$= \left(w^6 + w^5 + w^4 + w^3 + w^2 + w\right) + 3$ ***M1***

$= 3 - 1$ ***(A1)***

OR

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4 \left(= w^4\left(1 + w + w^3\right)\left(w^3 + w^2 + 1\right)\right)$ ***A1***

$= w^4\left(w^6 + w^5 + w^4 + w^2 + w + 1 + 3w^3\right)$ ***M1***

$= w^4\left(w^6 + w^5 + w^4 + w^3 + w^2 + w + 1 + 2w^3\right)$

$= w^4\left(2w^3\right)$ ***(A1)***

THEN

$= 2$ ***A1***

[10 marks]

(e) $z^2 + z + 2 = 0 \Rightarrow z = \dfrac{-1 \pm \mathrm{i}\sqrt{7}}{2}$ ***M1A1***

$\operatorname{Im}\left(w + w^2 + w^4\right) > 0$ ***R1***

$\operatorname{Im}\alpha = \dfrac{\sqrt{7}}{2}$ ***A1***

Note: Final ***A*** mark is independent of previous ***R*** mark.

[4 marks]

Total [23 marks]

Question 39

(a) $u_1 = 1$ **A1**

[1 mark]

(b) $u_6 = S_6 - S_5 = 31$ **M1A1**

[2 marks]

(c) $u_n = S_n - S_{n-1}$ **M1**

$= \left(3n^2 - 2n\right) - \left(3(n-1)^2 - 2(n-1)\right)$

$= \left(3n^2 - 2n\right) - \left(3n^2 - 6n + 3 - 2n + 2\right)$

$= 6n - 5$ **A1**

$d = u_{n+1} - u_n$ **R1**

$= 6n + 6 - 5 - 6n + 5$

$= \left(6(n+1) - 5\right) - (6n - 5)$

$= 6$ (constant) **A1**

Notes: Award ***R1*** only if candidate provides a clear argument that proves that the difference between **ANY** two consecutive terms of the sequence is constant. Do not accept examples involving particular terms of the sequence nor circular reasoning arguments (*eg* use of formulas of APs to prove that it is an AP). Last ***A1*** is independent of ***R1***.

[4 marks]

Total [7 marks]

Question 40

attempt to form a quadratic in 2^x **M1**

$\left(2^x\right)^2 + 4 \cdot 2^x - 3 = 0$ **A1**

$2^x = \dfrac{-4 \pm \sqrt{16 + 12}}{2} \left(= -2 \pm \sqrt{7}\right)$ **M1**

$2^x = -2 + \sqrt{7}$ (as $-2 - \sqrt{7} < 0$) **R1**

$x = \log_2\left(-2 + \sqrt{7}\right) \quad \left(x = \dfrac{\ln\left(-2 + \sqrt{7}\right)}{\ln 2}\right)$ **A1**

te: Award ***R0 A1*** if final answer is $x = \log_2\left(-2 \pm \sqrt{7}\right)$.

[5 marks]

Question 41

(a) (i) **METHOD 1**

$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=0$ ***A1***

as $\omega \neq 1$ ***R1***

METHOD 2

solutions of $1-\omega^3=0$ are $\omega=1,\ \omega=\frac{-1\pm\sqrt{3}\mathrm{i}}{2}$ ***A1***

verification that the sum of these roots is 0 ***R1***

(ii) $1+\omega^*+(\omega^*)^2=0$ ***A2***

[4 marks]

(b) $(\omega-3\omega^2)(\omega^2-3\omega)=-3\omega^4+10\omega^3-3\omega^2$ ***M1A1***

EITHER

$=-3\omega^2(\omega^2+\omega+1)+13\omega^3$ ***M1***

$=-3\omega^2\times 0+13\times 1$ ***A1***

OR

$=-3\omega+10-3\omega^2=-3(\omega^2+\omega+1)+13$ ***M1***

$=-3\times 0+13$ ***A1***

OR

substitution by $\omega=\frac{-1\pm\sqrt{3}\mathrm{i}}{2}$ in any form ***M1***

numerical values of each term seen ***A1***

THEN

$=13$ ***AG***

[4 marks]

(c) $|p|=|q|\Rightarrow\sqrt{1^2+3^2}=\sqrt{x^2+(2x+1)^2}$ ***(M1)(A1)***

$5x^2+4x-9=0$ ***A1***

$(5x+9)(x-1)=0$ ***(M1)***

$x=1,\ x=-\frac{9}{5}$ ***A1***

[5 marks]

(d) $pq = (1 - 3i)\left(x + (2x + 1)i\right) = (7x + 3) + (1 - x)i$ **M1A1**

$\text{Re}(pq) + 8 < \left(\text{Im}(pq)\right)^2 \Rightarrow (7x + 3) + 8 < (1 - x)^2$ **M1**

$\Rightarrow x^2 - 9x - 10 > 0$ **A1**

$\Rightarrow (x+1)(x-10) > 0$ **M1**

$x < -1,\ x > 10$ **A1**

[6 marks]

Total [19 marks]

Question 42

$$\log_2 x - \log_2 5 = 2 + \log_2 3$$

collecting at least two log terms ***(M1)***

eg $\log_2 \frac{x}{5} = 2 + \log_2 3$ or $\log_2 \frac{x}{15} = 2$

obtaining a correct equation without logs ***(M1)***

eg $\frac{x}{5} = 12$ **OR** $\frac{x}{15} = 2^2$ ***(A1)***

$x = 60$ ***A1***

[4 marks]

Question 43

(a) $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

(i) $|w| = \sqrt{2}$ ***A1***

(ii) $\arg w = \frac{\pi}{12}$ ***A1***

Notes: Allow ***FT*** from incorrect answers for z_1 and z_2 in modulus-argument form.

[4 marks]

(b) **EITHER**

$\sin\left(\frac{\pi n}{12}\right) = 0$ ***(M1)***

OR

$\arg\left(w^n\right) = \pi$ ***(M1)***

$\frac{n\pi}{12} = \pi$

THEN

$\therefore n = 12$ ***A1***

[2 marks]

Total [6 marks]

Question 44

Question 51

(a) use of $u_n = u_1 + (n-1)d$ **M1**

$(1+2d)^2 = (1+d)(1+5d)$ (or equivalent) **M1A1**

$d = -2$ **A1**

[4 marks]

(b) $1 + (N-1) \times -2 = -15$

$N = 9$ ***(A1)***

$\sum_{r=1}^{9} u_r = \frac{9}{2}(2 + 8 \times -2)$ ***(M1)***

$= -63$ **A1**

[3 marks]

Total [7 marks]

Question 45

let $P(n)$ be the proposition that $4^n + 15n - 1$ is divisible by 9

showing true for $n = 1$ **A1**

ie for $n = 1$, $4^1 + 15 \times 1 - 1 = 18$

which is divisible by 9, therefore $P(1)$ is true

assume $P(k)$ is true so $4^k + 15k - 1 = 9A, \left(A \in \mathbb{Z}^+\right)$ **M1**

Note: Only award ***M1*** if "truth assumed" or equivalent.

consider $4^{k+1} + 15(k+1) - 1$

$= 4 \times 4^k + 15k + 14$

$= 4(9A - 15k + 1) + 15k + 14$ **M1**

$= 4 \times 9A - 45k + 18$ **A1**

$= 9(4A - 5k + 2)$ which is divisible by 9 **R1**

Note: Award ***R1*** for either the expression or the statement above.

since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, therefore (by the principle of mathematical induction) $P(n)$ is true for $n \in \mathbb{Z}^+$ **R1**

Note: Only award the final ***R1*** if the 2 ***M1***s have been awarded.

[6 marks]

Question 46

attempt at binomial expansion, relevant row of Pascal's triangle or use of general term with binomial coefficient must be seen ***(M1)***

term independent of x is $\binom{10}{4}\left(2x^2\right)^6\left(\frac{1}{2x^3}\right)^4$ (or equivalent) ***(A1)(A1)(A1)***

Notes: x's may be omitted. Also accept $\binom{10}{6}$ or 210.

$= 840$ **A1**

[5 marks]

Question 47

(a) **EITHER**

the first three terms of the geometric sequence are 9, $9r$ and $9r^2$ ***(M1)***

$9 + 3d = 9r$ $(\Rightarrow 3 + d = 3r)$ and $9 + 7d = 9r^2$ ***(A1)***

attempt to solve simultaneously ***(M1)***

$$9 + 7d = 9\left(\frac{3+d}{3}\right)^2$$

OR

the 1st, 4th and 8th terms of the arithmetic sequence are

9, $9 + 3d$, $9 + 7d$ ***(M1)***

$$\frac{9+7d}{9+3d} = \frac{9+3d}{9}$$ ***(A1)***

attempt to solve ***(M1)***

THEN

$d = 1$ ***A1***

[4 marks]

(b) $r = \frac{4}{3}$ ***A1***

Note:	Accept answers where a candidate obtains d by finding r first. The first two marks in either method for part (a) are awarded for the same ideas and the third mark is awarded for attempting to solve an equation in r.

[1 mark]

Total [5 marks]

Question 48

C represents the complex number $1 - 2i$ ***A2***

D represents the complex number $3 + 2i$ ***A2***

[4 marks]

Question 49

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$$

show true for $n=3$ ***(M1)***

LHS $=\binom{2}{2}=1$ RHS $=\binom{3}{3}=1$ ***A1***

hence true for $n=3$

assume true for $n=k$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$ ***M1***

consider for $n=k+1$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}$ ***(M1)***

$=\binom{k}{3}+\binom{k}{2}$ ***A1***

$=\frac{k!}{(k-3)!3!}+\frac{k!}{(k-2)!2!}\left(=\frac{k!}{3!}\left[\frac{1}{(k-3)!}+\frac{3}{(k-2)!}\right]\right)$ or any correct expression with a visible common factor ***(A1)***

$=\frac{k!}{3!}\left[\frac{k-2+3}{(k-2)!}\right]$ or any correct expression with a common denominator ***(A1)***

$=\frac{k!}{3!}\left[\frac{k+1}{(k-2)!}\right]$

Note: At least one of the above three lines or equivalent must be seen.

$=\frac{(k+1)!}{3!(k-2)!}$ or equivalent ***A1***

$=\binom{k+1}{3}$

Result is true for $k=3$. If result is true for k it is true for $k+1$. Hence result is true for all $k \geq 3$. Hence proved by induction. ***R1***

Note: In order to award the ***R1*** at least ***[5 marks]*** must have been awarded.

[9 marks]

Question 50

(c) (i) **EITHER**

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1-\cos 2\theta)}$ ***A1***

$= \sqrt{2\left(2\sin^2\theta\right)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\dfrac{\sin 2\theta}{1-\cos 2\theta}$ ***M1***

$= \dfrac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan(\dfrac{\pi}{2} - \theta)\right)$ ***A1***

$= \theta - \dfrac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2i\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta(\sin\theta - i\cos\theta)$ ***(A1)***

$= -2i\sin\theta(\cos\theta + i\sin\theta)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + i\sin\left(\theta - \dfrac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \dfrac{\pi}{2}$ ***A1***

(ii) attempt to apply De Moivre's theorem ***M1***

$$(1-\cos 2\theta - \mathrm{i}\sin 2\theta)^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta - \frac{\pi}{2} + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\theta - \frac{\pi}{2} + 2n\pi}{3}\right)\right]$$

A1A1A1

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n = -1, 0, 1$.
Equivalent forms are acceptable. ***A1***

[14 marks]

Question 51

$\log_2(x+3) + \log_2(x-3) = 4$

$\log_2(x^2 - 9) = 4$ ***(M1)***

$x^2 - 9 = 2^4\ (=16)$ ***M1A1***

$x^2 = 25$

$x = \pm 5$ ***(A1)***

$x = 5$ ***A1***

[5 marks]

Question 52

each term is of the form $\binom{7}{r}(x^2)^{7-r}\left(\frac{-2}{x}\right)^r$ ***(M1)***

$= \binom{7}{r}x^{14-2r}(-2)^r x^{-r}$

so $14 - 3r = 8$ ***(A1)***

$r = 2$

so require $\binom{7}{2}(x^2)^5\left(\frac{-2}{x}\right)^2$ (or simply $\binom{7}{2}(-2)^2$) ***A1***

$= 21 \times 4$

$= 84$ ***A1***

e: Candidates who attempt a full expansion, including the correct term, may only be awarded ***M1A0A0A0***.

[4 marks]

Question 53

METHOD 1

$216\text{i} = 216\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$ ***A1***

$z + 2\text{i} = \sqrt[3]{216}\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \text{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)^{\frac{1}{3}}$ ***(M1)***

$z + 2\text{i} = 6\left(\cos\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right) + \text{i}\sin\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right)\right)$ ***A1***

$z_1 + 2\text{i} = 6\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2} + \frac{\text{i}}{2}\right) = 3\sqrt{3} + 3\text{i}$

$z_2 + 2\text{i} = 6\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right) = 6\left(\frac{-\sqrt{3}}{2} + \frac{\text{i}}{2}\right) = -3\sqrt{3} + 3\text{i}$

$z_3 + 2\text{i} = 6\left(\cos\frac{3\pi}{2} + \text{i}\sin\frac{3\pi}{2}\right) = -6\text{i}$ ***A2***

: Award ***A1A0*** for one correct root.

so roots are $z_1 = 3\sqrt{3} + \text{i}$, $z_2 = -3\sqrt{3} + \text{i}$ and $z_3 = -8\text{i}$ ***M1A1***

: Award ***M1*** for subtracting 2i from their three roots.

[7 marks]

METHOD 2

$\left(a\sqrt{3} + (b+2)\text{i}\right)^3 = 216\text{i}$

$\left(a\sqrt{3}\right)^3 + 3\left(a\sqrt{3}\right)^2(b+2)\text{i} - 3\left(a\sqrt{3}\right)(b+2)^2 - \text{i}(b+2)^3 = 216\text{i}$ ***M1A1***

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 + \text{i}\left(3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3\right) = 216\text{i}$

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 = 0$ and $3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3 = 216$ ***M1A1***

$a\left(a^2 - (b+2)^2\right) = 0$ and $9a^2(b+2) - (b+2)^3 = 216$

$a = 0$ or $a^2 = (b+2)^2$

if $a = 0$, $-(b+2)^3 = 216 \Rightarrow b + 2 = -6$

$\therefore b = -8$ ***A1***

$(a, b) = (0, -8)$

if $a^2 = (b+2)^2$, $9(b+2)^2(b+2) - (b+2)^3 = 216$

$8(b+2)^3 = 216$

$(b+2)^3 = 27$

$b + 2 = 3$

$b = 1$

$\therefore a^2 = 9 \Rightarrow a = \pm 3$

$\therefore (a, b) = (\pm 3, 1)$ ***A1A1***

so roots are $z_1 = 3\sqrt{3} + \text{i}$, $z_2 = -3\sqrt{3} + \text{i}$ and $z_3 = -8\text{i}$

METHOD 3

$(z+2i)^3-(-6i)^3=0$

attempt to factorise: ***M1***

$((z+2i)-(-6i))((z+2i)^2+(z+2i)(-6i)+(-6i)^2)=0$ ***A1***

$(z+8i)(z^2-2iz-28)=0$ ***A1***

$z+8i=0 \Rightarrow z=-8i$ ***A1***

$z^2-2iz-28=0 \Rightarrow z=\dfrac{2i\pm\sqrt{-4-(4\times1\times-28)}}{2}$ ***M1***

$z=\dfrac{2i\pm\sqrt{108}}{2}$

$z=\dfrac{2i\pm6\sqrt{3}}{2}$

$z=i\pm3\sqrt{3}$ ***A1A1***

Question 54

(a) **METHOD 1**

$\log_{r^2}x=\dfrac{\log_r x}{\log_r r^2}\left(=\dfrac{\log_r x}{2\log_r r}\right)$ ***M1A1***

$=\dfrac{\log_r x}{2}$ ***AG***

[2 marks]

METHOD 2

$\log_{r^2}x=\dfrac{1}{\log_x r^2}$ ***M1***

$=\dfrac{1}{2\log_x r}$ ***A1***

$=\dfrac{\log_r x}{2}$ ***AG***

[2 marks]

(b) **METHOD 1**

$\log_2 y+\log_4 x+\log_4 2x=0$

$\log_2 y+\log_4 2x^2=0$ ***M1***

$\log_2 y+\dfrac{1}{2}\log_2 2x^2=0$ ***M1***

$\log_2 y=-\dfrac{1}{2}\log_2 2x^2$

$\log_2 y=\log_2\left(\dfrac{1}{\sqrt{2}x}\right)$ ***M1A1***

$y=\dfrac{1}{\sqrt{2}}x^{-1}$ ***A1***

METHOD 2

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \frac{1}{2}\log_2 x + \frac{1}{2}\log_2 2x = 0$ ***M1***

$\log_2 y + \log_2 x^{\frac{1}{2}} + \log_2 (2x)^{\frac{1}{2}} = 0$ ***M1***

$\log_2 \left(\sqrt{2}xy\right) = 0$ ***M1***

$\sqrt{2}xy = 1$ ***A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

Question 55

(a) $\frac{z+w}{z-w} = \frac{(a+c)+\mathrm{i}(b+d)}{(a-c)+\mathrm{i}(b-d)}$

$= \frac{(a+c)+\mathrm{i}(b+d)}{(a-c)+\mathrm{i}(b-d)} \times \frac{(a-c)-\mathrm{i}(b-d)}{(a-c)-\mathrm{i}(b-d)}$ ***M1A1***

real part $= \frac{(a+c)(a-c)+(b+d)(b-d)}{(a-c)^2+(b-d)^2} \left(= \frac{a^2-c^2+b^2-d^2}{(a-c)^2+(b-d)^2}\right)$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

[4 marks]

(b) $|z| = |w| \Rightarrow a^2+b^2 = c^2+d^2$ ***R1***

hence real part $= 0$ ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [6 marks]

Question 56

(a) **METHOD 1**

state that $u_n = u_1 r^{n-1}$ (or equivalent) ***A1***

attempt to consider a_n and use of at least one log rule ***M1***

$\log_2 |u_n| = \log_2 |u_1| + (n-1)\log_2 |r|$ ***A1***

(which is an AP) with $d = \log_2 |r|$ (and 1st term $\log_2 |u_1|$) ***A1***

so A is an arithmetic sequence ***AG***

Note: Condone absence of modulus signs.

Note: The final ***A*** mark may be awarded independently.

Note: Consideration of the first two or three terms only will score ***M0***.

[4 marks]

METHOD 2

consideration of $(d =) a_{n+1} - a_n$ ***M1***

$(d) = \log_2 |u_{n+1}| - \log_2 |u_n|$

$(d) = \log_2 \left| \frac{u_{n+1}}{u_n} \right|$ ***M1***

$(d) = \log_2 |r|$ ***A1***

which is constant ***R1***

Note: Condone absence of modulus signs.

Note: the final ***A*** mark may be awarded independently.

Note: Consideration of the first two or three terms only will score ***M0***.

(b) attempting to solve $\frac{3}{1-r} = 4$ ***M1***

$r = \frac{1}{4}$ ***A1***

$d = -2$ ***A1***

[3 marks]

Total [7 marks]

Question 57

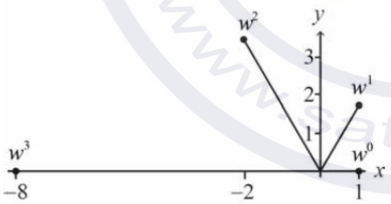
(a) (i) $w^2 = 4cis\left(\frac{2\pi}{3}\right)$; $w^3 = 8\text{cis}(\pi)$ ***(M1)A1A1***

Note: Accept Euler form.

Note: ***M1*** can be awarded for either both correct moduli or both correct arguments.

Note: Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

(ii)



A1A1

[5 marks]

(b) use of area $= \frac{1}{2}ab\sin C$ ***M1***

$\frac{1}{2}\times 1 \times 2 \times \sin\frac{\pi}{3} + \frac{1}{2} \times 2 \times 4 \times \sin\frac{\pi}{3} + \frac{1}{2} \times 4 \times 8 \times \sin\frac{\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $C = \frac{\pi}{3}$, ***A1*** for correct moduli.

$= \frac{21\sqrt{3}}{2}$ ***AG***

Note: Other methods of splitting the area may receive full marks.

[3 marks]

(c) $\frac{1}{2}\times 2^0 \times 2^1\times \sin\frac{\pi}{n} + \frac{1}{2}\times 2^1\times 2^2\times \sin\frac{\pi}{n} + \frac{1}{2}\times 2^2\times 2^3\times \sin\frac{\pi}{n} + \ldots + \frac{1}{2}\times 2^{n-1}\times 2^n\times \sin\frac{\pi}{n}$ ***M1A1***

Note: Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n}\times\left(2^0+2^2+2^4+\ldots+2^{2n-2}\right)$

identifying a geometric series with common ratio $2^2(=4)$ ***(M1)A1***

$=\frac{1-2^{2n}}{1-4}\times\sin\frac{\pi}{n}$ ***M1***

Note: Award ***M1*** for use of formula for sum of geometric series.

$=\frac{1}{3}\left(4^n-1\right)\sin\frac{\pi}{n}$ ***A1***

[6 marks]

Total [14 marks]

Question 58

if $n=1$

$\text{LHS}=1;\text{RHS}=4-\frac{3}{2^0}=4-3=1$ ***M1***

hence true for $n=1$

assume true for $n=k$ ***M1***

te: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

if $n=k+1$

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$

$=4-\frac{2(k+2)-(k+1)}{2^k}=4-\frac{k+3}{2^k}\left(=4-\frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n=k$ then also true for $n=k+1$, as true for $n=1$, so true (for all $n\in\mathbb{Z}^+$) ***R1***

te: Award the final ***R1*** only if the first four marks have been awarded.

[7 marks]

Question 59

$(\ln x)^2 - (\ln 2)(\ln x) - 2(\ln 2)^2 (= 0)$

EITHER

$\ln x = \dfrac{\ln 2 \pm \sqrt{(\ln 2)^2 + 8(\ln 2)^2}}{2}$ ***M1***

$= \dfrac{\ln 2 \pm 3\ln 2}{2}$ ***A1***

OR

$(\ln x - 2\ln 2)(\ln x + \ln 2)(=0)$ ***M1A1***

THEN

$\ln x = 2\ln 2$ or $-\ln 2$ ***A1***

$\Rightarrow x = 4$ or $x = \dfrac{1}{2}$ ***(M1)A1***

ote: ***(M1)*** is for an appropriate use of a log law in either case, dependent on the previous ***M1*** being awarded, ***A1*** for both correct answers.

solution is $\dfrac{1}{2} < x < 4$ ***A1***

[6 marks]

Question 60

attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

$1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

attempt to solve their two equations ***M1***

$p = -3$, $q = 2$ ***A1***

[5 marks]

Question 61

(a) $\left(r(\cos\theta + \mathrm{i}\sin\theta)\right)^{24} = 1(\cos 0 + \mathrm{i}\sin 0)$

use of De Moivre's theorem ***(M1)***

$r^{24} = 1 \Rightarrow r = 1$ ***(A1)***

$24\theta = 2\pi n \Rightarrow \theta = \dfrac{\pi n}{12}, (n \in \mathbb{Z})$ ***(A1)***

$0 < \arg z < \dfrac{\pi}{2} \Rightarrow n = 1, 2, 3, 4, 5$

$z = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{2\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{3\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{4\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{5\pi \mathrm{i}}{12}}$ ***A2***

Note: Award ***A1*** if additional roots are given or if three correct roots are given with no incorrect (or additional) roots.

[5 marks]

(b) (i) $\mathrm{Re}\,S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\mathrm{Im}\,S = \sin\frac{\pi}{12} + \sin\frac{2\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{4\pi}{12} + \sin\frac{5\pi}{12}$ ***A1***

Note: Award ***A1*** for both parts correct.

but $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$, $\sin\frac{4\pi}{12} = \cos\frac{2\pi}{12}$, $\sin\frac{3\pi}{12} = \cos\frac{3\pi}{12}$,

$\sin\frac{2\pi}{12} = \cos\frac{4\pi}{12}$ and $\sin\frac{\pi}{12} = \cos\frac{5\pi}{12}$ ***M1A1***

$\Rightarrow \mathrm{Re}\,S = \mathrm{Im}\,S$ ***AG***

Note: Accept a geometrical method.

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ ***M1A1***

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ ***A1***

(iii) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$ ***(M1)***

Note: Allow alternative methods *eg* $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)***

$\mathrm{Re}\,S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\mathrm{Re}\,S = \frac{\sqrt{2}+\sqrt{6}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}$ ***A1***

$= \frac{1}{2}\left(\sqrt{6} + 1 + \sqrt{2} + \sqrt{3}\right)$ ***A1***

$= \frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)$

$S = \mathrm{Re}(S)(1+\mathrm{i})$ since $\mathrm{Re}\,S = \mathrm{Im}\,S$, ***R1***

$S = \frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)(1+\mathrm{i})$ ***AG***

[11 marks]

Total [16 marks]

Question 62

$-\mathrm{i}\sqrt{3}$ is a root **(A1)**

$3+\log_2 3-\log_2 6\left(=3+\log_2\frac{1}{2}=3-1=2\right)$ is a root **(A1)**

sum of roots: $-a=3+\log_2 3 \Rightarrow a=-3-\log_2 3$ ***M1***

: Award ***M1*** for use of $-a$ is equal to the sum of the roots, do not award if minus is missing.

: If expanding the factored form of the equation, award ***M1*** for equating a to the coefficient of z^3.

product of roots: $(-1)^4 d \quad = 2(\log_2 6)\left(\mathrm{i}\sqrt{3}\right)\left(-\mathrm{i}\sqrt{3}\right)$ ***M1***

$= 6\log_2 6$ ***A1***

: Award ***M1A0*** for $d=-6\log_2 6$.

$6a+d+12=-18-6\log_2 3+6\log_2 6+12$

EITHER

$=-6+6\log_2 2=0$ ***M1A1AG***

: ***M1*** is for a correct use of one of the log laws.

OR

$=-6-6\log_2 3+6\log_2 3+6\log_2 2=0$ ***M1A1AG***

: ***M1*** is for a correct use of one of the log laws.

[7 marks]

Question 63

consider $n=1$. $1(1!)=1$ and $2!-1=1$ therefore true for $n=1$ ***R1***

: There must be evidence that $n=1$ has been substituted into both expressions, or an expression such LHS=RHS=1 is used. "therefore true for $n=1$" or an equivalent statement must be seen.

assume true for $n=k$, (so that $\sum_{r=1}^{k} r(r!)=(k+1)!-1$) ***M1***

: Assumption of truth must be present.

consider $n=k+1$

$\sum_{r=1}^{k+1} r(r!)=\sum_{r=1}^{k} r(r!)+(k+1)(k+1)!$ ***(M1)***

$=(k+1)!-1+(k+1)(k+1)!$ ***A1***

$=(k+2)(k+1)!-1$ ***M1***

: ***M1*** is for factorising $(k+1)!$

$=(k+2)!-1$

$=\left((k+1)+1\right)!-1$

so if true for $n=k$, then also true for $n=k+1$, and as true for $n=1$ then true for all $n\left(\in\mathbb{Z}^+\right)$ ***R1***

: Only award final ***R1*** if all three method marks have been awarded.
Award ***R0*** if the proof is developed from both LHS and RHS.

Total [6 marks]

Question 64

(a) an attempt at a valid method *eg* by inspection or row reduction **(M1)**

$2 \times R_2 = R_1 \Rightarrow 2a = -1$

$\Rightarrow a = -\frac{1}{2}$ **A1**

[2 marks]

(b) using elimination or row reduction to eliminate one variable **(M1)**

correct pair of equations in 2 variables, such as

$\left.\begin{array}{l} 5x+10y=25 \\ 5x+12y=4 \end{array}\right\}$ **A1**

Note: Award ***A1*** for $z = 0$ and one other equation in two variables.

attempting to solve for these two variables **(M1)**

$x = 26, y = -10.5, z = 0$ **A1A1**

Note: Award ***A1A0*** for only two correct values, and ***A0A0*** for only one.

Note: Award marks in part (b) for equivalent steps seen in part (a).

[5 marks]

Total [7 marks]

Question 65

(a) **METHOD 1**

$\binom{8}{4}$ **(A1)**

$= \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5$ **(M1)**

$= 70$ **A1**

METHOD 2

recognition that they need to count the teams with 0 boys, 1 boy… 4 boys **M1**

$1 + \binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} + 1$

$= 1 + (4 \times 4) + (6 \times 6) + (4 \times 4) + 1$ **(A1)**

$= 70$ **A1**

[3 marks]

(b) **EITHER**

recognition that the answer is the total number of teams minus the number of teams with all girls or all boys **(M1)**

$70 - 2$

OR

recognition that the answer is the total of the number of teams with 1 boy, 2 boys, 3 boys **(M1)**

$\binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} = (4 \times 4) + (6 \times 6) + (4 \times 4)$

THEN

$= 68$ **A1**

[2 marks]

Question 65

(a) **METHOD 1**

$\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ ***(A1)***

$= \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5$ ***(M1)***

$= 70$ ***A1***

METHOD 2

recognition that they need to count the teams with 0 boys, 1 boy… 4 boys ***M1***

$1 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} + 1$

$= 1 + (4 \times 4) + (6 \times 6) + (4 \times 4) + 1$ ***(A1)***

$= 70$ ***A1***

[3 marks]

(b) **EITHER**

recognition that the answer is the total number of teams minus the number of teams with all girls or all boys ***(M1)***

$70 - 2$

OR

recognition that the answer is the total of the number of teams with 1 boy, 2 boys, 3 boys ***(M1)***

$\begin{pmatrix} 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} = (4 \times 4) + (6 \times 6) + (4 \times 4)$

THEN

$= 68$ ***A1***

[2 marks]

Total [5 marks]

Question 66

use of at least one "log rule" applied correctly for the first equation **M1**

$\log_2 6x = \log_2 2 + 2\log_2 y$

$= \log_2 2 + \log_2 y^2$

$= \log_2 (2y^2)$

$\Rightarrow 6x = 2y^2$ **A1**

use of at least one "log rule" applied correctly for the second equation **M1**

$\log_6 (15y - 25) = 1 + \log_6 x$

$= \log_6 6 + \log_6 x$

$= \log_6 6x$

$\Rightarrow 15y - 25 = 6x$ **A1**

attempt to eliminate x (or y) from their two equations **M1**

$2y^2 = 15y - 25$

$2y^2 - 15y + 25 = 0$

$(2y - 5)(y - 5) = 0$

$x = \frac{25}{12}$, $y = \frac{5}{2}$, **A1**

or $x = \frac{25}{3}$, $y = 5$ **A1**

te: x, y values do not have to be "paired" to gain either of the final two ***A*** marks.

[7 marks]

Question 67

$g(x) = f(x+2)\left(= (x+2)^4 - 6(x+2)^2 - 2(x+2) + 4\right)$ **M1**

attempt to expand $(x+2)^4$ **M1**

$(x+2)^4 = x^4 + 4(2x^3) + 6(2^2x^2) + 4(2^3x) + 2^4$ ***(A1)***

$= x^4 + 8x^3 + 24x^2 + 32x + 16$ **A1**

$g(x) = x^4 + 8x^3 + 24x^2 + 32x + 16 - 6(x^2 + 4x + 4) - 2x - 4 + 4$

$= x^4 + 8x^3 + 18x^2 + 6x - 8$ **A1**

e: For correct expansion of $f(x-2) = x^4 - 8x^3 + 18x^2 - 10x$ award max ***M0M1(A1)A0A1***.

[5 marks]

Question 68

attempting to form two equations involving u_1 and d **M1**

$(u_1 + 2d) + (u_1 + 7d) = 1$ and $\frac{7}{2}[2u_1 + 6d] = 35$

$2u_1 + 9d = 1$

$14u_1 + 42d = 70$ $(2u_1 + 6d = 10)$ **A1**

e: Award ***A1*** for any two correct equations

attempting to solve their equations: **M1**

$u_1 = 14$, $d = -3$ **A1**

[4 marks]

Question 69

(a) (i) $p(2)=8-12+16-24$ **(M1)**

Note: Award ***M1*** for a valid attempt at remainder theorem or polynomial division.

$=-12$ ***A1***

remainder $=-12$

(ii) $p(3)=27-27+24-24=0$ ***A1***

remainder $=0$

[3 marks]

(b) $x=3$ (is a zero) ***A1***

Note: Can be seen anywhere.

EITHER

factorise to get $(x-3)(x^2+8)$ ***(M1)A1***

$x^2+8\neq 0$ (for $x\in\mathbb{R}$) (or equivalent statement) ***R1***

Note: Award ***R1*** if correct two complex roots are given.

OR

$p'(x)=3x^2-6x+8$ ***A1***

attempting to show $p'(x)\neq 0$ ***M1***

eg discriminant $=36-96<0$, completing the square

no turning points ***R1***

THEN

only one real zero (as the curve is continuous) ***AG***

[4 marks]

(c) new graph is $y=p(2x)$ ***(M1)***

stretch parallel to the x-axis (with $x=0$ invariant), scale factor 0.5 ***A1***

[2 marks]

[Total 9 marks]

Question 70

attempt at binomial expansion ***M1***

$1+\binom{11}{1}(-2x)+\binom{11}{2}(-2x)^2+\ldots$ ***(A1)***

$\binom{11}{2}=55$

$1-22x+220x^2$ ***A1A1***

te: ***A1*** for first two terms, ***A1*** for final term.

te: Award ***M1(A1)A0A0*** for $(-2x)^{11}+\binom{11}{10}(-2x)^{10}+\binom{11}{9}(-2x)^9+\ldots,$

Total [4 marks]

Question 71

(a) **METHOD 1**

$|z| = \sqrt[4]{4}\left(= \sqrt{2}\right)$ ***(A1)***

$\arg(z_1) = \dfrac{\pi}{4}$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

METHOD 2

$z^4 = -4$

$(a+\mathrm{i}b)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$\left(a^4 - 6a^4 + a^4 = -4 \Rightarrow\right) a = \pm 1$ ***and*** $\left(4a^3b - 4ab^3 = 0 \Rightarrow\right) a = \pm b$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/- their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

(b) complete method to find area of 'rectangle' ***(M1)***

$= 4$ ***A1***

[2 marks]

Total [7 marks]

Question 72

attempt to eliminate a variable (or attempt to find det A) ***M1***

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 3 & -1 & 4 \\ 3 & -5 & a & b \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & -14 & a+3 & b-12 \end{array}\right)$$ (or $\det A = 14(a-3)$)

(or two correct equations in two variables) ***A1***

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & 0 & a-3 & b-6 \end{array}\right)$$ (or solving $\det A = 0$)

(or attempting to reduce to one variable, e.g. $(a-3)z = b-6$) ***M1***

$a = 3,\ b \neq 6$ ***A1A1***

[5 marks]