Subject – Math(Higher Level)
Topic - Circular trigonometry
Year - Nov 2011 – Nov 2019
Paper 2

Question 1

[Maximum mark: 5]

(a) Given that $\arctan\frac{1}{2} - \arctan\frac{1}{3} = \arctan a$, $a \in \mathbb{Q}^+$, find the value of a. *[3 marks]*

(b) Hence, or otherwise, solve the equation $\arcsin x = \arctan a$. *[2 marks]*

Question 2

[Maximum mark: 8]

Consider a triangle ABC with $\hat{BAC} = 45.7^\circ$, $AB = 9.63$ cm and $BC = 7.5$ cm.

(a) By drawing a diagram, show why there are two triangles consistent with this information. *[2 marks]*

(b) Find the possible values of AC. *[6 marks]*

Question 3

[Maximum mark: 8]

Two discs, one of radius 8 cm and one of radius 5 cm, are placed such that they touch each other. A piece of string is wrapped around the discs. This is shown in the diagram below.

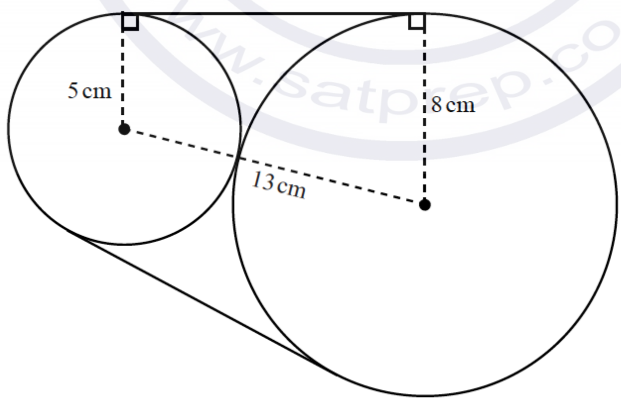


diagram not to scale

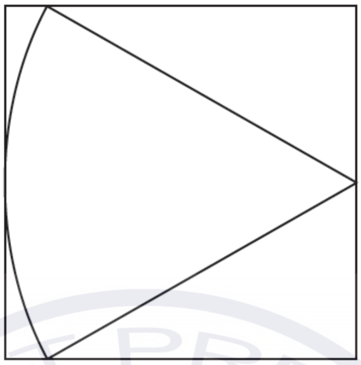
Calculate the length of string needed to go around the discs.

Question 4

[Maximum mark: 6]

A rectangle is drawn around a sector of a circle as shown. If the angle of the sector is 1 radian and the area of the sector is $7\,\text{cm}^2$, find the dimensions of the rectangle, giving your answers to the nearest millimetre.

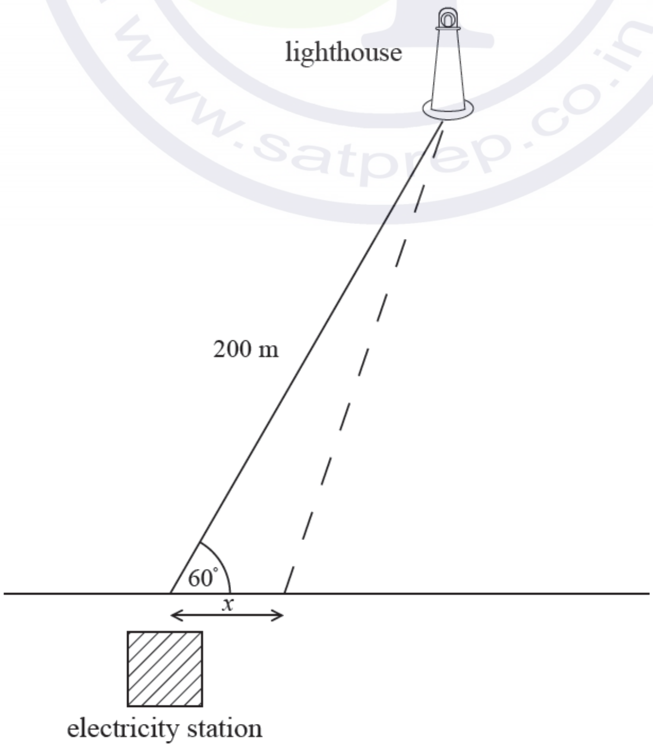
diagram not to scale



Question 5

[Maximum mark: 6]

An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60°. A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is x metres. This information is illustrated in the diagram below.



The cost of laying the cable along the sea bed is US$80 per metre, and the cost of laying it on land is US$20 per metre.

(a) Find, in terms of x, an expression for the cost of laying the cable. *[4 marks]*

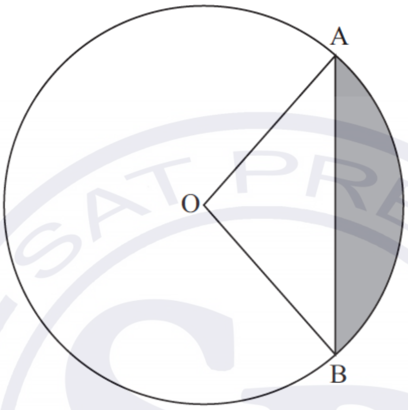
(b) Find the value of x, to the nearest metre, such that this cost is minimized. *[2 marks]*

Question 6

[Maximum mark: 5]

A circle of radius 4 cm, centre O, is cut by a chord [AB] of length 6 cm.

diagram not to scale



(a) Find $A\hat{O}B$, expressing your answer in radians correct to four significant figures. *[2 marks]*

(b) Determine the area of the shaded region. *[3 marks]*

Question 7

[Maximum mark: 6]

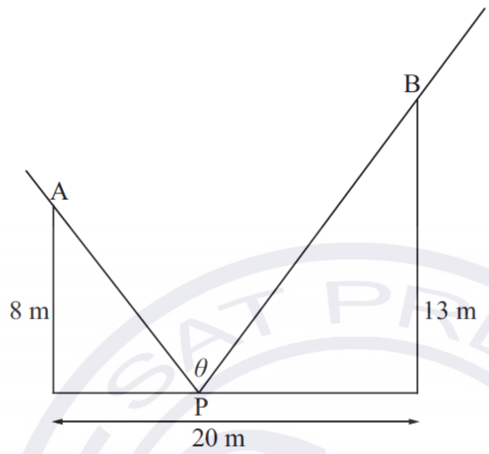
(a) Solve the equation $3\cos^2 x - 8\cos x + 4 = 0$, where $0 \le x \le 180°$, expressing your answer(s) to the nearest degree. *[3 marks]*

(b) Find the exact values of $\sec x$ satisfying the equation $3\sec^4 x - 8\sec^2 x + 4 = 0$. *[3 marks]*

Question 8

[Maximum mark: 19]

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \hat{APB}$, as shown in the diagram.



(a) Find an expression for θ in terms of x, where x is the distance of P from the base of the wall of height 8m. *[2 marks]*

(b) (i) Calculate the value of θ when $x = 0$.

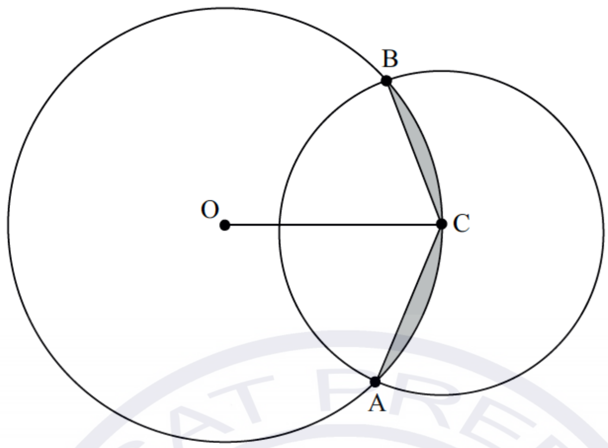
(ii) Calculate the value of θ when $x = 20$. *[2 marks]*

(c) Sketch the graph of θ, for $0 \le x \le 20$. *[2 marks]*

Question 9

[Maximum mark: 6]

The following diagram shows two intersecting circles of radii 4 cm and 3 cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.



Find:

(a) $B\hat{O}C$; *[2]*

(b) the area of the shaded region. *[4]*

Question 10

[Maximum mark: 8]

Compactness is a measure of how compact an enclosed region is.

The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where A is the area of the region and d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of diameter x units.

(a) If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$. *[3]*

If $n > 1$ and odd, it can be shown that $C = \dfrac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

(b) Find the regular polygon with the least number of sides for which the compactness is more than 0.99. *[4]*

(c) Comment briefly on whether C is a good measure of compactness. *[1]*

Question 11

[Maximum mark: 11]

In triangle ABC,

$$3\sin B + 4\cos C = 6 \text{ and}$$
$$4\sin C + 3\cos B = 1.$$

(a) Show that $\sin(B+C) = \frac{1}{2}$. *[6]*

Robert conjectures that $\text{C}\hat{\text{A}}\text{B}$ can have two possible values.

(b) Show that Robert's conjecture is incorrect by proving that $\text{C}\hat{\text{A}}\text{B}$ has only one possible value. *[5]*

Question 12

[Maximum mark: 4]

In triangle ABC, $\text{AB} = 5\,\text{cm}$, $\text{BC} = 12\,\text{cm}$ and $\text{A}\hat{\text{B}}\text{C} = 100°$.

(a) Find the area of the triangle. [2]

(b) Find AC. [2]

Question 13

[Maximum mark: 10]

Farmer Bill owns a rectangular field, $10\,\text{m}$ by $4\,\text{m}$. Bill attaches a rope to a wooden post at one corner of his field, and attaches the other end to his goat Gruff.

(a) Given that the rope is $5\,\text{m}$ long, calculate the percentage of Bill's field that Gruff is able to graze. Give your answer correct to the nearest integer. [4]

(b) Bill replaces Gruff's rope with another, this time of length a, $4 < a < 10$, so that Gruff can now graze exactly one half of Bill's field.

Show that a satisfies the equation

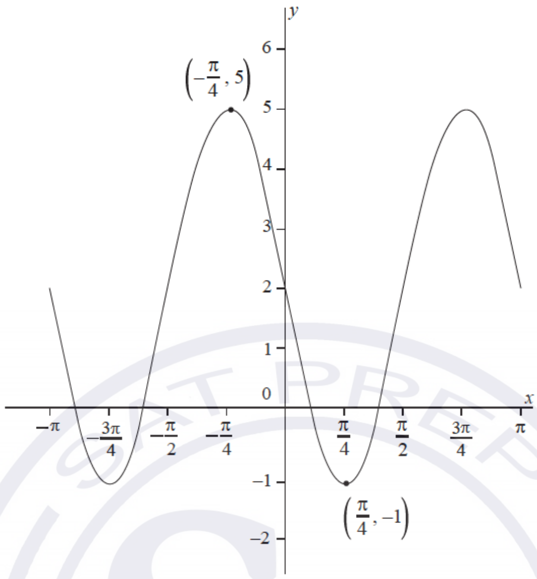
$$a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40.$$ [4]

(c) Find the value of a. [2]

Question 13

[Maximum mark: 6]

A function is defined by $f(x) = A\sin(Bx) + C$, $-\pi \le x \le \pi$, where $A, B, C \in \mathbb{Z}$. The following diagram represents the graph of $y = f(x)$.



(a) Find the value of

(i) A;

(ii) B;

(iii) C. [4]

(b) Solve $f(x) = 3$ for $0 \le x \le \pi$. [2]

Question 15

[Maximum mark: 6]

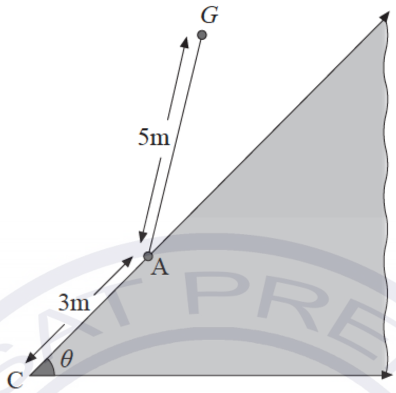
Triangle ABC has area $21\,\text{cm}^2$. The sides AB and AC have lengths $6\,\text{cm}$ and $11\,\text{cm}$ respectively. Find the two possible lengths of the side BC.

Question 16

[Maximum mark: 6]

The diagram below shows a fenced triangular enclosure in the middle of a large grassy field. The points A and C are $3\,\mathrm{m}$ apart. A goat G is tied by a $5\,\mathrm{m}$ length of rope at point A on the outside edge of the enclosure.

Given that the corner of the enclosure at C forms an angle of θ radians and the area of field that can be reached by the goat is $44\,\mathrm{m}^2$, find the value of θ.



Question 17

[Maximum mark: 5]

ABCD is a quadrilateral where $\mathrm{AB} = 6.5$, $\mathrm{BC} = 9.1$, $\mathrm{CD} = 10.4$, $\mathrm{DA} = 7.8$ and $\mathrm{C\hat{D}A} = 90^\circ$. Find $\mathrm{A\hat{B}C}$, giving your answer correct to the nearest degree.

Question 18

[Maximum mark: 8]

In a triangle ABC, $\mathrm{AB} = 4\,\mathrm{cm}$, $\mathrm{BC} = 3\,\mathrm{cm}$ and $\mathrm{B\hat{A}C} = \frac{\pi}{9}$.

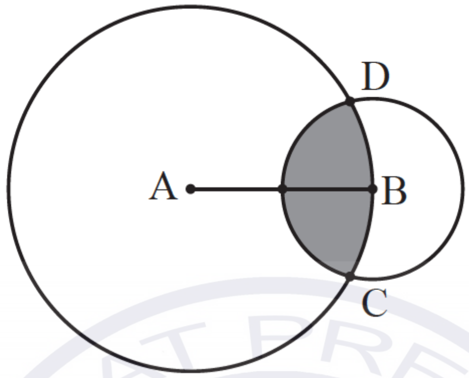
(a) Use the cosine rule to find the two possible values for AC. [5]

(b) Find the difference between the areas of the two possible triangles ABC. [3]

Question 19

[Maximum mark: 8]

The diagram shows two circles with centres at the points A and B and radii $2r$ and r, respectively. The point B lies on the circle with centre A. The circles intersect at the points C and D.



Let α be the measure of the angle CAD and θ be the measure of the angle CBD in radians.

(a) Find an expression for the shaded area in terms of α, θ and r. [3]

(b) Show that $\alpha = 4\arcsin\frac{1}{4}$. [2]

(c) Hence find the value of r given that the shaded area is equal to 4. [3]

Question 20

[Maximum mark: 6]

Consider the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$, where $-\pi \le x \le 2\pi$.

(a) Write down the equations of the vertical asymptotes of the graph. [2]

The graph is reflected in the y-axis, then stretched parallel to the y-axis by a factor $\frac{1}{2}$, then translated by $\begin{pmatrix} \frac{\pi}{4} \\ -3 \end{pmatrix}$.

(b) Give the equation of the transformed graph. [4]

Question 21

[Maximum mark: 15]

In triangle PQR, $\text{PR} = 12\,\text{cm}$, $\text{QR} = p\,\text{cm}$, $\text{PQ} = r\,\text{cm}$ and $\text{Q}\hat{\text{P}}\text{R} = 30^\circ$.

(a) Use the cosine rule to show that $r^2 - 12\sqrt{3}\,r + 144 - p^2 = 0$. [2]

Consider the possible triangles with $\text{QR} = 8\,\text{cm}$.

(b) Calculate the two corresponding values of PQ. [3]

(c) Hence, find the area of the smaller triangle. [3]

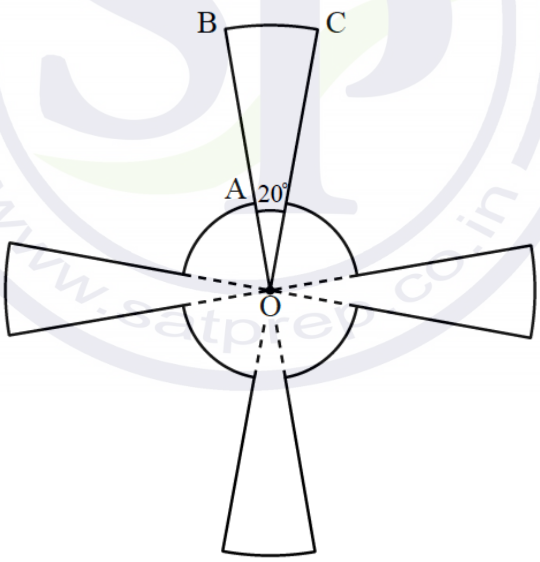
Consider the case where p, the length of QR is not fixed at $8\,\text{cm}$.

(d) Determine the range of values of p for which is possible to form two triangles. [7]

Question 22

[Maximum mark: 4]

This diagram shows a metallic pendant made out of four equal sectors of a larger circle of radius $\text{OB} = 9\,\text{cm}$ and four equal sectors of a smaller circle of radius $\text{OA} = 3\,\text{cm}$.
The angle $\text{BOC} = 20^\circ$.

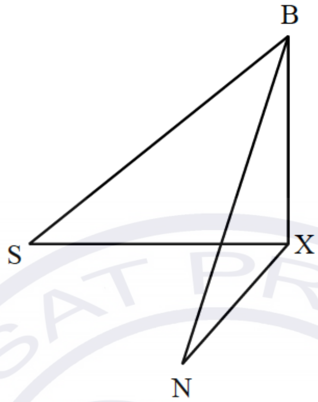


Find the area of the pendant.

Question 23

[Maximum mark: 6]

Barry is at the top of a cliff, standing $80\,\mathrm{m}$ above sea level, and observes two yachts in the sea.
"*Seaview*" (S) is at an angle of depression of 25°.
"*Nauti Buoy*" (N) is at an angle of depression of 35°.
The following three dimensional diagram shows Barry and the two yachts at S and N.
X lies at the foot of the cliff and angle $\mathrm{SXN} = 70^\circ$.

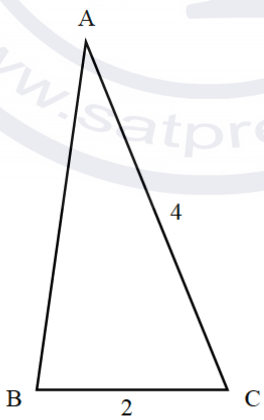


Find, to 3 significant figures, the distance between the two yachts.

Question 24

(a) Find the set of values of k that satisfy the inequality $k^2 - k - 12 < 0$. [2]

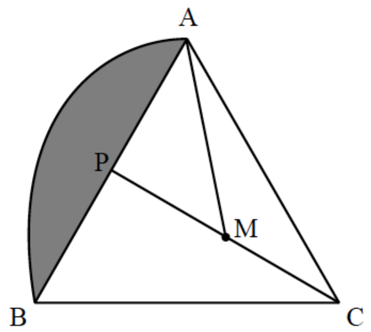
(b) The triangle ABC is shown in the following diagram. Given that $\cos B < \frac{1}{4}$, find the range of possible values for AB. [4]



Question 25

[Maximum mark: 8]

Consider the following diagram.



The sides of the equilateral triangle ABC have lengths 1 m. The midpoint of [AB] is denoted by P. The circular arc AB has centre, M, the midpoint of [CP].

(a) (i) Find AM.

(ii) Find $\hat{AMP}$ in radians. [5]

(b) Find the area of the shaded region. [3]

Question 26

[Maximum mark: 5]

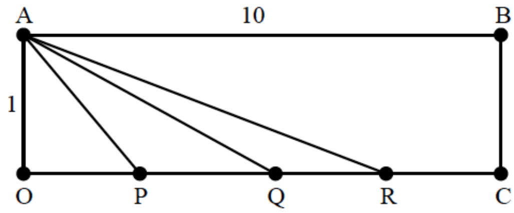
Let $f(x) = \tan(x + \pi)\cos\left(x - \frac{\pi}{2}\right)$ where $0 < x < \frac{\pi}{2}$.

Express $f(x)$ in terms of $\sin x$ and $\cos x$.

Question 27

[Maximum mark: 13]

Consider the rectangle OABC such that $AB = OC = 10$ and $BC = OA = 1$, with the points P, Q and R placed on the line OC such that $OP = p$, $OQ = q$ and $OR = r$, such that $0 < p < q < r < 10$.



Let θ_p be the angle APO, θ_q be the angle AQO and θ_r be the angle ARO.

(a) Find an expression for θ_p in terms of p. [3]

Consider the case when $\theta_p = \theta_q + \theta_r$ and $QR = 1$.

(b) Show that $p = \dfrac{q^2 + q - 1}{2q + 1}$. [6]

(c) By sketching the graph of p as a function of q, determine the range of values of p for which there are possible values of q. [4]

Question 28

[Maximum mark: 5]

In triangle ABC, $AB = 5$, $BC = 14$ and $AC = 11$.
Find all the interior angles of the triangle. Give your answers in degrees to one decimal place.

Question 29

[Maximum mark: 6]

(a) Prove the identity $\dfrac{1+\sin 2x}{\cos 2x} \equiv \dfrac{1+\tan x}{1-\tan x}$. [4]

(b) Solve the equation $\dfrac{1+\sin 2x}{\cos 2x} = \sqrt{3}$ for $0 \le x < 2\pi$. [2]

Question 30

[Maximum mark: 20]

The voltage v in a circuit is given by the equation

$$v(t) = 3\sin(100\pi t),\ t \geq 0 \text{ where } t \text{ is measured in seconds.}$$

(a) Write down the maximum and minimum value of v. [2]

The current i in this circuit is given by the equation

$$i(t) = 2\sin(100\pi(t + 0.003)).$$

(b) Write down two transformations that will transform the graph of $y = v(t)$ onto the graph of $y = i(t)$. [2]

The power p in this circuit is given by $p(t) = v(t) \times i(t)$.

(c) Sketch the graph of $y = p(t)$ for $0 \leq t \leq 0.02$, showing clearly the coordinates of the first maximum and the first minimum. [3]

(d) Find the total time in the interval $0 \leq t \leq 0.02$ for which $p(t) \geq 3$. [3]

The average power p_{av} in this circuit from $t = 0$ to $t = T$ is given by the equation

$$p_{av}(T) = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t, \text{ where } T > 0.$$

(e) Find $p_{av}(0.007)$. [2]

(f) With reference to your graph of $y = p(t)$ explain why $p_{av}(T) > 0$ for all $T > 0$. [2]

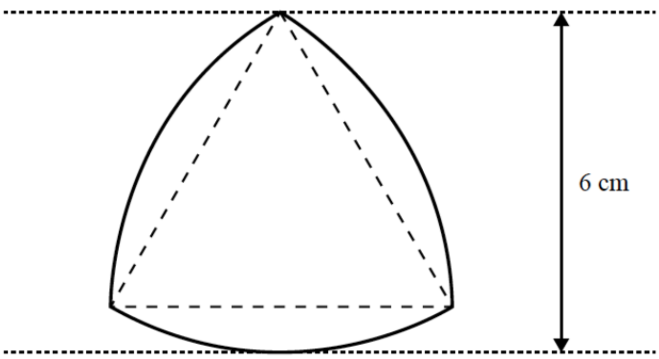
(g) Given that $p(t)$ can be written as $p(t) = a\sin(b(t - c)) + d$ where $a, b, c, d > 0$, use your graph to find the values of a, b, c and d. [6]

Question 31

[Maximum mark: 7]

The following shape consists of three arcs of a circle, each with centre at the opposite vertex of an equilateral triangle as shown in the diagram.

diagram not to scale



For this shape, calculate