# Subject – Math (Higher Level)
# Topic - Functions and Equations
# Year - Nov 2011 – Nov 2019
# Paper -2

## Question 1

*[Maximum mark: 5]*

Consider the graph of $y = x + \sin(x-3)$, $-\pi \le x \le \pi$.

(a) Sketch the graph, clearly labelling the $x$ and $y$ intercepts with their values. *[3 marks]*

## Question 2

*[Maximum mark: 7]*

Given that $f(x) = \dfrac{1}{1+e^{-x}}$,

(a) find $f^{-1}(x)$, stating its domain; *[6 marks]*

(b) find the value of $x$ such that $f(x) = f^{-1}(x)$. *[1 mark]*

## Question 3

*[Maximum mark: 5]*

Let $f(x) = \ln x$. The graph of $f$ is transformed into the graph of the function $g$ by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the $x$-axis. Find an expression for $g(x)$, giving your answer as a single logarithm.

## Question 4

*[Maximum mark: 8]*

(a) Sketch the curve $y = \dfrac{\cos x}{\sqrt{x^2+1}}$, $-4 \le x \le 4$ showing clearly the coordinates of the $x$-intercepts, any maximum points and any minimum points. *[4 marks]*

## Question 5

*[Maximum mark: 4]*

Show that the quadratic equation $x^2 - (5-k)\,x - (k+2) = 0$ has two distinct real roots for all real values of $k$.

Question 6

*[Maximum mark: 7]*

Consider $f(x) = \ln x - e^{\cos x}$, $0 < x \le 10$.

(a) Sketch the graph of $y = f(x)$, stating the coordinates of any maximum and minimum points and points of intersection with the $x$-axis. *[5]*

(b) Solve the inequality $\ln x \le e^{\cos x}$, $0 < x \le 10$. *[2]*

Question 7

*[Maximum mark: 10]*

Let $f(x) = x(x+2)^6$.

(a) Solve the inequality $f(x) > x$. *[5]*

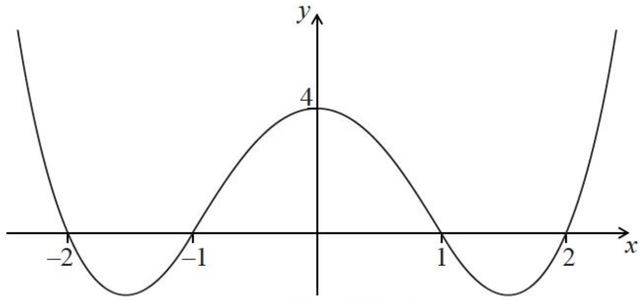
(b) Find $\int f(x)\,dx$. *[5]*

## Question 8

*[Maximum mark: 18]*

Let $f(x) = |x| - 1$.

(a) The graph of $y = g(x)$ is drawn below.




(i) Find the value of $(f \circ g)(1)$.

(ii) Find the value of $(f \circ g \circ g)(1)$.

(iii) Sketch the graph of $y = (f \circ g)(x)$. *[5]*

(b) (i) Sketch the graph of $y = f(x)$.

(ii) State the zeros of $f$. *[3]*

(c) (i) Sketch the graph of $y = (f \circ f)(x)$.

(ii) State the zeros of $f \circ f$. *[3]*

(d) Given that we can denote $\underbrace{f \circ f \circ f \circ \ldots \circ f}_{n \text{ times}}$ as $f^n$,

(i) find the zeros of $f^3$;

(ii) find the zeros of $f^4$;

(iii) deduce the zeros of $f^8$. *[3]*

(e) The zeros of $f^{2n}$ are $a_1, a_2, a_3, \ldots, a_N$.

(i) State the relation between $n$ and $N$;

(ii) Find, and simplify, an expression for $\sum_{r=1}^{N} |a_r|$ in terms of $n$. *[4]*

## Question 9

*[Maximum mark: 8]*

The function $f$ is defined as $f(x) = -3 + \frac{1}{x-2}$, $x \neq 2$.

(a) (i) Sketch the graph of $y = f(x)$, clearly indicating any asymptotes and axes intercepts.

(ii) Write down the equations of any asymptotes and the coordinates of any axes intercepts. *[4]*

(b) Find the inverse function $f^{-1}$, stating its domain. *[4]*

## Question 10

*[Maximum mark: 6]*

Consider $p(x) = 3x^3 + ax + 5a$, $a \in \mathbb{R}$.

The polynomial $p(x)$ leaves a remainder of $-7$ when divided by $(x-a)$.

Show that only one value of $a$ satisfies the above condition and state its value.

## Question 11

[Maximum mark: 10]

A function $f$ is defined by $f(x) = (x+1)(x-1)(x-5)$, $x \in \mathbb{R}$.

(a) Find the values of $x$ for which $f(x) < |f(x)|$. [3]

A function $g$ is defined by $g(x) = x^2 + x - 6$, $x \in \mathbb{R}$.

(b) Find the values of $x$ for which $g(x) < \frac{1}{g(x)}$. [7]

## Question 12

[Maximum mark: 5]

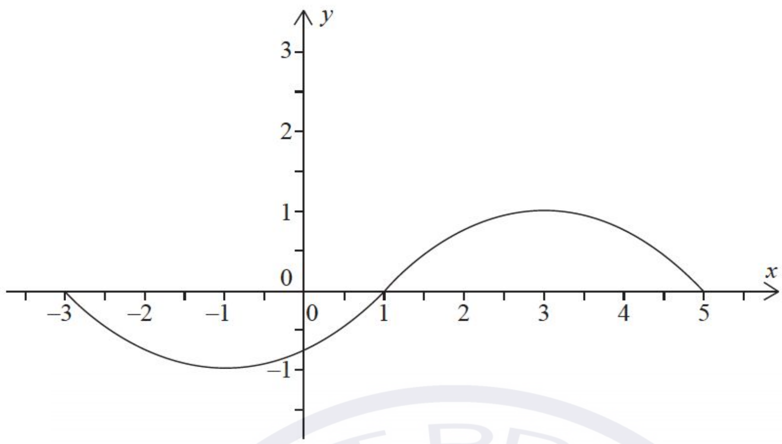
(a) Sketch the graph of $y = (x-5)^2 - 2|x-5| - 9$, for $0 \leq x \leq 10$. [3]

(b) Hence, or otherwise, solve the equation $(x-5)^2 - 2|x-5| - 9 = 0$. [2]

## Question 13

[Maximum mark: 21]

The following graph represents a function $y = f(x)$, where $-3 \le x \le 5$.
The function has a maximum at $(3, 1)$ and a minimum at $(-1, -1)$.



(a) The functions $u$ and $v$ are defined as $u(x) = x - 3$, $v(x) = 2x$ where $x \in \mathbb{R}$.

(i) State the range of the function $u \circ f$.

(ii) State the range of the function $u \circ v \circ f$.

(iii) Find the largest possible domain of the function $f \circ v \circ u$. [7]

(b) (i) Explain why $f$ does not have an inverse.

(ii) The domain of $f$ is restricted to define a function $g$ so that it has an inverse $g^{-1}$. State the largest possible domain of $g$.

(iii) Sketch a graph of $y = g^{-1}(x)$, showing clearly the $y$-intercept and stating the coordinates of the endpoints. [6]

Consider the function defined by $h(x) = \dfrac{2x-5}{x+d}$, $x \ne -d$ and $d \in \mathbb{R}$.

(c) (i) Find an expression for the inverse function $h^{-1}(x)$.

(ii) Find the value of $d$ such that $h$ is a self-inverse function.

For this value of $d$, there is a function $k$ such that $h \circ k(x) = \dfrac{2x}{x+1}$, $x \ne -1$.

(iii) Find $k(x)$. [8]

## Question 14

[Maximum mark: 6]

The graph of $y = \ln(5x + 10)$ is obtained from the graph of $y = \ln x$ by a translation of $a$ units in the direction of the $x$-axis followed by a translation of $b$ units in the direction of the $y$-axis.

(a) Find the value of $a$ and the value of $b$. [4]

## Question 15

[Maximum mark: 4]

(a) Express $x^2 + 4x - 2$ in the form $(x + a)^2 + b$ where $a, b \in \mathbb{Z}$. [2]

(b) If $f(x) = x + 2$ and $(g \circ f)(x) = x^2 + 4x - 2$ write down $g(x)$. [2]

## Question 16

[Maximum mark: 7]

The function $f$ is given by $f(x) = \dfrac{3x^2 + 10}{x^2 - 4}$, $x \in \mathbb{R}$, $x \neq 2$, $x \neq -2$.

(a) Prove that $f$ is an even function. [2]

(b) (i) Sketch the graph $y = f(x)$.

(ii) Write down the range of $f$. [5]

## Question 17

[Maximum mark: 22]

Let $f(x) = x^4 + 0.2x^3 - 5.8x^2 - x + 4$, $x \in \mathbb{R}$.

(a) Find the solutions of $f(x) > 0$. [3]

(b) For the curve $y = f(x)$.

(i) Find the coordinates of both local minimum points.

(ii) Find the $x$-coordinates of the points of inflexion. [5]

The domain of $f$ is now restricted to $[0, a]$.

(c) (i) Write down the largest value of $a$ for which $f$ has an inverse. Give your answer correct to 3 significant figures.

(ii) For this value of $a$ sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing clearly the coordinates of the end points of each curve.

(iii) Solve $f^{-1}(x) = 1$. [6]

Let $g(x) = 2\sin(x-1) - 3$, $-\frac{\pi}{2} + 1 \le x \le \frac{\pi}{2} + 1$.

(d) (i) Find an expression for $g^{-1}(x)$, stating the domain.

(ii) Solve $\left(f^{-1} \circ g\right)(x) < 1$. [8]

## Question 18

[Maximum mark: 9]

Consider the function $f$ defined by $f(x) = 3x\arccos(x)$ where $-1 \le x \le 1$.

(a) Sketch the graph of $f$ indicating clearly any intercepts with the axes and the coordinates of any local maximum or minimum points. [3]

(b) State the range of $f$. [2]

(c) Solve the inequality $|3x\arccos(x)| > 1$. [4]

## Question 19

[Maximum mark: 18]

Consider $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$.

(a) Find the largest possible domain $D$ for $f$ to be a function. [2]

The function $f$ is defined by $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$, $x \in D$.

(b) Sketch the graph of $y = f(x)$ showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]

(c) Explain why $f$ is an even function. [1]

(d) Explain why the inverse function $f^{-1}$ does not exist. [1]

The function $g$ is defined by $g(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$, $x \in ]1, \infty[$.

(e) Find the inverse function $g^{-1}$ and state its domain. [4]

## Question 20

[Maximum mark: 13]

It is given that $f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$ where $a$ and $b$ are positive integers.

(a) Given that $x^2 - 1$ is a factor of $f(x)$ find the value of $a$ and the value of $b$. [4]

(b) Factorize $f(x)$ into a product of linear factors. [3]

(c) Sketch the graph of $y = f(x)$, labelling the maximum and minimum points and the $x$ and $y$ intercepts. [3]

(d) Using your graph state the range of values of $c$ for which $f(x) = c$ has exactly two distinct real roots. [3]

Question 21

[Maximum mark: 18]

Consider the expression $f(x) = \tan\left(x + \frac{\pi}{4}\right)\cot\left(\frac{\pi}{4} - x\right)$.

(a) (i) Sketch the graph of $y = f(x)$ for $-\frac{5\pi}{8} \le x \le \frac{\pi}{8}$.

(ii) With reference to your graph, explain why $f$ is a function on the given domain.

(iii) Explain why $f$ has no inverse on the given domain.

(iv) Explain why $f$ is not a function for $-\frac{3\pi}{4} \le x \le \frac{\pi}{4}$. [5]

The expression $f(x)$ can be written as $g(t)$ where $t = \tan x$.

(b) Show that $g(t) = \left(\frac{1+t}{1-t}\right)^2$. [3]

(c) Sketch the graph of $y = g(t)$ for $t \le 0$. Give the coordinates of any intercepts and the equations of any asymptotes. [3]

(d) Let $\alpha$, $\beta$ be the roots of $g(t) = k$, where $0 < k < 1$.

(i) Find $\alpha$ and $\beta$ in terms of $k$.

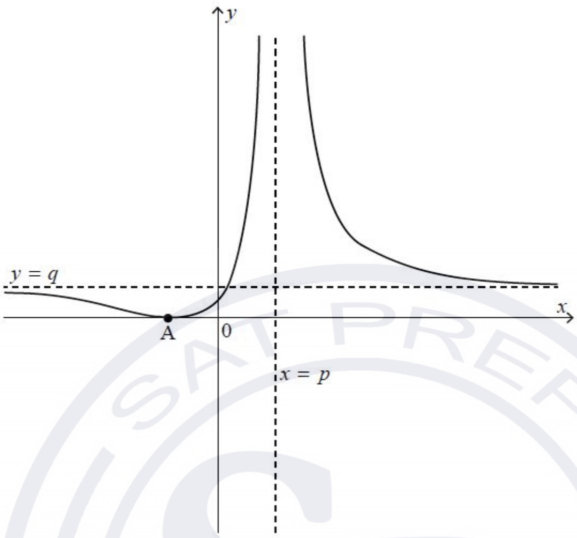
(ii) Show that $\alpha + \beta < -2$. [7]

## Question 22

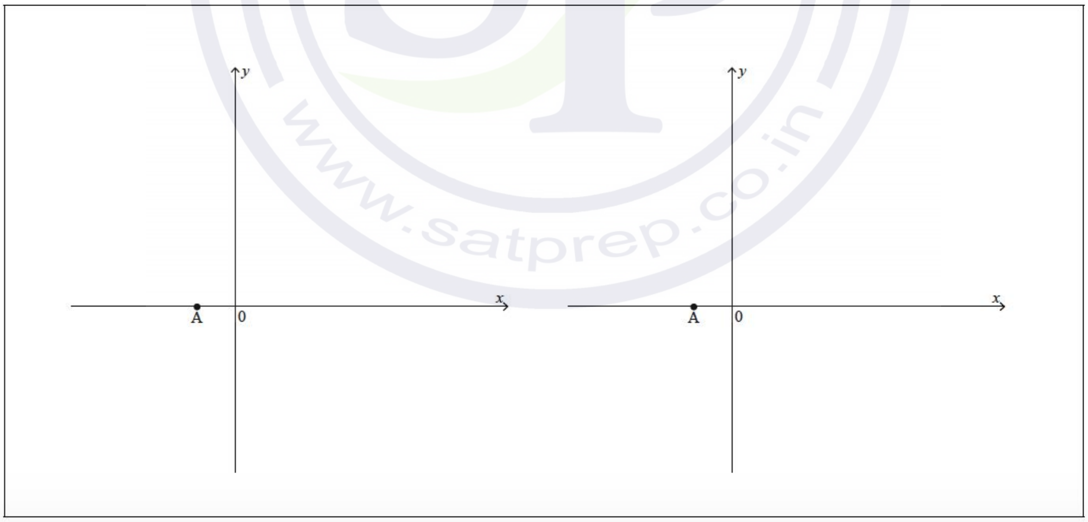
[Maximum mark: 8]

Consider the function $f(x) = \dfrac{ax+1}{bx+c},\ x \neq -\dfrac{c}{b}$, where $a, b, c \in \mathbb{Z}$.

The following graph shows the curve $y = (f(x))^2$. It has asymptotes at $x = p$ and $y = q$ and meets the $x$-axis at A.




(a) On the following axes, sketch the two possible graphs of $y = f(x)$ giving the equations of any asymptotes in terms of $p$ and $q$. **[4]**




(b) Given that $p = \dfrac{4}{3}$, $q = \dfrac{4}{9}$ and A has coordinates $\left(-\dfrac{1}{2},\ 0\right)$, determine the possible sets of values for $a$, $b$ and $c$. **[4]**

## Question 23

[Maximum mark: 19]

The function $f$ is defined by $f(x) = \dfrac{2\ln x + 1}{x - 3}$, $0 < x < 3$.

(a) Find $f'(x)$. [4]

(b) Hence, or otherwise, find the coordinates of the point of inflexion on the graph of $y = f(x)$. [4]

(c) Draw a set of axes showing $x$ and $y$ values between $-3$ and $3$. On these axes

(i) sketch the graph of $y = f(x)$, showing clearly any axis intercepts and giving the equations of any asymptotes.

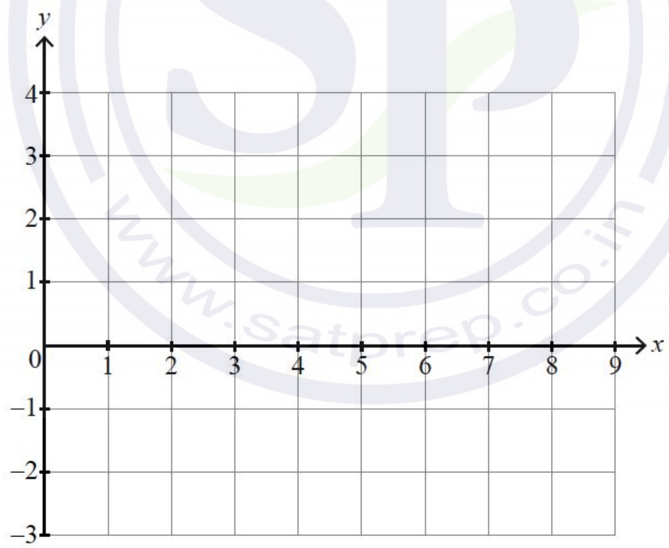
(ii) sketch the graph of $y = f^{-1}(x)$, showing clearly any axis intercepts and giving the equations of any asymptotes. [8]

(d) Hence, or otherwise, solve the inequality $f(x) > f^{-1}(x)$. [3]

## Question 24

[Maximum mark: 6]

(a) Sketch the graphs of $y = \sin^3 x + \ln x$ and $y = 1 + \cos x$ on the following axes for $0 < x \le 9$. [2]




(b) Hence solve $\sin^3 x + \ln x - \cos x - 1 < 0$ in the range $0 < x \le 9$. [4]

## Question 25

[Maximum mark: 5]

The function $f$ is defined by $f(x) = \sec x + 2$, $0 \le x < \dfrac{\pi}{2}$.

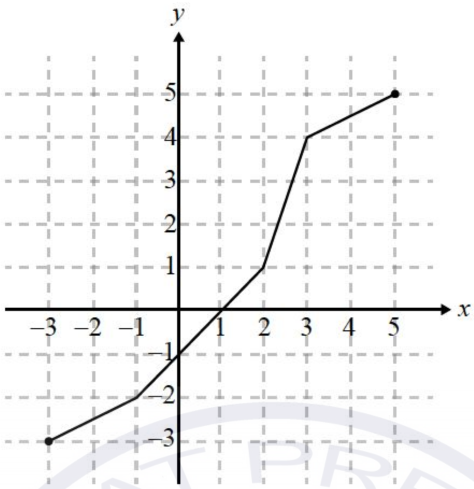
(a) Write down the range of $f$. [1]

(b) Find $f^{-1}(x)$, stating its domain. [4]

## Question 26

[Maximum mark: 6]

The following diagram shows the graph of $y = f(x)$, $-3 \le x \le 5$.




(a) Find the value of $(f \circ f)(1)$. [2]

(b) Given that $f^{-1}(a) = 3$, determine the value of $a$. [2]

(c) Given that $g(x) = 2f(x-1)$, find the domain and range of $g$. [2]