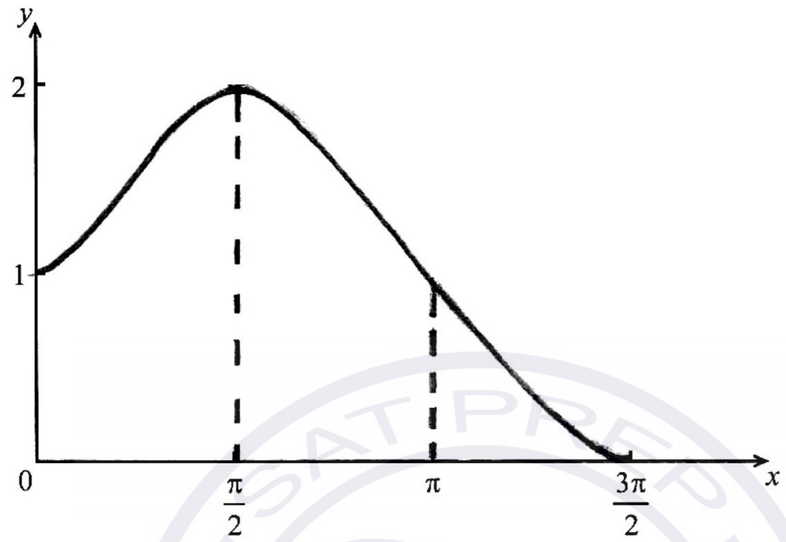
Subject – Math(Higher Level)
Topic - Functions and Equations
Year - Nov 2011 – Nov 2019

Question 1

(a)



A1

(b) $(1+\sin x)^2 = 1+2\sin x+\sin^2 x$

$= 1+2\sin x+\frac{1}{2}(1-\cos 2x)$ *A1*

$= \frac{3}{2}+2\sin x-\frac{1}{2}\cos 2x$ *AG*

Question 2

(a) for the equation to have real roots

$(y-1)^2-4y(y-1)\geq 0$ *M1*

$\Rightarrow 3y^2-2y-1\leq 0$

(by sign diagram, or algebraic method) *M1*

$-\frac{1}{3}\leq y\leq 1$ *A1A1*

Note: Award first ***A1*** for $-\frac{1}{3}$ and 1, and second ***A1*** for inequalities. These are independent marks.

(b) $f: x\rightarrow \frac{x+1}{x^2+x+1} \Rightarrow x+1 = yx^2+yx+y$ *(M1)*

$\Rightarrow 0 = yx^2+(y-1)x+(y-1)$ *A1*

hence, from (a) range is $-\frac{1}{3}\leq y\leq 1$ *A1*

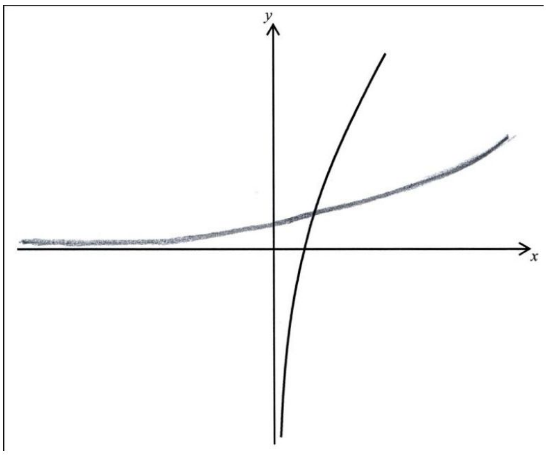
(c) a value for y would lead to 2 values for x from (a) *R1*

Note: Do not award ***R1*** if (b) has not been tackled.

[8 marks]

Question 3

(a)



A1A1

Note: Award ***A1*** for correct asymptote with correct behaviour and ***A1*** for shape.

[2 marks]

(b) intersect on $y = x$ ***(M1)***

$x + \ln x = x \Rightarrow \ln x = 0$ ***(A1)***

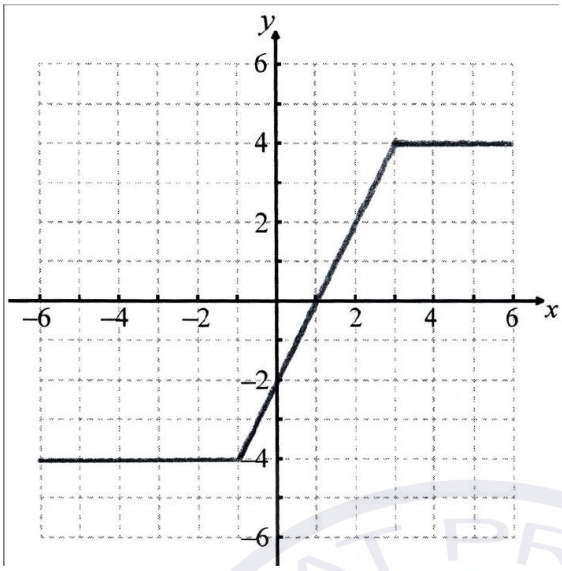
intersect at $(1, 1)$ ***A1A1***

[4 marks]

Total [6 marks]

Question 4

(a)



M1A1A1A1

Note:	Award ***M1*** for any of the three sections completely correct, ***A1*** for each correct segment of the graph.

[4 marks]

(b) (i) 0 ***A1***

(ii) 2 ***A1***

(iii) finding area of rectangle ***(M1)***
−4 ***A1***

Note:	Award ***M1A0*** for the answer 4.

[4 marks]

Total [8 marks]

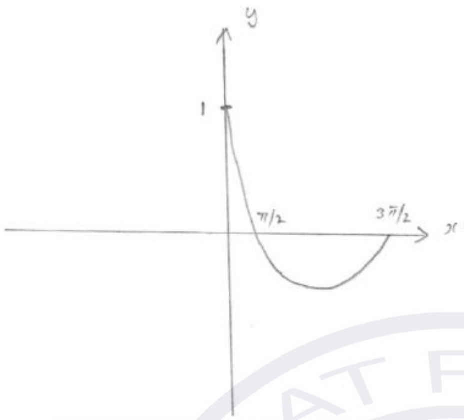
Question 5

(a) $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ ***A1***

[1 mark]

(b)



A1

Note:	Accept any form of concavity for $x \in [0, \frac{\pi}{2}]$.

Note:	Do not penalize unmarked zeros if given in part (a).

Note:	Zeros written on diagram can be used to allow the mark in part (a) to be awarded retrospectively.

[1 mark]

Question 6

(a) $f(x) \geq \frac{1}{25}$ ***A1***

$g(x) \in \mathbb{R}, g(x) \geq 0$ ***A1***

[2 marks]

(b) $f \circ g(x) = \dfrac{2\left(\frac{3x-4}{10}\right)^2 + 3}{75}$ ***M1A1***

$= \dfrac{\frac{2(9x^2-24x+16)}{100} + 3}{75}$ ***(A1)***

$= \dfrac{9x^2-24x+166}{3750}$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

$y = \dfrac{2x^2+3}{75}$

$x^2 = \dfrac{75y-3}{2}$ ***M1***

$x = \sqrt{\dfrac{75y-3}{2}}$ ***(A1)***

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x-3}{2}}$ ***A1***

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

METHOD 2

$y = \dfrac{2x^2+3}{75}$

$x = \dfrac{2y^2+3}{75}$ ***M1***

$y = \sqrt{\dfrac{75x-3}{2}}$ ***(A1)***

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x-3}{2}}$ ***A1***

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

(ii) domain: $x \geq \frac{1}{25}$; range: $f^{-1}(x) \geq 0$ ***A1***

[4 marks]

Question 7

let $f(x) = 2x^3 + kx^2 + 6x + 32$
let $g(x) = x^4 - 6x^2 - k^2x + 9$
$f(-1) = -2 + k - 6 + 32 (= 24 + k)$ *A1*
$g(-1) = 1 - 6 + k^2 + 9 (= 4 + k^2)$ *A1*
$\Rightarrow 24 + k = 4 + k^2$ *M1*
$\Rightarrow k^2 - k - 20 = 0$
$\Rightarrow (k-5)(k+4) = 0$ *(M1)*
$\Rightarrow k = 5, -4$ *A1A1*

[6 marks]

Question 8

(a) **METHOD 1**

$f(x) = (x+1)(x-1)(x-2)$ *M1*
$= x^3 - 2x^2 - x + 2$ *A1A1A1*
$a = -2$, $b = -1$ and $c = 2$

METHOD 2

from the graph or using $f(0) = 2$
$c = 2$ *A1*
setting up linear equations using $f(1) = 0$ and $f(-1) = 0$ (or $f(2) = 0$) *M1*
obtain $a = -2$, $b = -1$ *A1A1*

[4 marks]

(b) (i) $(1, 0)$, $(3, 0)$ and $(4, 0)$ *A1*

(ii) $g(0)$ occurs at $3f(-2)$ *(M1)*
$= -36$ *A1*

[3 marks]

Total [7 marks]

Question 9

METHOD 1

(a) $\det\begin{pmatrix}1 & 3 & a-1\\ 2 & 2 & a-2\\ 3 & 1 & a-3\end{pmatrix}$ *M1*

$=1\big(2(a-3)-(a-2)\big)-3\big(2(a-3)-3(a-2)\big)+(a-1)(2-6)$

(or equivalent) *A1*

$=0$ (therefore there is no unique solution) *A1*

[3 marks]

(b) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right):\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ *M1A1*

$:\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ *A1*

$b=1$ *A1* ***N2***

Note: Award ***M1*** for an attempt to use row operations.

[4 marks]

METHOD 2

(a) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right):\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ *M1A1*

$:\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ (and 3 zeros imply no unique solution) *A1*

[3 marks]

(b) $b=1$ *A4*

Note: Award ***A4*** only if "$b-1$" seen in (a).

[4 marks]

Total [7 marks]

Question 10

(a) **EITHER**

$f(x)-1=\dfrac{1+3^{-x}}{3^{x}-3^{-x}}$ ***M1A1***

>0 as both numerator and denominator are positive ***R1***

OR

$3^{x}+1>3^{x}>3^{x}-3^{-x}$ ***M1A1***

Note: Accept a convincing valid argument the numerator is greater than the denominator.

numerator and denominator are positive ***R1***

hence $f(x)>1$ ***AG***

[3 marks]

(b) one line equation to solve, for example, $4(3^{x}-3^{-x})=3^{x}+1$, or equivalent ***A1***

$(3y^{2}-y-4=0)$

attempt to solve a three-term equation ***M1***

obtain $y=\dfrac{4}{3}$ ***A1***

$x=\log_{3}\left(\dfrac{4}{3}\right)$ or equivalent ***A1***

Note: Award ***A0*** if an extra solution for x is given.

[4 marks]

Total [7 marks]

Question 11

(a) $4(x-0.5)^2+4$ *A1A1*

Note: *A1* for two correct parameters, *A2* for all three correct.

[2 marks]

(b) translation $\begin{pmatrix}0.5\\0\end{pmatrix}$ (allow "0.5 to the right") *A1*

stretch parallel to y-axis, scale factor 4 (allow vertical stretch or similar) *A1*

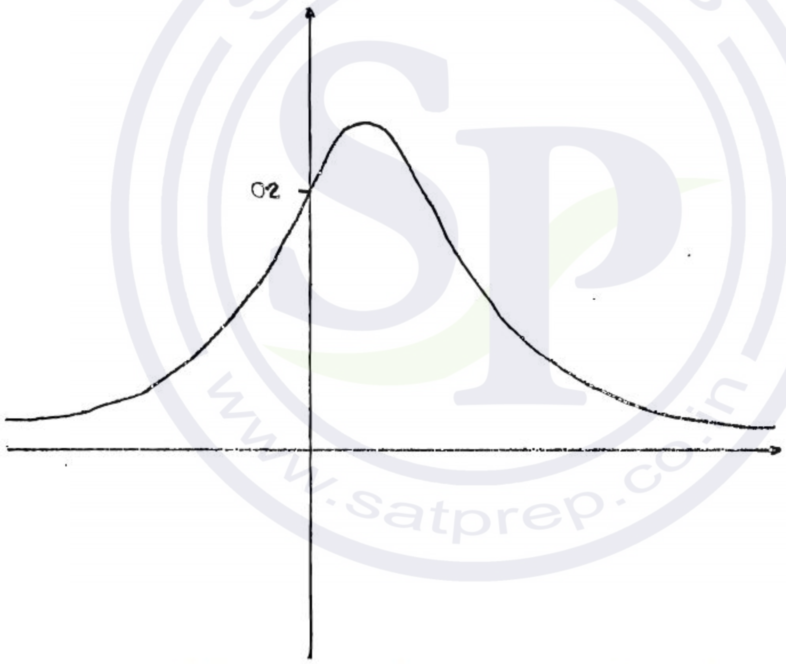
translation $\begin{pmatrix}0\\4\end{pmatrix}$ (allow "4 up") *A1*

Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.
It could be a stretch followed by a single translation of $\begin{pmatrix}0.5\\4\end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix}0\\1\end{pmatrix}$.

[3 marks]

(c)



general shape (including asymptote and single maximum in first quadrant), *A1*

intercept $\left(0,\frac{1}{5}\right)$ **or** maximum $\left(\frac{1}{2},\frac{1}{4}\right)$ shown *A1*

[2 marks]

(d) $0<f(x)\le\frac{1}{4}$ *A1A1*

Note: *A1* for $\le\frac{1}{4}$, *A1* for $0<$.

[2 marks]

Question 12

$f(-2)=0$ $(\Rightarrow -24+4p-2q-2=0)$ ***M1***
$f(-1)=4$ $(\Rightarrow -3+p-q-2=4)$ ***M1***

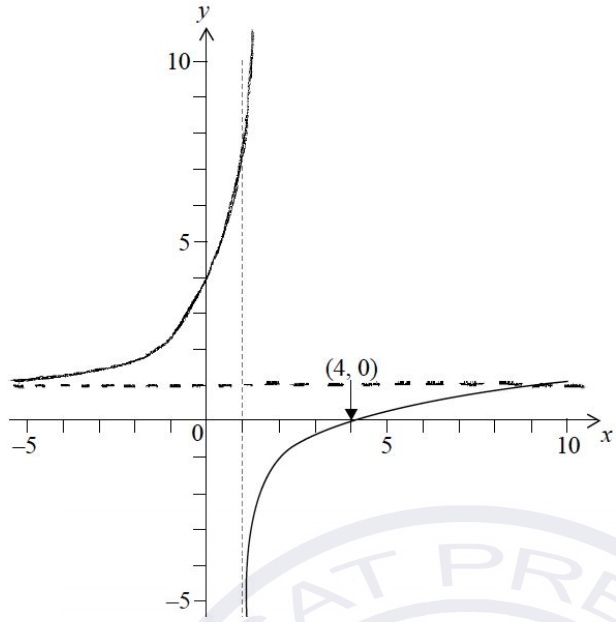
ote:	In each case award the ***M*** marks if correct substitution attempted and right-hand side correct.

attempt to solve simultaneously $(2p-q=13,\ p-q=9)$ ***M1***
$p=4$ ***A1***
$q=-5$ ***A1***

Total [5 marks]

Question 13

(a)



shape with y-axis intercept $(0, 4)$ *A1*

Note: Accept curve with an asymptote at $x = 1$ suggested.

correct asymptote $y = 1$ *A1*

[2 marks]

(b) range is $f^{-1}(x) > 1$ (or $]1, \infty[$) *A1*

Note: Also accept $]1, 10]$ or $]1, 10[$.

Note: Do not allow follow through from incorrect asymptote in (a).

[1 mark]

(c) $(4, 0) \Rightarrow \ln(4a + b) = 0$ *M1*

$\Rightarrow 4a + b = 1$ *A1*

asymptote at $x = 1 \Rightarrow a + b = 0$ *M1*

$\Rightarrow a = \frac{1}{3}, \ b = -\frac{1}{3}$ *A1*

[4 marks]

Total [7 marks]

Question 14

$P(2) = 24 + 2a + b = 2, \ P(-1) = -3 - a + b = 5$ *M1A1A1*

$(2a + b = -22, -a + b = 8)$

Note: Award ***M1*** for substitution of 2 or –1 and equating to remainder, ***A1*** for each correct equation.

attempt to solve simultaneously *M1*

$a = -10 \ , \ b = -2$ *A1*

[5 marks]

Question 15

(a) $1-2(2)=-3$ and $\frac{3}{4}(2-2)^2-3=-3$ ***A1***

both answers are the same, hence f is continuous (at $x=2$) ***R1***

Note: ***R1*** may be awarded for justification using a graph or referring to limits. Do not award ***A0R1***.

[2 marks]

(b) reflection in the y-axis

$$f(-x)=\begin{cases} 1+2x, & x\geq -2 \\ \frac{3}{4}(x+2)^2-3, & x<-2 \end{cases}$$ ***(M1)***

Note: Award ***M1*** for evidence of reflecting a graph in y-axis.

translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$g(x)=\begin{cases} 2x-3, & x\geq 0 \\ \frac{3}{4}x^2-3, & x<0 \end{cases}$$ ***(M1)A1A1***

Note: Award ***(M1)*** for attempting to substitute $(x-2)$ for x, or translating a graph along positive x-axis. Award ***A1*** for the correct domains (this mark can be awarded independent of the ***M1***). Award ***A1*** for the correct expressions.

[4 marks]

Total [6 marks]

Question 16

(a) using the formulae for the sum and product of roots:

$\alpha + \beta = -2$ *A1*

$\alpha\beta = -\frac{1}{2}$ *A1*

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ *M1*

$= (-2)^2 - 2\left(-\frac{1}{2}\right)$

$= 5$ *A1*

[4 marks]

Note: Award ***M0*** for attempt to solve quadratic equation.

(b) $(x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2$ *M1*

$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0$ *A1*

$x^2 - 5x + \frac{1}{4} = 0$

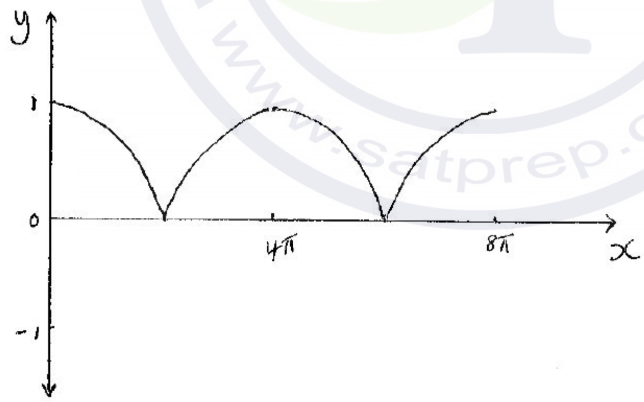
Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

Question 18

(a)



A1A1

Note: Award ***A1*** for correct shape and ***A1*** for correct domain and range.

[2 marks]

(b) $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$ ***A1***

attempting to find any other solutions ***M1***

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$

$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$

$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$ ***A1***

Note: Award ***A1*** for all other three solutions correct and no extra solutions.

Note: If working in degrees, then max ***A0M1A0***.

[3 marks]

Total [5 marks]

Question 19

(a) $g(x) = \frac{1}{x+3} + 1$ ***A1A1***

Note: Award ***A1*** for $x+3$ in the denominator and ***A1*** for the "$+1$".

[2 marks]

(b) $x = -3$ ***A1***

$y = 1$ ***A1***

[2 marks]

Total [4 marks]

Question 20

(a) using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$ ***A1***

(ii) $\alpha\beta = \frac{1}{2}$ ***A1***

te: Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

[2 m

(b) **METHOD 1**

required quadratic is of the form $x^2 - \left(\frac{2}{\alpha} + \frac{2}{\beta}\right)x + \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$ ***(M1)***

$q = \frac{4}{\alpha\beta}$

$q = 8$ ***A1***

$p = -\left(\frac{2}{\alpha} + \frac{2}{\beta}\right)$

$= -\frac{2(\alpha + \beta)}{\alpha\beta}$ ***M1***

$= -\frac{2 \times 4}{\frac{1}{2}}$

$p = -16$ ***A1***

METHOD 2

replacing x with $\frac{2}{x}$ ***M1***

$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$

$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$ ***(A1)***

$x^2 - 16x + 8 = 0$

$p = -16$ and $q = 8$ ***A1A1***

Note: Award ***A1A0*** for $x^2 - 16x + 8 = 0$ *ie*, if $p = -16$ and $q = 8$ are not explicitly stated.

[4 marks]

Question 21

(a) **EITHER**

$f(-x) = f(x)$ ***M1***

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0,\ (\forall x \in \mathbb{R})$ ***A1***

OR

y-axis is eqn of symmetry ***M1***

so $\frac{-b}{2a} = 0$ ***A1***

THEN

$\Rightarrow b = 0$ ***AG***

[2 marks]

(b) $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$ ***M1***

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$ ***A1***

Note: In (a) and (b) allow substitution of a particular value of x

[2 marks]

(c) $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0,\ (\forall x)$ ***M1A1***

Note: Accept geometrical explanations.

[2 marks]

Total [6 marks]

Question 23

(a) $f: x \to y = \dfrac{3x-2}{2x-1}$ $f^{-1}: y \to x$

$y = \dfrac{3x-2}{2x-1} \Rightarrow 3x - 2 = 2xy - y$ **M1**

$\Rightarrow 3x - 2xy = -y + 2$ **M1**

$x(3-2y) = 2 - y$

$x = \dfrac{2-y}{3-2y}$ **A1**

$\left(f^{-1}(y) = \dfrac{2-y}{3-2y}\right)$

$f^{-1}(x) = \dfrac{2-x}{3-2x}$ $\left(x \neq \dfrac{3}{2}\right)$ **A1**

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

[4 marks]

(b) $\dfrac{3x-2}{2x-1} = A + \dfrac{B}{2x-1} \Rightarrow 3x - 2 = A(2x-1) + B$

equating coefficients $3 = 2A$ and $-2 = -A + B$ ***(M1)***

$A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$ **A1**

Note: Could also be done by division or substitution of values.

[2 marks]

(c) $\int f(x)\,\mathrm{d}x = \dfrac{3}{2}x - \dfrac{1}{4}\ln|2x-1| + c$ **A1**

Note: accept equivalent e.g. ln $|4x-2|$

[1 mark]

Total [7 marks]

Question 23

(a) **EITHER**

$f(-x) = f(x)$ ***M1***

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0, (\forall x \in \mathbb{R})$ ***A1***

OR

y-axis is eqn of symmetry ***M1***

so $\frac{-b}{2a} = 0$ ***A1***

THEN

$\Rightarrow b = 0$ ***AG***

[2 marks]

(b) $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$ ***M1***

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$ ***A1***

Note: In (a) and (b) allow substitution of a particular value of x

[2 marks]

(c) $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0, (\forall x)$ ***M1A1***

Note: Accept geometrical explanations.

[2 marks]

Total [6 marks]

Question 24

(a) $g \circ f(x) = g(f(x))$ *M1*

$= g\left(2x + \frac{\pi}{5}\right)$

$= 3\sin\left(2x + \frac{\pi}{5}\right) + 4$ *AG*

[1 mark]

(b) since $-1 \le \sin\theta \le +1$, range is $[1, 7]$ ***(R1)A1***

[2 marks]

(c) $3\sin\left(2x + \frac{\pi}{5}\right) + 4 = 7 \Rightarrow 2x + \frac{\pi}{5} = \frac{\pi}{2} + 2n\pi \Rightarrow x = \frac{3\pi}{20} + n\pi$ ***(M1)***

so next biggest value is $\frac{23\pi}{20}$ ***A1***

Note: Allow use of period.

[2 marks]

(d) **Note:** Transformations can be in any order but see notes below.

stretch scale factor 3 parallel to y axis (vertically) ***A1***
vertical translation of 4 up ***A1***

Note: Vertical translation is $\frac{4}{3}$ up if it occurs before stretch parallel to y axis.

stretch scale factor $\frac{1}{2}$ parallel to x axis (horizontally) ***A1***

horizontal translation of $\frac{\pi}{10}$ to the left ***A1***

Note: Horizontal translation is $\frac{\pi}{5}$ to the left if it occurs before stretch parallel to x axis.

Note: Award ***A1*** for magnitude and direction in each case. Accept any correct terminology provided that the meaning is clear *eg* shift for translation.

[4 marks]

Total [9 marks]

Question 25

(a) $a > 0$ **A1**

$a \neq 1$ **A1**

[2 marks]

(b) **METHOD 1**

$\log_x y = \frac{\ln y}{\ln x}$ and $\log_y x = \frac{\ln x}{\ln y}$ ***M1A1***

> **Note:** Use of any base is permissible here, not just "e".

$\left(\frac{\ln y}{\ln x}\right)^2 = 4$ ***A1***

$\ln y = \pm 2 \ln x$ ***A1***

$y = x^2$ or $\frac{1}{x^2}$ ***A1A1***

METHOD 2

$\log_y x = \frac{\log_x x}{\log_x y} = \frac{1}{\log_x y}$ ***M1A1***

$(\log_x y)^2 = 4$ ***A1***

$\log_x y = \pm 2$ ***A1***

$y = x^2$ or $y = \frac{1}{x^2}$ ***A1A1***

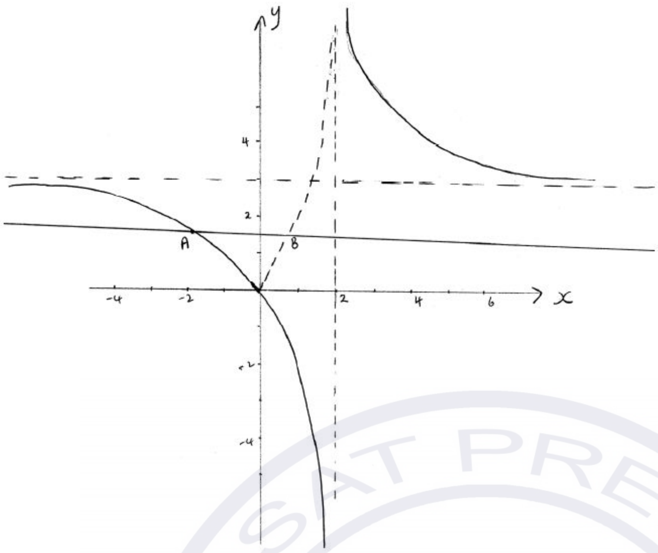
> **Note:** The final two ***A*** marks are independent of the one coming before.

[6 marks]

Total [8 marks]

Question 26

(a)



> **Note:** In the diagram, points marked A and B refer to part (d) and do not need to be seen in part (a).

shape of curve ***A1***

> **Note:** This mark can only be awarded if there appear to be both horizontal and vertical asymptotes.

intersection at $(0,\ 0)$ ***A1***

horizontal asymptote at $y = 3$ ***A1***

vertical asymptote at $x = 2$ ***A1***

[4 marks]

(b) $y = \dfrac{3x}{x-2}$

$xy - 2y = 3x$ ***M1A1***

$xy - 3x = 2y$

$x = \dfrac{2y}{y-3}$

$\left(f^{-1}(x)\right) = \dfrac{2x}{x-3}$ ***M1A1***

(c) **METHOD 1**

attempt to solve $\frac{2x}{x-3} = \frac{3x}{x-2}$ ***(M1)***

$2x(x-2) = 3x(x-3)$

$x[2(x-2) - 3(x-3)] = 0$

$x(5-x) = 0$

$x = 0$ or $x = 5$ ***A1A1***

METHOD 2

$x = \frac{3x}{x-2}$ or $x = \frac{2x}{x-3}$ ***(M1)***

$x = 0$ or $x = 5$ ***A1A1***

[3 marks]

(d) **METHOD 1**

at $A: \frac{3x}{x-2} = \frac{3}{2}$ AND at $B: \frac{3x}{x-2} = -\frac{3}{2}$ ***M1***

$6x = 3x - 6$

$x = -2$ ***A1***

$6x = 6 - 3x$

$x = \frac{2}{3}$ ***A1***

solution is $-2 < x < \frac{2}{3}$ ***A1***

[4 marks]

METHOD 2

$\left(\frac{3x}{x-2}\right)^2 < \left(\frac{3}{2}\right)^2$ ***M1***

$9x^2 < \frac{9}{4}(x-2)^2$

$3x^2 + 4x - 4 < 0$

$(3x-2)(x+2) < 0$

$x = -2$ ***(A1)***

$x = \frac{2}{3}$ ***(A1)***

solution is $-2 < x < \frac{2}{3}$ ***A1***

[4 marks]

(e) $-2 < x < 2$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

Note: If working is shown, then ***A*** marks may only be awarded following correct working.

[2 marks]

Total [17 marks]

Question 27

(a) $g \circ f(x) = \dfrac{\tan x + 1}{\tan x - 1}$ **A1**

$x \neq \dfrac{\pi}{4},\ 0 \leq x < \dfrac{\pi}{2}$ **A1**

[2 marks]

(b) $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{\frac{\sin x}{\cos x} + 1}{\frac{\sin x}{\cos x} - 1}$ **M1A1**

$= \dfrac{\sin x + \cos x}{\sin x - \cos x}$ **AG**

[2 marks]

(c) **METHOD 1**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ **M1(A1)**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\left(2\sin x\cos x - \cos^2 x - \sin^2 x\right) - \left(2\sin x\cos x + \cos^2 x + \sin^2 x\right)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \dfrac{-2}{1 - \sin 2x}$

Substitute $\dfrac{\pi}{6}$ into any formula for $\dfrac{dy}{dx}$ **M1**

$\dfrac{-2}{1 - \sin\frac{\pi}{3}}$

$= \dfrac{-2}{1 - \frac{\sqrt{3}}{2}}$ **A1**

$= \dfrac{-4}{2 - \sqrt{3}}$

$= \dfrac{-4}{2 - \sqrt{3}}\left(\dfrac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ **M1**

$= \dfrac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ **A1**

(d) Area $= \left|\int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x}\,dx\right|$ ***M1***

$= \left|\left[\ln\left|\sin x - \cos x\right|\right]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln\left|\sin 0 - \cos 0\right|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3}-1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3}-1}{2}\right) = \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$ ***M1***

$= \ln\left(\sqrt{3}+1\right)$ ***AG***

[6 marks]

Total [16 marks]

Question 28

(a) (i)–(iii) given the three roots α, β, γ, we have

$x^3 + px^2 + qx + c = (x-\alpha)(x-\beta)(x-\gamma)$ **M1**

$= \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)(x-\gamma)$ **A1**

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$ **A1**

comparing coefficients:

$p = -(\alpha+\beta+\gamma)$ **AG**

$q = (\alpha\beta+\beta\gamma+\gamma\alpha)$ **AG**

$c = -\alpha\beta\gamma$ **AG**

[3 marks]

(b) **METHOD 1**

i) Given $-\alpha-\beta-\gamma = -6$

And $\alpha\beta+\beta\gamma+\gamma\alpha = 18$

Let the three roots be α, β, γ.

So $\beta - \alpha = \gamma - \beta$ **M1**

or $2\beta = \alpha + \gamma$

Attempt to solve simultaneous equations: **M1**

$\beta + 2\beta = 6$ **A1**

$\beta = 2$ **AG**

ii) $\alpha + \gamma = 4$

$2\alpha + 2\gamma + \alpha\gamma = 18$

$\Rightarrow \gamma^2 - 4\gamma + 10 = 0$

$\Rightarrow \gamma = \dfrac{4 \pm \mathrm{i}\sqrt{24}}{2}$ ***(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\dfrac{4+\mathrm{i}\sqrt{24}}{2}\right)\left(\dfrac{4-\mathrm{i}\sqrt{24}}{2}\right)2 = -20$ **A1**

[5 marks]

METHOD 2

(i)	let the three roots be α, $\alpha - d$, $\alpha + d$	***M1***
	adding roots	***M1***
	to give $3\alpha = 6$	***A1***
	$\alpha = 2$	***AG***
(ii)	α is a root, so $2^3 - 6\times 2^2 + 18\times 2 + c = 0$	***M1***
	$8 - 24 + 36 + c = 0$	
	$c = -20$	***A1***

[5 marks]

METHOD 3

(i)	let the three roots be α, $\alpha - d$, $\alpha + d$	***M1***
	adding roots	***M1***
	to give $3\alpha = 6$	***A1***
	$\alpha = 2$	***AG***
(ii)	$q = 18 = 2(2 - d) + (2 - d)(2 + d) + 2(2 + d)$	***M1***
	$d^2 = -6 \Rightarrow d = \sqrt{6}\,\mathrm{i}$	
	$\Rightarrow c = -20$	***A1***

[5 marks]

METHOD 1

Given $-\alpha-\beta-\gamma=-6$
And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be α,β,γ.

So $\dfrac{\beta}{\alpha}=\dfrac{\gamma}{\beta}$ ***M1***

or $\beta^2=\alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta+\gamma\beta+\beta^2=18$

$\beta(\alpha+\beta+\gamma)=18$

$6\beta=18$

$\beta=3$ ***A1***

$\alpha+\gamma=3$, $\alpha=\dfrac{9}{\gamma}$

$\Rightarrow \gamma^2-3\gamma+9=0$

$\Rightarrow \gamma=\dfrac{3\pm \mathrm{i}\sqrt{27}}{2}$ ***(A1)(A1)***

Therefore $c=-\alpha\beta\gamma=-\left(\dfrac{3+\mathrm{i}\sqrt{27}}{2}\right)\left(\dfrac{3-\mathrm{i}\sqrt{27}}{2}\right)3=-27$ ***A1***

[6 marks]

METHOD 2

let the three roots be a, ar, ar^2 ***M1***

attempt at substitution of a, ar, ar^2 and *p* and *q* into equations from (a) ***M1***

$6=a+ar+ar^2\left(=a\left(1+r+r^2\right)\right)$ ***A1***

$18=a^2r+a^2r^3+a^2r^2\left(=a^2r\left(1+r+r^2\right)\right)$ ***A1***

therefore $3=ar$ ***A1***

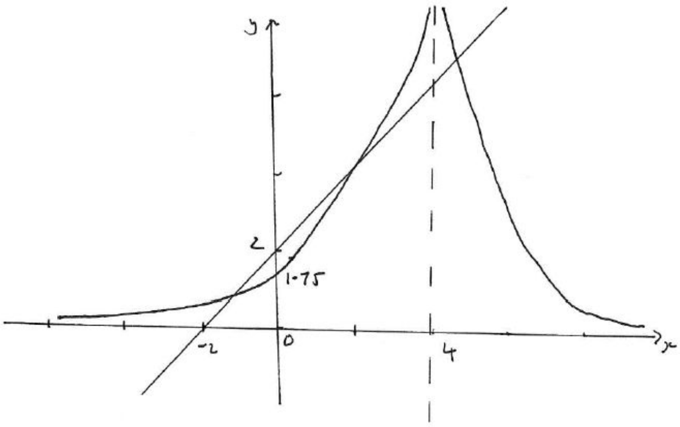
therefore $c=-a^3r^3=-3^3=-27$ ***A1***

[6 marks]

Total [14 marks]

Question 29

(a)



A1 for vertical asymptote and for the y-intercept $\frac{7}{4}$

A1 for general shape of $y = \left|\frac{7}{x-4}\right|$ including the x-axis as asymptote

A1 for straight line with y-intercept 2 and x-intercept of -2 ***A1A1A1***

(b) **METHOD 1**

for $x > 4$

$(x + 2)(x - 4) = 7$ ***(M1)***

$x^2 - 2x - 8 = 7 \Rightarrow x^2 - 2x - 15 = 0$

$(x - 5)(x + 3) = 0$

(as $x > 4$ then) $x = 5$ ***A1***

Note: Award ***A0*** if $x = -3$ is also given as a solution.

for $x < 4$

$(x + 2)(x - 4) = -7$ ***M1***

$\Rightarrow x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ***(M1)A1***

Note: Second ***M1*** is dependent on first ***M1***.

[5 marks]

METHOD 2

$(x+2)^2 = \frac{49}{(x-4)^2}$ ***M1***

$x^4 - 4x^3 - 12x^2 + 32x + 15 = 0$ ***A1***

$(x+3)(x-5)(x^2 - 2x - 1) = 0$

$x = 5$ ***A1***

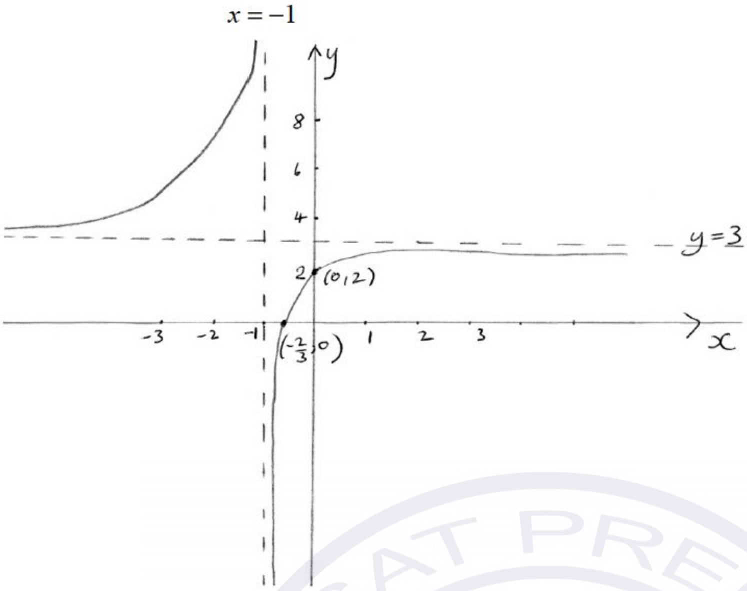
Note: Award ***A0*** if $x = -3$ is also given as a solution.

$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ***(M1)A1***

[5 marks]

Total [8 marks]

Question 30



A1A1A1A1A1

: Award **A1** for correct shape, **A1** for $x = -1$ clearly stated and asymptote shown,

A1 for $y = 3$ clearly stated and asymptote shown, **A1** for $\left(-\frac{2}{3}, 0\right)$ and **A1** for $(0, 2)$.

[5 marks]

Question 31

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{ax_2^2 + bx_2 + c - (ax_1^2 + bx_1 + c)}{x_2 - x_1}$$ **(M1)**

$$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1}$$ **A1**

$$= \frac{a(x_2 - x_1)(x_2 + x_1) + b(x_2 - x_1)}{x_2 - x_1}$$ **(A1)**

$$= a(x_2 + x_1) + b \qquad (x_1 \neq x_2)$$ **A1**

$$\frac{f'(x_2) + f'(x_1)}{2} = \frac{(2ax_2 + b) + (2ax_1 + b)}{2}$$ **M1**

$$= \frac{2a(x_2 + x_1) + 2b}{2}$$

$$= a(x_2 + x_1) + b$$ **A1**

so Hayley's conjecture is correct **AG**

[6 marks]

Question 32

(a) $a = 1$ **A1**

$c = 3$ **A1**

[2 marks]

(b) use the coordinates of $(1, 0)$ on the graph **M1**

$f(1) = 0 \Rightarrow 1 + \dfrac{b}{1-3} = 0 \Rightarrow b = 2$ **A1**

[2 marks]

Total [4 marks]

Question 33

$\alpha + \beta = 2k$ **A1**

$\alpha\beta = k - 1$ **A1**

$(\alpha + \beta)^2 = 4k^2 \Rightarrow \alpha^2 + \beta^2 + 2\underbrace{\alpha\beta}_{k-1} = 4k^2$ ***(M1)***

$\alpha^2 + \beta^2 = 4k^2 - 2k + 2$

$\alpha^2 + \beta^2 = 4 \Rightarrow 4k^2 - 2k - 2 = 0$ **A1**

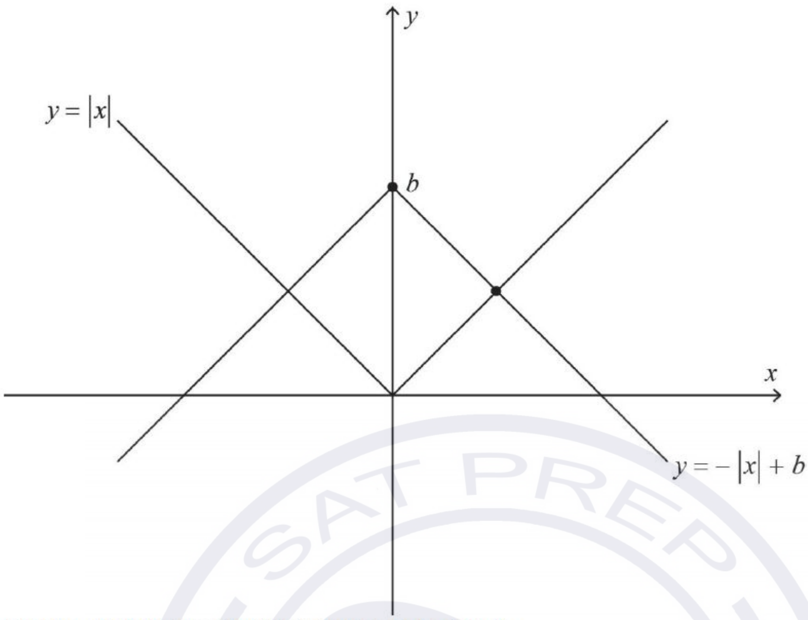
attempt to solve quadratic ***(M1)***

$k = 1, -\dfrac{1}{2}$ **A1**

[6 marks]

Question 34

(a)



graphs sketched correctly (condone missing b) **A1A1**

[2 marks]

(b) $\frac{b^2}{2} = 18$ ***(M1)A1***

$b = 6$ ***A1***

[3 marks]

Total [5 marks]

Question 35

(a) sum = 0 **A1**

product = 6 **A1**

[2 marks]

(b) $P(1) = 1 - 10 + 15 - 6 = 0$ **M1A1**

$\Rightarrow (z-1)$ is a factor of $P(z)$ **AG**

Note: Accept use of division to show remainder is zero.

[2 marks]

(c) **METHOD 1**

$(z-1)^3\left(z^2+bz+c\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)***

by inspection $c = 6$ ***A1***

$\left(z^3 - 3z^2 + 3z - 1\right)\left(z^2 + bz + 6\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)(A1)***

$b = 3$ ***A1***

METHOD 2

α, β are two roots of the quadratic

$b = -(\alpha + \beta)$, $c = \alpha\beta$ ***(A1)***

from part (a) $1 + 1 + 1 + \alpha + \beta = 0$ ***(M1)***

$\Rightarrow b = 3$ ***A1***

$1 \times 1 \times 1 \times \alpha\beta = 6$ ***(M1)***

$\Rightarrow c = 6$ ***A1***

Note: Award ***FT*** if $b = -7$ following through from their sum = 10.

METHOD 3

$\left(z^5 - 10z^2 + 15z - 6\right) \div (z-1) = z^4 + z^3 + z^2 - 9z + 6$ ***(M1)A1***

Note: This may have been seen in part (b).

$z^4 + z^3 + z^2 - 9z + 6 \div (z-1) = z^3 + 2z^2 + 3z - 6$ ***(M1)***

$z^3 + 2z^2 + 3z - 6 \div (z-1) = z^2 + 3z + 6$ ***A1A1***

[5 marks]

(d) $z^2 + 3z + 6 = 0$ **M1**

$z = \frac{-3 \pm \sqrt{9 - 4 \cdot 6}}{2}$ **M1**

$= \frac{-3 \pm \sqrt{-15}}{2}$

$z = -\frac{3}{2} \pm \frac{i\sqrt{15}}{2}$ **A1**

(or $z = 1$)

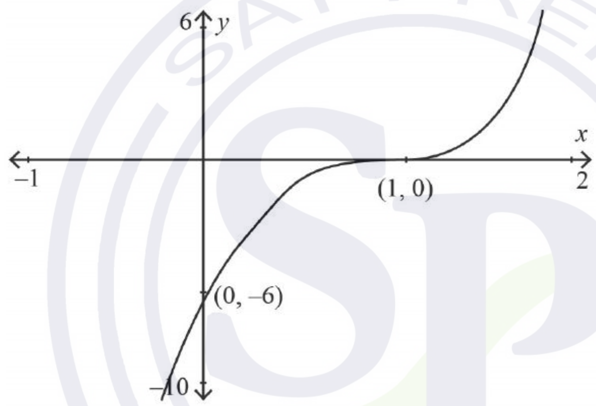
Notes:	Award the second ***M1*** for an attempt to use the quadratic formula or to complete the square. Do not award ***FT*** from (c).

[3 marks]

(e) (i) $\frac{d^2y}{dx^2} = 20x^3 - 20$ **M1A1**

for $x > 1$, $20x^3 - 20 > 0 \Rightarrow$ concave up **R1AG**

(ii)



x-intercept at $(1, 0)$ **A1**

y-intercept at $(0, -6)$ **A1**

stationary point of inflexion at $(1, 0)$ with correct curvature either side **A1**

[6 marks]

Total [18 marks]

Question 36

(a) $-11 \le f(x) \le 21$ **A1A1**

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

[2 marks]

(b) $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$ ***(M1)A1***

[2 marks]

(c) $-11 \le x \le 21$, $-2 \le f^{-1}(x) \le 2$ **A1A1**

[2 marks]

Total [6 marks]

Question 37

(a) $q(4) = 0$ **(M1)**

$192 - 176 + 4k + 8 = 0 \ (24 + 4k = 0)$ **A1**

$k = -6$ **A1**

[3 marks]

(b) $3x^3 - 11x^2 - 6x + 8 = (x - 4)\left(3x^2 + px - 2\right)$

equate coefficients of x^2: **(M1)**

$-12 + p = -11$

$p = 1$

$(x - 4)\left(3x^2 + x - 2\right)$ **(A1)**

$(x - 4)(3x - 2)(x + 1)$ **A1**

Note: Allow part (b) marks if any of this work is seen in part (a).

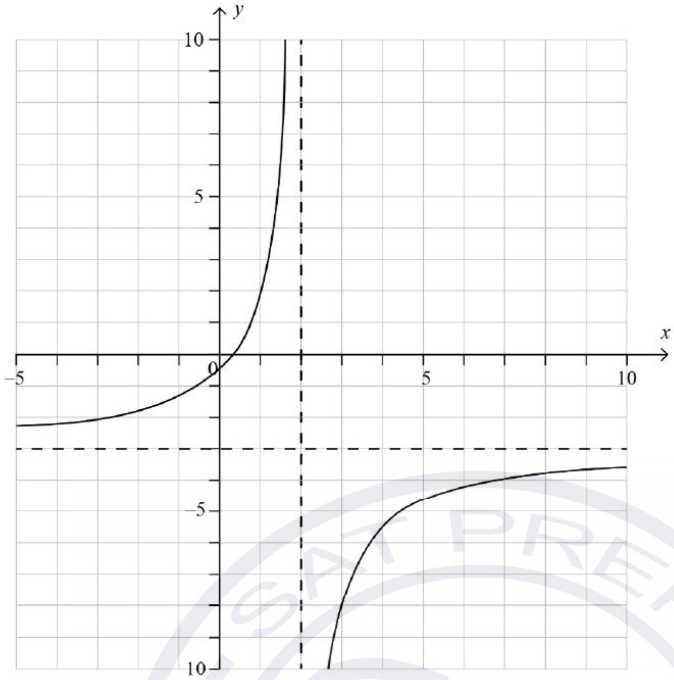
Note: Allow equivalent methods (*eg*, synthetic division) for the ***M*** marks in each part.

[3 marks]

Total [6 marks]

Question 38

(a)



correct vertical asymptote *A1*

shape including correct horizontal asymptote *A1*

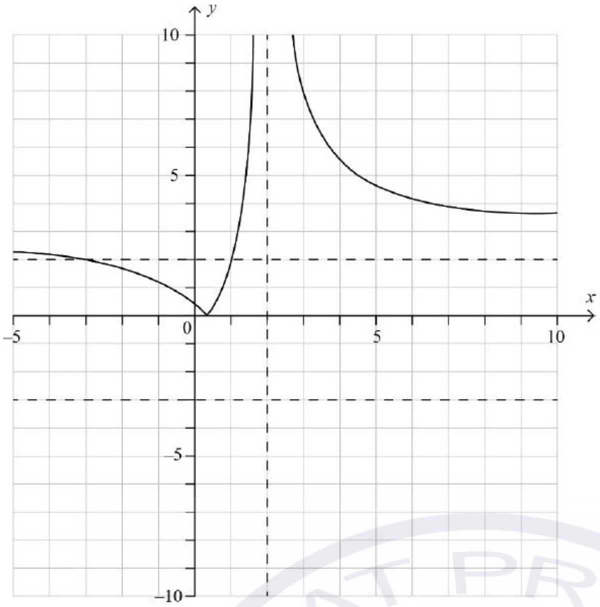
$\left(\frac{1}{3}, 0\right)$ *A1*

$\left(0, -\frac{1}{2}\right)$ *A1*

Note: Accept $x = \frac{1}{3}$ and $y = -\frac{1}{2}$ marked on the axes.

[4 marks]

(b) **METHOD 1**



$\dfrac{1-3x}{x-2}=2$ ***(M1)***

$\Rightarrow x=1$ ***A1***

$-\left(\dfrac{1-3x}{x-2}\right)=2$ ***(M1)***

Note: Award this ***M1*** for the line above or a correct sketch identifying a second critical value.

$\Rightarrow x=-3$ ***A1***

solution is $-3<x<1$ ***A1***

[5 marks]

METHOD 2

$|1-3x|<2|x-2|,\ x\neq 2$

$1-6x+9x^2<4\left(x^2-4x+4\right)$ ***(M1)A1***

$1-6x+9x^2<4x^2-16x+16$

$5x^2+10x-15<0$

$x^2+2x-3<0$ ***A1***

$(x+3)(x-1)<0$ ***(M1)***

solution is $-3<x<1$ ***A1***

[5 marks]

Question 39

(a) attempt to make x the subject of $y = \frac{ax+b}{cx+d}$ **M1**

$y(cx+d) = ax + b$ **A1**

$x = \frac{dy-b}{a-cy}$ **A1**

$f^{-1}(x) = \frac{dx-b}{a-cx}$, **A1**

Note: Do not allow $y =$ in place of $f^{-1}(x)$.

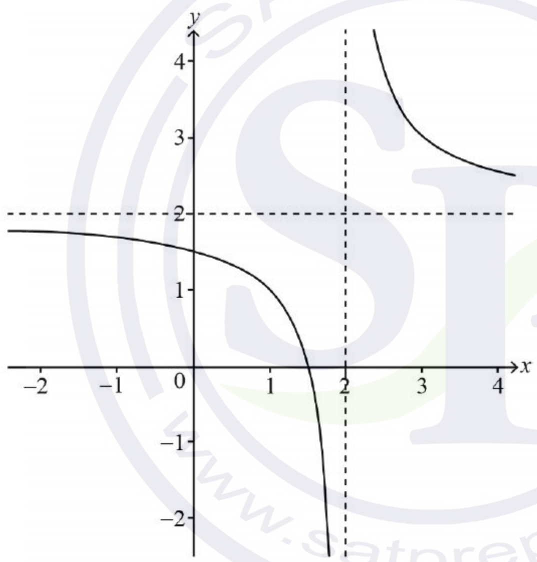
$x \neq \frac{a}{c}$, $(x \in \mathbb{R})$ **A1**

Note: The final ***A*** mark is independent.

[5 marks]

(b) (i) $g(x) = 2 + \frac{1}{x-2}$ **A1A1**

(ii)



hyperbola shape, with single curves in second and fourth quadrants and third quadrant blank, including vertical asymptote $x = 2$ **A1**

horizontal asymptote $y = 2$ **A1**

intercepts $\left(\frac{3}{2}, 0\right), \left(0, \frac{3}{2}\right)$ **A1**

[5 marks]

(c) the domain of $h \circ g$ is $x \le \frac{3}{2}$, $x > 2$ **A1A1**

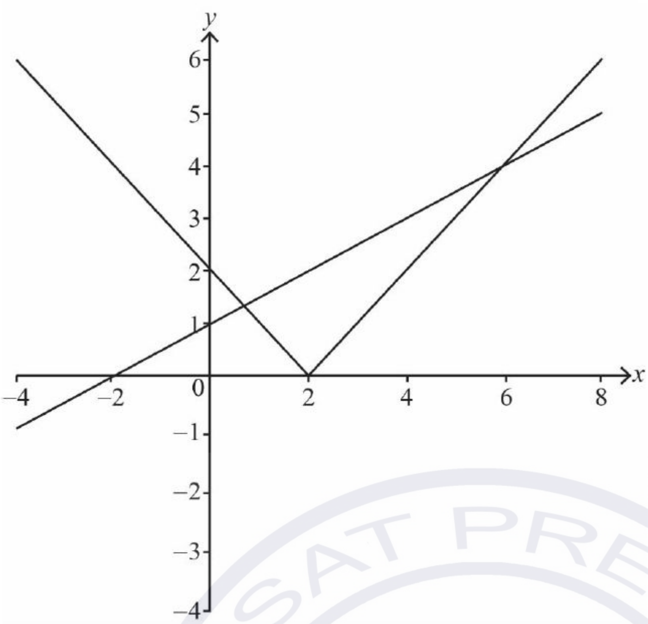
the range of $h \circ g$ is $y \ge 0$, $y \neq \sqrt{2}$ **A1A1**

[4 marks]

Total [14 marks]

Question 40

(a)



straight line graph with correct axis intercepts **A1**

modulus graph: V shape in upper half plane **A1**

modulus graph having correct vertex and y-intercept **A1**

[3 marks]

(b) $\left(\frac{x}{2}+1\right)^2 = (x-2)^2$ **M1**

$\frac{x^2}{4}+x+1 = x^2-4x+4$

$0 = \frac{3x^2}{4}-5x+3$

$3x^2-20x+12=0$

attempt to factorise (or equivalent) **M1**

$(3x-2)(x-6)=0$

$x = \frac{2}{3}$ **A1**

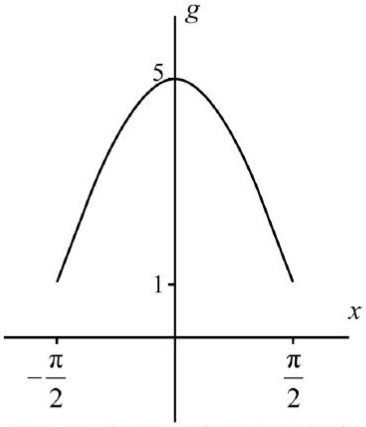
$x = 6$ **A1**

[4 marks]

Total [7 marks]

Question 41

(a)



concave down and symmetrical over correct domain **A1**

indication of maximum and minimum values of the function (correct range) **A1A1**

[3 marks]

(b) $a = 0$ **A1**

Note: Award **A1** for $a = 0$ only if consistent with their graph.

[1 mark]

(c) (i) $1 \le x \le 5$ **A1**

Note: Allow FT from their graph.

(ii) $y = 4\cos x + 1$

$x = 4\cos y + 1$

$\frac{x-1}{4} = \cos y$ ***(M1)***

$\Rightarrow y = \arccos\left(\frac{x-1}{4}\right)$

$\Rightarrow g^{-1}(x) = \arccos\left(\frac{x-1}{4}\right)$ **A1**

[3 marks]

Total [7 marks]

Question 42

(a) translation k units to the left (or equivalent) **A1**

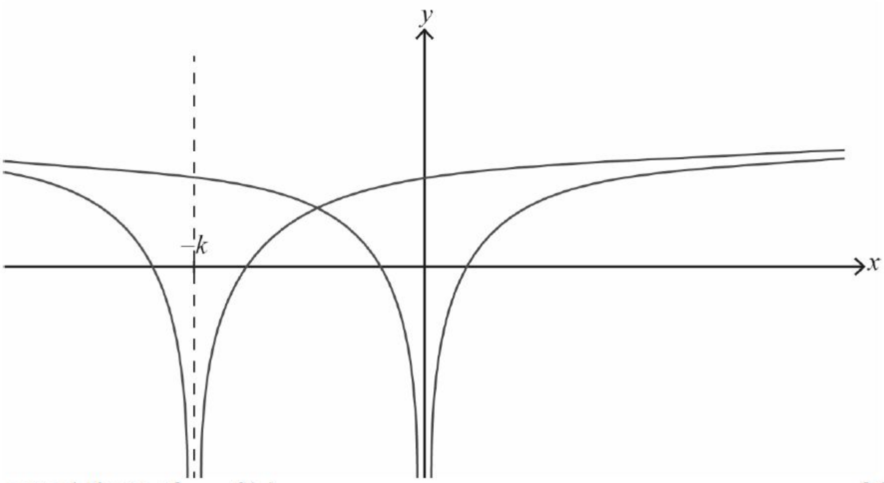
[1 mark]

(b) range is $(g(x) \in)\mathbb{R}$ **A1**

[1 mark]

continued…

(c)



correct shape of $y = f(x)$ **A1**

their $f(x)$ translated k units to left (possibly shown by $x = -k$ marked on x-axis) **A1**

asymptote included and marked as $x = -k$ **A1**

$f(x)$ intersects x-axis at $x = -1,\ x = 1$ **A1**

$g(x)$ intersects x-axis at $x = -k-1,\ x = -k+1$ **A1**

$g(x)$ intersects y-axis at $y = \ln k$ **A1**

te: Do not penalise candidates if their graphs "cross" as $x \to \pm\infty$.

te: Do not award ***FT*** marks from the candidate's part (a) to part (c).

[6 marks]

(d) at P $\ln(x+k) = \ln(-x)$

attempt to solve $x + k = -x$ (or equivalent) ***(M1)***

$x = -\frac{k}{2} \Rightarrow y = \ln\left(\frac{k}{2}\right)$ (or $y = \ln\left|\frac{k}{2}\right|$) **A1**

$\mathrm{P}\left(-\frac{k}{2}, \ln\frac{k}{2}\right)$ (or $\mathrm{P}\left(-\frac{k}{2}, \ln\left|\frac{k}{2}\right|\right)$)

[2 marks]

(e) attempt to differentiate $\ln(-x)$ or $\ln|x|$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ **A1**

at P, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2}{k}$ **A1**

recognition that tangent passes through origin $\Rightarrow \frac{y}{x} = \frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\ln\left(\frac{k}{2}\right)}{-\frac{k}{2}} = \frac{-2}{k}$ **A1**

$\ln\left(\frac{k}{2}\right) = 1$ ***(A1)***

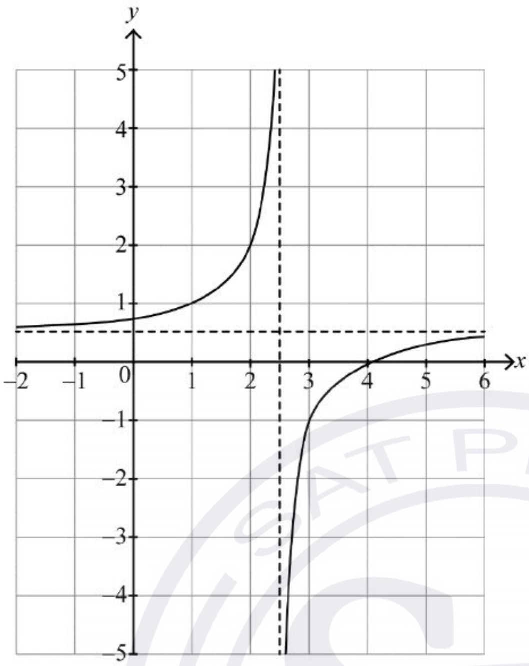
$\Rightarrow k = 2\mathrm{e}$ **A1**

[7 marks]

Total [17 marks]

Question 43

(a)



correct shape: two branches in correct quadrants with asymptotic behaviour **A1**

crosses at $(4, 0)$ and $\left(0, \frac{4}{5}\right)$ **A1A1**

asymptotes at $x = \frac{5}{2}$ and $y = \frac{1}{2}$ **A1A1**

[5 marks]

(b) (i) $x < \frac{5}{2}, x \geq 4$ **A1A1**

(ii) $f(x) \geq 0, f(x) \neq \frac{1}{\sqrt{2}} \left(f(x) \in \mathbb{R}\right)$ **A1**

Note: Follow through from their graph, as long as it is a rectangular hyperbola.

Note: Allow range expressed in terms of y.

[3 marks]

Total [8 marks]