Subject – Math(Higher Level)
Topic - Calculus
Year - Nov 2011 – Nov 2019
Paper -2

Question -1

$$V = \frac{\pi}{3}r^2h$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$ ***M1A1A1***

at the given instant

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2(40)(200)\left(-\frac{1}{2}\right) + 40^2(3)\right]$$ ***M1***

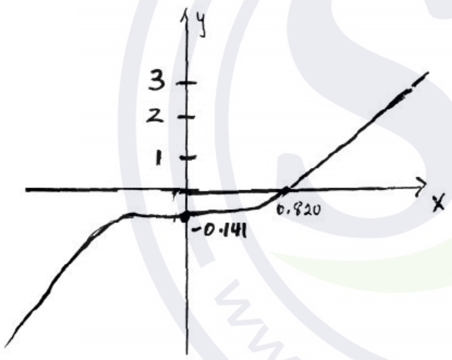
$$= \frac{-3200\pi}{3} = -3351.03\ldots \approx -3350$$ ***A1***

hence, the volume is decreasing (at approximately 3350 mm^3 per century) ***R1***

[6 marks]

Question -2

(a)



A1A1A1

Note:	Award ***A1*** for shape, ***A1*** for x-intercept is 0.820, accept $\sin(-3)$ or $-\sin(3)$ ***A1*** for y-intercept is -0.141.

(b) $A = \int_0^{0.8202} \lvert x + \sin(x-3) \rvert \, dx \approx 0.0816$ sq units ***(M1)A1***

[5 marks]

Question-3

(a) $\frac{dv}{dt} = -v^2 - 1$

attempt to separate the variables **M1**

$\int \frac{1}{1+v^2} dv = \int -1 dt$ **A1**

$\arctan v = -t + k$ **A1A1**

Note: Do not penalize the lack of constant at this stage.

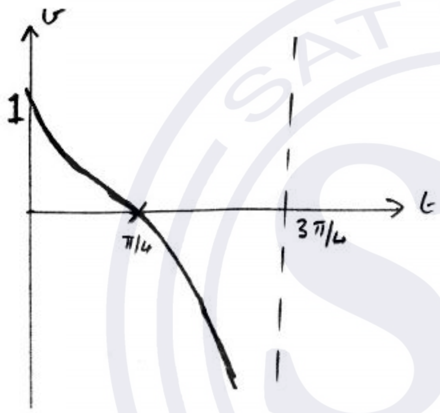
when $t = 0,\ v = 1$ **M1**

$\Rightarrow k = \arctan 1 = \left(\frac{\pi}{4}\right) = \left(45^\circ\right)$ **A1**

$\Rightarrow v = \tan\left(\frac{\pi}{4} - t\right)$ **A1**

[7 marks]

(b)



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for asymptote, ***A1*** for correct t and v intercept.

Note: Do not penalise if a larger domain is used.

[3 marks]

(c) (i) $T = \frac{\pi}{4}$ **A1**

(ii) area under curve $= \int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - t\right) dt$ ***(M1)***

$= 0.347 \left(= \frac{1}{2}\ln 2\right)$ **A1**

[3 marks]

(d) $v=\tan\left(\frac{\pi}{4}-t\right)$

$s=\int \tan\left(\frac{\pi}{4}-t\right)\mathrm{d}t$ **M1**

$\int \frac{\sin\left(\frac{\pi}{4}-t\right)}{\cos\left(\frac{\pi}{4}-t\right)}\mathrm{d}t$ ***(M1)***

$=\ln\cos\left(\frac{\pi}{4}-t\right)+k$ ***A1***

when $t=0,\ s=0$

$k=-\ln\cos\frac{\pi}{4}$ ***A1***

$s=\ln\cos\left(\frac{\pi}{4}-t\right)-\ln\cos\frac{\pi}{4}\ \left(=\ln\left[\sqrt{2}\cos\left(\frac{\pi}{4}-t\right)\right]\right)$ ***A1***

[5 marks]

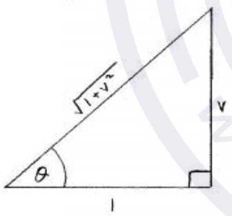
(e) **METHOD 1**

$\frac{\pi}{4}-t=\arctan v$ ***M1***

$t=\frac{\pi}{4}-\arctan v$

$s=\ln\left[\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{\pi}{4}+\arctan v\right)\right]$

$s=\ln\left[\sqrt{2}\cos(\arctan v)\right]$ ***M1A1***



$s=\ln\left[\sqrt{2}\cos\left(\arccos\frac{1}{\sqrt{1+v^2}}\right)\right]$ ***A1***

$=\ln\frac{\sqrt{2}}{\sqrt{1+v^2}}$

$=\frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

Question - 4

$x = r - \frac{r}{h}y$ or $x = \frac{r}{h}(h - y)$ (or equivalent) **_(A1)_**

$\int \pi x^2 \, dy$

$= \pi \int_0^h \left(r - \frac{r}{h}y \right)^2 dy$ **_M1A1_**

e: Award ***M1*** for $\int x^2 \, dy$ and ***A1*** for correct expression.
Accept $\pi \int_0^h \left(\frac{r}{h}y - r \right)^2 dy$ and $\pi \int_0^h \left(\pm \left(r - \frac{r}{h}x \right) \right)^2 dx$

$= \pi \int_0^h \left(r^2 - \frac{2r^2}{h}y + \frac{r^2}{h^2}y^2 \right) dy$ **_A1_**

e: Accept substitution method and apply markscheme to corresponding steps.

$= \pi \left[r^2 y - \frac{r^2 y^2}{h} + \frac{r^2 y^3}{3h^2} \right]_0^h$ **_M1A1_**

e: Award ***M1*** for attempted integration of any quadratic trinomial.

$= \pi \left(r^2 h - r^2 h + \frac{1}{3} r^2 h \right)$ **_M1A1_**

e: Award ***M1*** for attempted substitution of limits in a trinomial.

$= \frac{1}{3} \pi r^2 h$ **_A1_**

e: Throughout the question do not penalize missing dx/dy as long as the integrations are done with respect to correct variable

[9 marks]

Question -5

(a) $(3.79, -5)$ **A1**

[1 mark]

(b) $p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$ **A1A1**

[2 marks]

(c) $f'(x) = 3\cos x - 4\sin x$ **(M1)(A1)**

$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43\ldots$ **(A1)**

$(y = -4)$ **A1**

Coordinates are $(4.43, -4)$

[4 marks]

(d) $m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}$ **(M1)**

gradient at P is -4 so gradient of normal at P is $\frac{1}{4}$ **(A1)**

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$ **(A1)**

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570\ldots)$ (or $y = 0.25x + 2.60\ldots$) **(M1)**

equation of normal at Q is $y - 3 = -\frac{1}{4}\left(x - 5.999\ldots\right)\left(\text{or } y = -0.25x + \underbrace{4.499\ldots}\right)$ **(M1)**

Note: Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where (a, b) are coordinates of P and Q (or in $y = mx + c$ with c determined using coordinates of Pand Q.

intersect at $(3.79, 3.55)$ **A1A1**

Note: Award ***N2*** for 3.79 without other working.

[7 marks]

Total [14 marks]

Question -6

$\frac{dy}{dx} = 3x^2 - 12x + k$ **M1A1**

For use of discriminant $b^2 - 4ac = 0$ or completing the square $3(x-2)^2 + k - 12$ **(M1)**

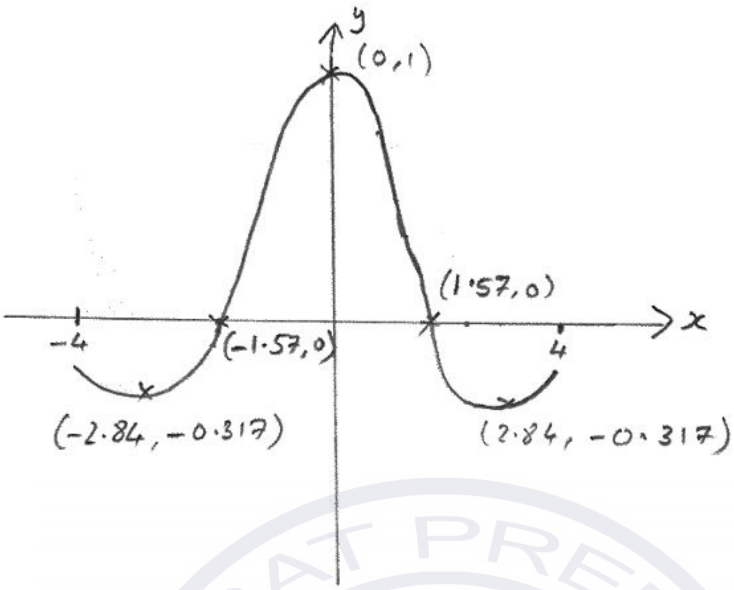
$144 - 12k = 0$ **(A1)**

te: Accept trial and error, sketches of parabolas with vertex (2,0) or use of second derivative.

$k = 12$ **A1**

Question – 7

(a)



A1A1A1A1

Note: Award ***A1*** for correct shape. Do not penalise if too large a domain is used,
A1 for correct *x*-intercepts,
A1 for correct coordinates of two minimum points,
A1 for correct coordinates of maximum point.

Accept answers which correctly indicate the position of the intercepts, maximum point and minimum points.

[4 marks]

(b) gradient at $x=1$ is -0.786 ***A1***

[1 mark]

(c) gradient of normal is $\frac{-1}{-0.786}\,(=1.272\ldots)$ ***(A1)***

when $x=1,\ y=0.3820\ldots$ ***(A1)***

Equation of normal is $y-0.382=1.27(x-1)$ ***A1***

$(\Rightarrow y=1.27x-0.890)$

[3 marks]

Total [8 marks]

Question – 8

$$2s\frac{\mathrm{d}s}{\mathrm{d}t}+\frac{\mathrm{d}s}{\mathrm{d}t}-2=0$$ ***M1A1***

$$v=\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{2}{2s+1}$$ ***A1***

EITHER

$$a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$$ ***(M1)***

$$\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{-4}{(2s+1)^2}$$ ***(A1)***

$$a=\frac{-4}{(2s+1)^2}\frac{\mathrm{d}s}{\mathrm{d}t}$$

OR

$$2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2+2s\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=0$$ ***(M1)***

$$\underbrace{\frac{\mathrm{d}^2s}{\mathrm{d}t^2}}_{a}=\frac{-2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{2s+1}$$ ***(A1)***

THEN

$$a=\frac{-8}{(2s+1)^3}$$ ***A1***

[6 marks]

Question - 9

$x = \sin t,\ \mathrm{d}x = \cos t\,\mathrm{d}t$

$\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\sin^3 t}{\sqrt{1-\sin^2 t}} \cos t\ \mathrm{d}t$ ***M1***

$= \int \sin^3 t\,\mathrm{d}t$ ***(A1)***

$= \int \sin^2 t\ \sin t\,\mathrm{d}t$

$= \int \left(1-\cos^2 t\right)\sin t\,\mathrm{d}t$ ***M1A1***

$= \int \sin t\,\mathrm{d}t - \int \cos^2 t \sin t\,\mathrm{d}t$

$= -\cos t + \frac{\cos^3 t}{3} + C$ ***A1A1***

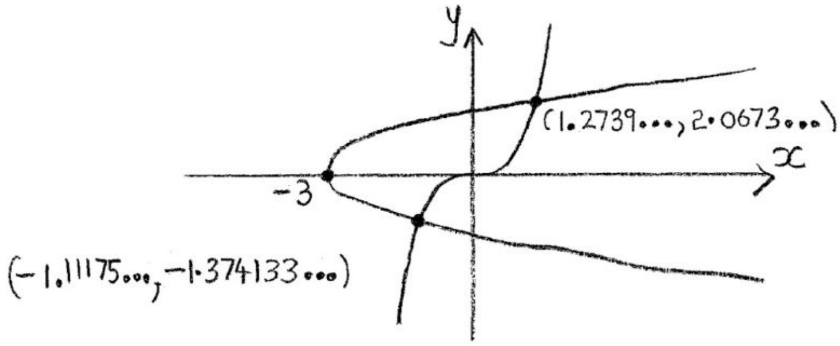
$= -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$ ***A1***

$\left(= -\sqrt{1-x^2}\left(1-\frac{1}{3}(1-x^2)\right)+C\right)$

$\left(= -\frac{1}{3}\sqrt{1-x^2}\,(2+x^2)+C\right)$

[7 marks]

Question -10



intersection points **_A1A1_**

te: Only either the x-coordinate or the y-coordinate is needed.

EITHER

$x = y^2 - 3 \Rightarrow y = \pm\sqrt{x+3}$ $\quad \left(\text{accept } y = \sqrt{x+3}\right)$ **_(M1)_**

$A = \int_{-3}^{-1.111...} 2\sqrt{x+3}\,dx + \int_{-1.111...}^{1.2739...} \sqrt{x+3} - x^3\,dx$ **_(M1)A1A1_**

$= 3.4595... + 3.8841...$

$= 7.34$ (3sf) **_A1_**

OR

$y = x^3 \Rightarrow x = \sqrt[3]{y}$ **_(M1)_**

$A = \int_{-1.374...}^{2.067...} \sqrt[3]{y} - \left(y^2 - 3\right)dy$ **_(M1)A1_**

$= 7.34$ (3sf) **_A2_**

[7 marks]

Question -11

(a) $L = \text{CA} + \text{AD}$ *M1*

$\sin\alpha = \frac{a}{\text{CA}} \Rightarrow \text{CA} = \frac{a}{\sin\alpha}$ *A1*

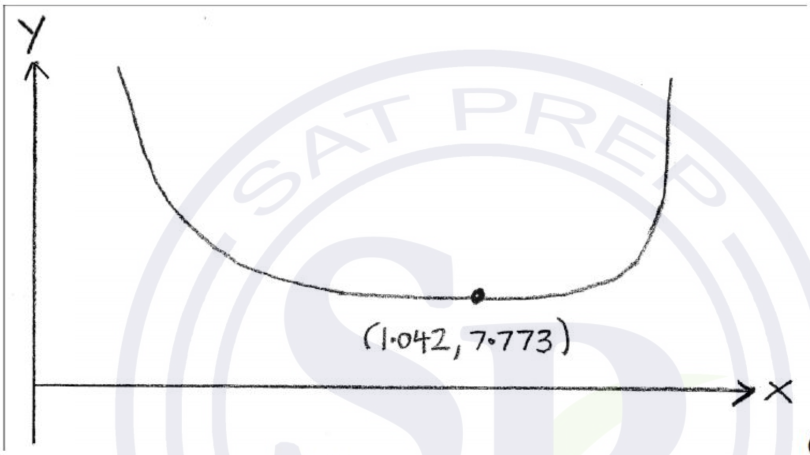
$\cos\alpha = \frac{b}{\text{AD}} \Rightarrow \text{AD} = \frac{b}{\cos\alpha}$ *A1*

$L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$ *AG*

[3 marks]

(b) $a = 5$ and $b = 1 \Rightarrow L = \frac{5}{\sin\alpha} + \frac{1}{\cos\alpha}$

METHOD 1



(M1)

minimum from graph $\Rightarrow L = 7.77$ *(M1)A1*

minimum of L gives the max length of the painting *R1*

[4 marks]

METHOD 2

$\frac{dL}{d\alpha} = \frac{-5\cos\alpha}{\sin^2\alpha} + \frac{\sin\alpha}{\cos^2\alpha}$ *(M1)*

$\frac{dL}{d\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = 5 \Rightarrow \tan\alpha = \sqrt[3]{5} \quad (\alpha = 1.0416\ldots)$ *(M1)*

minimum of L gives the max length of the painting *R1*

maximum length = 7.77 *A1*

[4 marks]

(c) $\frac{dL}{d\alpha} = \frac{-3k\cos\alpha}{\sin^2\alpha} + \frac{k\sin\alpha}{\cos^2\alpha}$ (or equivalent) *M1A1A1*

[3 marks]

(d) $\frac{dL}{d\alpha} = \frac{-3k\cos^3\alpha + k\sin^3\alpha}{\sin^2\alpha\cos^2\alpha}$ *(A1)*

$\frac{dL}{d\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = \frac{3k}{k} \Rightarrow \tan\alpha = \sqrt[3]{3}$ $(\alpha = 0.96454\ldots)$ ***M1A1***

$\tan\alpha = \sqrt[3]{3} \Rightarrow \frac{1}{\cos\alpha} = \sqrt{1+\sqrt[3]{9}}$ $(1.755\ldots)$ *(A1)*

and $\frac{1}{\sin\alpha} = \frac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}}$ $(1.216\ldots)$ *(A1)*

$L = 3k\left(\frac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}}\right) + k\sqrt{1+\sqrt[3]{9}}$ $(L = 5.405598\ldots k)$ ***A1*** ***N4***

[6 marks]

(e) $L \le 8 \Rightarrow k \ge 1.48$ ***M1A1***

the minimum value is 1.48

[2 marks]

Total [18 marks]

Question 12

volume $= \pi\int x^2 dy$ *(M1)*

$x = \arcsin y + 1$ *(M1)(A1)*

volume $= \pi\int_0^1 (\arcsin y + 1)^2 dy$ ***A1***

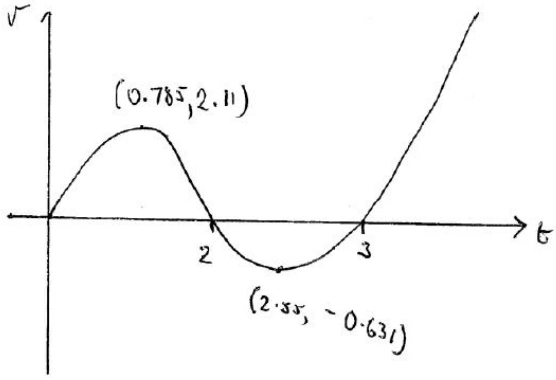
te: ***A1*** is for the limits, provided a correct integration of y.

$= 2.608993\ldots\pi = 8.20$ ***A2*** ***N5***

[6 marks]

Question 13

(a)



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maximum and minimum, ***A1*** for intercepts.

Note: Follow through applies to (b) and (c).

[3 marks]

(b) $0 \le t < 0.785, \left(\text{or } 0 \le t < \frac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

and $t > 2.55 \left(\text{or } t > \frac{5+\sqrt{7}}{3}\right)$ ***A1***

[2 marks]

(c) $0 \le t < 0.785, \left(\text{or } 0 \le t < \frac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

$2 < t < 2.55, \left(\text{or } 2 < t < \frac{5+\sqrt{7}}{3}\right)$ ***A1***

$t > 3$ ***A1***

[3 marks]

(d) position of A: $x_A = \int t^3 - 5t^2 + 6t \, \mathrm{d}t$ ***(M1)***

$x_A = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 \quad (+c)$ ***A1***

when $t = 0$, $x_A = 0$ so $c = 0$ ***R1***

[3 marks]

(e) $\frac{dv_B}{dt} = -2v_B \Rightarrow \int \frac{1}{v_B} dv_B = \int -2dt$ ***(M1)***

$\ln|v_B| = -2t + c$ ***(A1)***

$v_B = Ae^{-2t}$ ***(M1)***

$v_B = -20$ when $t = 0$ so $v_B = -20e^{-2t}$ ***A1***

[4 marks]

(f) $x_B = 10e^{-2t}(+c)$ ***(M1)(A1)***

$x_B = 20$ when $t = 0$ so $x_B = 10e^{-2t} + 10$ ***(M1)A1***

meet when $\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 = 10e^{-2t} + 10$ ***(M1)***

$t = 4.41(290\ldots)$ ***A1***

[6 marks]

Total: [21 marks]

Question 14

(a) $\int x\sec^2 x\, dx = x\tan x - \int 1 \times \tan x\, dx$ ***M1A1***

$= x\tan x + \ln|\cos x|\ (+c)\ \ (= x\tan x - \ln|\sec x|(+c))$ ***M1A1***

[4 marks]

(b) attempting to solve an appropriate equation *eg* $m\tan m + \ln(\cos m) = 0.5$ ***(M1)***

$m = 0.822$ ***A1***

Note: Award ***A1*** if $m = 0.822$ is specified with other positive solutions.

[2 marks]

Total [6 marks]

Question 15

METHOD 1

$\frac{dv}{dt} = \frac{1}{40}(60 - v)$ *(M1)*

attempting to separate variables $\int \frac{dv}{60 - v} = \int \frac{dt}{40}$ ***M1***

$-\ln(60 - v) = \frac{t}{40} + c$ ***A1***

$c = -\ln 60$ (or equivalent) ***A1***

attempting to solve for v when $t = 30$ *(M1)*

$v = 60 - 60e^{-\frac{3}{4}}$

$v = 31.7\ (\text{m s}^{-1})$ ***A1***

METHOD 2

$\frac{dv}{dt} = \frac{1}{40}(60 - v)$ *(M1)*

$\frac{dt}{dv} = \frac{40}{60 - v}$ (or equivalent) ***M1***

$\int_0^{v_f} \frac{40}{60 - v}\, dv = 30$ where v_f is the velocity of the car after 30 seconds. ***A1A1***

attempting to solve $\int_0^{v_f} \frac{40}{60 - v}\, dv = 30$ for v_f *(M1)*

$v = 31.7\ (\text{m s}^{-1})$ ***A1***

Total [6 marks]

Question 16

(a) **EITHER**

$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent) ***M1A1***

Note:	Accept $\theta = 180° - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent).

OR

$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$ (or equivalent) ***M1A1***

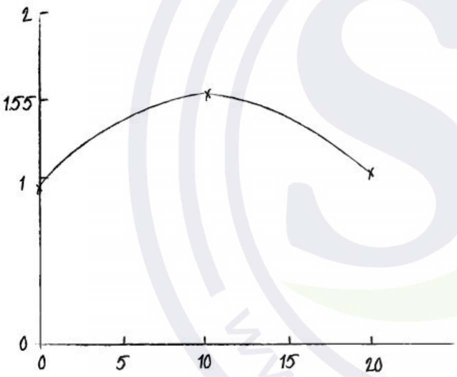
[2 marks]

(b) (i) $\theta = 0.994$ $\left(= \arctan\frac{20}{13}\right)$ ***A1***

(ii) $\theta = 1.19$ $\left(= \arctan\frac{5}{2}\right)$ ***A1***

[2 marks]

(c) correct shape. ***A1***
correct domain indicated. ***A1***



[2 marks]

(d) attempting to differentiate one $\arctan(f(x))$ term **M1**

EITHER

$$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$$

$$\frac{d\theta}{dx} = \frac{8}{x^2} \times \frac{1}{1+\left(\frac{8}{x}\right)^2} - \frac{13}{(20-x)^2} \times \frac{1}{1+\left(\frac{13}{20-x}\right)^2}$$ **A1A1**

OR

$$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$$

$$\frac{d\theta}{dx} = \frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2} + \frac{-\frac{1}{13}}{1+\left(\frac{20-x}{13}\right)^2}$$ **A1A1**

THEN

$$= \frac{8}{x^2+64} - \frac{13}{569-40x+x^2}$$ **A1**

$$= \frac{8(569-40x+x^2)-13(x^2+64)}{(x^2+64)(x^2-40x+569)}$$ **M1A1**

$$= \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$$ **AG**

[6 marks]

(e) Maximum light intensity at P occurs when $\frac{d\theta}{dx} = 0$. ***(M1)***

either attempting to solve $\frac{d\theta}{dx} = 0$ for x or using the graph of either θ or $\frac{d\theta}{dx}$ ***(M1)***

$x = 10.05$ (m) **A1**

[3 marks]

(f) $\frac{dx}{dt} = 0.5$ ***(A1)***

At $x = 10$, $\frac{d\theta}{dx} = 0.000453$ $\left(= \frac{5}{11\,029}\right)$. ***(A1)***

use of $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$ **M1**

$\frac{d\theta}{dt} = 0.000227$ $\left(= \frac{5}{22058}\right)$ (rad s^{-1}) **A1**

Note:	Award ***(A1)*** for $\frac{dx}{dt} = -0.5$ and ***A1*** for $\frac{d\theta}{dt} = -0.000227$ $\left(= -\frac{5}{22058}\right)$.

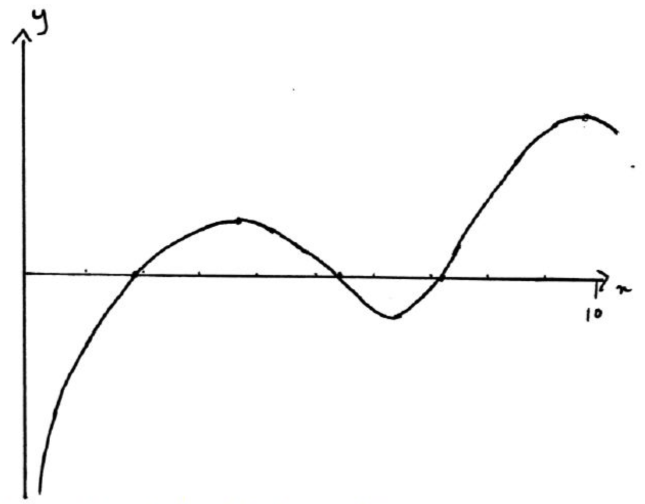
Note:	Implicit differentiation can be used to find $\frac{d\theta}{dt}$. Award as above.

[4 marks]

Total [19 marks]

Question 17

(a)



A correct graph shape for $0 < x \le 10$. *A1*

maxima (3.78, 0.882) and (9.70, 1.89) *A1*

minimum (6.22, −0.885) *A1*

x-axis intercepts (1.97, 0), (5.24, 0) and (7.11, 0) *A2*

Note: Award ***A1*** if two x-axis intercepts are correct.

[5 marks]

(b) $0 < x \le 1.97$ *A1*

$5.24 \le x \le 7.11$ *A1*

[2 marks]

Total [7 marks]

Question 18

EITHER

$\frac{dx}{du} = 2\sec^2 u$ ***A1***

$\int \frac{2\sec^2 u \, du}{4\tan^2 u\sqrt{4+4\tan^2 u}}$ ***(M1)***

$= \int \frac{2\sec^2 u \, du}{4\tan^2 u \times 2\sec u}$ $\left(= \int \frac{du}{4\sin^2 u\sqrt{\tan^2 u+1}} \text{ or } = \int \frac{2\sec^2 u \, du}{4\tan^2 u\sqrt{4\sec^2 u}}\right)$ ***A1***

OR

$u = \arctan \frac{x}{2}$

$\frac{du}{dx} = \frac{2}{x^2+4}$ ***A1***

$\int \frac{\sqrt{4\tan^2 u + 4}\, du}{2 \times 4\tan^2 u}$ ***(M1)***

$\int \frac{2\sec u \, du}{2 \times 4\tan^2 u}$ ***A1***

THEN

$= \frac{1}{4}\int \frac{\sec u \, du}{\tan^2 u}$

$= \frac{1}{4}\int \operatorname{cosec} u \cot u \, du$ $\left(= \frac{1}{4}\int \frac{\cos u}{\sin^2 u} du\right)$ ***A1***

$= -\frac{1}{4}\operatorname{cosec} u \ (+C)$ $\left(= -\frac{1}{4\sin u} \ (+C)\right)$ ***A1***

use of either $\tan u = \frac{x}{2}$ or an appropriate trigonometric identity ***M1***

either $\sin u = \frac{x}{\sqrt{x^2+4}}$ or $\operatorname{cosec} u = \frac{\sqrt{x^2+4}}{x}$ (or equivalent) ***A1***

$= \frac{-\sqrt{x^2+4}}{4x} (+C)$ ***AG***

Total [7 marks]

Question 19

(a) (i) **METHOD 1**

$v = \int 3\cos\frac{t}{4}\,dt$ ***M1***

$= 12\sin\frac{t}{4} + c$ ***A1***

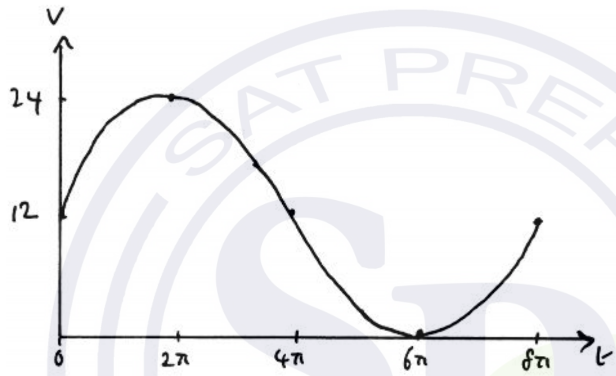
$t = 0,\ v = 12 \Rightarrow v = 12\sin\frac{t}{4} + 12$ ***A1***

METHOD 2

$v - 12 = \int_0^t 3\cos\frac{t}{4}\,dt$ ***M1A1***

$v = 12\sin\frac{t}{4} + 12$ ***A1***

(ii)



A1A1A1

Note: Award ***A1*** for shape and domain $0 \le t \le 8\pi$.
Award ***A1*** for $(0, 12)$ and $(6\pi, 0)$ $\left((18.8, 0)\right)$.
Award ***A1*** for $(2\pi, 24)$ $\left((6.28, 24)\right)$.

(iii) **METHOD 1**

$\int_0^{6\pi} \left(12\sin\frac{t}{4} + 12\right) dt$ ***M1***

$= 274\ \text{(m)}\ \left(= 72\pi + 48\ \text{(m)}\right)$ ***A1***

METHOD 2

$s = \int 12\sin\frac{t}{4} + 12\,dt$

$= -48\cos\frac{t}{4} + 12t + c$ ***M1***

When $t = 0$, $s = 0$ and so $c = 48$.

When $t = 6\pi$, $s = 274$ (m) $\left(= 72\pi + 48\ \text{(m)}\right)$. ***A1***

[8 marks]

(b) (i) **METHOD 1**

$\frac{\mathrm{d}v}{\mathrm{d}t} = -(v^2+4)$ ***(A1)***

$\int \frac{\mathrm{d}v}{v^2+4} = -\int \mathrm{d}t$ ***M1***

$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = -t + c$ ***A1***

EITHER

$t=0,\ v=2 \Rightarrow c = \frac{\pi}{8}$ ***M1***

$\arctan\left(\frac{v}{2}\right) = \frac{\pi}{4} - 2t$ ***A1***

OR

$v = 2\tan(2c-2t)$ ***A1***

$t=0,\ v=2 \Rightarrow \tan 2c = 1$ and so $c = \frac{\pi}{8}$ ***M1***

THEN

$v = 2\tan\left(\frac{\pi}{4} - 2t\right)$ ***A1***

$v = 2\tan\left(\frac{\pi-8t}{4}\right)$ ***AG***

METHOD 2

$\frac{dv}{dt} = -4\sec^2\left(\frac{\pi-8t}{4}\right)$ ***M1A1***

Substituting $v = 2\tan\left(\frac{\pi-8t}{4}\right)$ into $\frac{dv}{dt} = -(v^2+4)$:

$\frac{dv}{dt} = -\left(4\tan^2\left(\frac{\pi-8t}{4}\right)+4\right)$ ***M1***

$= -4\left(\tan^2\left(\frac{\pi-8t}{4}\right)+1\right)$ ***(A1)***

$= -4\sec^2\left(\frac{\pi-8t}{4}\right)$ ***A1***

Verifying that $v=2$ when $t=0$. ***A1***

(ii) **METHOD 1**

$v\dfrac{\mathrm{d}v}{\mathrm{d}s} = -\left(v^2+4\right)$ ***A1***

$\Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}s} = -\dfrac{\left(v^2+4\right)}{v}$ ***AG***

METHOD 2

$\dfrac{\mathrm{d}v}{\mathrm{d}s} = \dfrac{\mathrm{d}v}{\mathrm{d}t} \times \dfrac{\mathrm{d}t}{\mathrm{d}s}$ ***A1***

$\Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}s} = -\dfrac{\left(v^2+4\right)}{v}$ ***AG***

(iii) **METHOD 1**

When $v = 0$, $t = \dfrac{\pi}{8}$ $(t = 0.392\ldots)$. ***(M1)A1***

$s = \int_0^{\frac{\pi}{8}} 2\tan\left(\dfrac{\pi - 8t}{4}\right)\mathrm{d}t$ ***(M1)***

$s = 0.347$ (m) $\left(s = \dfrac{1}{2}\ln 2 \text{ (m)}\right)$ ***A2***

METHOD 2

$\int \dfrac{v}{4+v^2}\,\mathrm{d}v = -\int \mathrm{d}s$ ***M1***

EITHER

$\dfrac{1}{2}\ln\left(v^2+4\right) = -s + c$ (or equivalent) ***A1***

$v = 2,\ s = 0 \Rightarrow c = \dfrac{1}{2}\ln 8$ ***M1***

$s = -\dfrac{1}{2}\ln\left(v^2+4\right) + \dfrac{1}{2}\ln 8$ $\left(s = \dfrac{1}{2}\ln\left(\dfrac{8}{v^2+4}\right)\right)$ ***(A1)***

$v = 0 \Rightarrow s = \dfrac{1}{2}\ln 2$ (m) $\left(s = 0.347 \text{ (m)}\right)$ ***A1***

OR

$-\int_2^0 \dfrac{v}{4+v^2}\,\mathrm{d}v = s$ (or equivalent) ***M1A1***

Note: Award ***M1*** for setting up a definite integral and award ***A1*** for stating correct limits.

$s = 0.347$ (m) $\left(s = \dfrac{1}{2}\ln 2 \text{ (m)}\right)$ ***A2***

[12 marks]
Total [20 marks]

Question 20

(a) (i) either counterexample or sketch or recognising that $y=k\ (k>1)$ intersects the graph of $y=f(x)$ twice ***M1***

function is not 1–1 (does not obey horizontal line test) ***R1***

so f^{-1} does not exist ***AG***

(ii) $f'(x)=\frac{1}{2}\left(e^{x}-e^{-x}\right)$ ***(A1)***

$f'(\ln 3)=\frac{4}{3}\ (=1.33)$ ***(A1)***

$m=-\frac{3}{4}$ ***M1***

$f(\ln 3)=\frac{5}{3}\ (=1.67)$ ***A1***

EITHER

$\frac{y-\frac{5}{3}}{x-\ln 3}=-\frac{3}{4}$ ***M1***

$4y-\frac{20}{3}=-3x+3\ln 3$ ***A1***

OR

$\frac{5}{3}=-\frac{3}{4}\ln 3+c$ ***M1***

$c=\frac{5}{3}+\frac{3}{4}\ln 3$

$y=-\frac{3}{4}x+\frac{5}{3}+\frac{3}{4}\ln 3$ ***A1***

$12y=-9x+20+9\ln 3$

THEN

$9x+12y-9\ln 3-20=0$ ***AG***

(iii) The tangent at $(a, f(a))$ has equation $y-f(a)=f'(a)(x-a)$. ***(M1)***

$f'(a)=\frac{f(a)}{a}$ (or equivalent) ***(A1)***

$e^{a}-e^{-a}=\frac{e^{a}+e^{-a}}{a}$ (or equivalent) ***A1***

attempting to solve for a ***(M1)***

$a=\pm 1.20$ ***A1A1***

[14 marks]

(b) (i) $2y = e^x + e^{-x}$

$e^{2x} - 2ye^x + 1 = 0$ ***M1A1***

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$e^x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^x = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi\int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \mathrm{d}y$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use $V = \pi\int_c^d x^2 \mathrm{d}y$.

$= 37.1 \left(\text{units}^3\right)$ ***A1***

[8 marks]

Total [22 marks]

Question 21

(a) $\dfrac{\pi}{2}(1.57), \dfrac{3\pi}{2}(4.71)$ ***A1A1***

hence the coordinates are $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$ ***A1***

[3 marks]

(b) (i) $\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(x^2 - (x + 2\cos x)^2\right)\mathrm{d}x$ ***A1A1A1***

Note: Award ***A1*** for $x^2 - (x + 2\cos x)^2$, ***A1*** for correct limits and ***A1*** for π.

(ii) $6\pi^2 (= 59.2)$ ***A2***

Notes: Do not award **ft** from (b)(i).

[5 marks]

Total [8 marks]

Question 22

METHOD 1

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{dV}{dh} = \pi h^2$ ***(A1)***

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ (using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$) ***M1A1***

$\frac{dV}{dt} = 8\pi$ $(= 25.1)$ $(\text{cm}^3\,\text{min}^{-1})$ ***A1***

METHOD 2

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{dV}{dt} = \frac{1}{3}\pi \times 3h^2 \times \frac{dh}{dt}$ ***A1***

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ ***M1A1***

$\frac{dV}{dt} = 8\pi$ $(= 25.1)$ $(\text{cm}^3\,\text{min}^{-1})$ ***A1***

METHOD 3

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$ ***M1A1***

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when $h = 4$, $r = 4$, $\frac{dV}{dt} = \frac{1}{3}\pi\left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5\right)$ ***M1A1***

$\frac{dV}{dt} = 8\pi$ $(= 25.1)$ $(\text{cm}^3\,\text{min}^{-1})$ ***A1***

[5 marks]

Question 23

(a) **METHOD 1**

expanding the brackets first:

$x^4+2x^2y^2+y^4=4xy^2$ ***M1***

$4x^3+4xy^2+4x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+4y^3\frac{\mathrm{d}y}{\mathrm{d}x}=4y^2+8xy\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1***

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x^3-xy^2+y^2}{yx^2-2xy+y^3}$ or equivalent ***A1***

METHOD 2

$2\left(x^2+y^2\right)\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)=4y^2+8xy\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1***

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\left(x^2+y^2\right)\left(x+y\frac{\mathrm{d}y}{\mathrm{d}x}\right)=y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}$

$x^3+x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2x+y^3\frac{\mathrm{d}y}{\mathrm{d}x}=y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x^3-xy^2+y^2}{yx^2-2xy+y^3}$ or equivalent ***A1***

[5 marks]

(b) **METHOD 1**

at $(1, 1)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined ***M1A1***

$y=1$ ***A1***

METHOD 2

gradient of normal $=-\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}=-\frac{\left(yx^2-2xy+y^3\right)}{\left(-x^3-xy^2+y^2\right)}$ ***M1***

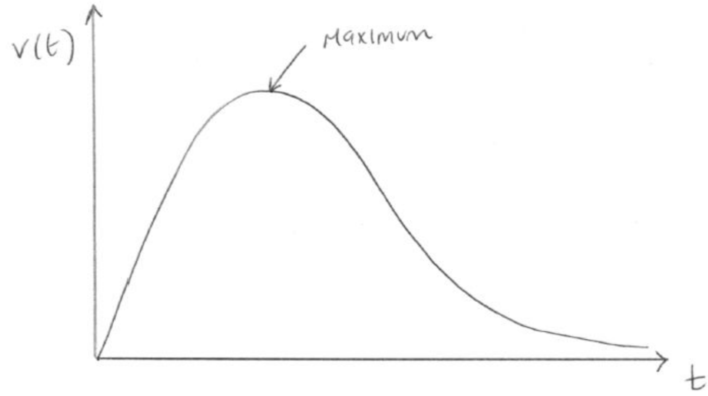
at $(1, 1)$ gradient $=0$ ***A1***

$y=1$ ***A1***

[3 marks]

Question 24

(a)



A1

A1 for correct shape and correct domain

$(1.41, 0.0884)\left(\sqrt{2}, \frac{\sqrt{2}}{16}\right)$ ***A1***

[2 marks]

(b) **EITHER**

$u = t^2$

$\frac{du}{dt} = 2t$ ***A1***

OR

$t = u^{\frac{1}{2}}$

$\frac{dt}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ ***A1***

THEN

$\int \frac{t}{12+t^4} dt = \frac{1}{2}\int \frac{du}{12+u^2}$ ***M1***

$= \frac{1}{2\sqrt{12}} \arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ ***M1***

$= \frac{1}{4\sqrt{3}} \arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent ***A1***

[4 marks]

(c) $\int_0^6 \frac{t}{12+t^4}\,dt$ *(M1)*

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)$(m) ***A1***

Note: Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

[3 marks]

(d) $\frac{dv}{ds} = \frac{1}{2\sqrt{s(1-s)}}$ *(A1)*

$a = v\frac{dv}{ds}$

$a = \arcsin\left(\sqrt{s}\right)\times\frac{1}{2\sqrt{s(1-s)}}$ *(M1)*

$a = \arcsin\left(\sqrt{0.1}\right)\times\frac{1}{2\sqrt{0.1\times 0.9}}$

$a = 0.536 \ \left(\text{ms}^{-2}\right)$ ***A1***

[3 marks]

Total [12 marks]

Question 25

METHOD 1

attempt to set up (diagram, vectors) ***(M1)***

correct distances $x = 15t,\ y = 20t$ ***(A1) (A1)***

the distance between the two cyclists at time t is $s = \sqrt{(15t)^2 + (20t)^2} = 25t$ (km) ***A1***

$\frac{ds}{dt} = 25\ (\text{km h}^{-1})$ ***A1***

hence the rate is independent of time ***AG***

METHOD 2

attempting to differentiate $x^2 + y^2 = s^2$ implicitly ***(M1)***

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$ ***(A1)***

the distance between the two cyclists at time t is $\sqrt{(15t)^2 + (20t)^2} = 25t$ (km) ***(A1)***

$2(15t)(15) + 2(20t)(20) = 2(25t)\frac{ds}{dt}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their equation involving $\frac{ds}{dt}$.

$\frac{ds}{dt} = 25\ (\text{km h}^{-1})$ ***A1***

hence the rate is independent of time ***AG***

Question 26

(a) $3-\frac{t}{2}=0 \Rightarrow t=6\text{(s)}$ ***(M1)A1***

[2 marks]

Note: Award ***A0*** if either $t=-0.236$ or $t=4.24$ or both are stated with $t=6$.

(b) let d be the distance travelled before coming to rest

$d=\int_0^4 5-(t-2)^2\,\mathrm{d}t+\int_4^6 3-\frac{t}{2}\,\mathrm{d}t$ ***(M1)(A1)***

Note: Award ***M1*** for two correct integrals even if the integration limits are incorrect. The second integral can be specified as the area of a triangle.

$d=\frac{47}{3}\ (=15.7)\text{(m)}$ ***(A1)***

attempting to solve $\int_6^T\left(\frac{t}{2}-3\right)\mathrm{d}t=\frac{47}{3}$ (or equivalent) for T ***M1***

$T=13.9\text{(s)}$ ***A1***

[5 marks]

Total [7 marks]

Question 27

(a) use of $A = \frac{1}{2}qr\sin\theta$ to obtain $A = \frac{1}{2}(x+2)(5-x)^2 \sin 30°$ *M1*

$= \frac{1}{4}(x+2)(25-10x+x^2)$ *A1*

$A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$ *AG*

[2 marks]

(b) (i) $\frac{dA}{dx} = \frac{1}{4}\left(3x^2 - 16x + 5\right) = \frac{1}{4}(3x-1)(x-5)$ *A1*

(ii) **METHOD 1**

EITHER

$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5\right) = 0$ *M1A1*

OR

$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right) - 1\right)\left(\left(\frac{1}{3}\right) - 5\right) = 0$ *M1A1*

THEN

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$ *AG*

METHOD 2

solving $\frac{dA}{dx} = 0$ for x *M1*

$-2 < x < 5 \Rightarrow x = \frac{1}{3}$ *A1*

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$ *AG*

METHOD 3

a correct graph of $\frac{dA}{dx}$ versus x *M1*

the graph clearly showing that $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$ *A1*

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$ *AG*

[3 marks]

(c) (i) $\frac{d^2A}{dx^2} = \frac{1}{2}(3x-8)$ ***A1***

for $x = \frac{1}{3}$, $\frac{d^2A}{dx^2} = -3.5 (< 0)$ ***R1***

so $x = \frac{1}{3}$ gives the maximum area of triangle PQR ***AG***

(ii) $A_{max} = \frac{343}{27} (= 12.7)(cm^2)$ ***A1***

(iii) $PQ = \frac{7}{3}$ (cm) and $PR = \left(\frac{14}{3}\right)^2$ (cm) ***(A1)***

$QR^2 = \left(\frac{7}{3}\right)^2 + \left(\frac{14}{3}\right)^4 - 2\left(\frac{7}{3}\right)\left(\frac{14}{3}\right)^2 \cos 30°$ ***(M1)(A1)***

$= 391.702\ldots$

$QR = 19.8$ (cm) ***A1***

[7 marks]

Total [12 marks]

Question 28

(a) attempting to use $V=\pi\int_a^b x^2 \mathrm{d}y$ *(M1)*

attempting to express x^2 in terms of y *ie* $x^2=4(y+16)$ *(M1)*

for $y=h$, $V=4\pi\int_0^h y+16\,\mathrm{d}y$ **A1**

$$V=4\pi\left(\frac{h^2}{2}+16h\right)$$ **AG**

[3 marks]

(b) (i) **METHOD 1**

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$ *(M1)*

$$\frac{\mathrm{d}V}{\mathrm{d}h}=4\pi(h+16)$$ *(A1)*

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{4\pi(h+16)}\times\frac{-250\sqrt{h}}{\pi(h+16)}$$ **M1A1**

Note: Award **M1** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$.

$$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$$ **AG**

METHOD 2

$$\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$$ (implicit differentiation) *(M1)*

$$\frac{-250\sqrt{h}}{\pi(h+16)}=4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$$ (or equivalent) **A1**

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{4\pi(h+16)}\times\frac{-250\sqrt{h}}{\pi(h+16)}$$ **M1A1**

$$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$$ **AG**

(ii) $$\frac{\mathrm{d}t}{\mathrm{d}h}=-\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$$ **A1**

$$t=\int-\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\,\mathrm{d}h$$ *(M1)*

$$t=\int-\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\,\mathrm{d}h$$ **A1**

$$t=\frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}}+32h^{\frac{1}{2}}+256h^{-\frac{1}{2}}\right)\mathrm{d}h$$ **AG**

(iii) **METHOD 1**

$t=\dfrac{-4\pi^2}{250}\displaystyle\int_{48}^{0}\left(h^{\frac{3}{2}}+32h^{\frac{1}{2}}+256h^{-\frac{1}{2}}\right)\mathrm{d}h$ ***(M1)***

$t=2688.756\ldots\text{(s)}$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$t=\dfrac{-4\pi^2}{250}\left(\dfrac{2}{5}h^{\frac{5}{2}}+\dfrac{64}{3}h^{\frac{3}{2}}+512h^{\frac{1}{2}}\right)+c$

when $t=0, h=48 \Rightarrow c=2688.756\ldots\left(c=\dfrac{4\pi^2}{250}\left(\dfrac{2}{5}\times48^{\frac{5}{2}}+\dfrac{64}{3}\times48^{\frac{3}{2}}+512\times48^{\frac{1}{2}}\right)\right)$ ***(M1)***

when $h=0$, $t=2688.756\ldots\left(t=\dfrac{4\pi^2}{250}\left(\dfrac{2}{5}\times48^{\frac{5}{2}}+\dfrac{64}{3}\times48^{\frac{3}{2}}+512\times48^{\frac{1}{2}}\right)\right)\text{(s)}$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

[10 marks]

(c) **EITHER**

the depth stabilises when $\dfrac{\mathrm{d}V}{\mathrm{d}t}=0$ *ie* $8.5-\dfrac{250\sqrt{h}}{\pi(h+16)}=0$ ***R1***

attempting to solve $8.5-\dfrac{250\sqrt{h}}{\pi(h+16)}=0$ for h ***(M1)***

OR

the depth stabilises when $\dfrac{\mathrm{d}h}{\mathrm{d}t}=0$ *ie* $\dfrac{1}{4\pi(h+16)}\left(8.5-\dfrac{250\sqrt{h}}{\pi(h+16)}\right)=0$ ***R1***

attempting to solve $\dfrac{1}{4\pi(h+16)}\left(8.5-\dfrac{250\sqrt{h}}{\pi(h+16)}\right)=0$ for h ***(M1)***

THEN

$h=5.06\text{ (cm)}$ ***A1***

[3 marks]

Total [16 marks]

Question 29

$\displaystyle\int_{-1}^{1}\pi\left(\mathrm{e}^{-x^2}\right)^2\mathrm{d}x \quad \left(\int_{-1}^{1}\pi\mathrm{e}^{-2x^2}\,\mathrm{d}x \text{ or } \int_{0}^{1}2\pi\mathrm{e}^{-2x^2}\,\mathrm{d}x\right)$ ***(M1)(A1)(A1)***

:e: Award ***M1*** for integral involving the function given; ***A1*** for correct limits; ***A1*** for π and $\left(\mathrm{e}^{-x^2}\right)^2$

$=3.758249\ldots=3.76$ ***A1***

[4 marks]

Question 30

$V = 200\pi r^2$ **(A1)**

> **ote:** Allow $V = \pi h r^2$ if value of h is substituted later in the question.

EITHER

$\frac{dV}{dt} = 200\pi 2r \frac{dr}{dt}$ **M1A1**

> **ote:** Award ***M1*** for an attempt at implicit differentiation.

at $r = 2$ we have $30 = 200\pi 4 \frac{dr}{dt}$ **M1**

OR

$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}}$ **M1**

$\frac{dV}{dr} = 400\pi r$ **M1**

$r = 2$ we have $\frac{dV}{dr} = 800\pi$ **A1**

THEN

$\frac{dr}{dt} = \frac{30}{800\pi}\left(= \frac{3}{80\pi} = 0.0119\right)$ (cm s^{-1}) **A1**

Total [5 marks]

Question 31

$f'(x) = 3x^2 + e^x$ **A1**

> **e:** Accept labelled diagram showing the graph $y = f'(x)$ above the x-axis; do not accept unlabelled graphs nor graph of $y = f(x)$.

EITHER

this is always > 0 **R1**
so the function is (strictly) increasing **R1**
and thus 1–1 **A1**

OR

this is always > 0 (accept $\neq 0$) **R1**
so there are no turning points **R1**
and thus 1–1 **A1**

> **te:** ***A1*** is dependent on the first ***R1***.

Total [4 marks]

Question 32

$x = 0 \Rightarrow y = 1$ **(A1)**

$y'(0) = 1.367879\ldots$ **(M1)(A1)**

ote: The exact answer is $y'(0) = \frac{e+1}{e} = 1 + \frac{1}{e}$.

so gradient of normal is $\frac{-1}{1.367879\ldots} (= -0.731058\ldots)$ **(M1)(A1)**

equation of normal is $y = -0.731058\ldots x + c$ **(M1)**

gives $y = -0.731x + 1$ **A1**

ote: The exact answer is $y = -\frac{e}{e+1}x + 1$.

Accept $y - 1 = -0.731058\ldots(x - 0)$

Total [7 marks]

Question 33

(a) $x \to -\infty \Rightarrow y \to -\frac{1}{2}$ so $y = -\frac{1}{2}$ is an asymptote **(M1)A1**

$e^x - 2 = 0 \Rightarrow x = \ln 2$ so $x = \ln 2 (= 0.693)$ is an asymptote **(M1)A1**

[4 marks]

(b) (i) [redacted]$(x) = \dfrac{2(e^x - 2)e^{2x} - (e^{2x} + 1)e^x}{(e^x - 2)^2}$ **M1A1**

$= \dfrac{e^{3x} - 4e^{2x} - e^x}{(e^x - 2)^2}$

(ii) [redacted]$'(x) = 0$ when $e^{3x} - 4e^{2x} - e^x = 0$ **M1**

$e^x(e^{2x} - 4e^x - 1) = 0$

$e^x = 0,\ e^x = -0.236,\ e^x = 4.24\ \left(\text{or } e^x = 2 \pm \sqrt{5}\right)$ **A1A1**

Note: Award ***A1*** for zero, ***A1*** for other two solutions.
Accept any answers which show a zero, a negative and a positive.

as $e^x > 0$ exactly one solution **R1**

Note: Do not award marks for purely graphical solution.

(iii) $(1.44, 8.47)$ **A1A1**

[8 marks]

(c) $f'(0) = -4$ ***(A1)***

so gradient of normal is $\frac{1}{4}$ ***(M1)***

[redacted]$0) = -2$ ***(A1)***

so equation of L_1 is $y = \frac{1}{4}x - 2$ **A1**

[4 marks]

(d) [redacted]$(x) = \frac{1}{4}$ [redacted]***1***

so $x = 1.46$ ***(M1)A1***

[redacted]$(1.46) = 8.47$ ***(A1)***

equation of [redacted] $y - 8.47 = \frac{1}{4}(x - 1.46)$ **A1**

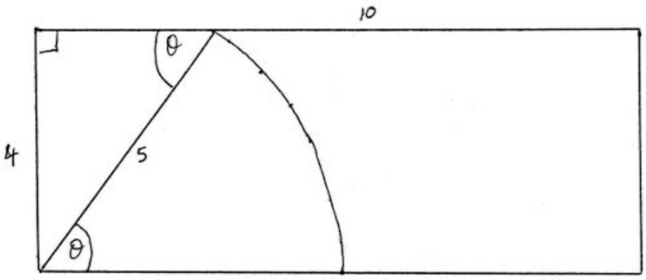
(or $y = \frac{1}{4}x + 8.11$)

[redacted]***5 marks]***

Total [21 marks]

Question 34

(a)



EITHER

area of triangle $= \frac{1}{2} \times 3 \times 4 \; (= 6)$ ***A1***

area of sector $= \frac{1}{2} \arcsin\left(\frac{4}{5}\right) \times 5^2 \; (= 11.5911\ldots)$ ***A1***

OR

$\int_0^4 \sqrt{25 - x^2}\, dx$ ***M1A1***

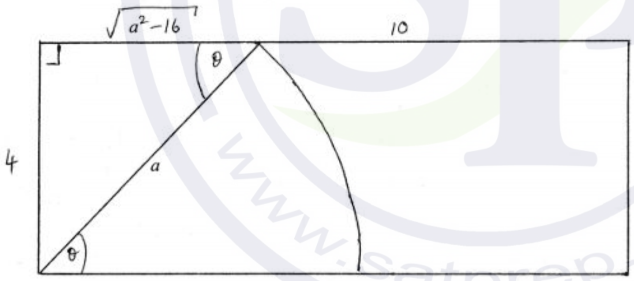
THEN

total area $= 17.5911\ldots \text{ m}^2$ ***(A1)***

percentage $= \frac{17.5911\ldots}{40} \times 100 = 44\%$ ***A1***

[4 marks]

(b) **METHOD 1**



area of triangle $= \frac{1}{2} \times 4 \times \sqrt{a^2 - 16}$ ***A1***

$\theta = \arcsin\left(\frac{4}{a}\right)$ ***(A1)***

area of sector $= \frac{1}{2} r^2 \theta = \frac{1}{2} a^2 \arcsin\left(\frac{4}{a}\right)$ ***A1***

therefore total area $= 2\sqrt{a^2 - 16} + \frac{1}{2} a^2 \arcsin\left(\frac{4}{a}\right) = 20$ ***A1***

rearrange to give: $a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40$ ***AG***

continued…

METHOD 2

$$\int_0^4 \sqrt{a^2 - x^2}\,dx = 20$$ ***M1***

use substitution $x = a\sin\theta,\ \frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta$

$$\int_0^{\arcsin\left(\frac{4}{a}\right)} a^2\cos^2\theta\,\mathrm{d}\theta = 20$$

$$\frac{a^2}{2}\int_0^{\arcsin\left(\frac{4}{a}\right)} (\cos 2\theta + 1)\,\mathrm{d}\theta = 20$$ ***M1***

$$a^2\left[\left(\frac{\sin 2\theta}{2} + \theta\right)\right]_0^{\arcsin\left(\frac{4}{a}\right)} = 40$$ ***A1***

$$a^2\left[(\sin\theta\cos\theta + \theta)\right]_0^{\arcsin\left(\frac{4}{a}\right)} = 40$$

$$a^2\arcsin\left(\frac{4}{a}\right) + a^2\left(\frac{4}{a}\right)\sqrt{\left(1 - \left(\frac{4}{a}\right)^2\right)} = 40$$ ***A1***

$$a^2\arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40$$ ***AG***

[4 marks]

(c) solving using $\text{GDC} \Rightarrow a = 5.53\,\text{cm}$ ***A2***

[2 marks]

Total [10 marks]

Question 35

(a) attempt at implicit differentiation **M1**

$$2x - 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 5y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$ ***A1A1***

> **Note:** ***A1*** for differentiation of $x^2 - 5xy$, ***A1*** for differentiation of y^2 and 7.

$$2x - 5y + \frac{\mathrm{d}y}{\mathrm{d}x}(2y - 5x) = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y - 2x}{2y - 5x}$$ ***AG***

[3 marks]

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5 \times 1 - 2 \times 6}{2 \times 1 - 5 \times 6} = \frac{1}{4}$ ***A1***

gradient of normal $= -4$ ***A1***

equation of normal $y = -4x + c$ **M1**

substitution of $(6, 1)$

$y = -4x + 25$ ***A1***

> **Note:** Accept $y - 1 = -4(x - 6)$

[4 marks]

(c) setting $\frac{5y - 2x}{2y - 5x} = 1$ **M1**

$y = -x$ ***A1***

substituting into original equation **M1**

$x^2 + 5x^2 + x^2 = 7$ ***(A1)***

$7x^2 = 7$

$x = \pm 1$ ***A1***

points $(1, -1)$ and $(-1, 1)$ ***(A1)***

distance $= \sqrt{8}\left(= 2\sqrt{2}\right)$ ***(M1)A1***

[8 marks]

Total [15 marks]

Question 36

(a) **METHOD 1**

$s=\int\left(9t-3t^2\right)\mathrm{d}t=\frac{9}{2}t^2-t^3(+c)$ ***(M1)***

$t=0,\ s=3\Rightarrow c=3$ ***(A1)***

$t=4\Rightarrow s=11$ ***A1***

[3 marks]

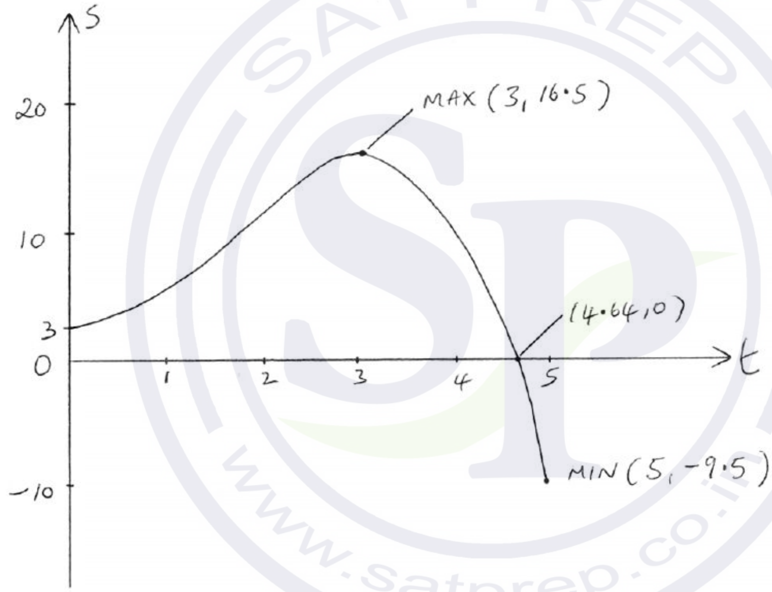
METHOD 2

$s=3+\int_0^4\left(9t-3t^2\right)dt$ ***(M1)(A1)***

$s=11$ ***A1***

[3 marks]

(b) $s=3+\frac{9}{2}t^2-t^3$ ***(A1)***



correct shape over correct domain ***A1***

maximum at $(3,\ 16.5)$ ***A1***

t intercept at 4.64, *s* intercept at 3 ***A1***

minimum at $(5,\ -9.5)$ ***A1***

[5 marks]

(c) $-9.5 = a + b\cos 2\pi$

$16.5 = a + b\cos 3\pi$ *(M1)*

Note: Only award ***M1*** if two simultaneous equations are formed over the correct domain.

$a = \frac{7}{2}$ ***A1***

$b = -13$ ***A1***

[3 marks]

(d) at t_1:

$3 + \frac{9}{2}t^2 - t^3 = 3$ *(M1)*

$t^2\left(\frac{9}{2} - t\right) = 0$

$t_1 = \frac{9}{2}$ ***A1***

solving $\frac{7}{2} - 13\cos\frac{2\pi t}{5} = 3$ *(M1)*

$\text{GDC} \Rightarrow t_2 = 6.22$ ***A1***

Note: Accept graphical approaches.

[4 marks]

Total [15 marks]

Question 37

(a) (i) $a(t)=\frac{\mathrm{d}v}{\mathrm{d}t}=-10\ (\mathrm{m\,s^{-2}})$ **A1**

(ii) $t=10\Rightarrow v=-100\ (\mathrm{m\,s^{-1}})$ **A1**

(iii) $s=\int -10t\,\mathrm{d}t=-5t^2\ (+c)$ **M1A1**

$s=1000$ for $t=0\Rightarrow c=1000$ **(M1)**

$s=-5t^2+1000$ **A1**

at $t=10,\ s=500\ (\mathrm{m})$ **AG**

Note: Accept use of definite integrals.

[6 marks]

(b) $\frac{\mathrm{d}t}{\mathrm{d}v}=\frac{1}{(-10-5v)}$ **A1**

[1 mark]

(c) **METHOD 1**

$t=\int \frac{1}{-10-5v}\,\mathrm{d}v=-\frac{1}{5}\ln(-10-5v)(+c)$ **M1A1**

Note: Accept equivalent forms using modulus signs.

$t=10,\ v=-100$

$10=-\frac{1}{5}\ln(490)+c$ **M1**

$c=10+\frac{1}{5}\ln(490)$ **A1**

$t=10+\frac{1}{5}\ln 490-\frac{1}{5}\ln(-10-5v)$ **A1**

Note: Accept equivalent forms using modulus signs.

$t=10+\frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ **AG**

Note: Accept use of definite integrals.

METHOD 2

$t = \int \frac{1}{-10-5v}\,\mathrm{d}v = -\frac{1}{5}\int \frac{1}{2+v}\,\mathrm{d}v = -\frac{1}{5}\ln\left|2+v\right| (+c)$ ***M1A1***

Note: Accept equivalent forms.

$t = 10,\ v = -100$

$10 = -\frac{1}{5}\ln\left|-98\right| + c$ ***M1***

Note: If $\ln(-98)$ is seen do not award further A marks.

$c = 10 + \frac{1}{5}\ln 98$ ***A1***

$t = 10 + \frac{1}{5}\ln 98 - \frac{1}{5}\ln\left|2+v\right|$ ***A1***

Note: Accept equivalent forms.

$t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

[5 marks]

(d) $5(t-10) = \ln\frac{98}{(-2-v)}$

$\frac{2+v}{98} = -\mathrm{e}^{-5(t-10)}$ ***(M1)***

$v = -2 - 98\mathrm{e}^{-5(t-10)}$ ***A1***

[2 marks]

(e) $\frac{\mathrm{d}s}{\mathrm{d}t} = -2 - 98\mathrm{e}^{-5(t-10)}$

$s = -2t + \frac{98}{5}\mathrm{e}^{-5(t-10)} (+k)$ ***M1A1***

at $t = 10,\ s = 500 \Rightarrow 500 = -20 + \frac{98}{5} + k \Rightarrow k = 500.4$ ***M1A1***

$s = -2t + \frac{98}{5}\mathrm{e}^{-5(t-10)} + 500.4$ ***A1***

Note: Accept use of definite integrals.

[5 marks]

continued…

(f) $t = 250$ for $s = 0$ ***(M1)A1***

[2 marks]

Total [21 marks]

Question 38

(a) (i) area $= \int_2^4 \sqrt{y-2}\,dy$ ***M1A1***

(ii) $= 1.886$ (4 sf only) ***A1***

[3 marks]

(b) volume $= \pi\int_2^4 (y-2)\,dy$ ***(M1)***

$= \pi\left[\frac{y^2}{2} - 2y\right]_2^4$ ***(A1)***

$= 2\pi$ (exact only) ***A1***

[3 marks]

Total [6 marks]

Question 39

(a) $t_1 = 1.77(\text{s}) \left(= \sqrt{\pi}(\text{s})\right)$ and $t_2 = 2.51(\text{s}) \left(= \sqrt{2\pi}(\text{s})\right)$ ***A1A1***

[2 marks]

(b) (i) attempting to find (graphically or analytically) the first $t_{\max}$ ***(M1)***

$t = 1.25(\text{s}) \left(= \sqrt{\frac{\pi}{2}}(\text{s})\right)$ ***A1***

(ii) attempting to find (graphically or analytically) the first $t_{\min}$ ***(M1)***

$t = 2.17(\text{s}) \left(= \sqrt{\frac{3\pi}{2}}(\text{s})\right)$ ***A1***

[4 marks]

(c) distance travelled $= \left|\int_{1.772...}^{2.506...} 1 - e^{-\sin t^2}\,dt\right|$ (or equivalent) ***(M1)***

$= 0.711(\text{m})$ ***A1***

[2 marks]

Total [8 marks]

Question 40

(a) $f'(x) = 30e^{-\frac{x^2}{400}} \cdot -\frac{2x}{400} \left(= -\frac{3x}{20} e^{-\frac{x^2}{400}} \right)$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f''(x) = -\frac{3}{20} e^{-\frac{x^2}{400}} + \frac{3x^2}{4000} e^{-\frac{x^2}{400}} \left(= \frac{3}{20} e^{-\frac{x^2}{400}} \left(\frac{x^2}{200} - 1 \right) \right)$ ***M1A1***

Note: Award ***M1*** for attempting to use the product rule.

[4 marks]

(b) the roof function has maximum gradient when $f''(x) = 0$ ***(M1)***

Note: Award ***(M1)*** for attempting to find $f''\left(-\sqrt{200}\right)$.

EITHER

$= 0$ ***A1***

OR

$f''(x) = 0 \Rightarrow x = \pm\sqrt{200}$ ***A1***

THEN

valid argument for maximum such as reference to an appropriate graph or change in the sign of $f''(x)$ *eg* $f''(-15) = 0.010...(> 0)$ and $f''(-14) = -0.001...(< 0)$ ***R1***

$\Rightarrow x = -\sqrt{200}$ ***AG***

[3 marks]

(c) $A = 2a \cdot 30\mathrm{e}^{-\frac{a^2}{400}} \left(= 60a\mathrm{e}^{-\frac{a^2}{400}} = -400 f'(a) \right)$ ***(M1)(A1)***

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}a} = 60a\mathrm{e}^{-\frac{a^2}{400}} \cdot -\frac{a}{200} + 60\mathrm{e}^{-\frac{a^2}{400}} = 0 \Rightarrow a = \sqrt{200} \quad \left(-400 f''(a) = 0 \Rightarrow a = \sqrt{200}\right)$

M1A1

OR

by symmetry *eg* $a = -\sqrt{200}$ found in (b) or $A_{\max}$ coincides with $f''(a) = 0$ ***R1***

$\Rightarrow a = \sqrt{200}$ ***A1***

THEN

$A_{\max} = 60 \cdot \sqrt{200}\mathrm{e}^{-\frac{200}{400}}$ ***M1***

$= 600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$ ***AG***

[5 marks]

(d) (i) perimeter $= 4a + 60\mathrm{e}^{-\frac{a^2}{400}}$ ***A1A1***

Note: Condone use of x.

(ii) $I(a) = \frac{4a + 60\mathrm{e}^{-\frac{a^2}{400}}}{60a\mathrm{e}^{-\frac{a^2}{400}}}$ ***(A1)***

graphing $I(a)$ or other valid method to find the minimum ***(M1)***

$a = 12.6$ ***A1***

(iii) area under roof $= \int_{-20}^{20} 30\mathrm{e}^{-\frac{x^2}{400}}\,\mathrm{d}x$ ***M1***

$= 896.18\ldots$ ***(A1)***

area of living space $= 60 \cdot (12.6\ldots) \cdot \mathrm{e}^{-\frac{(12.6\ldots)^2}{400}} = 508.56\ldots$ ***(A1)***

percentage of empty space $= 43.3\%$ ***A1***

[9 marks]

Total [21 marks]

Question 41

(a) $v=\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{e}^{-t}}{2-\mathrm{e}^{-t}}\left(=\frac{1}{2\mathrm{e}^{t}-1}\text{ or }-1+\frac{2}{2-\mathrm{e}^{-t}}\right)$ ***M1A1***

[2 marks]

(b) $a=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{-\mathrm{e}^{-t}\left(2-\mathrm{e}^{-t}\right)-\mathrm{e}^{-t}\times\mathrm{e}^{-t}}{\left(2-\mathrm{e}^{-t}\right)^2}\left(=\frac{-2\mathrm{e}^{-t}}{\left(2-\mathrm{e}^{-t}\right)^2}\right)$ ***M1A1***

Note: If simplified in part (a) award ***(M1)A1*** for $a=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{-2\mathrm{e}^{t}}{\left(2\mathrm{e}^{t}-1\right)^2}$.

Note: Award ***M1A1*** for $a=-\mathrm{e}^{-t}\left(2-\mathrm{e}^{-t}\right)^{-2}\left(\mathrm{e}^{-t}\right)-\mathrm{e}^{-t}\left(2-\mathrm{e}^{-t}\right)^{-1}$.

[2 marks]

(c) $a=-2\ (\mathrm{ms}^{-2})$ ***A1***

[1 mark]

Question 42

(a) valid method *eg*, sketch of curve or critical values found **(M1)**

$x < -2.24,\ x > 2.24,$ **A1**

$-1 < x < 0.8$ **A1**

Note: Award ***M1A1A0*** for correct intervals but with inclusive inequalities.

[3 marks]

(b) (i) $(1.67, -5.14), (-1.74, -3.71)$ **A1A1**

Note: Award ***A1A0*** for any two correct terms.

(ii) $f'(x) = 4x^3 + 0.6x^2 - 11.6x - 1$

$f''(x) = 12x^2 + 1.2x - 11.6 = 0$ **(M1)**

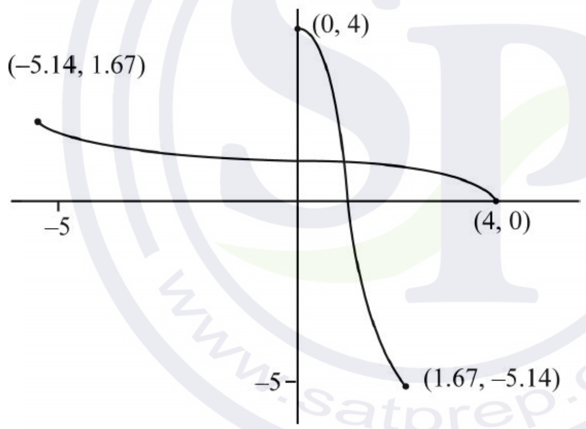
$-1.03,\ 0.934$ **A1A1**

Note: ***M1*** should be awarded if graphical method to find zeros of $f''(x)$ or turning points of $f'(x)$ is shown.

[5 marks]

(c) (i) 1.67 **A1**

(ii)



M1A1A1

Note: Award ***M1*** for reflection of their $y = f(x)$ in the line $y = x$ provided their f is one-one.

A1 for (0, 4), (4,0) (Accept axis intercept values) ***A1*** for the other two sets of coordinates of other end points

(iii) $x = f(1)$ **M1**

$= -1.6$ **A1**

(d) (i) $y = 2\sin(x-1) - 3$

$x = 2\sin(y-1) - 3$ ***(M1)***

$\left(g^{-1}(x) =\right) \arcsin\left(\frac{x+3}{2}\right) + 1$ ***A1***

$-5 \le x \le -1$ ***A1A1***

Note: Award ***A1*** for −5 and −1, and ***A1*** for correct inequalities if numbers are reasonable.

(ii) $f^{-1}(g(x)) < 1$

$g(x) > -1.6$ ***(M1)***

$x > g^{-1}(-1.6) = 1.78$ ***(A1)***

Note: Accept = in the above.

$1.78 < x \le \frac{\pi}{2} + 1$ ***A1A1***

Note: ***A1*** for $x > 1.78$ (allow ≥) and ***A1*** for $x \le \frac{\pi}{2} + 1$.

[8 marks]

Total [22 marks]

Question 43

(a) $a^2 = 5 - 1$ ***(M1)***

$a = 2$ ***A1***

[2 marks]

(b) $2y\frac{dy}{dx} - \left(2x\frac{dy}{dx} + 2y\right) = -e^x$ ***M1A1A1A1***

Note: Award ***M1*** for an attempt at implicit differentiation, ***A1*** for each part.

$\frac{dy}{dx} = \frac{2y - e^x}{2(y-x)}$ ***AG***

[4 marks]

(c) at $x = 0$, $\frac{dy}{dx} = \frac{3}{4}$ ***(A1)***

finding the negative reciprocal of a number ***(M1)***

gradient of normal is $-\frac{4}{3}$

$y = -\frac{4}{3}x + 2$ ***A1***

[3 marks]

(d) substituting linear expression **(M1)**

$\left(-\frac{4}{3}x+2\right)^2 - 2x\left(-\frac{4}{3}x+2\right) + e^x - 5 = 0$ or equivalent

$x = 1.56$ **(M1)A1**

$y = -0.0779$ **A1**

$(1.56, -0.0779)$

[4 marks]

(e) $\frac{dv}{dx} = 3y^2\frac{dy}{dx}$ **M1A1**

$\frac{dv}{dx} = 3 \times 4 \times \frac{3}{4} = 9$ **A1**

[3 marks]

Total [16 marks]

Question 44

(a) $3x^2 + 3y^2\frac{dy}{dx} = 4\left(y + x\frac{dy}{dx}\right)$ **M1A1**

$(3y^2 - 4x)\frac{dy}{dx} = 4y - 3x^2$ **A1**

$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$ **AG**

[3 marks]

(b) $\frac{dy}{dx} = 0 \Rightarrow 4y - 3x^2 = 0$ **(M1)**

substituting $x = k$ and $y = \frac{3}{4}k^2$ into $x^3 + y^3 = 4xy$ **M1**

$k^3 + \frac{27}{64}k^6 = 3k^3$ **A1**

attempting to solve $k^3 + \frac{27}{64}k^6 = 3k^3$ for k **(M1)**

$k = 1.68\left(= \frac{4}{3}\sqrt[3]{2}\right)$ **A1**

Note: Condone substituting $y = \frac{3}{4}x^2$ into $x^3 + y^3 = 4xy$ and solving for x.

[5 marks]

Total [8 marks]

Question 45

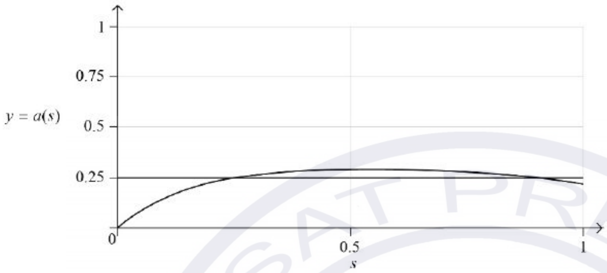
(a) $\frac{dv}{ds} = \frac{\cos s}{\sin^2 s + 1}$ ***M1A1***

$a = v\frac{dv}{ds}$ ***(M1)***

$a = \frac{\arctan(\sin s)\cos s}{\sin^2 s + 1}$ ***A1***

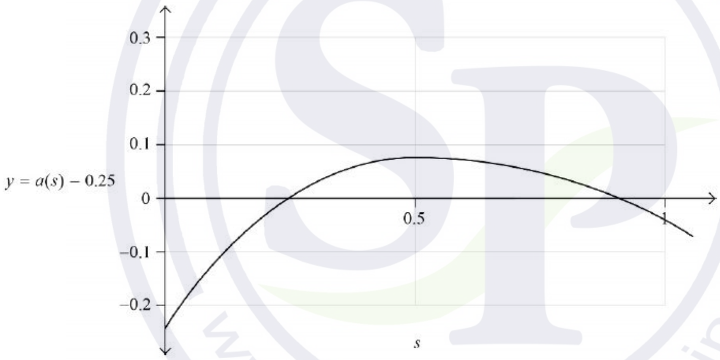
[4 marks]

(b) **EITHER**



(M1)

OR



(M1)

THEN

$s = 0.296, 0.918$ (m) ***A1***

[2 marks]

Total [6 marks]

Question 46

(a) **EITHER**

$$\alpha = \arctan\frac{7}{10} - \arctan\frac{5}{10} \quad (= 34.992\ldots^\circ - 26.5651\ldots^\circ)$$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{A\hat{P}T} - \mathrm{B\hat{P}T}$, ***(A1)*** for a correct $\mathrm{A\hat{P}T}$ and ***(A1)*** for a correct $\mathrm{B\hat{P}T}$.

OR

$$\alpha = \arctan 2 - \arctan\frac{10}{7} \quad (= 63.434\ldots^\circ - 55.008\ldots^\circ)$$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{P\hat{B}T} - \mathrm{P\hat{A}T}$, ***(A1)*** for a correct $\mathrm{P\hat{B}T}$ and ***(A1)*** for a correct $\mathrm{P\hat{A}T}$.

OR

$$\alpha = \arccos\left(\frac{125 + 149 - 4}{2 \times \sqrt{125} \times \sqrt{149}}\right)$$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of cosine rule, ***(A1)*** for a correct numerator and ***(A1)*** for a correct denominator.

THEN

$= 8.43^\circ$ ***A1***

[4 marks]

(b) **EITHER**

$$\tan\alpha = \frac{\frac{7}{x} - \frac{5}{x}}{1 + \left(\frac{7}{x}\right)\left(\frac{5}{x}\right)}$$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2}{x}}{1 + \frac{35}{x^2}}$$ ***M1***

OR

$$\tan\alpha = \frac{\frac{x}{5} - \frac{x}{7}}{1 + \left(\frac{x}{5}\right)\left(\frac{x}{7}\right)}$$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2x}{35}}{1 + \frac{x^2}{35}}$$ ***M1***

OR

$$\cos\alpha = \frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***M1A1***

Note: Award ***M1*** for either use of the cosine rule or use of $\cos(A - B)$.

$$\sin\alpha = \frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***A1***

$$\tan\alpha = \frac{\frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}}{\frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}}$$ ***M1***

THEN

$$\tan\alpha = \frac{2x}{x^2 + 35}$$ ***AG***

[4 marks]

(c) (i) $\frac{d}{dx}(\tan\alpha) = \frac{2\left(x^2+35\right)-(2x)(2x)}{\left(x^2+35\right)^2} \left(= \frac{70-2x^2}{\left(x^2+35\right)^2}\right)$ ***M1A1A1***

Note: Award ***M1*** for attempting product or quotient rule differentiation, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

(ii) **METHOD 1**

EITHER

$\frac{d}{dx}(\tan\alpha) = 0 \Rightarrow 70 - 2x^2 = 0$ ***(M1)***

$x = \sqrt{35}(\text{m})\left(= 5.9161\ldots(\text{m})\right)$ ***A1***

$\tan\alpha = \frac{1}{\sqrt{35}}(= 0.16903\ldots)$ ***(A1)***

OR

attempting to locate the stationary point on the graph of

$\tan\alpha = \frac{2x}{x^2+35}$ ***(M1)***

$x = 5.9161\ldots\ (\text{m})\left(= \sqrt{35}\,(\text{m})\right)$ ***A1***

$\tan\alpha = 0.16903\ldots\left(= \frac{1}{\sqrt{35}}\right)$ ***(A1)***

THEN

$\alpha = 9.59^\circ$ ***A1***

METHOD 2

EITHER

$\alpha = \arctan\left(\dfrac{2x}{x^2+35}\right) \Rightarrow \dfrac{d\alpha}{dx} = \dfrac{70-2x^2}{\left(x^2+35\right)^2+4x^2}$ ***M1***

$\dfrac{d\alpha}{dx} = 0 \Rightarrow x = \sqrt{35}(\text{m})(= 5.9161\ldots(\text{m}))$ ***A1***

OR

attempting to locate the stationary point on the graph of

$\alpha = \arctan\left(\dfrac{2x}{x^2+35}\right)$ ***(M1)***

$x = 5.9161\ldots(\text{m})\ \left(= \sqrt{35}\,(\text{m})\right)$ ***A1***

THEN

$\alpha = 0.1674\ldots\left(= \arctan\dfrac{1}{\sqrt{35}}\right)$ ***(A1)***

$= 9.59^\circ$ ***A1***

(iii) $\dfrac{d^2}{dx^2}(\tan\alpha) = \dfrac{\left(x^2+35\right)^2(-4x)-(2)(2x)\left(x^2+35\right)\left(70-2x^2\right)}{\left(x^2+35\right)^4}\ \left(=\dfrac{4x\left(x^2-105\right)}{\left(x^2+35\right)^3}\right)$

M1A1

substituting $x = \sqrt{35}\ (= 5.9161\ldots)$ into $\dfrac{d^2}{dx^2}(\tan\alpha)$ ***M1***

$\dfrac{d^2}{dx^2}(\tan\alpha) < 0\ (= -0.004829\ldots)$ and so $\alpha = 9.59^\circ$ is the maximum value of α ***R1***

α never exceeds 10° ***AG***

[11 marks]

(d) attempting to solve $\dfrac{2x}{x^2+35} \geq \tan 7^\circ$ ***(M1)***

Note: Award ***(M1)*** for attempting to solve $\dfrac{2x}{x^2+35} = \tan 7^\circ$.

$x = 2.55$ and $x = 13.7$ ***(A1)***

$2.55 \leq x \leq 13.7\ (\text{m})$ ***A1***

[3 marks]

Total [22 marks]

Question 47

(a) (i) $\dfrac{1}{4\left(\dfrac{e^x+e^{-x}}{2}\right)-2\left(\dfrac{e^x-e^{-x}}{2}\right)}$ ***(M1)***

$= \dfrac{1}{2\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)}$ ***(A1)***

$= \dfrac{1}{e^x+3e^{-x}}$ ***A1***

$= \dfrac{e^x}{e^{2x}+3}$ ***AG***

(ii) $u = e^x \Rightarrow du = e^x dx$ ***A1***

$\int \dfrac{e^x}{e^{2x}+3}dx = \int \dfrac{1}{u^2+3}du$ ***M1***

(when $x = 0$, $u = 1$ and when $x = \ln 3$, $u = 3$)

$\int_1^3 \dfrac{1}{u^2+3}du = \left[\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{u}{\sqrt{3}}\right)\right]_1^3$ ***M1A1***

$\left(= \left[\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{e^x}{\sqrt{3}}\right)\right]_0^{\ln 3}\right)$

$= \dfrac{\pi\sqrt{3}}{9} - \dfrac{\pi\sqrt{3}}{18}$ ***(M1)***

$= \dfrac{\pi\sqrt{3}}{18}$ ***A1***

[9 marks]

(b) (i) $(n+1)e^{2x} - 2ke^{x} + (n-1) = 0$ ***M1A1***

$$e^{x} = \frac{2k \pm \sqrt{4k^2 - 4\left(n^2 - 1\right)}}{2(n+1)}$$ ***M1***

$$x = \ln\left(\frac{k \pm \sqrt{k^2 - n^2 + 1}}{n+1}\right)$$ ***M1A1***

(ii) for two real solutions, we require $k > \sqrt{k^2 - n^2 + 1}$ ***R1***

and we also require $k^2 - n^2 + 1 > 0$ ***R1***

$k^2 > n^2 - 1$ ***A1***

$\Rightarrow k > \sqrt{n^2 - 1}$ $(k \in \mathbb{R}^+)$ ***AG***

[8 marks]

(c) (i) **METHOD 1**

$$t(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$t'(x) = \frac{\left(e^{x} + e^{-x}\right)^2 - \left(e^{x} - e^{-x}\right)^2}{\left(e^{x} + e^{-x}\right)^2}$$ ***M1A1***

$$t'(x) = \frac{\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2}{\left(\frac{e^{x} + e^{-x}}{2}\right)^2}$$ ***A1***

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$ ***AG***

METHOD 2

$$t'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$ ***M1A1***

$g'(x) = f(x)$ and $f'(x) = g(x)$ ***A1***

$$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$$ ***AG***

METHOD 3

$t(x) = \left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)^{-1}$

$t'(x) = 1 - \dfrac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$ ***M1A1***

$= 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$ ***A1***

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

METHOD 4

$t'(x) = \dfrac{g'(x)}{f(x)} - \dfrac{g(x)f'(x)}{[f(x)]^2}$ ***M1A1***

$g'(x) = f(x)$ and $f'(x) = g(x)$ gives $t'(x) = 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$ ***A1***

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

(ii) **METHOD 1**

$[f(x)]^2 > [g(x)]^2$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \dfrac{e^x + e^{-x}}{2}$ and $g(x) = \dfrac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 2

$[f(x)]^2 - [g(x)]^2 = 1$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \frac{e^x + e^{-x}}{2}$ and $g(x) = \frac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 3

$t'(x) = \frac{4}{(e^x + e^{-x})^2}$

$(e^x + e^{-x})^2 > 0$ ***M1A1***

$\frac{4}{(e^x + e^{-x})^2} > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

[6 marks]

Total [23 marks]

Question 48

METHOD 1

substituting for x and attempting to solve for y (or vice versa) ***(M1)***

$y = (\pm)\ 0.11821\ldots$ ***(A1)***

EITHER

$145x + 143y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \ \left(\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{145x}{143y}\right)$ ***M1A1***

OR

$145x\frac{\mathrm{d}x}{\mathrm{d}t} + 143y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***M1A1***

THEN

attempting to find $\frac{\mathrm{d}y}{\mathrm{d}t}$ $\left(\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{145\left(3.2\times10^{-3}\right)}{143\left((\pm)\ 0.11821\ldots\right)}\times\left(7.75\times10^{-5}\right)\right)$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

METHOD 2

$y = (\pm)\sqrt{\frac{1-72.5x^2}{71.5}}$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)0.0274\ldots$ ***(M1)(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = (\pm)0.0274\ldots\times7.75\times10^{-5}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

[6 marks]

Question 49

(a) attempting to solve either $2e^x - 1 = 0$ or $2e^x - 1 \neq 0$ for x **(M1)**

$D = \mathbb{R} \setminus \{-\ln 2\}$ (or equivalent *eg* $x \neq -\ln 2$) **A1**

> **Note:** Accept $D = \mathbb{R} \setminus \{-0.693\}$ or equivalent *eg* $x \neq -0.693$.

[2 marks]

(b) considering $\lim_{x \to -\ln 2} f(x)$ **(M1)**

$x = -\ln 2 \ (x = -0.693)$ **A1**

considering one of $\lim_{x \to -\infty} f(x)$ or $\lim_{x \to +\infty} f(x)$ **M1**

$\lim_{x \to -\infty} f(x) = -2 \Rightarrow y = -2$ **A1**

$\lim_{x \to +\infty} f(x) = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$ **A1**

> **Note:** Award ***A0A0*** for $y = -2$ and $y = -\frac{1}{2}$ stated without any justification.

[5 marks]

(c) $f'(x) = \dfrac{-e^x\left(2e^x - 1\right) - 2e^x\left(2 - e^x\right)}{\left(2e^x - 1\right)^2}$ ***M1A1A1***

$= -\dfrac{3e^x}{\left(2e^x - 1\right)^2}$ ***AG***

[3 marks]

(d) $f'(x) < 0$ (for all $x \in D$) $\Rightarrow f$ is (strictly) decreasing **R1**

> **Note:** Award ***R1*** for a statement such as $f'(x) \neq 0$ and so the graph of f has no turning points.

one branch is above the upper horizontal asymptote and the other branch is below the lower horizontal asymptote **R1**

f has an inverse ***AG***

$-\infty < x < -2 \cup -\frac{1}{2} < x < \infty$ **A2**

> **Note:** Award ***A2*** if the domain of the inverse is seen in either part (d) or in part (e).

[4 marks]

(e) $x = \frac{2 - e^y}{2e^y - 1}$ **M1**

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$2xe^y - x = 2 - e^y$ **M1**

$e^y(2x + 1) = x + 2$ **A1**

$f^{-1}(x) = \ln\left(\frac{x+2}{2x+1}\right)$ $\left(f^{-1}(x) = \ln(x + 2) - \ln(2x + 1)\right)$ **A1**

[4 marks]

(f) use of $V = \pi\int_a^b x^2\,dy$ ***(M1)***

$= \pi\int_0^1 \left(\ln\left(\frac{y+2}{2y+1}\right)\right)^2 dy$ ***(A1)(A1)***

Note: Award ***(A1)*** for the correct integrand and ***(A1)*** for the limits.

$= 0.331$ ***A1***

[4 marks]

Total [22 marks]

Question 50

(a) $y + x\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = 0$ ***M1A1A1***

Note: Award ***A1*** for the first two terms, ***A1*** for the third term and the 0.

$\frac{dy}{dx} = \frac{y^2}{1 - xy}$ ***A1***

Note: Accept $\frac{-y^2}{\ln y}$.

Note: Accept $\frac{-y}{x - \frac{1}{y}}$.

[4 marks]

(b) $m_T = \frac{e^2}{1 - e \times \frac{2}{e}}$ ***(M1)***

$m_T = -e^2$ ***(A1)***

$y - e = -e^2x + 2e$

$-e^2x - y + 3e = 0$ or equivalent ***A1***

Note: Accept $y = -7.39x + 8.15$.

[3 marks]

Total [7 marks]

Question 51

(a) **METHOD 1**

$2\arcsin(x-1)-\frac{\pi}{4}=\frac{\pi}{4}$ ***(M1)***

$x=1+\frac{1}{\sqrt{2}}(=1.707\ldots)$ ***(A1)***

$\int_0^{1+\frac{1}{\sqrt{2}}}\frac{\pi}{4}-\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1A1***

Note: Award ***M1*** for an attempt to find the difference between two functions, ***A1*** for all correct.

METHOD 2

when $x=0,\ y=\frac{-5\pi}{4}\ (=-3.93)$ ***A1***

$x=1+\sin\left(\frac{4y+\pi}{8}\right)$ ***M1A1***

Note: Award ***M1*** for an attempt to find the inverse function.

$\int_{\frac{-5\pi}{4}}^{\frac{\pi}{4}}\left(1+\sin\left(\frac{4y+\pi}{8}\right)\right)\mathrm{d}y$ ***A1***

METHOD 3

$\left|\int_0^{1.38\ldots}\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x\right|+\int_0^{1.71\ldots}\frac{\pi}{4}\mathrm{d}x-\int_{1.38\ldots}^{1.71\ldots}\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1A1A1A1***

Note: Award ***M1*** for considering the area below the x-axis and above the x-axis and ***A1*** for each correct integral.

[4 marks]

(b) area = 3.30 (square units) ***A2***

[2 marks]

Total [6 marks]

Question 52

(a) area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ **M1A1**

$V =$ area of segment $\times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ **A1**

[3 marks]

(b) **METHOD 1**

$\frac{dV}{dt} = \frac{5}{4}(1 - \cos\theta)\frac{d\theta}{dt}$ **M1A1**

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{d\theta}{dt}$ **(M1)**

$\frac{d\theta}{dt} = 0.00128 \text{ (rad } s^{-1})$ **A1**

METHOD 2

$\frac{d\theta}{dt} = \frac{d\theta}{dV} \times \frac{dV}{dt}$ **(M1)**

$\frac{dV}{d\theta} = \frac{5}{4}(1 - \cos\theta)$ **A1**

$\frac{d\theta}{dt} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ **(M1)**

$\frac{d\theta}{dt} = 0.00128 \left(\frac{4}{3125}\right) \text{(rad } s^{-1})$ **A1**

[4 marks]

Total [7 marks]

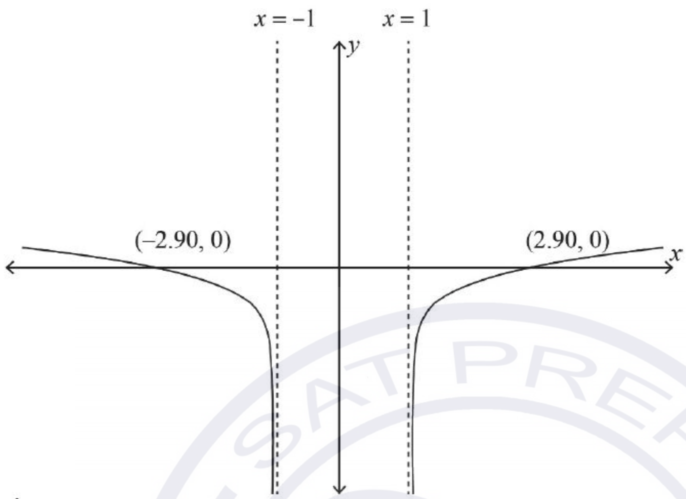
Question 53

(a) $x^2 - 1 > 0$ *(M1)*

$x < -1$ or $x > 1$ *A1*

[2 marks]

(b)



shape *A1*

$x = 1$ and $x = -1$ *A1*

x-intercepts *A1*

[3 marks]

(c) **EITHER**

f is symmetrical about the y-axis *R1*

OR

$f(-x) = f(x)$ *R1*

[1 mark]

(d) **EITHER**

f is not one-to-one function *R1*

OR

horizontal line cuts twice *R1*

Note: Accept any equivalent correct statement.

[1 mark]

(e) $x = -1 + \ln\left(\sqrt{y^2 - 1}\right)$ **M1**

$e^{2x+2} = y^2 - 1$ **M1**

$g^{-1}(x) = \sqrt{e^{2x+2} + 1},\ x \in \mathbb{R}$ **A1A1**

[4 marks]

(f) $g'(x) = \dfrac{1}{\sqrt{x^2 - 1}} \times \dfrac{2x}{2\sqrt{x^2 - 1}}$ **M1A1**

$g'(x) = \dfrac{x}{x^2 - 1}$ **A1**

[3 marks]

(g) (i) $g'(x) = \dfrac{x}{x^2 - 1} = 0 \Rightarrow x = 0$ **M1**

which is not in the domain of g (hence no solutions to $g'(x) = 0$) **R1**

(ii) $\left(g^{-1}\right)'(x) = \dfrac{e^{2x+2}}{\sqrt{e^{2x+2} + 1}}$ **M1**

as $e^{2x+2} > 0 \Rightarrow \left(g^{-1}\right)'(x) > 0$ so no solutions to $\left(g^{-1}\right)'(x) = 0$ **R1**

Note: Accept: equation $e^{2x+2} = 0$ has no solutions.

[4 marks]

Total [18 marks]

Question 54

(a) **METHOD 1**

$4x^2 + y^2 = 7$

$8x + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(M1)(A1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{4x}{y}$

Note: Award ***M1A1*** for finding $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \dfrac{y}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\dfrac{\sqrt{3}}{4}(= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \dfrac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \dfrac{\sqrt{3}}{4}x + \dfrac{3\sqrt{3}}{4}\right)$ $\left(y = 0.433x + 1.30\right)$

METHOD 2

$4x^2 + y^2 = 7$

$y = \sqrt{7 - 4x^2}$ ***(M1)***

$\frac{dy}{dx} = -\frac{4x}{\sqrt{7-4x^2}}$ ***(A1)***

Note: Award ***M1A1*** for finding $\frac{dy}{dx} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{\sqrt{7-4x^2}}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right) \left(y = 0.433x + 1.30\right)$

[6 marks]

(b) Use of $V = \pi \int_0^{\frac{\sqrt{7}}{2}} y^2 dx$

$V = \pi \int_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) dx$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark. Do not condone absence of or multiples of π.

$= 19.4 \left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

Total [9 marks]

Question 55

EITHER

$x^2 = 2\sec\theta$

$2x\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec\theta\tan\theta$ ***M1A1***

$\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^4 - 4}}$

$\displaystyle = \int \frac{\sec\theta\tan\theta\mathrm{d}\theta}{2\sec\theta\sqrt{4\sec^2\theta - 4}}$ ***M1A1***

OR

$x = \sqrt{2}(\sec\theta)^{\frac{1}{2}}\left(= \sqrt{2}(\cos\theta)^{-\frac{1}{2}}\right)$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \dfrac{\sqrt{2}}{2}(\sec\theta)^{\frac{1}{2}}\tan\theta\left(= \dfrac{\sqrt{2}}{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta\right)$ ***M1A1***

$\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^4 - 4}}$

$\displaystyle = \int \frac{\sqrt{2}(\sec\theta)^{\frac{1}{2}}\tan\theta\mathrm{d}\theta}{2\sqrt{2}(\sec\theta)^{\frac{1}{2}}\sqrt{4\sec^2\theta - 4}}\left(= \int \frac{\sqrt{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta\mathrm{d}\theta}{2\sqrt{2}(\cos\theta)^{-\frac{1}{2}}\sqrt{4\sec^2\theta - 4}}\right)$ ***M1A1***

THEN

$\displaystyle = \frac{1}{2}\int \frac{\tan\theta\mathrm{d}\theta}{2\tan\theta}$ ***(M1)***

$\displaystyle = \frac{1}{4}\int \mathrm{d}\theta$

$= \dfrac{\theta}{4} + c$ ***A1***

$x^2 = 2\sec\theta \Rightarrow \cos\theta = \dfrac{2}{x^2}$ ***M1***

te: This ***M1*** may be seen anywhere, including a sketch of an appropriate triangle.

so $\dfrac{\theta}{4} + c = \dfrac{1}{4}\arccos\left(\dfrac{2}{x^2}\right) + c$ ***AG***

[7 marks]

Question 56

(a) (i) attempt to use quotient rule or product rule **M1**

$$f'(x) = \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x}\left(= \frac{1}{2\sqrt{x}\sin x} - \frac{\sqrt{x}\cos x}{\sin^2 x}\right)$$ **A1A1**

Note: Award **A1** for $\frac{1}{2\sqrt{x}\sin x}$ or equivalent and **A1** for $-\frac{\sqrt{x}\cos x}{\sin^2 x}$ or equivalent.

setting $f'(x) = 0$ **M1**

$\frac{\sin x}{2\sqrt{x}} - \sqrt{x}\cos x = 0$

$\frac{\sin x}{2\sqrt{x}} = \sqrt{x}\cos x$ or equivalent **A1**

$\tan x = 2x$ **AG**

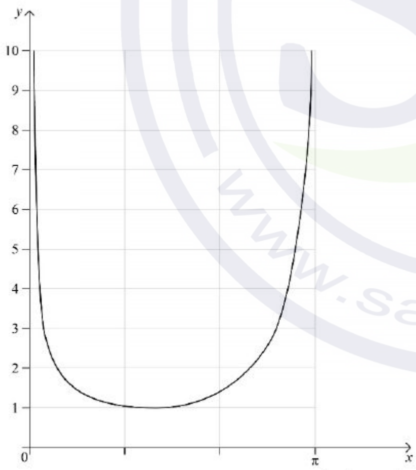
(ii) $x = 1.17$

$0 < x \le 1.17$ **A1A1**

Note: Award **A1** for $0 < x$ and **A1** for $x \le 1.17$. Accept $x < 1.17$.

[7 marks]

(b)



concave up curve over correct domain with one minimum point above the *x*-axis. **A1**

approaches $x = 0$ asymptotically **A1**

approaches $x = \pi$ asymptotically **A1**

Note: For the final **A1** an asymptote must be seen, and π must be seen on the x-axis or in an equation.

[3 marks]

(c) $f'(x)\left(=\dfrac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\sqrt{x}\cos x}{\sin^2 x}\right)=1$ **(A1)**

attempt to solve for x **(M1)**

$x = 1.96$ **A1**

$y = f(1.96...)$

$= 1.51$ **A1**

[4 marks]

(d) $V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{x\,\mathrm{d}x}{\sin^2 x}$ **(M1)(A1)**

Note: ***M1*** is for an integral of the correct squared function (with or without limits and/or π).

$= 2.68 (= 0.852\pi)$ **A1**

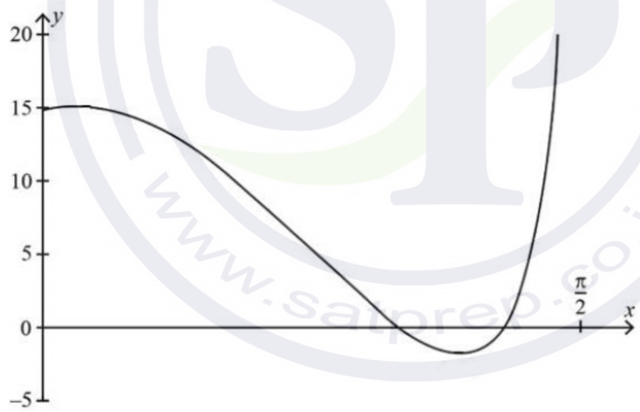
[3 marks]

Total [17 marks]

Question 57

(a) (i) $f'(x) = 4\sin x\cos x + 14\cos 2x + \sec^2 x$ (or equivalent) **(M1)A1**

(ii)



A1A1A1A1

Note: Award ***A1*** for correct behaviour at $x = 0$, ***A1*** for correct domain and correct behaviour for $x \to \frac{\pi}{2}$, ***A1*** for two clear intersections with x-axis and minimum point, ***A1*** for clear maximum point.

(iii) $x = 0.0736$ **A1**

$x = 1.13$ **A1**

[8 marks]

(b) (i) attempt to write $\sin x$ in terms of u only ***(M1)***

$\sin x = \frac{u}{\sqrt{1 + u^2}}$ **A1**

(ii) $\cos x = \frac{1}{\sqrt{1 + u^2}}$ ***(A1)***

attempt to use $\sin 2x = 2\sin x \cos x \left(= 2\frac{u}{\sqrt{1 + u^2}}\frac{1}{\sqrt{1 + u^2}} \right)$ ***(M1)***

$\sin 2x = \frac{2u}{1 + u^2}$ **A1**

(iii) $2\sin^2 x + 7\sin 2x + \tan x - 9 = 0$

$\frac{2u^2}{1 + u^2} + \frac{14u}{1 + u^2} + u - 9 \, (= 0)$ **M1**

$\frac{2u^2 + 14u + u(1 + u^2) - 9(1 + u^2)}{1 + u^2} = 0$ (or equivalent) **A1**

$u^3 - 7u^2 + 15u - 9 = 0$ **AG**

[7 marks]

(c) $u = 1$ or $u = 3$ ***(M1)***

$x = \arctan(1)$ **A1**

$x = \arctan(3)$ **A1**

Note: Only accept answers given the required form.

[3 marks]

Total [18 marks]

Question 58

(a) attempt to solve $v(t) = 0$ for t or equivalent ***(M1)***

$t_1 = 0.441(\text{s})$ **A1**

[2 marks]

(b) (i) $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{e}^{-t} - 16t\mathrm{e}^{-2t} + 16t^2\mathrm{e}^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

(ii) $a(t_1) = -2.28\left(\text{ms}^{-2}\right)$ **A1**

[3 marks]

Total [5 marks]

Question59

(a) attempt at implicit differentiation *M1*

$1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})\sin(xy) = 0$ *A1M1A1*

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{dy}{dx} = -1 - y\sin(xy)$ or equivalent *A1*

$\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$ *AG*

[5 marks]

(b) (i) **EITHER**

when $xy = -\frac{\pi}{2}$, $\cos xy = 0$ *M1*

$\Rightarrow x + y = 0$ *(A1)*

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent *M1*

$x - \frac{\pi}{2x} = 0$ *(A1)*

THEN

therefore $x^2 = \frac{\pi}{2}\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ *A1*

$P\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right), Q\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ or $P(1.25, -1.25)$, $Q(-1.25, 1.25)$ *A1*

(ii) $m_1 = -\left(\frac{1 - \sqrt{\frac{\pi}{2}} \times -1}{1 + \sqrt{\frac{\pi}{2}} \times -1}\right)$ *M1A1*

$m_2 = -\left(\frac{1 + \sqrt{\frac{\pi}{2}} \times -1}{1 - \sqrt{\frac{\pi}{2}} \times -1}\right)$ *A1*

$m_1 m_2 = 1$ *AG*

Note: Award ***M1A0A0*** if decimal approximations are used.

Note: No ***FT*** applies.

[7 marks]

(c) equate derivative to -1 **M1**

$(y-x)\sin(xy)=0$ ***(A1)***

$y=x,\ \sin(xy)=0$ **R1**

in the first case, attempt to solve $2x=\cos\left(x^2\right)$ **M1**

$(0.486, 0.486)$ **A1**

in the second case, $\sin(xy)=0 \Rightarrow xy=0$ and $x+y=1$ ***(M1)***

$(0,1), (1,0)$ **A1**

[7 marks]

Total [19 marks]

Question 60

(a) $2x^3-3x+1=Ax\left(x^2+1\right)+Bx+C$

$A=2,\ C=1,$ **A1**

$A+B=-3 \Rightarrow B=-5$ **A1**

[2 marks]

(b) $\int \frac{2x^3-3x+1}{x^2+1}\mathrm{d}x=\int\left(2x-\frac{5x}{x^2+1}+\frac{1}{x^2+1}\right)\mathrm{d}x$ **M1M1**

Note: Award ***M1*** for dividing by (x^2+1) to get $2x$, ***M1*** for separating the $5x$ and 1.

$=x^2-\frac{5}{2}\ln\left(x^2+1\right)+\arctan x(+c)$ ***(M1)A1A1***

Note: Award ***(M1)A1*** for integrating $\frac{5x}{x^2+1}$, ***A1*** for the other two terms.

[5 marks]

Total [7 marks]

Question 61

(a) differentiating implicitly: **M1**

$2xy + x^2\frac{dy}{dx} = -4y^3\frac{dy}{dx}$ **A1A1**

Note: Award ***A1*** for each side.

if $\frac{dy}{dx} = 0$ then either $x = 0$ or $y = 0$ **M1A1**

$x = 0 \Rightarrow$ two solutions for $y\left(y = \pm\sqrt[4]{5}\right)$ **R1**

$y = 0$ not possible (as $0 \neq 5$) **R1**

hence exactly two points **AG**

Note: For a solution that only refers to the graph giving two solutions at $x = 0$ and no solutions for $y = 0$ award ***R1*** only.

[7 marks]

(b) at $(2, 1)$ $4 + 4\frac{dy}{dx} = -4\frac{dy}{dx}$ **M1**

$\frac{dy}{dx} = -\frac{1}{2}$ ***(A1)***

gradient of normal is 2 **M1**

$1 = 4 + c$ ***(M1)***

equation of normal is $y = 2x - 3$ **A1**

[5 marks]

(c) substituting ***(M1)***

$x^2(2x-3) = 5 - (2x-3)^4$ or $\left(\frac{y+3}{2}\right)^2 y = 5 - y^4$ ***(A1)***

$x = 0.724$ **A1**

[3 marks]

(d) recognition of two volumes ***(M1)***

volume 1 $= \pi\int_1^{\sqrt[4]{5}} \frac{5-y^4}{y}\,dy (= 1.01\pi = 3.178\ldots)$ ***M1A1A1***

Note: Award ***M1*** for attempt to use $\pi\int x^2 dy$, ***A1*** for limits, ***A1*** for $\frac{5-y^4}{y}$. Condone omission of π at this stage.

volume 2

EITHER

$= \frac{1}{3}\pi \times 2^2 \times 4 (= 16.75\ldots)$ ***(M1)(A1)***

OR

$= \pi\int_{-3}^{1}\left(\frac{y+3}{2}\right)^2 dy \left(= \frac{16\pi}{3} = 16.75...\right)$ ***(M1)(A1)***

THEN

total volume $= 19.9$ **A1**

[7 marks]

Total [22 marks]

Question 62

$$f'(x) = \int\left(15\sqrt{x} + \frac{1}{(x+1)^2}\right)\mathrm{d}x = 10x^{\frac{3}{2}} - \frac{1}{x+1} \ (+\ c)$$ ***(M1)A1A1***

e: ***A1*** for first term, ***A1*** for second term. Withhold one ***A1*** if extra terms are seen.

$$f(x) = \int\left(10x^{\frac{3}{2}} - \frac{1}{x+1} + c\right)\mathrm{d}x = 4x^{\frac{5}{2}} - \ln(x+1) + cx + d$$ ***A1***

e: Allow FT from incorrect $f'(x)$ if it is of the form $f'(x) = Ax^{\frac{3}{2}} + \frac{B}{x+1} + c$.

Accept $\ln|x+1|$.

attempt to use at least one boundary condition in their $f(x)$ ***(M1)***

$x = 0, y = -4$

$\Rightarrow d = -4$ ***A1***

$x = 1, y = 0$

$\Rightarrow 0 = 4 - \ln 2 + c - 4$

$\Rightarrow c = \ln 2 (= 0.693)$ ***A1***

$f(x) = 4x^{\frac{5}{2}} - \ln(x+1) + x\ln 2 - 4$

[7 marks]

Question 63

$$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{\left(3(x+h)^3 - (x+h)\right) - \left(3x^3 - x\right)}{h}$$ ***M1***

$$= \frac{3\left((x+h)^3 - x^3\right) + \left(x - (x+h)\right)}{h}$$ ***(A1)***

$$= \frac{3h\left((x+h)^2 + x(x+h) + x^2\right) - h}{h}$$ ***A1***

cancelling h ***M1***

$$= 3\left((x+h)^2 + x(x+h) + x^2\right) - 1$$

then $\lim_{h\to 0}\left(3\left((x+h)^2 + x(x+h) + x^2\right) - 1\right)$

$$= 9x^2 - 1$$ ***A1***

te: Final ***A1*** dependent on all previous marks.

[5 marks]

$$\frac{f(x+h)-f(x)}{h}$$

$= \dfrac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ **M1**

$= \dfrac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ **(A1)**

$= \dfrac{9x^2h+9xh^2+3h^3-h}{h}$ **A1**

cancelling h **M1**

$= 9x^2+9xh+3h^2-1$

then $\lim\limits_{h\to 0}\left(9x^2+9xh+3h^2-1\right)$

$= 9x^2-1$ **A1**

Question 64

attempt to integrate a to find v **M1**

$v=\int a\ \mathrm{d}t=\int(2t-1)\ \mathrm{d}t$

$=t^2-t+c$ **A1**

$s=\int v\ \mathrm{d}t=\int\left(t^2-t+c\right)\mathrm{d}t$

$=\dfrac{t^3}{3}-\dfrac{t^2}{2}+ct+d$ **A1**

attempt at substitution of given values **(M1)**

at $t=6,\ 18.25=72-18+6c+d$

at $t=15,\ 922.75=1125-112.5+15c+d$

solve simultaneously: **(M1)**

$c=-6$; $d=0.25$ **A1**

$\Rightarrow s=\dfrac{t^3}{3}-\dfrac{t^2}{2}-6t+\dfrac{1}{4}$

[6 marks]

Question 65

METHOD 1

equation of tangent is $y=22.167...\,x-14.778...$ **OR** $y-7.389...=22.167...\,(x-1)$

(M1)(A1)

meets the x-axis when $y=0$

$x=0.667$

meets x-axis at $(0.667,0)\left(=\left(\dfrac{2}{3},0\right)\right)$ **A1A1**

continue….

METHOD 2

Attempt to differentiate *(M1)*

$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$

when $x = 1, \frac{dy}{dx} = 3e^2$ *(M1)*

equation of the tangent is $y - e^2 = 3e^2(x-1)$

$y = 3e^2x - 2e^2$

meets x-axis at $x = \frac{2}{3}$

$\left(\frac{2}{3}, 0\right)$ ***A1A1***

te: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates $(x, 0)$ given.

Total [4 marks]

Question 66

METHOD 1

write as $\int 1 \times (\ln x)^2 dx$ *(M1)*

$= x(\ln x)^2 - \int x \times \frac{2(\ln x)}{x} dx \left(= x(\ln x)^2 - \int 2\ln x\right)$ *M1A1*

$= x(\ln x)^2 - 2x\ln x + \int 2dx$ *(M1)(A1)*

$= x(\ln x)^2 - 2x\ln x + 2x + c$ *A1*

METHOD 2

let $u = \ln x$ *M1*

$\frac{du}{dx} = \frac{1}{x}$

$\int u^2 e^u du$ *A1*

$= u^2e^u - \int 2ue^u du$ *M1*

$= u^2e^u - 2ue^u + \int 2e^u du$ *A1*

$= u^2e^u - 2ue^u + 2e^u + c$

$= x(\ln x)^2 - 2x\ln x + 2x + c$ *M1A1*

METHOD 3

Setting up $u = \ln x$ and $\frac{dv}{dx} = \ln x$ *M1*

$\ln x(x\ln x - x) - \int(\ln x - 1)dx$ *M1A1*

$= x(\ln x)^2 - x\ln x - (x\ln x - x) + x + c$ *M1A1*

$= x(\ln x)^2 - 2x\ln x + 2x + c$ *A1*

Total [6 marks]

Question 67

volume $= \pi\int_0^9 \left(y^{\frac{1}{2}}+1\right)^2 \mathrm{d}y - \pi\int_1^9 (y-1)\,\mathrm{d}y$ ***(M1)(M1)(M1)(A1)(A1)***

: Award ***(M1)*** for use of formula for rotating about y-axis, ***(M1)*** for finding at least one inverse, ***(M1)*** for subtracting volumes, ***(A1)(A1)*** for each correct expression, including limits.

$= 268.6\ldots - 100.5\ldots\ (85.5\pi - 32\pi)$

$= 168\ (= 53.5\pi)$ ***A2***

Total [7 marks]

Question 68

(a) (i) $\mathrm{A}(7.47, 2.28)$ and $\mathrm{B}(43.4, -2.45)$ ***A1A1A1A1***

(ii) maximum speed is $2.45\ (\mathrm{m\,s^{-1}})$ ***A1***

[5 marks]

(b) (i) $v = 0 \Rightarrow t_1 = 25.1\ (\mathrm{s})$ ***(M1)A1***

(ii) $\int_0^{t_1} v\,\mathrm{d}t$ ***(M1)***

$= 41.0\ (\mathrm{m})$ ***A1***

(iii) $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ at $t = t_1 = 25.1$ ***(M1)***

$a = -0.200\ (\mathrm{m\,s^{-2}})$ ***A1***

Note: Accept $a = -0.2$.

[6 marks]

(c) attempt to integrate between 0 and 30 ***(M1)***

Note: An unsupported answer of 38.6 can imply integrating from 0 to 30.

EITHER

$\int_0^{30} |v|\,\mathrm{d}t$ ***(A1)***

OR

$41.0 - \int_{t_1}^{30} v\,\mathrm{d}t$ ***(A1)***

THEN

$= 43.3\ (\mathrm{m})$ ***A1***

[3 marks]

Total [14 marks]

Question 69

(a) (i) valid attempt to differentiate implicitly **(M1)**

$$4x = 3\sin^2 y \cos y \frac{dy}{dx}$$ **A1A1**

$$\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y}$$ **A1**

(ii) at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ **(M1)**

$$\Rightarrow \frac{dy}{dx} = -\frac{8}{3\sqrt{3}} (= -1.54)$$ **A1**

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ **(M1)A1**

Note: Accept $y = -1.54x + 3$.

[8 marks]

(b) $x = \sqrt{\frac{1}{2}\sin^3 y}$ **(M1)**

$$\int_0^{\pi} \sqrt{\frac{1}{2}\sin^3 y}\, dy$$ **(A1)**

$= 1.24$ **A1**

[3 marks]

(c) use of volume $= \int \pi x^2 dy$ **(M1)**

$$= \int_0^{\pi} \frac{1}{2}\pi \sin^3 y\, dy$$ **A1**

$$= \frac{1}{2}\pi \int_0^{\pi} \left(\sin y - \sin y \cos^2 y\right) dy$$

Note: Condone absence of limits up to this point.

reasonable attempt to integrate **(M1)**

$$= \frac{1}{2}\pi\left[-\cos y + \frac{1}{3}\cos^3 y\right]_0^{\pi}$$ **A1A1**

Note: Award ***A1*** for correct limits (not to be awarded if previous ***M1*** has not been awarded) and ***A1*** for correct integrand.

$$= \frac{1}{2}\pi\left(1 - \frac{1}{3}\right) - \frac{1}{2}\pi\left(-1 + \frac{1}{3}\right)$$ **A1**

$$= \frac{2\pi}{3}$$ ***AG***

[6 marks]

Total [17 marks]