Subject – Math(Higher Level)
Topic - Calculus
Year - Nov 2011 – Nov 2019

Question 1

to find the points of intersection of the two curves

$-x^2+2=x^3-x^2-bx+2$ ***M1***

$x^3-bx=x(x^2-b)=0$

$\Rightarrow x=0\ ;\ x=\pm\sqrt{b}$ ***A1A1***

$A_1=\int_{-\sqrt{b}}^{0}\left[(x^3-x^2-bx+2)-(-x^2+2)\right]\mathrm{d}x\left(=\int_{-\sqrt{b}}^{0}(x^3-bx)\,\mathrm{d}x\right)$ ***M1***

$=\left[\frac{x^4}{4}-\frac{bx^2}{2}\right]_{-\sqrt{b}}^{0}$

$=-\left(\frac{(-\sqrt{b})^4}{4}-\frac{b(-\sqrt{b})^2}{2}\right)=-\frac{b^2}{4}+\frac{b^2}{2}=\frac{b^2}{4}$ ***A1***

$A_2=\int_{0}^{\sqrt{b}}\left[(-x^2+2)-(x^3-x^2-bx+2)\right]\mathrm{d}x$ ***M1***

$=\int_{0}^{\sqrt{b}}(-x^3+bx)\,\mathrm{d}x$

$=\left[-\frac{x^4}{4}+\frac{bx^2}{2}\right]_{0}^{\sqrt{b}}=\frac{b^2}{4}$ ***A1***

therefore $A_1=A_2=\frac{b^2}{4}$ ***AG***

[7 marks]

Question 2

(a) angle APB is a right angle

$\Rightarrow \cos\theta = \frac{\text{AP}}{4} \Rightarrow \text{AP} = 4\cos\theta$ ***A1***

Note: Allow correct use of cosine rule.

arc $\text{PB} = 2\times 2\theta = 4\theta$ ***A1***

$t = \frac{\text{AP}}{3} + \frac{\text{PB}}{6}$ ***M1***

Note: Allow use of their AP and their PB for the ***M1***.

$\Rightarrow t = \frac{4\cos\theta}{3} + \frac{4\theta}{6} = \frac{4\cos\theta}{3} + \frac{2\theta}{3} = \frac{2}{3}(2\cos\theta + \theta)$ ***AG***

(b) $\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{2}{3}(-2\sin\theta + 1)$ ***A1***

$\frac{2}{3}(-2\sin\theta + 1) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ (or 30 degrees) ***A1***

(c) $\frac{\mathrm{d}^2 t}{\mathrm{d}\theta^2} = -\frac{4}{3}\cos\theta < 0 \left(\text{at } \theta = \frac{\pi}{6}\right)$ ***M1***

$\Rightarrow t$ is maximized at $\theta = \frac{\pi}{6}$ ***R1***

time needed to walk along arc AB is $\frac{2\pi}{6}$ (≈ 1 hour)

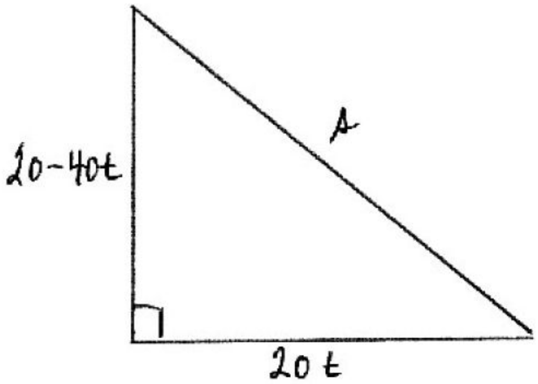
time needed to row from A to B is $\frac{4}{3}$ (≈ 1.33 hour)

hence, time is minimized in walking from A to B ***R1***

[8 marks]

Question 3

(a)



(M1)

$s^2 = (20t)^2 + (20 - 40t)^2$ **M1**

$s^2 = 2000t^2 - 1600t + 400$ **A1**

to minimize s it is enough to minimize s^2

$f'(t) = 4000t - 1600$ **A1**

setting $f'(t)$ equal to 0 **M1**

$4000t - 1600 = 0 \Rightarrow t = \frac{2}{5}$ or 24 minutes **A1**

$f''(t) = 4000 > 0$ **M1**

$\Rightarrow$ at $t = \frac{2}{5}$, $f(t)$ is minimized

hence, the ships are closest at 12:24 **A1**

Note: accept solution based on s.

[8 marks]

(b) $f\left(\frac{2}{5}\right) = \sqrt{80}$ **M1A1**

since $\sqrt{80} < 9$, the captains can see one another **R1**

[3 marks]

Total [11 marks]

Question 4

(a) $\frac{dy}{dx} = \frac{e}{\ln e}(2+2) = 4e$ *A1*

at $(2, e)$ the tangent line is $y - e = 4e(x-2)$ *M1*

hence $y = 4ex - 7e$ *A1*

[3 marks]

(b) $\frac{dy}{dx} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}dy = (x+2)dx$ *M1*

$\int \frac{\ln y}{y} dy = \int (x+2) dx$

using substitution $u = \ln y$; $du = \frac{1}{y} dy$ *(M1)(A1)*

$\Rightarrow \int \frac{\ln y}{y} dy = \int u \, du = \frac{1}{2}u^2$ *(A1)*

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$ *A1A1*

at $(2, e)$, $\frac{(\ln e)^2}{2} = 6 + c$ *M1*

$\Rightarrow c = -\frac{11}{2}$ *A1*

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

$\ln y = \pm\sqrt{x^2+4x-11} \Rightarrow y = e^{\pm\sqrt{x^2+4x-11}}$ *M1A1*

since $y > 1$, $f(x) = e^{\sqrt{x^2+4x-11}}$ *R1*

[11 marks]

ote: *M1* for attempt to make y the subject.

(c) **EITHER**

$x^2 + 4x - 11 > 0$ *A1*

using the quadratic formula *M1*

critical values are $\frac{-4 \pm \sqrt{60}}{2}$ $\left(= -2 \pm \sqrt{15}\right)$ *A1*

using a sign diagram or algebraic solution *M1*

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ *A1A1*

OR

$x^2 + 4x - 11 > 0$ *A1*

by methods of completing the square *M1*

$(x+2)^2 > 15$ *A1*

$\Rightarrow x + 2 < -\sqrt{15}$ or $x + 2 > \sqrt{15}$ *(M1)*

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ *A1A1*

[6 marks]

continued …

(d) $f(x) = f'(x) \Rightarrow f(x) = \dfrac{f(x)}{\ln f(x)}(x+2)$ **M1**

$\Rightarrow \ln(f(x)) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2+4x-11}\right)$ **A1**

$\Rightarrow (x+2)^2 = x^2+4x-11 \Rightarrow x^2+4x+4 = x^2+4x-11$ **A1**

$\Rightarrow 4 = -11$, hence $f(x) \neq f'(x)$ **R1AG**

[4 marks]

Total [24 marks]

Question 5

(a) $\int_{\frac{1}{6}}^{1} \dfrac{k}{x} - \dfrac{1}{x}\,\mathrm{d}x = (k-1)[\ln x]_{\frac{1}{6}}^{1}$ **M1A1**

Note: Award ***M1*** for $\int \dfrac{k}{x} - \dfrac{1}{x}\,\mathrm{d}x$ or $\int \dfrac{1}{x} - \dfrac{k}{x}\,\mathrm{d}x$ and ***A1*** for $(k-1)\ln x$ seen in part (a) or later in part (b).

$= (1-k)\ln\dfrac{1}{6}$ **A1**

[3 marks]

(b) $\int_{1}^{\sqrt{6}} \dfrac{k}{x} - \dfrac{1}{x}\,\mathrm{d}x = (k-1)[\ln x]_{1}^{\sqrt{6}}$ ***(A1)***

Note: Award ***A1*** for correct change of limits.

$= (k-1)\ln\sqrt{6}$ **A1**

[2 marks]

(c) $(1-k)\ln\dfrac{1}{6} = (k-1)\ln 6$ **A1**

$(k-1)\ln\sqrt{6} = \dfrac{1}{2}(k-1)\ln 6$ **A1**

Note: This simplification could have occurred earlier, and marks should still be awarded.

ratio is 2 (or 2:1) **A1**

[3 marks]

Total [8 marks]

Question 6

$4x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{2x}{y}$ ***M1A1***

Note: Allow follow through on incorrect $\frac{dy}{dx}$ from this point.

gradient of normal at (a, b) is $\frac{b}{2a}$

Note: No further A marks are available if a general point is not used

equation of normal at (a, b) is $y-b=\frac{b}{2a}(x-a)$ $\left(\Rightarrow y=\frac{b}{2a}x+\frac{b}{2}\right)$ ***M1A1***

substituting $(1, 0)$ ***M1***

$b=0$ or $a=-1$ ***A1A1***

four points are $(3, 0)$, $(-3, 0)$, $(-1, 4)$, $(-1, -4)$ ***A1A1***

Note: Award ***A1A0*** for any two points correct.

[9 marks]

Question 7

(a) **EITHER**

derivative of $\frac{x}{1-x}$ is $\frac{(1-x)-x(-1)}{(1-x)^2}$ ***M1A1***

$f'(x)=\frac{1}{2}\left(\frac{x}{1-x}\right)^{-\frac{1}{2}}\frac{1}{(1-x)^2}$ ***M1A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x)>0$ (for all $0<x<1$) so the function is increasing ***R1***

OR

$f(x)=\frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$

$f'(x)=\frac{(1-x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}(-1)}{1-x}$ ***M1A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}[1-x+x]$ ***M1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x)>0$ (for all $0<x<1$) so the function is increasing ***R1***

[5 marks]

(b) $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

$\Rightarrow f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}}+\frac{3}{4}x^{-\frac{1}{2}}(1-x)^{-\frac{5}{2}}$ ***M1A1***

$=-\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{5}{2}}[1-4x]$

$f''(x)=0\Rightarrow x=\frac{1}{4}$ ***M1A1***

$f''(x)$ changes sign at $x=\frac{1}{4}$ hence there is a point of inflexion ***R1***

$x=\frac{1}{4}\Rightarrow y=\frac{1}{\sqrt{3}}$ ***A1***

the coordinates are $\left(\frac{1}{4},\frac{1}{\sqrt{3}}\right)$

[6 marks]

(c) $x = \sin^2\theta \Rightarrow \dfrac{dx}{d\theta} = 2\sin\theta\cos\theta$ ***M1A1***

$\int\sqrt{\dfrac{x}{1-x}}\,dx = \int\sqrt{\dfrac{\sin^2\theta}{1-\sin^2\theta}}\,2\sin\theta\cos\theta\,d\theta$ ***M1A1***

$= \int 2\sin^2\theta\,d\theta$ ***A1***

$= \int 1-\cos 2\theta\,d\theta$ ***M1A1***

$= \theta - \dfrac{1}{2}\sin 2\theta + c$ ***A1***

$\theta = \arcsin\sqrt{x}$ ***A1***

$\dfrac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sqrt{x}\sqrt{1-x} = \sqrt{x-x^2}$ ***M1A1***

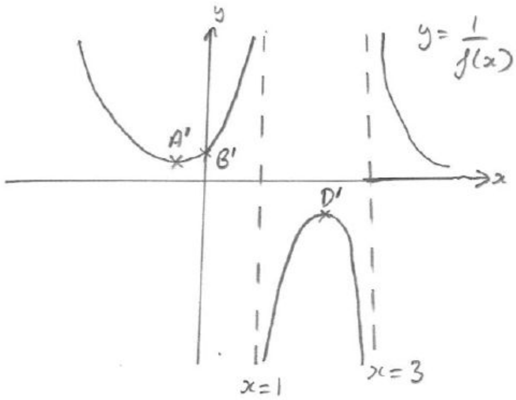
hence $\int\sqrt{\dfrac{x}{1-x}}\,dx = \arcsin\sqrt{x} - \sqrt{x-x^2} + c$ ***AG***

[11 marks]

Total [22 marks]

Question 8

(a)

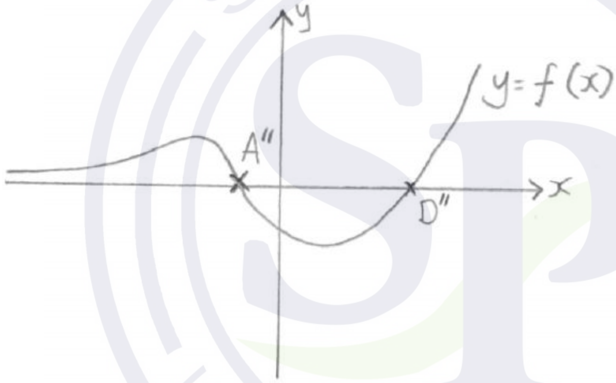


A1A1A1

Note:	Award ***A1*** for correct shape. Award ***A1*** for two correct asymptotes, $x = 1$ and $x = 3$. Award ***A1*** for correct coordinates, $A'\left(-1, \frac{1}{4}\right)$, $B'\left(0, \frac{1}{3}\right)$ and $D'\left(2, -\frac{1}{3}\right)$.

[3 marks]

(b)



A1A1A1

Note:	Award ***A1*** for correct general shape including the horizontal asymptote. Award ***A1*** for recognition of 1 maximum point and 1 minimum point. Award ***A1*** for correct coordinates, $A''(-1, 0)$ and $D''(2, 0)$.

[3 marks]

Total [6 marks]

Question 9

$x^3 y = a\sin nx$

attempt to differentiate implicitly **M1**

$\Rightarrow 3x^2 y + x^3 \frac{dy}{dx} = an\cos nx$ **A2**

te: Award ***A1*** for two out of three correct, ***A0*** otherwise.

$\Rightarrow 6xy + 3x^2 \frac{dy}{dx} + 3x^2 \frac{dy}{dx} + x^3 \frac{d^2y}{dx^2} = -an^2 \sin nx$ **A2**

te: Award ***A1*** for three or four out of five correct, ***A0*** otherwise.

$\Rightarrow 6xy + 6x^2 \frac{dy}{dx} + x^3 \frac{d^2y}{dx^2} = -an^2 \sin nx$

$\Rightarrow x^3 \frac{d^2y}{dx^2} + 6x^2 \frac{dy}{dx} + 6xy + n^2 x^3 y = 0$ **A1**

$\Rightarrow x^3 \frac{d^2y}{dx^2} + 6x^2 \frac{dy}{dx} + (n^2x^2 + 6)xy = 0$ **AG**

[6 marks]

Question 10

(c) attempt at integration by parts *M1*

EITHER

$I = \int e^{-x}\cos x dx = -e^{-x}\cos x dx - \int e^{-x}\sin x dx$ *A1*

$\Rightarrow I = -e^{-x}\cos x dx - \left[-e^{-x}\sin x + \int e^{-x}\cos x dx\right]$ *A1*

$\Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$ *A1*

Note: Do not penalize absence of C.

OR

$I = \int e^{-x}\cos x dx = e^{-x}\sin x + \int e^{-x}\sin x dx$ *A1*

$\Rightarrow I = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x dx$ *A1*

$\Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$ *A1*

Note: Do not penalize absence of C.

THEN

$\int_0^{\frac{\pi}{2}} e^{-x}\cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} = \frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2}$ *A1*

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x}\cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{e^{-\frac{3\pi}{2}}}{2} - \frac{e^{-\frac{\pi}{2}}}{2}$ *A1*

ratio of A:B is $\dfrac{\frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2}}{\frac{e^{-\frac{3\pi}{2}}}{2} + \frac{e^{-\frac{\pi}{2}}}{2}}$

$= \dfrac{e^{\frac{3\pi}{2}}\left(e^{-\frac{\pi}{2}} + 1\right)}{e^{\frac{3\pi}{2}}\left(e^{-\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}\right)}$ *M1*

$= \dfrac{e^{\pi}\left(e^{\frac{\pi}{2}} + 1\right)}{e^{\pi} + 1}$ *AG*

[7 marks]

Total [9 marks]

Question 11

(a) $f'(x) = (\ln x)^2 + \frac{2x\ln x}{x}\left(= (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)\right)$ **M1A1**

$f'(x) = 0\ (\Rightarrow x = 1,\ x = e^{-2})$ **M1**

Note: Award **M1** for an attempt to solve $f'(x) = 0$.

$A(e^{-2}, 4e^{-2})$ **and** $B(1, 0)$ **A1A1**

Note: The final **A1** is independent of prior working.

[5 marks]

(b) $f''(x) = \frac{2}{x}(\ln x + 1)$ **A1**

$f''(x) = 0\ \left(\Rightarrow x = e^{-1}\right)$ **(M1)**

inflexion point (e^{-1}, e^{-1}) **A1**

Note: **M1** for attempt to solve $f''(x) = 0$.

[3 marks]

Total [8 marks]

Question 12

(a) attempt to differentiate implicitly **M1**

$2x + \cos y\frac{dy}{dx} - y - x\frac{dy}{dx} = 0$ **A1A1**

Note: **A1** for differentiating x^2 and $\sin y$; **A1** for differentiating xy.

substitute x and y by π **M1**

$2\pi - \frac{dy}{dx} - \pi - \pi\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{\pi}{1+\pi}$ **M1A1**

Note: **M1** for attempt to make dy/dx the subject. This could be seen earlier.

[6 marks]

(b) $\theta = \frac{\pi}{4} - \arctan\frac{\pi}{1+\pi}$ (or seen the other way) **M1**

$\tan\theta = \tan\left(\frac{\pi}{4} - \arctan\frac{\pi}{1+\pi}\right) = \frac{1 - \frac{\pi}{1+\pi}}{1 + \frac{\pi}{1+\pi}}$ **M1A1**

$\tan\theta = \frac{1}{1+2\pi}$ **AG**

[3 marks]

Total [9 marks]

Question 13

(a) $(f \circ f)(x) = f\left(\dfrac{x}{2-x}\right) = \dfrac{\dfrac{x}{2-x}}{2-\dfrac{x}{2-x}}$ **M1A1**

$(f \circ f)(x) = \dfrac{x}{4-3x}$ **A1**

[3 marks]

(b) $P(n): \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$

$P(1):\ f(x) = F_1(x)$

$LHS = f(x) = \dfrac{x}{2-x}$ and $RHS = F_1(x) = \dfrac{x}{2^1-(2^1-1)x} = \dfrac{x}{2-x}$ **A1A1**

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$ **M1**

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right)$ ***(M1)***

$= f\left(\dfrac{x}{2^k-(2^k-1)x}\right) = \dfrac{\dfrac{x}{2^k-(2^k-1)x}}{2-\dfrac{x}{2^k-(2^k-1)x}}$ **A1**

$= \dfrac{x}{2\left(2^k-(2^k-1)x\right)-x} = \dfrac{x}{2^{k+1}-(2^{k+1}-2)x-x}$ **A1**

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$ ***(M1)***

$= F_k\left(\dfrac{x}{2-x}\right) = \dfrac{\dfrac{x}{2-x}}{2^k-(2^k-1)\dfrac{x}{2-x}}$ **A1**

$= \dfrac{x}{2^{k+1}-2^k x-2^k x+x}$ **A1**

THEN

$= \dfrac{x}{2^{k+1}-(2^{k+1}-1)x} = F_{k+1}(x)$ **A1**

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ **R1**

[8 marks]

(c) **METHOD 1**

$$x = \frac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$$ ***M1A1***

$$\Rightarrow 2^n x = \left((2^n - 1)x + 1\right)y \Rightarrow y = \frac{2^n x}{(2^n - 1)x + 1}$$ ***A1***

$$F_n^{-1}(x) = \frac{2^n x}{(2^n - 1)x + 1}$$ ***A1***

$$F_n^{-1}(x) = \frac{x}{\frac{2^n - 1}{2^n}x + \frac{1}{2^n}}$$ ***M1***

$$F_n^{-1}(x) = \frac{x}{(1 - 2^{-n})x + 2^{-n}}$$ ***A1***

$$F_n^{-1}(x) = \frac{x}{2^{-n} - (2^{-n} - 1)x}$$ ***AG***

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$ ***M1***

$$= F_{-n}\left(\frac{x}{2^n - (2^n - 1)x}\right) = \frac{\frac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\frac{x}{2^n - (2^n - 1)x}}$$ ***A1A1***

$$= \frac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$$= \frac{x}{1} = x$$ ***A1AG***

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$ ***M1***

$$= F_n\left(\frac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \frac{\frac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\frac{x}{2^{-n} - (2^{-n} - 1)x}}$$ ***A1A1***

$$= \frac{x}{2^n\left(2^{-n} - (2^{-n} - 1)x\right) - (2^n - 1)x}$$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$$= \frac{x}{1} = x$$ ***A1AG***

[6 marks]

(d) (i) $F_n(0) = 0,\ F_n(1) = 1$ ***A1***

(ii) **METHOD 1**

$2^n - (2^n - 1)x - 1 = (2^n - 1)(1 - x)$ ***(M1)***

> 0 if $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***A1***

so $2^n - (2^n - 1)x > 1$ and $F_n(x) = \dfrac{x}{2^n - (2^n - 1)x} < \dfrac{x}{1}\ (< x)$ ***R1***

$F_n(x) = \dfrac{x}{2^n - (2^n - 1)x} < x$ for $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***AG***

METHOD 2

$\dfrac{x}{2^n - (2^n - 1)x} < x \Leftrightarrow 2^n - (2^n - 1)x > 1$ ***(M1)***

$\Leftrightarrow (2^n - 1)x < 2^n - 1$ ***A1***

$\Leftrightarrow x < \dfrac{2^n - 1}{2^n - 1} = 1$ true in the interval $]0, 1[$ ***R1***

(iii) $B_n = 2\left(A_n - \dfrac{1}{2}\right)\ (= 2A_n - 1)$ ***(M1)A1***

[6 marks]

Total [23 marks]

Question 14

$V = 0.5\pi r^2$ *(A1)*

EITHER

$\frac{dV}{dr} = \pi r$ ***A1***

$\frac{dV}{dt} = 4$ *(A1)*

applying chain rule ***M1***

for example $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

OR

$\frac{dV}{dt} = \pi r \frac{dr}{dt}$ ***M1A1***

$\frac{dV}{dt} = 4$ *(A1)*

THEN

$\frac{dr}{dt} = 4 \times \frac{1}{\pi r}$ ***A1***

when $r = 20$, $\frac{dr}{dt} = \frac{4}{20\pi}$ or $\frac{1}{5\pi}$ (cm s^{-1}) ***A1***

te: Allow h instead of 0.5 up until the final ***A1***.

[6 marks]

Question 15

$8y \times \frac{1}{x} + 8\frac{dy}{dx}\ln x - 4x + 8y\frac{dy}{dx} = 0$ ***M1A1A1***

Note: ***M1*** for attempt at implicit differentiation. ***A1*** for differentiating $8y\ln x$, ***A1*** for differentiating the rest.

when $x = 1$, $8y \times 0 - 2 \times 1 + 4y^2 = 7$ *(M1)*

$y^2 = \frac{9}{4} \Rightarrow y = \frac{3}{2}$ (as $y > 0$) ***A1***

at $\left(1, \frac{3}{2}\right)$ $\frac{dy}{dx} = -\frac{2}{3}$ ***A1***

$y - \frac{3}{2} = -\frac{2}{3}(x-1)$ or $y = -\frac{2}{3}x + \frac{13}{6}$ ***A1***

[7 marks]

Question 16

(a) $\sin\left(\pi x^{-1}\right)=0 \quad \frac{\pi}{x}=\pi, 2\pi\,(\ldots)$ *(A1)*

$x=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ ***A1***

[2 marks]

(b) $\left[\cos(\pi x^{-1})\right]_{\frac{1}{n+1}}^{\frac{1}{n}}$ ***M1***

$=\cos\left(\pi n\right)-\cos\left(\pi(n+1)\right)$ ***A1***

$=2$ when n is even and $=-2$ when n is odd ***A1***

[3 marks]

(c) $\int_{0.1}^{1}\left|\pi x^{-2}\sin(\pi x^{-1})\right|\mathrm{d}x=2+2+\ldots+2=18$ ***(M1)A1***

[2 marks]

Total [7 marks]

Question 17

(e) let $u = x - \frac{1}{2}$ *A1*

$\frac{du}{dx} = 1$ (or $du = dx$) *A1*

$\int \frac{1}{4x^2 - 4x + 5} dx = \int \frac{1}{4\left(x - \frac{1}{2}\right)^2 + 4} dx$ *A1*

$\int \frac{1}{4u^2 + 4} du = \frac{1}{4} \int \frac{1}{u^2 + 1} du$ *AG*

Note: If following through an incorrect answer to part (a), do not award final ***A1*** mark.

[3 marks]

(f) $\int_1^{3.5} \frac{1}{4x^2 - 4x + 5} dx = \frac{1}{4} \int_{0.5}^{3} \frac{1}{u^2 + 1} du$ *A1*

Note: ***A1*** for correct change of limits. Award also if they do not change limits but go back to x values when substituting the limit (even if there is an error in the integral).

$\frac{1}{4}\left[\arctan(u)\right]_{0.5}^{3}$ ***(M1)***

$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ *A1*

let the integral $= I$

$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ *M1*

$\frac{3 - 0.5}{1 + 3 \times 0.5} = \frac{2.5}{2.5} = 1$ ***(M1)A1***

$4I = \frac{\pi}{4} \quad \Rightarrow I = \frac{\pi}{16}$ *A1AG*

Question 18

$\left[\frac{1}{3}(x-2)^3 + \ln x - \frac{1}{\pi}\cos \pi x\right]_{(1)}^{(2)}$ *A1A1A1*

te: Accept $\frac{1}{3}x^3 - 2x^2 + 4x$ in place of $\frac{1}{3}(x-2)^3$.

$= \left(0 + \ln 2 - \frac{1}{\pi}\cos 2\pi\right) - \left(-\frac{1}{3} + \ln 1 - \frac{1}{\pi}\cos \pi\right)$ ***(M1)***

$= \frac{1}{3} + \ln 2 - \frac{2}{\pi}$ *A1A1*

Question 19

(a) $\frac{dy}{dx}=\frac{(x+\cos x)(\cos x - x\sin x)-x\cos x(1-\sin x)}{(x+\cos x)^2}$ ***M1A1A1***

Note:	Award ***M1*** for attempt at differentiation of a quotient and a product condoning sign errors in the quotient formula and the trig differentiations, ***A1*** for correct derivative of "*u*", ***A1*** for correct derivative of "*v*".

$=\frac{x\cos x+\cos^2 x - x^2\sin x - x\cos x\sin x - x\cos x + x\cos x\sin x}{(x+\cos x)^2}$ ***A1***

$=\frac{\cos^2 x - x^2\sin x}{(x+\cos x)^2}$ ***AG***

[4 marks]

(b) the derivative has value -1 ***(A1)***

the equation of the tangent line is $(y-0)=(-1)\left(x-\frac{\pi}{2}\right)\left(y=\frac{\pi}{2}-x\right)$ ***M1A1***

[3 marks]

Total [7 marks]

Question 20

(a) attempt at implicit differentiation *M1*

EITHER

$$\frac{2x}{y}-\frac{x^2}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}-2=\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$ *A1A1*

Note: Award ***A1*** for each side.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{2x}{y}-2}{\frac{1}{y}+\frac{x^2}{y^2}}\left(=\frac{2xy-2y^2}{x^2+y}\right)$$ *A1*

OR

after multiplication by y

$$2x-2y-2x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}\ln y+y\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$ *A1A1*

Note: Award ***A1*** for each side.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(x-y)}{1+2x+\ln y}$$ *A1*

[4 marks]

(b) for $y=1$, $x^2-2x=0$

$x=(0\text{ or})\ 2$ *A1*

for $x=2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{5}$ *A1*

[2 marks]

Total [6 marks]

Question 21

$$3x^2y^2 + 2x^3y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 9\frac{dy}{dx} = 0$$ ***M1M1A1***

Note: First ***M1*** for attempt at implicit differentiation, second ***M1*** for use of product rule.

$$\left(\frac{dy}{dx} = \frac{3x^2y^2 + 3x^2}{3y^2 - 2x^3y - 9}\right)$$

$\Rightarrow 3x^2 + 3x^2y^2 = 0$ ***(A1)***

$\Rightarrow 3x^2\left(1 + y^2\right) = 0$

$x = 0$ ***A1***

Note: Do not award ***A1*** if extra solutions given *eg* $y = \pm 1$.

substituting $x = 0$ into original equation ***(M1)***

$y^3 - 9y = 0$

$y(y+3)(y-3) = 0$

$y = 0,\ y = \pm 3$

coordinates $(0,\ 0),\ (0,\ 3),\ (0,\ -3)$ ***A1***

Total [7 marks]

Question 22

(a) (i) $f'(x) = e^{-x} - xe^{-x}$ **_M1A1_**

(ii) $f'(x) = 0 \Rightarrow x = 1$

coordinates $\left(1,\ e^{-1}\right)$ **_A1_**

[3 marks]

(b) $f''(x) = -e^{-x} - e^{-x} + xe^{-x}\left(= -e^{-x}(2-x)\right)$ **_A1_**

substituting $x = 1$ into $f''(x)$ **_M1_**

$f''(1)\left(= -e^{-1}\right) < 0$ hence maximum **_R1AG_**

[3 marks]

(c) $f''(x) = 0 \quad (\Rightarrow x = 2)$ **_M1_**

coordinates $\left(2,\ 2e^{-2}\right)$ **_A1_**

[2 marks]

(d) (i) $g(x) = \dfrac{x}{2}e^{-\frac{x}{2}}$ **_A1_**

(ii) coordinates of maximum $\left(2,\ e^{-1}\right)$ **_A1_**

(iii) equating $f(x) = g(x)$ and attempting to solve $xe^{-x} = \dfrac{x}{2}e^{-\frac{x}{2}}$

$\Rightarrow x\left(2e^{\frac{x}{2}} - e^{x}\right) = 0$ **_(A1)_**

$\Rightarrow x = 0$ **_A1_**

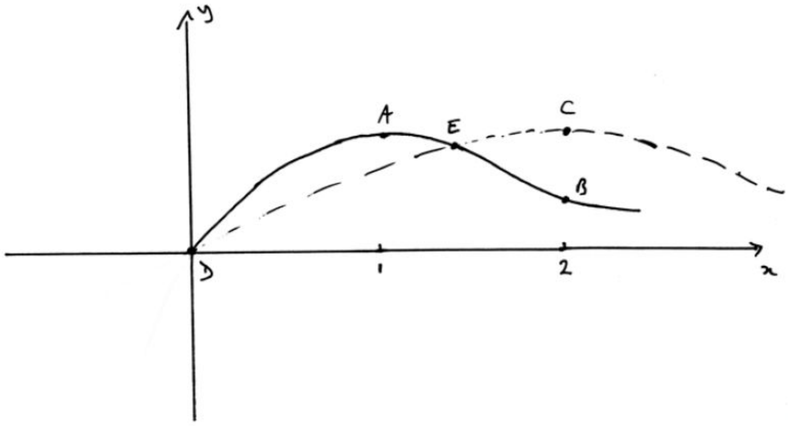
or $2e^{\frac{x}{2}} = e^{x}$

$\Rightarrow e^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2 \qquad (\ln 4)$ **_A1_**

[5 marks]

(e)



A4

> **Note:** Award ***A1*** for shape of f, including domain extending beyond $x=2$. Ignore any graph shown for $x<0$.
> Award ***A1*** for A and B correctly identified.
> Award ***A1*** for shape of g, including domain extending beyond $x=2$. Ignore any graph shown for $x<0$. Allow follow through from f.
> Award ***A1*** for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

(f) $A=\int_0^1 \frac{x}{2}e^{-\frac{x}{2}}\,dx$ ***M1***

$=\left[-xe^{-\frac{x}{2}}\right]_0^1-\int_0^1 -e^{-\frac{x}{2}}\,dx$ ***A1***

> **Note:** Condone absence of limits or incorrect limits.

$=-e^{-\frac{1}{2}}-\left[2e^{-\frac{x}{2}}\right]_0^1$

$=-e^{-\frac{1}{2}}-\left(2e^{-\frac{1}{2}}-2\right)=2-3e^{-\frac{1}{2}}$ ***A1***

[3 marks]

Question 23

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16} \right) d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta \right]_0^{\frac{\pi}{2}}$ ***M1A1***

$= \frac{5\pi}{32}$ ***A1***

[3 marks]

(f) $V = \pi \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$ ***M1***

$= \pi \int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \pi \int_0^{\frac{\pi}{2}} \cos^6 x \, dx$ ***M1***

$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3\pi}{16}$ ***A1***

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$ ***A1***

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k} \int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, d\theta = \frac{(2k)!}{(k!)^2} \frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, d\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}} \right)$ ***A1***

[3 marks]

Question 24

(a) $\cos x = 2\cos^2 \frac{1}{2}x - 1$

$\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ ***M1***

positive as $0 \le x \le \pi$ ***R1***

$\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$ ***AG***

[2 marks]

(b) $\cos 2\theta = 1 - 2\sin^2\theta$ ***(M1)***

$\sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ ***A1***

[2 marks]

(c) $\sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{1}{2}x + \sin\frac{1}{2}x\,\mathrm{d}x$ ***A1***

$= \sqrt{2}\left[2\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \sqrt{2}(0) - \sqrt{2}(0-2)$ ***(A1)***

$= 2\sqrt{2}$ ***A1***

[4 marks]

Total [8 marks]

Question 25

(a) $x = 1$ ***A1***

[1 mark]

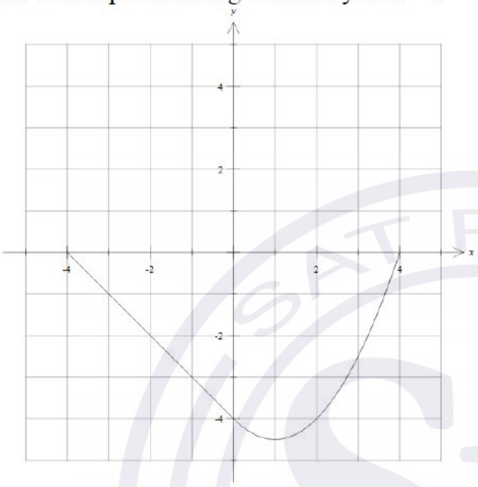
(b) ***A1*** for point $(-4, 0)$

A1 for $(0, -4)$

A1 for min at $x = 1$ in approximately the correct place

A1 for $(4, 0)$

A1 for shape including continuity at $x = 0$



[5 marks]

Total [6 marks]

Question 26

$$\frac{dv}{ds} = 2s^{-3}$$ ***M1A1***

> **Note:** Award ***M1*** for $2s^{-3}$ and ***A1*** for the whole expression.

$$a = v\frac{dv}{ds}$$ ***(M1)***

$$a = -\frac{1}{s^2} \times \frac{2}{s^3} \left(= -\frac{2}{s^5}\right)$$ ***(A1)***

when $s = \frac{1}{2}$, $a = -\frac{2}{(0.5)^5}$ $(= -64)$ (m s^{-2}) ***M1A1***

Question 27

(a) **METHOD 1**

$$\frac{2x}{1+x^4}+\frac{2y}{1+y^4}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$ ***M1A1A1***

Note: Award ***M1*** for implicit differentiation, ***A1*** for LHS and ***A1*** for RHS.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x\left(1+y^4\right)}{y\left(1+x^4\right)}$$ ***A1***

METHOD 2

$$y^2=\tan\left(\frac{\pi}{4}-\arctan x^2\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\left(\arctan x^2\right)}{1+\left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan x^2\right)\right)}$$ ***(M1)***

$$=\frac{1-x^2}{1+x^2}$$ ***A1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2x\left(1+x^2\right)-2x\left(1-x^2\right)}{\left(1+x^2\right)^2}$$ ***M1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x}{y\left(1+x^2\right)^2}$$ ***A1***

$$\left(=\frac{2x\sqrt{1+x^2}}{\sqrt{1-x^2}\left(1+x^2\right)^2}\right)$$

[4 marks]

Question 28

(a) $f'(x) = \dfrac{x \times \frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \dfrac{1 - \ln x}{x^2}$ ***AG***

[2 marks]

(b) $\dfrac{1 - \ln x}{x^2} = 0$ has solution $x = \mathrm{e}$ ***M1A1***

$y = \dfrac{1}{\mathrm{e}}$ ***A1***

hence maximum at the point $\left(\mathrm{e}, \dfrac{1}{\mathrm{e}}\right)$

[3 marks]

(c) $f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - 2x(1 - \ln x)}{x^4}$ ***M1A1***

$= \dfrac{2\ln x - 3}{x^3}$

Note: The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where $f''(x) = 0$ ***M1***

so $x = \mathrm{e}^{\frac{3}{2}}$, $y = \dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}$ ***A1A1***

C has coordinates $\left(\mathrm{e}^{\frac{3}{2}}, \dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}\right)$

[5 marks]

(d) $f(1) = 0$ ***A1***

$f'(1) = 1$ ***(A1)***

$y = x + c$ ***(M1)***

through $(1, 0)$

equation is $y = x - 1$ ***A1***

[4 marks]

(e) **METHOD 1**

area $= \int_1^e x - 1 - \frac{\ln x}{x}\,dx$ ***M1A1A1***

> **Note:** Award ***M1*** for integration of difference between line and curve, ***A1*** for correct limits, ***A1*** for correct expressions in either order.

$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$ ***(M1)A1***

$\int (x-1)\,dx = \frac{x^2}{2} - x(+c)$ ***A1***

$= \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^e$

$= \left(\frac{1}{2}e^2 - e - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}e^2 - e$ ***A1***

METHOD 2

area = area of triangle $- \int_1^e \frac{\ln x}{x}\,dx$ ***M1A1***

> **Note:** ***A1*** is for correct integral with limits and is dependent on the ***M1***.

$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$ ***(M1)A1***

area of triangle $= \frac{1}{2}(e-1)(e-1)$ ***M1A1***

$\frac{1}{2}(e-1)(e-1) - \left(\frac{1}{2}\right) = \frac{1}{2}e^2 - e$ ***A1***

[7 marks]

Total [21 marks]

(b) $y^2 = \tan\left(\frac{\pi}{4} - \arctan\frac{1}{2}\right)$ ***(M1)***

$$= \frac{\tan\frac{\pi}{4} - \tan\left(\arctan\frac{1}{2}\right)}{1 + \left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan\frac{1}{2}\right)\right)}$$ ***(M1)***

Note: The two ***M1***s may be awarded for working in part (a).

$$= \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}$$ ***A1***

$$y = -\frac{1}{\sqrt{3}}$$ ***A1***

substitution into $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$= \frac{4\sqrt{6}}{9}$$ ***A1***

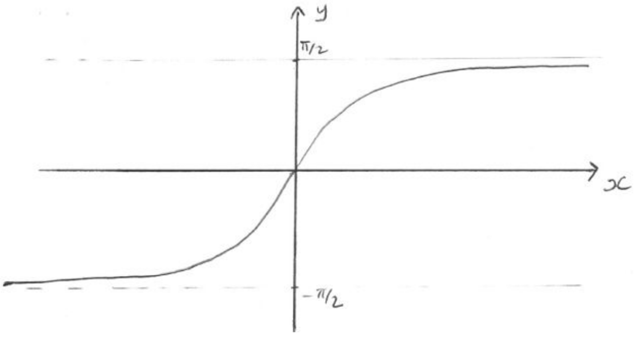
Note: Accept $\frac{8\sqrt{3}}{9\sqrt{2}}$ *etc.*

[5 marks]

Total [9 marks]

Question 29

(a)



A1A1

> **Note:** ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

[2 marks]

(b) $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ ***A1***

domain of $h \circ g$ is equal to the domain of g : $x \in$ ° , $x \neq 0$ ***A1***

[2 marks]

(c) (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$$ ***M1A1***

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$$ ***(A1)***

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$$

$$= 0$$ ***A1***

(ii) **METHOD 1**

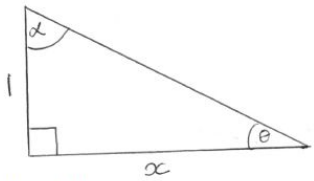
f is a constant *R1*

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ *M1A1*

$= \frac{\pi}{2}$ *AG*

METHOD 2



from diagram

$\theta = \arctan\frac{1}{x}$ *A1*

$\alpha = \arctan x$ *A1*

$\theta + \alpha = \frac{\pi}{2}$ *R1*

hence $f(x) = \frac{\pi}{2}$ *AG*

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ *M1*

$= \frac{x + \frac{1}{x}}{1 - x\left(\frac{1}{x}\right)}$ *A1*

denominator $= 0$, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) *R1*

[7 marks]

(d) (i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$f(-x)=\arctan(-x)+\arctan\left(-\frac{1}{x}\right)=-\arctan(x)-\arctan\left(\frac{1}{x}\right)=-f(x)$

therefore f is an odd function. ***R1***

(ii) $f(x)=-\frac{\pi}{2}$ ***A1***

[3 marks]

Total [14 marks]

Question 30

$x=a\sec\theta$

$\frac{dx}{d\theta}=a\sec\theta\tan\theta$ ***(A1)***

new limits:

$x=a\sqrt{2}\Rightarrow\theta=\frac{\pi}{4}$ and $x=2a\Rightarrow\theta=\frac{\pi}{3}$ ***(A1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{a\sec\theta\tan\theta}{a^3\sec^3\theta\sqrt{a^2\sec^2\theta-a^2}}\,d\theta$ ***M1***

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos^2\theta}{a^3}d\theta$ ***A1***

using $\cos^2\theta=\frac{1}{2}(\cos 2\theta+1)$ ***M1***

$\frac{1}{2a^3}\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or equivalent ***A1***

$=\frac{1}{4a^3}\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}-1-\frac{\pi}{2}\right)$ or equivalent ***A1***

$=\frac{1}{24a^3}\left(3\sqrt{3}+\pi-6\right)$ ***AG***

[7 marks]

Total [7 marks]

Question 31

(a) $f'(x)=\dfrac{\left(x^2+1\right)-2x(x+1)}{\left(x^2+1\right)^2}\left(=\dfrac{-x^2-2x+1}{\left(x^2+1\right)^2}\right)$ ***M1A1***

[2 marks]

(b) $\dfrac{-x^2-2x+1}{\left(x^2+1\right)^2}=0$

$x=-1\pm\sqrt{2}$ ***A1***

[1 mark]

(c) $f''(x) = \dfrac{(-2x-2)(x^2+1)^2 - 2(2x)(x^2+1)(-x^2-2x+1)}{(x^2+1)^4}$ *A1A1*

Note: Award ***A1*** for $(-2x-2)(x^2+1)^2$ or equivalent.

Note: Award ***A1*** for $-2(2x)(x^2+1)(-x^2-2x+1)$ or equivalent.

$= \dfrac{(-2x-2)(x^2+1) - 4x(-x^2-2x+1)}{(x^2+1)^3}$

$= \dfrac{2x^3+6x^2-6x-2}{(x^2+1)^3}$ *A1*

$\left(= \dfrac{2(x^3+3x^2-3x-1)}{(x^2+1)^3} \right)$

[3 marks]

(d) recognition that $(x-1)$ is a factor ***(R1)***

$(x-1)(x^2+bx+c) = (x^3+3x^2-3x-1)$ ***M1***

$\Rightarrow x^2+4x+1=0$ ***A1***

$x = -2 \pm \sqrt{3}$ ***A1***

Note: Allow long division / synthetic division.

[4 marks]

(e) $\int_{-1}^{0} \dfrac{x+1}{x^2+1}\,\mathrm{d}x$ ***M1***

$\int \dfrac{x+1}{x^2+1}\,\mathrm{d}x = \int \dfrac{x}{x^2+1}\,\mathrm{d}x + \int \dfrac{1}{x^2+1}\,\mathrm{d}x$ ***M1***

$= \dfrac{1}{2}\ln(x^2+1) + \arctan(x)$ ***A1A1***

$= \left[\dfrac{1}{2}\ln(x^2+1) + \arctan(x)\right]_{-1}^{0} = \dfrac{1}{2}\ln 1 + \arctan 0 - \dfrac{1}{2}\ln 2 - \arctan(-1)$ ***M1***

$= \dfrac{\pi}{4} - \ln\sqrt{2}$ ***A1***

[6 marks]

Total [16 marks]

Question 32

use of the quotient rule or the product rule **M1**

$$C'(t) = \frac{(3+t^2)\times 2 - 2t\times 2t}{(3+t^2)^2} \left(= \frac{6-2t^2}{(3+t^2)^2}\right) \text{ or } \frac{2}{3+t^2} - \frac{4t^2}{(3+t^2)^2}$$ **A1A1**

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator in the quotient rule, and ***A1*** for each correct term in the product rule.

attempting to solve $C'(t) = 0$ for t ***(M1)***

$t = \pm\sqrt{3}$ (minutes) ***A1***

$C(\sqrt{3}) = \frac{\sqrt{3}}{3}$ $(\text{mg}\,\text{l}^{-1})$ or equivalent. ***A1***

Total [6 marks]

Question 33

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$ ***A1***

$dx = 2(u-1)\,du$

ote: Award the ***A1*** for any correct relationship between dx and du.

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = 2\int \frac{(u-1)^2}{u}\,du$$ ***(M1)A1***

ote: Award the ***M1*** for an attempt at substitution resulting in an integral only involving u.

$$= 2\int u - 2 + \frac{1}{u}\,du$$ ***(A1)***

$$= u^2 - 4u + 2\ln u\,(+C)$$ ***A1***

$$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$$ ***A1***

ote: Award the ***A1*** for a correct expression in x, but not necessarily fully expanded/simplified.

Total [6 marks]

Question 34

(a) $p'(3) = f'(3)\,g(3) + g'(3)\,f(3)$ **(M1)**

te: Award ***M1*** if the derivative is in terms of x or 3.

$= 2 \times 4 + 3 \times 1$

$= 11$ ***A1***

[2 marks]

(b) $h'(x) = g'(f(x))\,f'(x)$ ***(M1)(A1)***

$h'(2) = g'(1)\,f'(2)$ ***A1***

$= 4 \times 4$

$= 16$ ***A1***

[4 marks]

Total [6 marks]

Question 35

(a) (i) $x = e^{3y+1}$ ***M1***

Note: The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$. ***A1***

[4 marks]

(b) $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent) ***M1A1***

$\ln x = -\frac{1}{2}$ (or equivalent) ***A1***

$x = e^{-\frac{1}{2}}$ ***A1***

coordinates of P are $\left(e^{-\frac{1}{2}}, -\frac{1}{2}\right)$ ***A1***

[5 marks]

(c) coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{dy}{dx} = \frac{1}{x}$ ***M1***

at Q, $\frac{dy}{dx} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

(d) let the required area be A

$A = \int_1^e x - 1\mathrm{d}x - \int_1^e \ln x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\mathrm{d}x$ ***(M1)***

$= \left[\frac{x^2}{2} - x\right]_1^e - [x\ln x - x]_1^e$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$= \frac{e^2}{2} - e - \frac{1}{2}\left(= \frac{e^2 - 2e - 1}{2}\right)$ ***A1***

[5 marks]

(e) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \frac{1}{x}$ ***(A1)***

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$ ***R1***

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$ ***R1***

so $g(x) \leq x - 1$, $x \in \mathbb{R}^+$ ***AG***

METHOD 2

$g''(x) = -\frac{1}{x^2}$ ***A1***

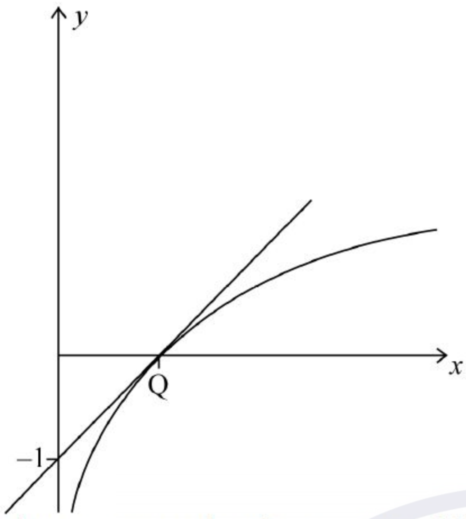
$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$ ***R1***

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$) ***R1***

so $g(x) \leq x - 1$, $x \in \mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1\left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \le \frac{1}{x} - 1\left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \ge 1 - \frac{1}{x}\left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \le g(x)$, $x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [23 marks]

Question 36

(b) (i) $\tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$ ***A1***

let $t = \tan\frac{\pi}{8}$

attempting to solve $t^2 + 2t - 1 = 0$ for t ***M1***

$t = -1 \pm \sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8} > 0$ ***R1***

$\tan\frac{\pi}{8} = \sqrt{2} - 1$ ***AG***

(ii) $\cos 4x = 2\cos^2 2x - 1$ ***A1***

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ ***M1***

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ ***A1***

$= 8\cos^4 x - 8\cos^2 x + 1$ ***AG***

Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,dx = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,dx$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,dx$ ***M1***

The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,dx$ ***A1***

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [23 marks]

Question 37

(a) $\int\left(1+\tan^2 x\right)dx = \int \sec^2 x\,dx = \tan x\,(+c)$ ***M1A1***

[2 marks]

(b) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$ ***M1A1***

$= \frac{x}{2} - \frac{\sin 2x}{4}\,(+c)$ ***A1***

Question 38

$\frac{du}{dx} = e^x$ **(A1)**

EITHER

integral is $\int \frac{e^x}{(e^x+3)^2+2^2}\,dx$ **M1A1**

$= \int \frac{1}{u^2+2^2}\,du$ **M1A1**

ote: Award ***M1*** only if the integral has completely changed to one in u.

ote: du needed for final ***A1***

OR

$e^x = u - 3$

integral is $\int \frac{1}{(u-3)^2+6(u-3)+13}\,du$ **M1A1**

ote: Award ***M1*** only if the integral has completely changed to one in u.

$= \int \frac{1}{u^2+2^2}\,du$ **M1A1**

ote: In both solutions the two method marks are independent.

THEN

$= \frac{1}{2}\arctan\left(\frac{u}{2}\right)(+c)$ **(A1)**

$= \frac{1}{2}\arctan\left(\frac{e^x+3}{2}\right)(+c)$ **A1**

Total [7 marks]

Question 39

(a) $\dfrac{dy}{dx} = 1 \times e^{3x} + x \times 3e^{3x} = \left(e^{3x} + 3xe^{3x}\right)$ ***M1A1***

[2 marks]

(b) let $P(n)$ be the statement $\dfrac{d^n y}{dx^n} = n3^{n-1}e^{3x} + x3^n e^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\dfrac{dy}{dx}$ which is $1 \times e^{3x} + x \times 3e^{3x}$ and RHS is $3^0 e^{3x} + x3^1 e^{3x}$ ***R1***

as LHS = RHS, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\dfrac{d^k y}{dx^k} = k3^{k-1}e^{3x} + x3^k e^{3x}$

$\dfrac{d^{k+1} y}{dx^{k+1}} = \dfrac{d}{dx}\left(\dfrac{d^k y}{dx^k}\right)$ ***(M1)***

$= k3^{k-1} \times 3e^{3x} + 1 \times 3^k e^{3x} + x3^k \times 3e^{3x}$ ***A1***

$= (k+1)3^k e^{3x} + x3^{k+1}e^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true ($\forall n \in \mathbb{Z}^+$) ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

(c) $e^{3x} + x \times 3e^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ **M1A1**

point is $\left(-\frac{1}{3}, -\frac{1}{3e}\right)$ **A1**

EITHER

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$

when $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} > 0$ therefore the point is a minimum **M1A1**

OR

x	$-\frac{1}{3}$
$\frac{dy}{dx}$	$-ve \quad 0 \quad +ve$

nature table shows point is a minimum **M1A1**

[5 marks]

(d) $\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$ **A1**

$2 \times 3e^{3x} + x \times 3^2 e^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ **M1A1**

point is $\left(-\frac{2}{3}, -\frac{2}{3e^2}\right)$ **A1**

x	$-\frac{2}{3}$
$\frac{d^2y}{dx^2}$	$-ve \quad 0 \quad +ve$

since the curvature does change (concave down to concave up) it is a point of inflection **R1**

Note: Allow 3rd derivative is not zero at $-\frac{2}{3}$

[5 marks]

Question 40

(a) attempt to differentiate $f(x) = x^3 - 3x^2 + 4$ **M1**

$f'(x) = 3x^2 - 6x$ **A1**

$= 3x(x-2)$

(Critical values occur at) $x = 0$, $x = 2$ ***(A1)***

so f decreasing on $x \in \left]0, 2\right[$ (or $0 < x < 2$) **A1**

[4 marks]

(b) $f''(x) = 6x - 6$ ***(A1)***

setting $f''(x) = 0$ **M1**

$\Rightarrow x = 1$

coordinate is $(1, 2)$ **A1**

[3 marks]

Total [7 marks]

Question 41

any attempt at integration by parts **M1**

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$ ***(A1)***

$\frac{dv}{dx} = x^3 \Rightarrow v = \frac{x^4}{4}$ ***(A1)***

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \int_1^2 \frac{x^3}{4}dx$ **A1**

Note: Condone absence of limits at this stage.

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \left[\frac{x^4}{16}\right]_1^2$ **A1**

Note: Condone absence of limits at this stage.

$= 4\ln 2 - \left(1 - \frac{1}{16}\right)$ **A1**

$= 4\ln 2 - \frac{15}{16}$ **AG**

[6 marks]

Question 42

(a) any attempt to use sine rule **M1**

$$\frac{\text{AB}}{\sin\frac{\pi}{3}} = \frac{\sqrt{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$ **A1**

$$= \frac{\sqrt{3}}{\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta}$$ **A1**

Note: Condone use of degrees.

$$= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$$ **A1**

$$\frac{\text{AB}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$$

$$\therefore \text{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$$ **AG**

[4 marks]

(b) **METHOD 1**

$$(\text{AB})' = \frac{-3\left(-\sqrt{3}\sin\theta + \cos\theta\right)}{\left(\sqrt{3}\cos\theta + \sin\theta\right)^2}$$ **M1A1**

setting $(\text{AB})' = 0$ **M1**

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$ **A1**

METHOD 2

$$\text{AB} = \frac{\sqrt{3}\sin\frac{\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

AB minimum when $\sin\left(\frac{2\pi}{3} - \theta\right)$ is maximum ***M1***

$\sin\left(\frac{2\pi}{3} - \theta\right) = 1$ ***(A1)***

$\frac{2\pi}{3} - \theta = \frac{\pi}{2}$ ***M1***

$\theta = \frac{\pi}{6}$ ***A1***

METHOD 3

shortest distance from B to AC is perpendicular to AC ***R1***

$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ ***M1A2***

[4 marks]

Total [8 marks]

Question 43

EITHER

$x = \arctan t$ ***(M1)***

$\frac{dx}{dt} = \frac{1}{1+t^2}$ ***A1***

OR

$t = \tan x$

$\frac{dt}{dx} = \sec^2 x$ ***(M1)***

$= 1 + \tan^2 x$ ***A1***

$= 1 + t^2$

THEN

$\sin x = \frac{t}{\sqrt{1+t^2}}$ ***(A1)***

Note: This ***A1*** is independent of the first two marks

$\int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{dt}{1+t^2}}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to obtain integral in terms of t and dt

$= \int \frac{dt}{\left(1+t^2\right)+t^2} = \int \frac{dt}{1+2t^2}$ ***A1***

$= \frac{1}{2}\int \frac{dt}{\frac{1}{2}+t^2} = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$ ***A1***

$= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x\right)(+c)$ ***A1***

[8 marks]

Question 44

(a) $g \circ f(x) = \dfrac{\tan x + 1}{\tan x - 1}$ **A1**

$x \neq \dfrac{\pi}{4},\ 0 \leq x < \dfrac{\pi}{2}$ **A1**

[2 marks]

(b) $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{\dfrac{\sin x}{\cos x} + 1}{\dfrac{\sin x}{\cos x} - 1}$ **M1A1**

$= \dfrac{\sin x + \cos x}{\sin x - \cos x}$ **AG**

[2 marks]

(c) **METHOD 1**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ **M1(A1)**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\left(2\sin x\cos x - \cos^2 x - \sin^2 x\right) - \left(2\sin x\cos x + \cos^2 x + \sin^2 x\right)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \dfrac{-2}{1 - \sin 2x}$

Substitute $\dfrac{\pi}{6}$ into any formula for $\dfrac{dy}{dx}$ **M1**

$\dfrac{-2}{1 - \sin\frac{\pi}{3}}$

$= \dfrac{-2}{1 - \frac{\sqrt{3}}{2}}$ **A1**

$= \dfrac{-4}{2 - \sqrt{3}}$

$= \dfrac{-4}{2 - \sqrt{3}}\left(\dfrac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ **M1**

$= \dfrac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ **A1**

(d) Area $= \left|\int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x} dx\right|$ ***M1***

$= \left|\left[\ln|\sin x - \cos x|\right]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln|\sin 0 - \cos 0|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3}-1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3}-1}{2}\right) = \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$ ***M1***

$= \ln\left(\sqrt{3}+1\right)$ ***AG***

[6 marks]

Total [16 marks]

Question 45

attempt to integrate one factor and differentiate the other, leading to a sum of two terms ***M1***

$\int x \sin x\, dx = x(-\cos x) + \int \cos x\, dx$ ***(A1)(A1)***

$= -x\cos x + \sin x + c$ ***A1***

Note: Only award final ***A1*** if $+c$ is seen.

[4 marks]

Question 46

(a) $\frac{dy}{dx} = (1-x)^{-2} \left(= \frac{1}{(1-x)^2}\right)$ ***(M1)A1***

(b) gradient of Tangent $= \frac{1}{4}$ ***(A1)***

gradient of Normal $= -4$ ***(M1)***

$y + \frac{1}{2} = -4(x-3)$ or attempt to find c in $y = mx + c$ ***M1***

$8x + 2y - 23 = 0$ ***A1***

[4 marks]

Total [6 marks]

Question 47

METHOD 1

$\int_{e}^{e^2} \frac{dx}{x \ln x} = [\ln(\ln x)]_{e}^{e^2}$ ***(M1)A1***

$= \ln(\ln e^2) - \ln(\ln e)$ $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

METHOD 2

$u = \ln x, \frac{du}{dx} = \frac{1}{x}$ ***M1***

$= \int_1^2 \frac{du}{u}$ ***A1***

$= [\ln u]_1^2$ or equivalent in x $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

Question 48

(a) $x\frac{dy}{dx} + y = 2y\frac{dy}{dx}$ ***M1A1***

a horizontal tangent occurs if $\frac{dy}{dx} = 0$ so $y = 0$ ***M1***

we can see from the equation of the curve that this solution is not possible $(0 = 4)$ and so there is not a horizontal tangent ***R1***

[4 marks]

(b) $\frac{dy}{dx} = \frac{y}{2y - x}$ or equivalent with $\frac{dx}{dy}$

the tangent is vertical when $2y = x$ ***M1***

substitute into the equation to give $2y^2 = y^2 + 4$ ***M1***

$y = \pm 2$ ***A1***

coordinates are $(4, 2), (-4, -2)$ ***A1***

[4 marks]

Total [8 marks]

Question 49

(a) $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta \cos\frac{\pi}{2} + \cos\theta \sin\frac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

(b) consider $n = 1$, $f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \frac{\mathrm{d}\left(f^{(k)}(x)\right)}{\mathrm{d}x}\left(= a\left(a^k \cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior M and R marks have been awarded.

[7 marks]

Total [8 marks]

Question 50

(a) $f(-x) = (-x)\sqrt{1-(-x)^2}$ **M1**

$= -x\sqrt{1-x^2}$

$= -f(x)$ **R1**

hence f is odd **AG**

[2 marks]

(b) $f'(x) = x \cdot \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot -2x + \left(1-x^2\right)^{\frac{1}{2}}$ **M1A1A1**

[3 marks]

(c) $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \left(= \frac{1-2x^2}{\sqrt{1-x^2}}\right)$ **A1**

Note: This may be seen in part (b).

$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$ **M1**

$x = \pm\frac{1}{\sqrt{2}}$ **A1**

[3 marks]

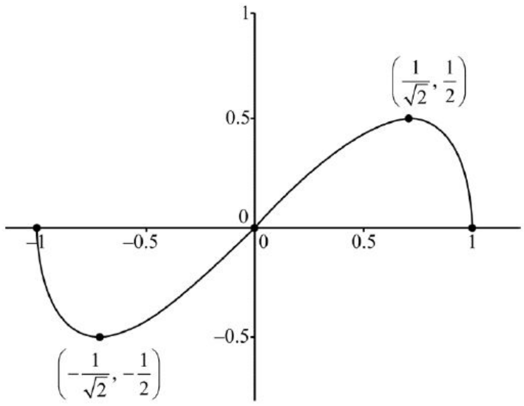
(d) y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$ **M1A1**

so range of $f(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ **A1**

Note: Allow FT from (c) if values of x, within the domain, are used.

[3 marks]

(e)



Shape: The graph of an odd function, on the given domain, s-shaped, where the max(min) is the right(left) of 0.5(-0.5) ***A1***

x-intercepts ***A1***

turning points ***A1***

[3 marks]

(f) area $= \int_0^1 x\sqrt{1-x^2}\,dx$ ***(M1)***

attempt at "backwards chain rule" or substitution ***M1***

$= -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\,dx$

$= \left[\frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} \cdot -\frac{1}{2}\right]_0^1$ ***A1***

$= \left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$

$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$ ***A1***

[4 marks]

(g) $\int_{-1}^1 \left| x\sqrt{1-x^2} \right| dx > 0$ ***R1***

$\left| \int_{-1}^1 x\sqrt{1-x^2}\,dx \right| = 0$ ***R1***

so $\int_{-1}^1 \left| x\sqrt{1-x^2} \right| dx > \left| \int_{-1}^1 x\sqrt{1-x^2}\,dx \right| = 0$ ***AG***

[2 marks]

Question 51

$$\frac{dy}{dx} = -\cos(\pi\cos x) \times \pi\sin x$$ **M1A1**

ote: Award follow through marks below if their answer is a multiple of the correct answer.

considering either $\sin x = 0$ or $\cos(\pi\cos x) = 0$ ***(M1)***

$x = 0, \pi$ ***A1***

$\pi\cos x = \frac{\pi}{2}, -\frac{\pi}{2}\left(\Rightarrow \cos x = \frac{1}{2}, -\frac{1}{2}\right)$ ***M1***

ote: Condone absence of $-\frac{\pi}{2}$.

$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$

$(0, 0), \left(\frac{\pi}{3}, 1\right), (\pi, 0)$ ***A1***

$\left(\frac{2\pi}{3}, -1\right)$ ***A1***

[7 marks]

Question 52

$\frac{dy}{dx} = 8x^3 + 18x^2 + 7x - 5$ ***A1***

when $x = -1, \frac{dy}{dx} = -2$ ***A1***

$8x^3 + 18x^2 + 7x - 5 = -2$ ***M1***

$8x^3 + 18x^2 + 7x - 3 = 0$

$(x + 1)$ is a factor ***A1***

$8x^3 + 18x^2 + 7x - 3 = (x + 1)(8x^2 + 10x - 3)$ ***(M1)***

Note: ***M1*** is for attempting to find the quadratic factor.

$(x + 1)(4x - 1)(2x + 3) = 0$

$(x = -1), x = 0.25, x = -1.5$ ***(M1)A1***

Note: ***M1*** is for an attempt to solve their quadratic factor.

[7 marks]

Question 53

(a) $a = 1$ **A1**

[1 mark]

(b) $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ ***(A1)***

$\int \frac{(\ln x)^2}{x}\mathrm{d}x = \int u^2 \mathrm{d}u$ ***M1A1***

area = $\left[\frac{1}{3}u^3\right]_0^1$ or $\left[\frac{1}{3}(\ln x)^3\right]_1^{\mathrm{e}}$ ***A1***

$= \frac{1}{3}$ ***A1***

[5 marks]

(c) (i) $I_0 = \left[-\frac{1}{x}\right]_1^{\mathrm{e}}$ ***(A1)***

$= 1 - \frac{1}{\mathrm{e}}$ ***A1***

(ii) use of integration by parts ***M1***

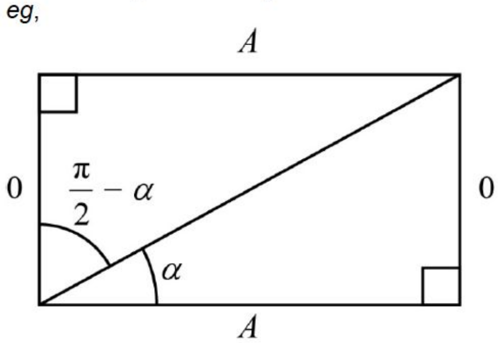
$I_n = \left[-\frac{1}{x}(\ln x)^n\right]_1^{\mathrm{e}} + \int_1^{\mathrm{e}} \frac{n(\ln x)^{n-1}}{x^2}\mathrm{d}x$ ***A1A1***

$= -\frac{1}{\mathrm{e}} + nI_{n-1}$ ***AG***

Question 54

(a) **EITHER**

use of a diagram and trig ratios
eg,



$\tan\alpha = \frac{O}{A} \Rightarrow \cot\alpha = \frac{A}{O}$

from diagram, $\tan\left(\frac{\pi}{2}-\alpha\right) = \frac{A}{O}$ ***R1***

OR

use of $\tan\left(\frac{\pi}{2}-\alpha\right) = \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)} = \frac{\cos\alpha}{\sin\alpha}$ ***R1***

THEN

$\cot\alpha = \tan\left(\frac{\pi}{2}-\alpha\right)$ ***AG***

[1 mark]

(b) $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx = [\arctan x]_{\tan\alpha}^{\cot\alpha}$ ***(A1)***

Note: Limits (or absence of such) may be ignored at this stage.

$= \arctan(\cot\alpha) - \arctan(\tan\alpha)$ ***(M1)***

$= \frac{\pi}{2} - \alpha - \alpha$ ***(A1)***

$= \frac{\pi}{2} - 2\alpha$ ***A1***

[4 marks]

Total [5 marks]

(iii) $I_1 = -\frac{1}{e} + 1 \times I_0$ ***(M1)***

$= 1 - \frac{2}{e}$ ***A1***

[7 marks]

(d) volume $= \pi \int_1^e \frac{(\ln x)^4}{x^2} dx \, (= \pi I_4)$ ***(A1)***

EITHER

$I_4 = -\frac{1}{e} + 4I_3$ ***M1A1***

$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3I_2\right)$ ***M1***

$= -\frac{5}{e} + 12I_2 = -\frac{5}{e} + 12\left(-\frac{1}{e} + 2I_1\right)$

OR

using parts $\int_1^e \frac{(\ln x)^4}{x^2} dx = -\frac{1}{e} + 4\int_1^e \frac{(\ln x)^3}{x^2} dx$ ***M1A1***

$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3\int_1^e \frac{(\ln x)^2}{x^2} dx\right)$ ***M1***

THEN

$= -\frac{17}{e} + 24\left(1 - \frac{2}{e}\right) = 24 - \frac{65}{e}$ ***A1***

volume $= \pi\left(24 - \frac{65}{e}\right)$

[5 marks]

Total [18 marks]

Question 55

(a) use of $\pi\int_a^b x^2 \, dy$ **(M1)**

Note: Condone any or missing limits.

$$V = \pi\int_0^{\pi} (3\cos 2y + 4)^2 \, dy$$ **(A1)**

$$= \pi\int_0^{\pi} \left(9\cos^2 2y + 24\cos 2y + 16\right) dy$$ **A1**

$$9\cos^2 2y = \frac{9}{2}(1 + \cos 4y)$$ **(M1)**

$$= \pi\left[\frac{9y}{2} + \frac{9}{8}\sin 4y + 12\sin 2y + 16y\right]_0^{\pi}$$ **M1A1**

$$= \pi\left(\frac{9\pi}{2} + 16\pi\right)$$ **(A1)**

$$= \frac{41\pi^2}{2} \, (\text{cm}^3)$$ **A1**

te: If the coefficient "π" is absent, or *eg*, "2π" is used, only ***M*** marks are available.

[8 marks]

(b) (i) attempting to use $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ with $\frac{dV}{dt} = 2$ **M1**

$$\frac{dh}{dt} = \frac{2}{\pi(3\cos 2h + 4)^2}$$ **A1**

(ii) substituting $h = \frac{\pi}{4}$ into $\frac{dh}{dt}$ **(M1)**

$$\frac{dh}{dt} = \frac{1}{8\pi} \, (\text{cm min}^{-1})$$ **A1**

Note: Do not allow FT marks for (b)(ii).

[4 marks]

(c) (i) $$\frac{d^2h}{dt^2} = \frac{d}{dt}\left(\frac{dh}{dt}\right) = \frac{dh}{dt} \times \frac{d}{dh}\left(\frac{dh}{dt}\right)$$ **(M1)**

$$= \frac{2}{\pi(3\cos 2h + 4)^2} \times \frac{24\sin 2h}{\pi(3\cos 2h + 4)^3}$$ **M1A1**

Note: Award ***M1*** for attempting to find $\frac{d}{dh}\left(\frac{dh}{dt}\right)$.

$$= \frac{48\sin 2h}{\pi^2(3\cos 2h + 4)^5}$$ **A1**

(ii) $\sin 2h = 0 \Rightarrow h = 0, \frac{\pi}{2}, \pi$ ***A1***

Note: Award ***A1*** for $\sin 2h = 0 \Rightarrow h = 0, \frac{\pi}{2}, \pi$ from an incorrect $\frac{d^2h}{dt^2}$.

(iii) **METHOD 1**

$\frac{dh}{dt}$ is a minimum at $h = 0, \pi$ and the container is widest at these values

R1

$\frac{dh}{dt}$ is a maximum at $h = \frac{\pi}{2}$ and the container is narrowest at this value

R1

[7 marks]

Total [19 marks]

Question 56

(a) attempt to differentiate implicitly ***M1***

$$3 - \left(4y\frac{dy}{dx} + 2y^2\right)e^{x-1} = 0$$ ***A1A1A1***

Note: Award ***A1*** for correctly differentiating each term.

$$\frac{dy}{dx} = \frac{3 \cdot e^{1-x} - 2y^2}{4y}$$ ***A1***

(b) $3 - 2y^2 = 2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$ ***A1***

$$\frac{dy}{dx} = \frac{3 - 2 \cdot \frac{1}{2}}{\pm 4\sqrt{\frac{1}{2}}} = \pm\frac{\sqrt{2}}{2}$$ ***M1***

at $\left(1, \sqrt{\frac{1}{2}}\right)$ the tangent is $y - \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}(x-1)$ and ***A1***

at $\left(1, -\sqrt{\frac{1}{2}}\right)$ the tangent is $y + \sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}(x-1)$ ***A1***

Question 57

(a) $\frac{dy}{dx} = e^x \sin x + e^x \cos x \left(= e^x (\sin x + \cos x)\right)$ ***M1A1***

[2 marks]

(b) $\frac{d^2y}{dx^2} = e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$ ***M1A1***

$= 2e^x \cos x$ ***AG***

[2 marks]

(c) $\frac{dy}{dx} = e^{\frac{3\pi}{4}} \left(\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} \right) = 0$ ***R1***

$\frac{d^2y}{dx^2} = 2e^{\frac{3\pi}{4}} \cos \frac{3\pi}{4} < 0$ ***R1***

hence maximum at $x = \frac{3\pi}{4}$ ***AG***

[2 marks]

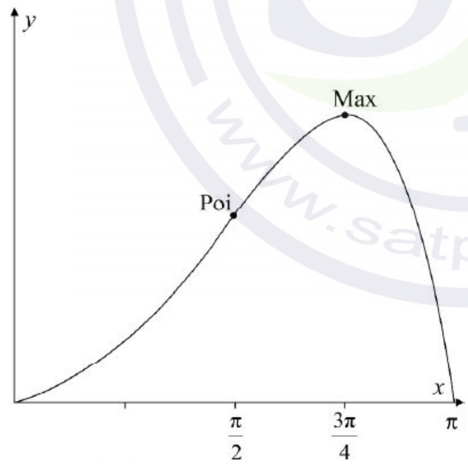
(d) $\frac{d^2y}{dx^2} = 0 \Rightarrow 2e^x \cos x = 0$ ***M1***

$\Rightarrow x = \frac{\pi}{2}$ ***A1***

Note: Award ***M1A0*** if extra zeros are seen.

[2 marks]

(e)



correct shape and correct domain ***A1***

max at $x = \frac{3\pi}{4}$, point of inflexion at $x = \frac{\pi}{2}$ ***A1***

zeros at $x = 0$ and $x = \pi$ ***A1***

(f) **EITHER**

$\int_0^{\pi} e^x \sin x \, dx = \left[e^x \sin x\right]_0^{\pi} - \int_0^{\pi} e^x \cos x \, dx$ ***M1A1***

$\int_0^{\pi} e^x \sin x \, dx = \left[e^x \sin x\right]_0^{\pi} - \left(\left[e^x \cos x\right]_0^{\pi} + \int_0^{\pi} e^x \sin x \, dx\right)$ ***A1***

OR

$\int_0^{\pi} e^x \sin x \, dx = \left[-e^x \cos x\right]_0^{\pi} + \int_0^{\pi} e^x \cos x \, dx$ ***M1A1***

$\int_0^{\pi} e^x \sin x \, dx = \left[-e^x \cos x\right]_0^{\pi} + \left(\left[e^x \sin x\right]_0^{\pi} - \int_0^{\pi} e^x \sin x \, dx\right)$ ***A1***

THEN

$\int_0^{\pi} e^x \sin x \, dx = \frac{1}{2}\left(\left[e^x \sin x\right]_0^{\pi} - \left[e^x \cos x\right]_0^{\pi}\right)$ ***M1A1***

$\int_0^{\pi} e^x \sin x \, dx = \frac{1}{2}\left(e^{\pi} + 1\right)$ ***A1***

[6 marks]

(g) $\frac{dy}{dx} = 0$ ***(A1)***

$\frac{d^2y}{dx^2} = 2e^{\frac{3\pi}{4}} \cos\frac{3\pi}{4} = -\sqrt{2}e^{\frac{3\pi}{4}}$ ***(A1)***

$\kappa = \frac{\left|-\sqrt{2}e^{\frac{3\pi}{4}}\right|}{1} = \sqrt{2}e^{\frac{3\pi}{4}}$ ***A1***

[3 marks]

(h) $\kappa = 0$ ***A1***

the graph is approximated by a straight line ***R1***

[2 marks]

Total [22 marks]

Question 58

attempt at integration by parts with $u = \arcsin x$ and $v' = 1$ ***M1***

$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$ ***A1A1***

Note: Award ***A1*** for $x \arcsin x$ and ***A1*** for $-\int \frac{x}{\sqrt{1 - x^2}} dx$.

solving $\int \frac{x}{\sqrt{1 - x^2}} dx$ by substitution with $u = 1 - x^2$ or inspection ***(M1)***

$\int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + c$ ***A1***

[5 marks]

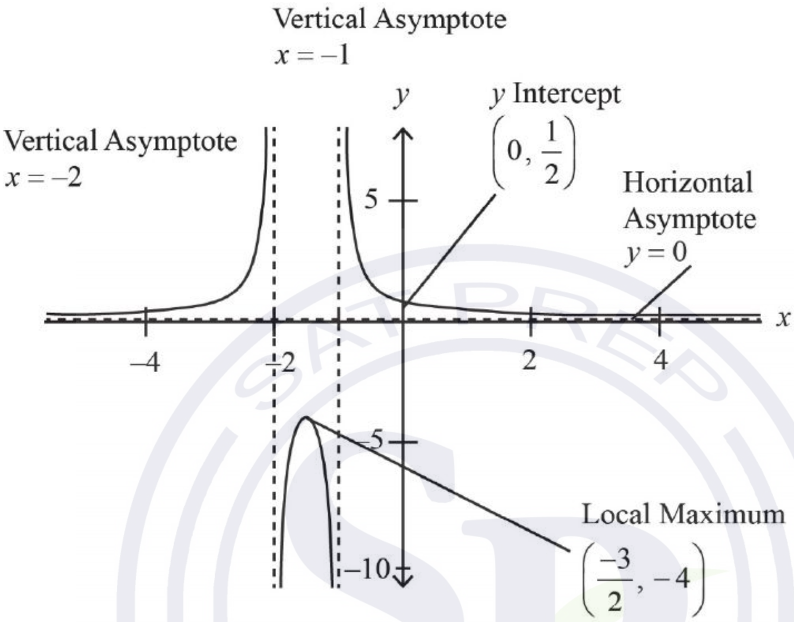
Question 59

(a) (i) $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ **A1**

(ii) $x^2 + 3x + 2 = (x + 2)(x + 1)$ **A1**

[2 marks]

(b)



A1 for the shape

A1 for the equation $y = 0$

A1 for asymptotes $x = -2$ and $x = -1$

A1 for coordinates $\left(-\frac{3}{2}, -4\right)$

A1 y-intercept $\left(0, \frac{1}{2}\right)$

[5 marks]

(c) $\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2)-(x+1)}{(x+1)(x+2)}$ **M1**

$= \frac{1}{x^2 + 3x + 2}$ **AG**

[1 mark]

(d) $\int_0^1 \frac{1}{x+1} - \frac{1}{x+2} \mathrm{d}x$

$= \left[\ln(x+1) - \ln(x+2)\right]_0^1$ **A1**

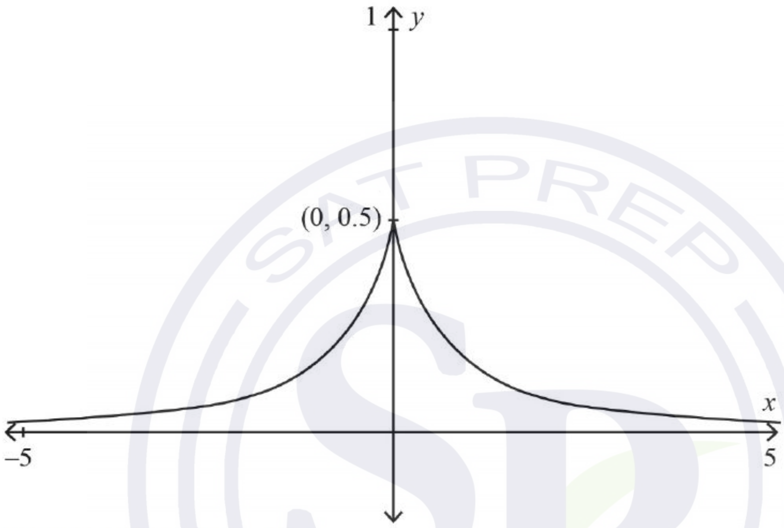
$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ **M1**

$= \ln\left(\frac{4}{3}\right)$ **M1A1**

$\therefore p = \frac{4}{3}$

[4 marks]

(e)



symmetry about the y-axis **M1**

correct shape **A1**

Note: Allow ***FT*** from part (b).

[2 marks]

(f) $2\int_0^1 f(x)\,\mathrm{d}x$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ **A1**

Note: Do not award ***FT*** from part (e).

[3 marks]

Total [17 marks]

Question 60

(a) $s = t + \cos 2t$

$\frac{ds}{dt} = 1 - 2\sin 2t$ ***M1A1***

$= 0$ ***M1***

$\Rightarrow \sin 2t = \frac{1}{2}$

$t_1 = \frac{\pi}{12}(s),\ t_2 = \frac{5\pi}{12}(s)$ ***A1A1***

Note: Award ***A0A0*** if answers are given in degrees.

[5 marks]

(b) $s = \frac{\pi}{12} + \cos\frac{\pi}{6}\left(s = \frac{\pi}{12} + \frac{\sqrt{3}}{2}(m)\right)$ ***A1A1***

[2 marks]

Total [7 marks]

Question 61

(a) let $x = \tan\theta$

$\Rightarrow \frac{dx}{d\theta} = \sec^2\theta$ ***(A1)***

$\int \frac{1}{(x^2+1)^2}dx = \int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}d\theta$ ***M1***

Note: The method mark is for an attempt to substitute for both x and dx.

$= \int \frac{1}{\sec^2\theta}d\theta$ (or equivalent) ***A1***

when $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{4}$ ***M1***

$= \int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$ ***AG***

[4 marks]

(b) $\left(\int_0^1 \frac{1}{(x^2+1)^2}dx = \int_0^{\frac{\pi}{4}}\cos^2\theta\, d\theta\right) = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)d\theta$ ***M1***

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$ ***A1***

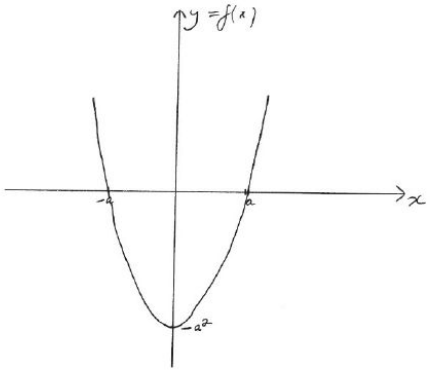
$= \frac{\pi}{8} + \frac{1}{4}$ ***A1***

[3 marks]

Total [7 marks]

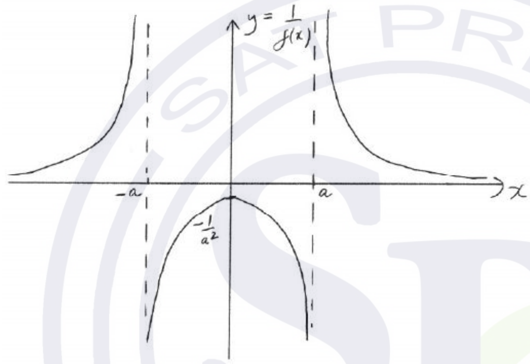
Question 62

(a) (i)



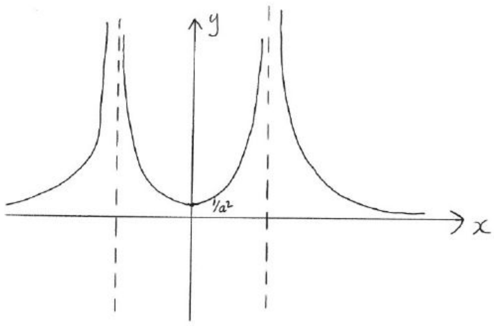
A1 for correct shape
A1 for correct x and y intercepts and minimum point

(ii)



A1 for correct shape
A1 for correct vertical asymptotes
A1 for correct implied horizontal asymptote
A1 for correct maximum point

(iii)



A1 for reflecting negative branch from (ii) in the x-axis
A1 for correctly labelled minimum point

[8 marks]

(b) **EITHER**

attempt at integration by parts ***(M1)***

$\int (x^2 - a^2)\cos x\,dx = (x^2 - a^2)\sin x - \int 2x\sin x\,dx$ ***A1A1***

$= (x^2 - a^2)\sin x - 2\left[-x\cos x + \int \cos x\,dx\right]$ ***A1***

$= (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

OR

$\int (x^2 - a^2)\cos x\,dx = \int x^2\cos x\,dx - \int a^2\cos x\,dx$

attempt at integration by parts ***(M1)***

$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$ ***A1A1***

$= x^2\sin x - 2\left[-x\cos x + \int \cos x\,dx\right]$ ***A1***

$= x^2\sin x + 2x\cos x - 2\sin x$

$-\int a^2\cos x\,dx = -a^2\sin x$

$\int (x^2 - a^2)\cos x\,dx = (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

[5 marks]

(c) $g(x) = x(x^2 - a^2)^{\frac{1}{2}}$

$g'(x) = (x^2 - a^2)^{\frac{1}{2}} + \frac{1}{2}x(x^2 - a^2)^{-\frac{1}{2}}(2x)$ ***M1A1A1***

Note: Method mark is for differentiating the product. Award ***A1*** for each correct term.

$g'(x) = (x^2 - a^2)^{\frac{1}{2}} + x^2(x^2 - a^2)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive ***R1***

and therefore g is an increasing function (for $|x| > a$) ***AG***

[4 marks]

Total [17 marks]

Question 63

(a) (i) the width of the rectangle is $2r$ and let the height of the rectangle be h

$\text{P} = 2r + 2h + \pi r$ ***(A1)***

$A = 2rh + \dfrac{\pi r^2}{2}$ ***(A1)***

$h = \dfrac{\text{P} - 2r - \pi r}{2}$

$A = 2r\left(\dfrac{\text{P} - 2r - \pi r}{2}\right) + \dfrac{\pi r^2}{2}\left(= \text{P}r - 2r^2 - \dfrac{\pi r^2}{2}\right)$ ***M1A1***

(ii) $\dfrac{\text{d}A}{\text{d}r} = \text{P} - 4r - \pi r$ ***A1***

$\dfrac{\text{d}A}{\text{d}r} = 0$ ***M1***

$\Rightarrow r = \dfrac{\text{P}}{4 + \pi}$ ***(A1)***

hence the width is $\dfrac{2\text{P}}{4 + \pi}$ ***A1***

$\dfrac{\text{d}^2 A}{\text{d}r^2} = -4 - \pi < 0$ ***R1***

hence maximum ***AG***

[9 marks]

(b) **EITHER**

$h = \dfrac{\text{P} - 2r - \pi r}{2}$

$h = \dfrac{\text{P} - \dfrac{2\text{P}}{4+\pi} - \dfrac{\text{P}\pi}{4+\pi}}{2}$ ***M1***

$h = \dfrac{4\text{P} + \pi\text{P} - 2\text{P} - \pi\text{P}}{2(4+\pi)}$ ***A1***

$h = \dfrac{\text{P}}{(4+\pi)} = r$ ***AG***

OR

$h = \dfrac{\text{P} - 2r - \pi r}{2}$

$P = r(4 + \pi)$ ***M1***

$h = \dfrac{\text{r}(4+\pi) - 2r - \pi r}{2}$ ***A1***

$h = \dfrac{4\text{r} + \pi r - 2r - \pi r}{2} = r$ ***AG***

[2 marks]

Total [11 marks]

Question 64

$s = \int_0^{\frac{1}{2}} 10te^{-2t}\,dt$

attempt at integration by parts **M1**

$= \left[-5te^{-2t}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -5e^{-2t}\,dt$ **A1**

$= \left[-5te^{-2t} - \frac{5}{2}e^{-2t}\right]_0^{\frac{1}{2}}$ ***(A1)***

:e: Condone absence of limits (or incorrect limits) and missing factor of 10 up to this point.

$s = \int_0^{\frac{1}{2}} 10te^{-2t}\,dt$ ***(M1)***

$= -5e^{-1} + \frac{5}{2}\left(= \frac{-5}{e} + \frac{5}{2}\right)\left(= \frac{5e-10}{2e}\right)$ **A1**

[5 marks]

Question 65

$x^3 + y^3 - 3xy = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$ ***M1A1***

:e: Differentiation wrt y is also acceptable.

$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}\left(= \frac{y - x^2}{y^2 - x}\right)$ ***(A1)***

:e: All following marks may be awarded if the ominator is correct, but the numerator incorrect.

$y^2 - x = 0$ ***M1***

EITHER

$x = y^2$

$y^6 + y^3 - 3y^3 = 0$ ***M1A1***

$y^6 - 2y^3 = 0$

$y^3(y^3 - 2) = 0$

$(y \neq 0) \quad \therefore y = \sqrt[3]{2}$ ***A1***

$x = \left(\sqrt[3]{2}\right)^2 \left(= \sqrt[3]{4}\right)$ ***A1***

OR

$x^3 + xy - 3xy = 0$ ***M1***

$x\left(x^2 - 2y\right) = 0$

$x \neq 0 \Rightarrow y = \dfrac{x^2}{2}$ ***A1***

$y^2 = \dfrac{x^4}{4}$

$x = \dfrac{x^4}{4}$

$x\left(x^3 - 4\right) = 0$

$\left(x \neq 0\ \right) \therefore x = \sqrt[3]{4}$ ***A1***

$y = \dfrac{\left(\sqrt[3]{4}\right)^2}{2} = \sqrt[3]{2}$ ***A1***

[8 marks]

Question 66

(a) even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x)\ldots = \left(\cos\left(-2x\right)\right)\left(\cos\left(-4x\right)\right)\ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

(b) consider the case $n = 1$

$\dfrac{\sin 4x}{2\sin 2x} = \dfrac{2\sin 2x \cos 2x}{2\sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x)\ldots\left(\cos 2^k x\right) = \dfrac{\sin 2^{k+1} x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)\left(\cos 2^{k+1} x\right)$ ***(M1)***

$= \dfrac{\sin 2^{k+1} x}{2^k \sin 2x} \cos 2^{k+1} x$ ***A1***

$= \dfrac{2\sin 2^{k+1} x \cos 2^{k+1} x}{2^{k+1} \sin 2x}$ ***A1***

$= \dfrac{\sin 2^{k+2} x}{2^{k+1} \sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

(c) attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) **M1**

$$f_n'(x) = \frac{(2^n \sin 2x)(2^{n+1}\cos 2^{n+1}x) - (\sin 2^{n+1}x)(2^{n+1}\cos 2x)}{(2^n \sin 2x)^2}$$ **A1A1**

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

(d) $$f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin\frac{\pi}{2}\right)\left(2^{n+1}\cos 2^{n+1}\frac{\pi}{4}\right) - \left(\sin 2^{n+1}\frac{\pi}{4}\right)\left(2^{n+1}\cos\frac{\pi}{2}\right)}{\left(2^n \sin\frac{\pi}{2}\right)^2}$$ ***(M1)(A1)***

$$f_n'\left(\frac{\pi}{4}\right) = \frac{(2^n)\left(2^{n+1}\cos 2^{n+1}\frac{\pi}{4}\right)}{(2^n)^2}$$ ***(A1)***

$$= 2\cos 2^{n+1}\frac{\pi}{4}\left(= 2\cos 2^{n-1}\pi\right)$$ ***A1***

$$f_n'\left(\frac{\pi}{4}\right) = 2$$ ***A1***

$$f_n\left(\frac{\pi}{4}\right) = 0$$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$$y = 2\left(x - \frac{\pi}{4}\right)$$ ***M1A1***

$$4x - 2y - \pi = 0$$ ***AG***

[8 marks]

Total [21 marks]

Question 67

(a)

$$\log_{r^2} x = \frac{\log_r x}{\log_r r^2}\left(= \frac{\log_r x}{2\log_r r}\right)$$ ***M1A1***

$$= \frac{\log_r x}{2}$$ ***AG***

[2 marks]

(b)

$$\log_2 y + \log_4 x + \log_4 2x = 0$$

$$\log_2 y + \log_4 2x^2 = 0$$ ***M1***

$$\log_2 y + \frac{1}{2}\log_2 2x^2 = 0$$ ***M1***

$$\log_2 y = -\frac{1}{2}\log_2 2x^2$$

$$\log_2 y = \log_2\left(\frac{1}{\sqrt{2}x}\right)$$ ***M1A1***

$$y = \frac{1}{\sqrt{2}}x^{-1}$$ ***A1***

(c) the area of R is $\int_1^{\alpha} \frac{1}{\sqrt{2}} x^{-1} \mathrm{d}x$ **M1**

$= \left[\frac{1}{\sqrt{2}} \ln x\right]_1^{\alpha}$ **A1**

$= \frac{1}{\sqrt{2}} \ln \alpha$ **A1**

$\frac{1}{\sqrt{2}} \ln \alpha = \sqrt{2}$ **M1**

$\alpha = \mathrm{e}^2$ **A1**

Note: Only follow through from part (b) if y is in the form $y = px^q$.

[5 marks]

Total [12 marks]

Question 68

(a) $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2} x^{-\frac{1}{2}}$ (accept $\mathrm{d}u = \frac{1}{2} x^{-\frac{1}{2}} \mathrm{d}x$ or equivalent) **A1**

substitution, leading to an integrand in terms of u **M1**

$\int \frac{2u\mathrm{d}u}{u^3 + u}$ or equivalent **A1**

$= 2\arctan(\sqrt{x})(+c)$ **A1**

[4 marks]

(b) $\frac{1}{2}\int_1^9 \frac{\mathrm{d}x}{x^{\frac{3}{2}} + x^{\frac{1}{2}}} = \arctan 3 - \arctan 1$ **A1**

$\tan(\arctan 3 - \arctan 1) = \frac{3 - 1}{1 + 3 \times 1}$ ***(M1)***

$\tan(\arctan 3 - \arctan 1) = \frac{1}{2}$

$\arctan 3 - \arctan 1 = \arctan \frac{1}{2}$ **A1**

[3 marks]

Total [7 marks]

Question 69

(a) (i) attempt at product rule **M1**

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$ **A1**

(ii) $g'(x) = -e^{-x}\cos x - e^{-x}\sin x$ **A1**

[3 marks]

$I = \int e^{-x}\sin x\,dx$

$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$ **OR** $= -e^{-x}\sin x + \int e^{-x}\cos x\,dx$ **M1A1**

$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$

$I = -\frac{1}{2}e^{-x}(\sin x + \cos x)$ **A1**

$\int_0^{\pi} e^{-x}\sin x\,dx = \frac{1}{2}\left(1 + e^{-\pi}\right)$ **A1**

[4 marks]

Total [7 marks]

(b)

Question 70

valid attempt to find $\frac{dy}{dx}$ **M1**

$\frac{dy}{dx} = \frac{1}{(1-x)^2} - \frac{4}{(x-4)^2}$ **A1A1**

attempt to solve $\frac{dy}{dx} = 0$ **M1**

$x = 2,\ x = -2$ **A1A1**

[6 marks]

Question 71

(a) attempt to differentiate **(M1)**

$f'(x) = -3x^{-4} - 3x$ **A1**

e: Award ***M1*** for using quotient or product rule award ***A1*** if correct derivative seen even in unsimplified form, for example $f'(x) = \dfrac{-15x^4 \times 2x^3 - 6x^2(2 - 3x^5)}{(2x^3)^2}$.

$-\dfrac{3}{x^4} - 3x = 0$ **M1**

$\Rightarrow x^5 = -1 \Rightarrow x = -1$ **A1**

$A\left(-1, -\dfrac{5}{2}\right)$ **A1**

[5 marks]

(b) (i) $f''(x) = 0$ **M1**

$f''(x) = 12x^{-5} - 3 (= 0)$ **A1**

Note: Award ***A1*** for correct derivative seen even if not simplified.

$\Rightarrow x = \sqrt[5]{4} \left(= 2^{\frac{2}{5}}\right)$ **A1**

hence (at most) one point of inflexion **R1**

Note: This mark is independent of the two ***A1*** marks above. If they have shown or stated their equation has only one solution this mark can be awarded.

$f''(x)$ changes sign at $x = \sqrt[5]{4} \left(= 2^{\frac{2}{5}}\right)$ **R1**

so exactly one point of inflexion

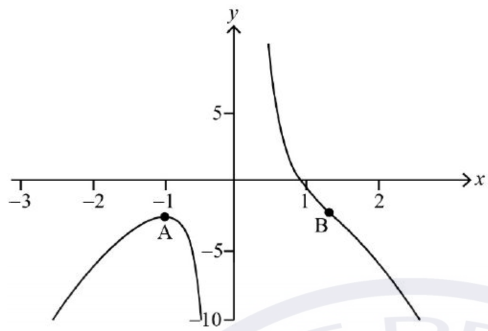
(ii) $x=\sqrt[5]{4}=2^{\frac{2}{5}}\left(\Rightarrow a=\frac{2}{5}\right)$ **A1**

$f\left(2^{\frac{2}{5}}\right)=\frac{2-3\times 2^2}{2\times 2^{\frac{6}{5}}}=-5\times 2^{-\frac{6}{5}}\ (\Rightarrow b=-5)$ **(M1)A1**

[8 marks]

Note: Award ***M1*** for the substitution of their value for x into $f(x)$.

(c)



A1A1A1A1

A1 for shape for $x<0$
A1 for shape for $x>0$
A1 for maximum at A
A1 for POI at B.

Note: Only award last two ***A1***s if A and B are placed in the correct quadrants, allowing for follow through.

[4 marks]

Total [17 marks]

Question 72

(a) $y = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \left(= -\frac{1}{\sqrt{4-x^2}}\right)$ ***M1A1***

Note: ***M1*** is for use of the chain rule.

[2 marks]

(b) attempt at integration by parts ***M1***

$u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{4-x^2}}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$ ***(A1)***

$\int_0^1 \arccos\left(\frac{x}{2}\right) dx = \left[x\arccos\left(\frac{x}{2}\right)\right]_0^1 + \int_0^1 \frac{x}{\sqrt{4-x^2}} dx$ ***A1***

using integration by substitution or inspection ***(M1)***

$\left[x\arccos\left(\frac{x}{2}\right)\right]_0^1 + \left[-\left(4-x^2\right)^{\frac{1}{2}}\right]_0^1$ ***A1***

Note: Award ***A1*** for $-\left(4-x^2\right)^{\frac{1}{2}}$ or equivalent.

Note: Condone lack of limits to this point.

attempt to substitute limits into their integral ***M1***

$= \frac{\pi}{3} - \sqrt{3} + 2$ ***A1***

[7 marks]

Total [9 marks]

Question 73

(a) $\int_{-2}^{0} f(x)dx = 10 - 12 = -2$ ***(M1)(A1)***

$\int_{-2}^{0} 2\,dx = [2x]_{-2}^{0} = 4$ ***A1***

$\int_{-2}^{0} (f(x)+2)dx = 2$ ***A1***

[4 marks]

(b) $\int_{-2}^{0} f(x+2)dx = \int_{0}^{2} f(x)dx$ ***(M1)***

$= 12$ ***A1***

[2 marks]

Total [6 marks]

Question 74

(a) attempt at chain rule or product rule *(M1)*

$\frac{dy}{d\theta} = 2\sin\theta\cos\theta$ **A1**

[2 marks]

(b) $2\sin\theta\cos\theta = 2\sin^2\theta$

$\sin\theta = 0$ *(A1)*

$\theta = 0,\ \pi$ **A1**

obtaining $\cos\theta = \sin\theta$ *(M1)*

$\tan\theta = 1$ *(M1)*

$\theta = \frac{\pi}{4}$ **A1**

[5 marks]

Total [7 marks]

Question 75

(a)

attempt at integration by parts with $u = \cos 2x,\ \frac{dv}{dx} = e^x$ **M1**

$\int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx$ **A1**

$= e^x\cos 2x + 2\left(e^x\sin 2x - 2\int e^x\cos 2x\,dx\right)$ **M1A1**

$= e^x\cos 2x + 2e^x\sin 2x - 4\int e^x\cos 2x\,dx$

$\therefore 5\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$ **M1**

$\int e^x\cos 2x\,dx = \frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x(+c)$ **AG**

[5 marks]

(b) $\int e^x\cos^2 x\,dx = \int \frac{e^x}{2}(\cos 2x + 1)\,dx$ **M1A1**

$= \frac{1}{2}\left(\frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x\right) + \frac{e^x}{2}$ **A1**

$= \frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2}\ (+c)$ **AG**

Note: Do not accept solutions where the RHS is differentiated.

[3 marks]

(c) $f'(x) = e^x \cos^2 x - 2e^x \sin x \cos x$ **M1A1**

Note: Award **M1** for an attempt at both the product rule and the chain rule.

$e^x \cos x(\cos x - 2\sin x) = 0$ **(M1)**

Note: Award **M1** for an attempt to factorise $\cos x$ or divide by $\cos x(\cos x \neq 0)$.

discount $\cos x = 0$ (as this would also be a zero of the function)

$\Rightarrow \cos x - 2\sin x = 0$

$\Rightarrow \tan x = \frac{1}{2}$ **(M1)**

$\Rightarrow x = \arctan\left(\frac{1}{2}\right)$ (at A) and $x = \pi + \arctan\left(\frac{1}{2}\right)$ (at C) **A1A1**

Note: Award **A1** for each correct answer. If extra values are seen award **A1A0**.

[6 marks]

(d) $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ **A1**

Note: The **A1** may be awarded for work seen in part (c).

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(e^x \cos^2 x\right) dx = \left[\frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ **M1**

$= \left(-\frac{e^{\frac{3\pi}{2}}}{10} + \frac{e^{\frac{3\pi}{2}}}{2}\right) - \left(-\frac{e^{\frac{\pi}{2}}}{10} + \frac{e^{\frac{\pi}{2}}}{2}\right)\left(= \frac{2e^{\frac{3\pi}{2}}}{5} - \frac{2e^{\frac{\pi}{2}}}{5}\right)$ **M1(A1)A1**

e: Award **M1** for substitution of the end points and subtracting, **(A1)** for $\sin 3\pi = \sin \pi = 0$ and $\cos 3\pi = \cos \pi = -1$ and **A1** for a completely correct answer.

[5 marks]

Total [19 marks]

Question 76

(a) $C_1 : y + x\frac{dy}{dx} = 0$ **(M1)**

Note: **M1** is for use of both product rule and implicit differentiation.

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ **A1**

Note: Accept $-\frac{4}{x^2}$.

$C_2 : 2y\frac{dy}{dx} - 2x = 0$ **(M1)**

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$ **A1**

Note: Accept $\pm\frac{x}{\sqrt{2+x^2}}$.

[4 marks]

(b) substituting a and b for x and y **M1**

product of gradients at P is $\left(-\frac{b}{a}\right)\left(\frac{a}{b}\right) = -1$ or equivalent reasoning **R1**

Note: The ***R1*** is dependent on the previous ***M1***.

so tangents are perpendicular **AG**

[2 marks]

Total [6 marks]

Question 77

(a) attempt to use Pythagoras in triangle OXB **M1**

$\Rightarrow r^2 = R^2 - (h-R)^2$ **A1**

substitution of their r^2 into formula for volume of cone $V = \frac{\pi r^2 h}{3}$ **M1**

$= \frac{\pi h}{3}\left(R^2 - (h-R)^2\right)$

$= \frac{\pi h}{3}\left(R^2 - (h^2 + R^2 - 2hR)\right)$ **A1**

e: This **A** mark is independent and may be seen anywhere for the correct expansion of $(h-R)^2$.

$= \frac{\pi h}{3}\left(2hR - h^2\right)$

$= \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ **AG**

[4 marks]

continued…

(b) at max, $\frac{dV}{dh} = 0$ **R1**

$\frac{dV}{dh} = \frac{\pi}{3}\left(4Rh - 3h^2\right)$

$\Rightarrow 4Rh = 3h^2$

$\Rightarrow h = \frac{4R}{3}$ (since $h \neq 0$) **A1**

EITHER

$V_{max} = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ from part (a)

$= \frac{\pi}{3}\left(2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right)$ **A1**

$= \frac{\pi}{3}\left(2R\frac{16R^2}{9} - \left(\frac{64R^3}{27}\right)\right)$ **A1**

OR

$$r^2 = R^2 - \left(\frac{4R}{3} - R\right)^2$$

$$r^2 = R^2 - \frac{R^2}{9} = \frac{8R^2}{9}$$ ***A1***

$$\Rightarrow V_{\max} = \frac{\pi r^2}{3}\left(\frac{4R}{3}\right)$$

$$= \frac{4\pi R}{9}\left(\frac{8R^2}{9}\right)$$ ***A1***

THEN

$$= \frac{32\pi R^3}{81}$$ ***AG***

[4 marks]

Total [8 marks]

Question 78

(a) attempt to differentiate implicitly ***M1***

$$\frac{dy}{dx} = x\sec^2\left(\frac{\pi xy}{4}\right)\left[\frac{\pi}{4}x\frac{dy}{dx} + \frac{\pi}{4}y\right] + \tan\left(\frac{\pi xy}{4}\right)$$ ***A1A1***

Note: Award ***A1*** for each term.

attempt to substitute $x=1,\ y=1$ into their equation for $\frac{dy}{dx}$ ***M1***

$$\frac{dy}{dx} = \frac{\pi}{2}\frac{dy}{dx} + \frac{\pi}{2} + 1$$

$$\frac{dy}{dx}\left(1 - \frac{\pi}{2}\right) = \frac{\pi}{2} + 1$$ ***A1***

$$\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$$ ***AG***

[5 marks]

(b) attempt to use gradient of normal $= \frac{-1}{\frac{dy}{dx}}$ ***(M1)***

$$= \frac{\pi-2}{\pi+2}$$

so equation of normal is $y-1 = \frac{\pi-2}{\pi+2}(x-1)$ or $y = \frac{\pi-2}{\pi+2}x + \frac{4}{\pi+2}$ ***A1***

[2 marks]

Total [7 marks]

Question 79

$u = \sin x \Rightarrow du = \cos x dx$ **(A1)**

valid attempt to write integral in terms of u and du **M1**

$\int \frac{\cos^3 x \, dx}{\sqrt{\sin x}} = \int \frac{(1-u^2)\, du}{\sqrt{u}}$ **A1**

$= \int \left(u^{-\frac{1}{2}} - u^{\frac{3}{2}} \right) du$

$= 2u^{\frac{1}{2}} - \frac{2u^{\frac{5}{2}}}{5} (+c)$ **(A1)**

$= 2\sqrt{\sin x} - \frac{2\left(\sqrt{\sin x}\right)^5}{5} (+c)$ or equivalent **A1**

[5 marks]

Question 80

(a) $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$ **M1A1**

Note: Do not award the ***M1*** for just $\sin^2 x + \cos^2 x$.

Note: Do not award ***A1*** if correct expression is followed by incorrect working.

$= 1 + \sin 2x$ **AG**

[2 marks]

(b) $\sec 2x + \tan 2x = \frac{1}{\cos 2x} + \frac{\sin 2x}{\cos 2x}$ **M1**

e: ***M1*** is for an attempt to change both terms into sine and cosine forms (with the same argument) or both terms into functions of $\tan x$.

$= \frac{1 + \sin 2x}{\cos 2x}$

$= \frac{(\sin x + \cos x)^2}{\cos^2 x - \sin^2 x}$ **A1A1**

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$= \frac{(\sin x + \cos x)^2}{(\cos x - \sin x)(\cos x + \sin x)}$ **M1**

$= \frac{\cos x + \sin x}{\cos x - \sin x}$ **AG**

Note: Apply MS in reverse if candidates have worked from RHS to LHS.

Note: Alternative method using $\tan 2x$ and $\sec 2x$ in terms of $\tan x$.

[4 marks]

(c) **METHOD 1**

$\int_0^{\frac{\pi}{6}}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)\mathrm{d}x$ ***A1***

Note: Award ***A1*** for correct expression with or without limits.

EITHER

$=\left[-\ln\left(\cos x-\sin x\right)\right]_0^{\frac{\pi}{6}}$ or $\left[\ln\left(\cos x-\sin x\right)\right]_{\frac{\pi}{6}}^{0}$ ***(M1)A1A1***

Note: Award ***M1*** for integration by inspection or substitution, ***A1*** for $\ln\left(\cos x-\sin x\right)$, ***A1*** for completely correct expression including limits.

$=-\ln\left(\cos\frac{\pi}{6}-\sin\frac{\pi}{6}\right)+\ln\left(\cos 0-\sin 0\right)$ ***M1***

Note: Award ***M1*** for substitution of limits into their integral and subtraction.

$=-\ln\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$ ***(A1)***

continued…

OR

let $u=\cos x-\sin x$ ***M1***

$\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x-\cos x=-(\sin x+\cos x)$

$-\int_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}}\left(\frac{1}{u}\right)\mathrm{d}u$ ***A1A1***

Note: Award ***A1*** for correct limits even if seen later, ***A1*** for integral.

$=\left[-\ln u\right]_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}}$ or $\left[\ln u\right]_{\frac{\sqrt{3}}{2}-\frac{1}{2}}^{1}$ ***A1***

$=-\ln\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)(+\ln 1)$ ***M1***

THEN

$=\ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***M1***

Award ***M1*** for both putting the expression over a common denominator and for correct use of law of logarithms.

$=\ln\left(1+\sqrt{3}\right)$ ***(M1)A1***

[9 marks]

METHOD 2

$$\left[\frac{1}{2}\ln(\tan 2x + \sec 2x) - \frac{1}{2}\ln(\cos 2x)\right]_0^{\frac{\pi}{6}}$$ ***A1A1***

$$= \frac{1}{2}\ln\left(\sqrt{3}+2\right) - \frac{1}{2}\ln\left(\frac{1}{2}\right) - 0$$ ***A1A1(A1)***

$$= \frac{1}{2}\ln\left(4+2\sqrt{3}\right)$$ ***M1***

$$= \frac{1}{2}\ln\left(\left(1+\sqrt{3}\right)^2\right)$$ ***M1A1***

$$= \ln\left(1+\sqrt{3}\right)$$ ***A1***

[9 marks]

Total [15 marks]

Question 81

(a) 3 ***A1***

[1 mark]

(b) attempt to use definite integral of $f'(x)$ ***(M1)***

$\int_0^1 f'(x)\mathrm{d}x = 0.5$

$f(1) - f(0) = 0.5$ ***(A1)***

$f(1) = 0.5 + 3$

$= 3.5$ ***A1***

[3 marks]

(c) $\int_1^4 f'(x)\mathrm{d}x = -2.5$ ***(A1)***

Note: *(A1)* is for -2.5.

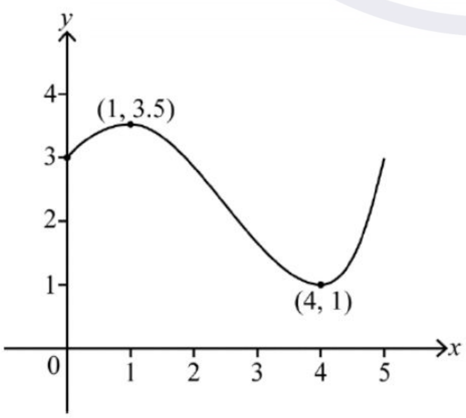
$f(4) - f(1) = -2.5$

$f(4) = 3.5 - 2.5$

$= 1$ ***A1***

[2 marks]

(d)



A1A1A1

A1 for correct shape over approximately the correct domain ***A1*** for maximum and minimum (coordinates or horizontal lines from 3.5 and 1 are required), ***A1*** for y-intercept at 3 ***[3 marks] Total [9 marks]***

Question 82

attempt at implicit differentiation **M1**

$3y^2\frac{dy}{dx}+3y^2+6xy\frac{dy}{dx}-3x^2=0$ **A1A1**

te: Award ***A1*** for the second & third terms, ***A1*** for the first term, fourth term & RHS equal to zero.

substitution of $\frac{dy}{dx}=0$ **M1**

$3y^2-3x^2=0$

$\Rightarrow y=\pm x$ **A1**

substitute either variable into original equation **M1**

$y=x\Rightarrow x^3=9\Rightarrow x=\sqrt[3]{9}$ (or $y^3=9\Rightarrow y=\sqrt[3]{9}$) **A1**

$y=-x\Rightarrow x^3=27\Rightarrow x=3$ (or $y^3=-27\Rightarrow y=-3$) **A1**

$\left(\sqrt[3]{9},\sqrt[3]{9}\right)$, $(3,-3)$ **A1**

Total [9 marks]

Question 83

let $OX=x$

METHOD 1

$\frac{dx}{dt}=24$ (or -24) ***(A1)***

$\frac{d\theta}{dt}=\frac{dx}{dt}\times\frac{d\theta}{dx}$ ***(M1)***

$3\tan\theta=x$ ***A1***

EITHER

$3\sec^2\theta=\frac{dx}{d\theta}$ ***A1***

$\frac{d\theta}{dt}=\frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta=0$ into their differential equation ***M1***

OR

$\theta=\arctan\left(\frac{x}{3}\right)$

$\frac{d\theta}{dx}=\frac{1}{3}\times\frac{1}{1+\frac{x^2}{9}}$ ***A1***

$\frac{d\theta}{dt}=24\times\frac{1}{3\left(1+\frac{x^2}{9}\right)}$

attempt to substitute for $x=0$ into their differential equation ***M1***

THEN

$\frac{d\theta}{dt}=\frac{24}{3}=8$ (rad s^{-1}) ***A1***

METHOD 2

$\frac{dx}{dt} = 24$ (or -24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{d\theta}{dt} = \frac{dx}{dt}$ ***A1***

$\frac{d\theta}{dt} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{d\theta}{dt} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept $-8\,\text{rad}\,s^{-1}$.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d - 24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3}\left(= \frac{d}{3} - 8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{d\theta}{dt} = -8$ ***A1***

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{d}{3} - 8t\right)$ ***M1***

$\frac{d\theta}{dt} = -\frac{8}{1+\left(\frac{d}{3} - 8t\right)^2}$ ***A1***

at O, $t = \frac{d}{24}$ ***A1***

THEN

$\frac{d\theta}{dt} = -8$ ***A1***

Total [6 marks]

Question 84

(a) (i) attempt to use quotient rule (or equivalent) ***(M1)***

$$f'(x) = \frac{(x^2-1)(2)-(2x-4)(2x)}{(x^2-1)^2}$$ ***A1***

$$= \frac{-2x^2+8x-2}{(x^2-1)^2}$$

(ii) $f'(x) = 0$

simplifying numerator (may be seen in part (i)) ***(M1)***

$\Rightarrow x^2-4x+1=0$ or equivalent quadratic equation ***A1***

EITHER

use of quadratic formula

$$\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$$ ***A1***

OR

use of completing the square

$(x-2)^2 = 3$ ***A1***

THEN

$x = 2-\sqrt{3}$ (since $2+\sqrt{3}$ is outside the domain) ***AG***

Note: Do not condone verification that $x = 2-\sqrt{3} \Rightarrow f'(x) = 0$.
Do not award the final ***A1*** as follow through from part (i).

[5 marks]

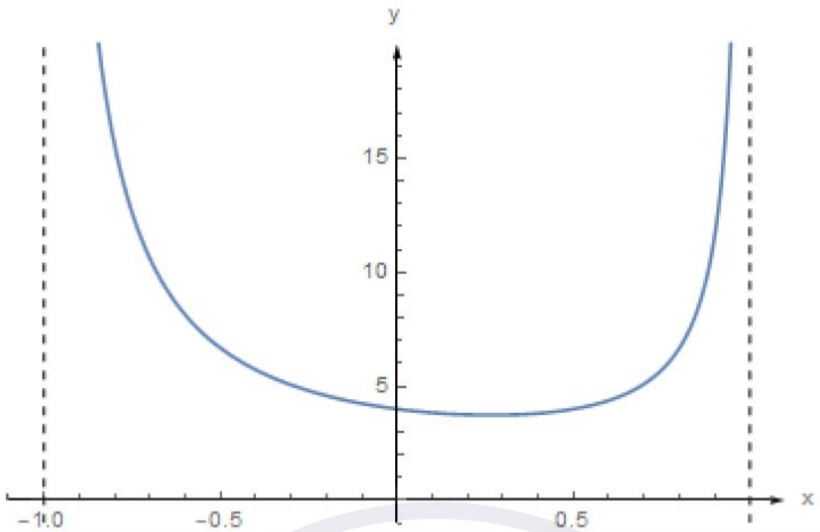
(b) (i) $(0, 4)$ ***A1***

(ii) $2x-4=0 \Rightarrow x=2$ ***A1***

outside the domain ***R1***

continued…

(iii)



A1A1

award ***A1*** for concave up curve over correct domain with one minimum point in the first quadrant
award ***A1*** for approaching $x = \pm 1$ asymptotically

[5 marks]

(c) valid attempt to combine fractions (using common denominator) ***M1***

$$\frac{3(x-1)-(x+1)}{(x+1)(x-1)}$$ ***A1***

$$=\frac{3x-3-x-1}{x^2-1}$$

$$=\frac{2x-4}{x^2-1}$$ ***AG***

[2 marks]

continued…

(d)

$$f(x)=4 \Rightarrow 2x-4=4x^2-4$$ ***M1***

$(x=0$ or$)$ $x=\frac{1}{2}$ ***A1***

area under the curve is $\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$$=\int_0^{\frac{1}{2}} \frac{3}{x+1}-\frac{1}{x-1}\,\mathrm{d}x$$

Note: Ignore absence of, or incorrect limits up to this point.

$= \left[3\ln|x+1| - \ln|x-1|\right]_0^{\frac{1}{2}}$ ***A1***

$= 3\ln\frac{3}{2} - \ln\frac{1}{2}(-0)$

$= \ln\frac{27}{4}$ ***A1***

area is $2 - \int_0^{\frac{1}{2}} f(x)\,dx$ or $\int_0^{\frac{1}{2}} 4\,dx - \int_0^{\frac{1}{2}} f(x)\,dx$ ***M1***

$= 2 - \ln\frac{27}{4}$

$= \ln\frac{4e^2}{27}$ ***A1***

$\left(\Rightarrow v = \frac{4e^2}{27}\right)$

[7 marks]

Total [19 marks]

Question 85

(a) attempt to complete the square or multiplication and equating coefficients ***(M1)***

$2x - x^2 = -(x-1)^2 + 1$ ***A1***

$a = -1,\ h = 1,\ k = 1$

[2 marks]

(b) use of their identity from part (a) $\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}}\,dx\right)$ ***(M1)***

$= \left[\arcsin(x-1)\right]_{\frac{1}{2}}^{\frac{3}{2}}$ or $\left[\arcsin(u)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ ***A1***

Note: Condone lack of, or incorrect limits up to this point.

$= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$ ***(M1)***

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$ ***(A1)***

$= \frac{\pi}{3}$ ***A1***

[5 marks]

Total [7 marks]

Question 86

$f'(x) = e^{2x} + 2xe^{2x}$ **A1**

e: This must be obtained from the candidate differentiating $f(x)$.

$= (2^1 x + 1 \times 2^{1-1})\, e^{2x}$ **A1**

(hence true for $n = 1$)

assume true for $n = k$: **M1**

$f^{(k)}(x) = (2^k x + k2^{k-1})e^{2x}$

e: Award ***M1*** if truth is assumed. Do not allow "let $n = k$".

consider $n = k + 1$:

$f^{(k+1)}(x) = \dfrac{d}{dx}\left((2^k x + k2^{k-1})e^{2x}\right)$

attempt to differentiate $f^{(k)}(x)$ **M1**

$f^{(k+1)}(x) = 2^k e^{2x} + 2(2^k x + k2^{k-1})e^{2x}$ **A1**

$f^{(k+1)}(x) = (2^k + 2^{k+1}x + k2^k)e^{2x}$

$f^{(k+1)}(x) = (2^{k+1}x + (k+1)2^k)e^{2x}$ **A1**

$= (2^{k+1}x + (k+1)2^{(k+1)-1})e^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n (\in \mathbb{Z}^+)$ **R1**

e: Do not award final ***R1*** if the two previous ***M1s*** are not awarded.
Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

Question 87

$\frac{1}{2}e^{2x}$ seen ***(A1)***

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}e^{2x}\right]_0^{\ln k} = \frac{1}{2}e^{2\ln k} - \frac{1}{2}e^0\right)$ ***(M1)***

$= \frac{1}{2}e^{\ln k^2} - \frac{1}{2}e^0$ ***(A1)***

Setting their equation $= 12$ **M1**

e: their equation must be an integrated expression with limits substituted.

$\frac{1}{2}k^2 - \frac{1}{2} = 12$ **A1**

$(k^2 = 25 \Rightarrow) k = 5$ **A1**

e: Do not award final ***A1*** for $k = \pm 5$.

[6 marks]