Subject – Math(Higher Level)
Topic - Algebra
Year - Nov 2011 – Nov 2019
Paper 2

Question 1

[Maximum mark: 7]

The complex numbers z_1 and z_2 have arguments between 0 and π radians. Given that $z_1 z_2 = -\sqrt{3} + \mathrm{i}$ and $\frac{z_1}{z_2} = 2\mathrm{i}$, find the modulus and argument of z_1 and of z_2.

Question 2

[Maximum mark: 6]

(a) Find the set of values of x for which the series $\sum_{n=1}^{\infty}\left(\frac{2x}{x+1}\right)^n$ has a finite sum. *[4 marks]*

(b) Hence find the sum in terms of x. *[2 marks]*

Question 3

[Maximum mark: 7]

Given that $z = \frac{2-\mathrm{i}}{1+\mathrm{i}} - \frac{6+8\mathrm{i}}{u+\mathrm{i}}$, find the values of u, $u \in \mathbb{R}$, such that $\operatorname{Re} z = \operatorname{Im} z$.

Question 4

[Maximum mark: 16]

(a) In an arithmetic sequence the first term is 8 and the common difference is $\frac{1}{4}$. If the sum of the first $2n$ terms is equal to the sum of the next n terms, find n. *[9 marks]*

(b) If $a_1, a_2, a_3, \ldots$ are terms of a geometric sequence with common ratio $r \neq 1$, show that $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \ldots + (a_n - a_{n+1})^2 = \frac{a_1^{\,2}(1-r)(1-r^{2n})}{1+r}$. *[7 marks]*

Question 5

[Maximum mark: 13]

(a) Show that $\left| e^{i\theta} \right| = 1$. *[1 mark]*

Consider the geometric series $1 + \frac{1}{3}e^{i\theta} + \frac{1}{9}e^{2i\theta} + \ldots$.

(b) Write down the common ratio, z, of the series, and show that $|z| = \frac{1}{3}$. *[2 marks]*

(c) Find an expression for the sum to infinity of this series. *[2 marks]*

(d) Hence, show that $\sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta + \ldots = \frac{9\sin\theta}{10 - 6\cos\theta}$. *[8 marks]*

Question 6

[Maximum mark: 7]

Find the constant term in the expansion of $\left(x - \frac{2}{x}\right)^4 \left(x^2 + \frac{2}{x}\right)^3$.

Question 7

[Maximum mark: 7]

A team of 6 players is to be selected from 10 volleyball players, of whom 8 are boys and 2 are girls.

(a) In how many ways can the team be selected? *[2 marks]*

(b) In how many of these selections is exactly one girl in the team? *[3 marks]*

Question 8

[Maximum mark: 7]

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

(a) Find the first term and the common difference. *[4 marks]*

(b) Find the smallest value of n such that the sum of the first n terms is greater than 600. *[3 marks]*

Question 9

[Maximum mark: 6]

Fifteen boys and ten girls sit in a single line.

(a) In how many ways can they be seated in a single line so that the boys and girls are in two separate groups? *[3 marks]*

(b) Two boys and three girls are selected to go the theatre. In how many ways can this selection be made? *[3 marks]*

Question 10

[Maximum mark: 4]

Find the sum of all the multiples of 3 between 100 and 500.

Question 11

[Maximum mark: 6]

A metal rod 1 metre long is cut into 10 pieces, the lengths of which form a geometric sequence. The length of the longest piece is 8 times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece.

Question 12

[Maximum mark: 7]

Let $\omega = \cos\theta + \mathrm{i}\sin\theta$. Find, in terms of θ, the modulus and argument of $(1-\omega^2)^*$.

Question 13

[Maximum mark: 7]

A polynomial $p(x)$ with real coefficients is of degree five. The equation $p(x) = 0$ has a complex root $2+\mathrm{i}$. The graph of $y = p(x)$ has the x-axis as a tangent at $(2, 0)$ and intersects the coordinate axes at $(-1, 0)$ and $(0, 4)$.

Find $p(x)$ in factorised form with real coefficients.

Question 14

[Maximum mark: 7]

Three boys and three girls are to sit on a bench for a photograph.

(a) Find the number of ways this can be done if the three girls must sit together. *[3 marks]*

(b) Find the number of ways this can be done if the three girls must all sit apart. *[4 marks]*

Question 15

[Maximum mark: 6]

The arithmetic sequence $\{u_n : n \in \mathbb{Z}^+\}$ has first term $u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence $\{v_n : n \in \mathbb{Z}^+\}$ has first term $v_1 = 3$ and common ratio $r = 1.2$.

(a) Find an expression for $u_n - v_n$ in terms of n. *[2 marks]*

(b) Determine the set of values of n for which $u_n > v_n$. *[3 marks]*

(c) Determine the greatest value of $u_n - v_n$. Give your answer correct to four significant figures. *[1 mark]*

Question 16

[Maximum mark: 7]

Use the method of mathematical induction to prove that $5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$.

Question 17

[Maximum mark: 19]

(a) (i) Express the sum of the first n positive odd integers using sigma notation.

(ii) Show that the sum stated above is n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers. *[4 marks]*

(b) A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points.

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible. *[7 marks]*

Question 18

[Maximum mark: 7]

The fourth term in an arithmetic sequence is 34 and the tenth term is 76.

(a) Find the first term and the common difference. *[3]*

(b) The sum of the first n terms exceeds 5000. Find the least possible value of n. *[4]*

Question 19

[Maximum mark: 6]

A complex number z is given by $z = \dfrac{a+\mathrm{i}}{a-\mathrm{i}}$, $a \in \mathbb{R}$.

(a) Determine the set of values of a such that

(i) z is real;

(ii) z is purely imaginary. *[4]*

(b) Show that $|z|$ is constant for all values of a. *[2]*

Question 20

[Maximum mark: 4]

One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where a, $b \in \mathbb{R}$. Find the value of a and the value of b.

Question 21

[Maximum mark: 4]

Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

Question 22

[Maximum mark: 8]

Prove, by mathematical induction, that $7^{8n+3} + 2$, $n \in \mathbb{N}$, is divisible by 5.

Question 23

[Maximum mark: 8]

(a) Find the term in x^5 in the expansion of $(3x + A)(2x + B)^6$. *[4]*

Question 24

[Maximum mark: 6]

(a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7.

(ii) Express the above sum using sigma notation. *[4]*

An arithmetic sequence has first term 1000 and common difference of -6. The sum of the first n terms of this sequence is negative.

(b) Find the least value of n. *[2]*

Question 25

[Maximum mark: 6]

Find the coefficient of x^{-2} in the expansion of $(x-1)^3\left(\frac{1}{x}+2x\right)^6$.

Question 26

[Maximum mark: 20]

Consider $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $z \in \mathbb{C}$.

(a) Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, $n \in \mathbb{Z}^+$. *[7]*

Given $u = 1+\sqrt{3}\mathrm{i}$ and $v = 1-\mathrm{i}$,

(b) (i) express u and v in modulus-argument form;

(ii) hence find u^3v^4. *[4]*

The complex numbers u and v are represented by point A and point B respectively on an Argand diagram.

(c) Plot point A and point B on the Argand diagram. *[1]*

Point A is rotated through $\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point A′. Point B is rotated through $\frac{\pi}{2}$ in the clockwise direction about O to become point B′.

(d) Find the area of triangle OA′B′. *[3]*

Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b, c, d, e \in \mathbb{R}$,

(e) find the values of b, c, d and e. *[5]*

Question 27

[Maximum mark: 9]

The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

(a) Show that $d = \frac{a}{2}$. *[3]*

The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200.

(b) Find the least value of n for which this occurs. *[6]*

Question 28

[Maximum mark: 19]

(a) (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$.

(ii) Hence use De Moivre's theorem to prove

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

(iii) State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$. [6]

Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

(b) Find the value of r and the value of α. [4]

(c) Using (a) (ii) and your answer from (b) show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$. [4]

(d) Hence express $\sin 72^\circ$ in the form $\frac{\sqrt{a + b\sqrt{c}}}{d}$ where $a, b, c, d \in \mathbb{Z}$. [5]

Question 29

[Maximum mark: 6]

From a group of five males and six females, four people are chosen.

(a) Determine how many possible groups can be chosen. [2]

(b) Determine how many groups can be formed consisting of two males and two females. [2]

(c) Determine how many groups can be formed consisting of at least one female. [2]

Question 30

[Maximum mark: 5]

When $x^2 + 4x - b$ is divided by $x - a$ the remainder is 2.
Given that $a, b \in \mathbb{R}$, find the smallest possible value for b.

Question 31

[Maximum mark: 7]

Two distinct roots for the equation $z^4 - 10z^3 + az^2 + bz + 50 = 0$ are $c + \mathrm{i}$ and $2 + \mathrm{i}d$ where $a, b, c, d \in \mathbb{R}, d > 0$.

(a) Write down the other two roots in terms of c and d. [1]

(b) Find the value of c and the value of d. [6]

Question 32

[Maximum mark: 6]

Solve the simultaneous equations

$$\ln \frac{y}{x} = 2$$

$$\ln x^2 + \ln y^3 = 7.$$

Question 33

[Maximum mark: 6]

The sum of the second and third terms of a geometric sequence is 96.

The sum to infinity of this sequence is 500.

Find the possible values for the common ratio, r.

Question 34

[Maximum mark: 6]

The function f is defined as $f(x) = \sqrt{\dfrac{1-x}{1+x}}, -1 < x \le 1$.

Find the inverse function, f^{-1} stating its domain and range.

Question 35

[Maximum mark: 5]

Find the constant term in the expansion of $\left(4x^2 - \dfrac{3}{2x}\right)^{12}$.

Question 36

[Maximum mark: 18]

On the day of her birth, 1st January 1998, Mary's grandparents invested $\$x$ in a savings account. They continued to deposit $\$x$ on the first day of each month thereafter.
The account paid a fixed rate of 0.4% interest per month. The interest was calculated on the last day of each month and added to the account.
Let $\$A_n$ be the amount in Mary's account on the last day of the nth month, immediately after the interest had been added.

(a) Find an expression for A_1 and show that $A_2 = 1.004^2 x + 1.004x$. [2]

(b) (i) Write down a similar expression for A_3 and A_4.

(ii) Hence show that the amount in Mary's account the day before she turned 10 years old is given by $251(1.004^{120} - 1)x$. [6]

Question 37

[Maximum mark: 6]

The coefficient of x^2 in the expansion of $\left(\frac{1}{x} + 5x\right)^8$ is equal to the coefficient of x^4 in the expansion of $(a + 5x)^7$, $a \in \mathbb{R}$. Find the value of a.

Question 38

[Maximum mark: 5]

Given that $\log_{10}\left(\frac{1}{2\sqrt{2}}(p + 2q)\right) = \frac{1}{2}(\log_{10} p + \log_{10} q)$, $p > 0$, $q > 0$, find p in terms of q.

Question 39

[Maximum mark: 6]

In a trial examination session a candidate at a school has to take 18 examination papers including the physics paper, the chemistry paper and the biology paper. No two of these three papers may be taken consecutively. There is no restriction on the order in which the other examination papers may be taken.

Find the number of different orders in which these 18 examination papers may be taken.

Question 40

[Maximum mark: 4]

Boxes of mixed fruit are on sale at a local supermarket.
Box A contains 2 bananas, 3 kiwifruit and 4 melons, and costs $\$6.58$.
Box B contains 5 bananas, 2 kiwifruit and 8 melons and costs $\$12.32$.
Box C contains 5 bananas and 4 kiwifruit and costs $\$3.00$.

Find the cost of each type of fruit.

Question 41

[Maximum mark: 5]

In the quadratic equation $7x^2 - 8x + p = 0$, $(p \in \mathbb{Q})$, one root is three times the other root.
Find the value of p.

Question 42

[Maximum mark: 6]

Twelve students are to take an exam in advanced combinatorics.
The exam room is set out in three rows of four desks, with the invigilator at the front of the room, as shown in the following diagram.

INVIGILATOR

Desk 1	Desk 2	Desk 3	Desk 4
Desk 5	Desk 6	Desk 7	Desk 8
Desk 9	Desk 10	Desk 11	Desk 12

(a) Find the number of ways the twelve students may be arranged in the exam hall. [1]

Two of the students, Helen and Nicky, are suspected of cheating in a previous exam.

(b) Find the number of ways the students may be arranged if Helen and Nicky must sit so that one is directly behind the other (with no desk in between). For example Desk 5 and Desk 9. [2]

(c) Find the number of ways the students may be arranged if Helen and Nicky must not sit next to each other in the same row. [3]

Question 43

[Maximum mark: 15]

Phil takes out a bank loan of $\$150\,000$ to buy a house, at an annual interest rate of 3.5%. The interest is calculated at the end of each year and added to the amount outstanding.

(a) Find the amount Phil would owe the bank after 20 years. Give your answer to the nearest dollar. [3]

To pay off the loan, Phil makes annual deposits of $\$P$ at the end of every year in a savings account, paying an annual interest rate of 2%. He makes his first deposit at the end of the first year after taking out the loan.

(b) Show that the total value of Phil's savings after 20 years is $\dfrac{(1.02^{20}-1)P}{(1.02-1)}$. [3]

(c) Given that Phil's aim is to own the house after 20 years, find the value for P to the nearest dollar. [3]

David visits a different bank and makes a single deposit of $\$Q$, the annual interest rate being 2.8%.

(d) (i) David wishes to withdraw $\$5000$ at the end of each year for a period of n years. Show that an expression for the minimum value of Q is $\dfrac{5000}{1.028}+\dfrac{5000}{1.028^2}+\ldots+\dfrac{5000}{1.028^n}$.

(ii) Hence or otherwise, find the minimum value of Q that would permit David to withdraw annual amounts of $\$5000$ indefinitely. Give your answer to the nearest dollar. [6]

Question 44

[Maximum mark: 6]

Consider the complex number $z=\dfrac{2+7\mathrm{i}}{6+2\mathrm{i}}$.

(a) Express z in the form $a+\mathrm{i}b$, where $a, b \in \mathbb{Q}$. [2]

(b) Find the exact value of the modulus of z. [2]

(c) Find the argument of z, giving your answer to 4 decimal places. [2]

Question 45

[Maximum mark: 5]

The polynomial $x^4+px^3+qx^2+rx+6$ is exactly divisible by each of $(x-1)$, $(x-2)$ and $(x-3)$.

Find the values of p, q and r.

Question 46

[Maximum mark: 6]

(a) Express the binomial coefficient $\binom{3n+1}{3n-2}$ as a polynomial in n. [3]

(b) Hence find the least value of n for which $\binom{3n+1}{3n-2} > 10^6$. [3]

Question 47

[Maximum mark: 7]

Use mathematical induction to prove that $(1-a)^n > 1 - na$ for $\{n : n \in \mathbb{Z}^+, n \geq 2\}$ where $0 < a < 1$.

Question 48

[Maximum mark: 7]

The 3rd term of an arithmetic sequence is 1407 and the 10th term is 1183.

(a) Find the first term and the common difference of the sequence. [4]

(b) Calculate the number of positive terms in the sequence. [3]

Question 49

[Maximum mark: 6]

The equation $x^2 - 5x - 7 = 0$ has roots α and β. The equation $x^2 + px + q = 0$ has roots $\alpha + 1$ and $\beta + 1$. Find the value of p and the value of q.

Question 50

[Maximum mark: 8]

It is known that the number of fish in a given lake will decrease by 7% each year unless some new fish are added. At the end of each year, 250 new fish are added to the lake. At the start of 2018, there are 2500 fish in the lake.

(a) Show that there will be approximately 2645 fish in the lake at the start of 2020. [3]

(b) Find the approximate number of fish in the lake at the start of 2042. [5]

Question 51

[Maximum mark: 5]

Consider a geometric sequence with a first term of 4 and a fourth term of -2.916.

(a) Find the common ratio of this sequence. [3]

(b) Find the sum to infinity of this sequence. [2]

Question 52

[Maximum mark: 5]

Find the value of the constant term in the expansion of $x^4\left(x+\dfrac{3}{x^2}\right)^5$.

Question 53

[Maximum mark: 6]

Let $P(x)=2x^4-15x^3+ax^2+bx+c$, where $a,b,c\in\mathbb{R}$.

(a) Given that $(x-5)$ is a factor of $P(x)$, find a relationship between a, b and c. [2]

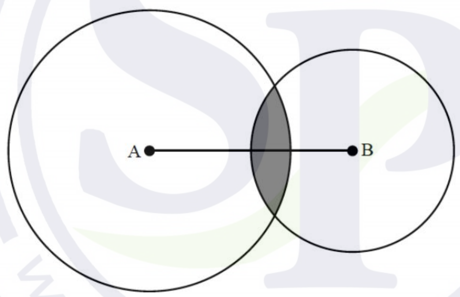
(b) Given that $(x-5)^2$ is a factor of $P(x)$, write down the value of $P'(5)$. [1]

(c) Given that $(x-5)^2$ is a factor of $P(x)$, and that $a=2$, find the values of b and c. [3]

Question 54

[Maximum mark: 6]

Boat A is situated 10km away from boat B, and each boat has a marine radio transmitter on board. The range of the transmitter on boat A is 7km, and the range of the transmitter on boat B is 5km. The region in which both transmitters can be detected is represented by the shaded region in the following diagram. Find the area of this region.



Question 55

[Maximum mark: 7]

Suppose that u_1 is the first term of a geometric series with common ratio r.
Prove, by mathematical induction, that the sum of the first n terms, S_n is given by
$S_n=\dfrac{u_1\left(1-r^n\right)}{1-r}$, where $n\in\mathbb{Z}^+$.

Question 56

[Maximum mark: 7]

(a) Find the roots of the equation $w^3=8\mathrm{i}$, $w\in\mathbb{C}$. Give your answers in Cartesian form. [4]

One of the roots w_1 satisfies the condition $\mathrm{Re}(w_1)=0$.

(b) Given that $w_1=\dfrac{z}{z-\mathrm{i}}$, express z in the form $a+b\mathrm{i}$ where $a,b\in\mathbb{Q}$. [3]

Question 57

[Maximum mark: 15]

Consider the polynomial $P(z) \equiv z^4 - 6z^3 - 2z^2 + 58z - 51$, $z \in \mathbb{C}$.

(a) Express $P(z)$ in the form $(z^2 + az + b)(z^2 + cz + d)$ where $a, b, c, d \in \mathbb{R}$. [7]

(b) Sketch the graph of $y = x^4 - 6x^3 - 2x^2 + 58x - 51$, stating clearly the coordinates of any maximum and minimum points and intersections with axes. [6]

(c) Hence, or otherwise, state the condition on $k \in \mathbb{R}$ such that all roots of the equation $P(z) = k$ are real. [2]

Question 58

[Maximum mark: 5]

Solve $z^2 = 4e^{\frac{\pi}{2}i}$, giving your answers in the form

(a) $re^{i\theta}$ where $r, \theta \in \mathbb{R}, r > 0$; [3]

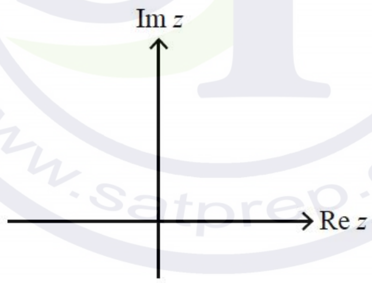
(b) $a + ib$ where $a, b \in \mathbb{R}$. [2]

Question 59

[Maximum mark: 7]

Let $z = a + bi$, $a, b \in \mathbb{R}^+$ and let $\arg z = \theta$.

(a) Show the points represented by z and $z - 2a$ on the following Argand diagram. [1]



(b) Find an expression in terms of θ for

(i) $\arg(z - 2a)$;

(ii) $\arg\left(\dfrac{z}{z-2a}\right)$. [3]

(c) Hence or otherwise find the value of θ for which $\mathrm{Re}\left(\dfrac{z}{z-2a}\right) = 0$. [3]

Question 60

[Maximum mark: 9]

(a) Solve the inequality $x^2 > 2x + 1$. [2]

(b) Use mathematical induction to prove that $2^{n+1} > n^2$ for $n \in \mathbb{Z}, n \geq 3$. [7]

Question 61

[Maximum mark: 17]

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.
The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.
The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

(a) Show that $abc = 8$. [5]

The values a, b, and c are consecutive terms in a geometric sequence.

(b) Show that one of the real roots is equal to 1. [3]

(c) Given that $q = 8d^2$, find the other two real roots. [9]

Question 62

[Maximum mark: 5]

A geometric sequence has $u_4 = -70$ and $u_7 = 8.75$. Find the second term of the sequence.

Question 63

[Maximum mark: 6]

Consider the expansion of $(2 + x)^n$, where $n \geq 3$ and $n \in \mathbb{Z}$.

The coefficient of x^3 is four times the coefficient of x^2. Find the value of n.

Question 64

[Maximum mark: 6]

Let $P(z) = az^3 - 37z^2 + 66z - 10$, where $z \in \mathbb{C}$ and $a \in \mathbb{Z}$.

One of the roots of $P(z) = 0$ is $3 + \mathrm{i}$. Find the value of a.

Question 65

[Maximum mark: 8]

Eight boys and two girls sit on a bench. Determine the number of possible arrangements, given that

(a) the girls do not sit together; [3]

(b) the girls do not sit on either end; [2]

(c) the girls do not sit on either end and do not sit together. [3]