Subject – Math(Higher Level)
Topic - Calculus
Year - Nov 2011 – Nov 2019
Paper -2

Question 1

[Maximum mark: 6]

A stalactite has the shape of a circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Determine if its volume is increasing or decreasing, and the rate at which the volume is changing.

Question 2

[Maximum mark: 5]

Consider the graph of $y = x + \sin(x-3)$, $-\pi \le x \le \pi$.

(a) Sketch the graph, clearly labelling the x and y intercepts with their values. *[3 marks]*

(b) Find the area of the region bounded by the graph and the x and y axes. *[2 marks]*

Question 3

[Maximum mark: 22]

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \le t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{dv}{dt} + v^2 + 1 = 0$. It is also given that $v = 1$ when $t = 0$.

(a) Find an expression for v in terms of t. *[7 marks]*

(b) Sketch the graph of v against t, clearly showing the coordinates of any intercepts, and the equations of any asymptotes. *[3 marks]*

(c) (i) Write down the time T at which the velocity is zero.

(ii) Find the distance travelled in the interval $[0, T]$. *[3 marks]*

(d) Find an expression for s, the displacement, in terms of t, given that $s = 0$ when $t = 0$. *[5 marks]*

(e) Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. *[4 marks]*

Question 4

[Maximum mark: 9]

A cone has height h and base radius r. Deduce the formula for the volume of this cone by rotating the triangular region, enclosed by the line $y = h - \frac{h}{r}x$ and the coordinate axes, through 2π about the y-axis.

Question 5

[Maximum mark: 14]

The function $f(x) = 3\sin x + 4\cos x$ is defined for $0 < x < 2\pi$.

(a) Write down the coordinates of the minimum point on the graph of f. *[1 mark]*

(b) The points $\mathrm{P}(p, 3)$ and $\mathrm{Q}(q, 3)$, $q > p$, lie on the graph of $y = f(x)$. Find p and q. *[2 marks]*

(c) Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. *[4 marks]*

(d) Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. *[7 marks]*

Question 6

[Maximum mark: 5]

Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k.

Question 7

[Maximum mark: 8]

(a) Sketch the curve $y = \dfrac{\cos x}{\sqrt{x^2+1}}$, $-4 \le x \le 4$ showing clearly the coordinates of the x-intercepts, any maximum points and any minimum points. *[4 marks]*

(b) Write down the gradient of the curve at $x = 1$. *[1 mark]*

(c) Find the equation of the normal to the curve at $x = 1$. *[3 marks]*

Question 8

[Maximum mark: 6]

A particle moves along a straight line so that after t seconds its displacement s, in metres, satisfies the equation $s^2 + s - 2t = 0$. Find, in terms of s, expressions for its velocity and its acceleration.

Question 9

[Maximum mark: 7]

By using the substitution $x = \sin t$, find $\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$.

Question 10

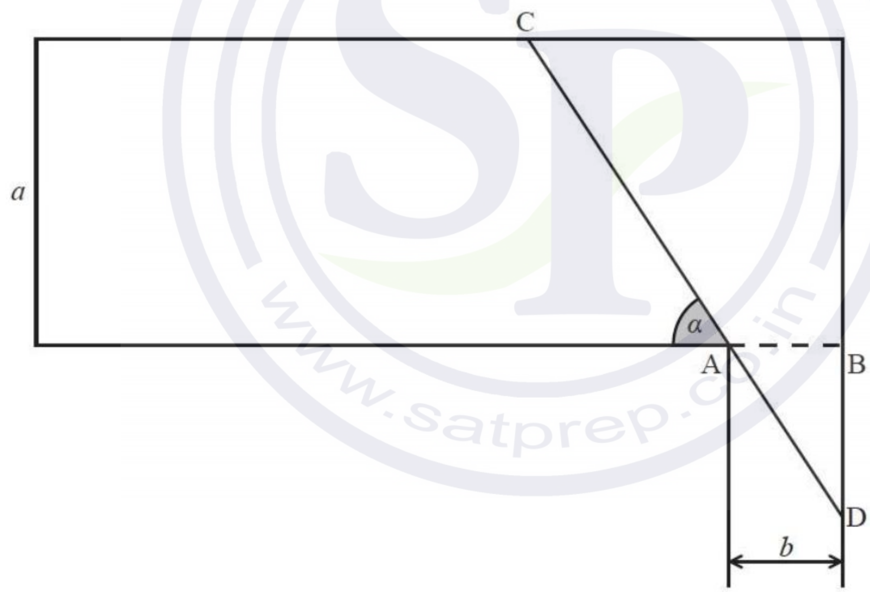
[Maximum mark: 7]

Find the area of the region enclosed by the curves $y = x^3$ and $x = y^2 - 3$.

Question 11

[Maximum mark: 18]

The diagram shows the plan of an art gallery a metres wide. [AB] represents a doorway, leading to an exit corridor b metres wide. In order to remove a painting from the art gallery, CD (denoted by L) is measured for various values of α, as represented in the diagram.



(a) If α is the angle between [CD] and the wall, show that $L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$, $0 < \alpha < \frac{\pi}{2}$. *[3 marks]*

(b) If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. *[4 marks]*

Let $a = 3k$ and $b = k$.

(c) Find $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha}$. *[3 marks]*

(d) Find, in terms of k, the maximum length of a painting that can be removed from the gallery through this doorway. *[6 marks]*

(e) Find the minimum value of k if a painting 8 metres long is to be removed through this doorway. *[2 marks]*

Question 12

[Maximum mark: 6]

Let $f(x) = \sin(x - 1)$, $0 \le x \le \dfrac{\pi}{2} + 1$

Find the volume of the solid formed when the region bounded by $y = f(x)$, and the lines $x = 0$, $y = 0$ and $y = 1$ is rotated by 2π about the y-axis.

Question 13

[Maximum mark: 21]

A particle, A, is moving along a straight line. The velocity, v_A ms^{-1}, of A t seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

(a) Sketch the graph of $v_A = t^3 - 5t^2 + 6t$ for $t \geq 0$, with v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t-axis. *[3 marks]*

(b) Write down the times for which the velocity of the particle is increasing. *[2 marks]*

(c) Write down the times for which the magnitude of the velocity of the particle is increasing. *[3 marks]*

At $t = 0$ the particle is at point O on the line.

(d) Find an expression for the particle's displacement, x_A m, from O at time t. *[3 marks]*

A second particle, B, moving along the same line, has position x_B m, velocity v_B ms^{-1} and acceleration, a_B ms^{-2}, where $a_B = -2v_B$ for $t \geq 0$. At $t = 0$, $x_B = 20$ and $v_B = -20$.

(e) Find an expression for v_B in terms of t. *[4 marks]*

(f) Find the value of t when the two particles meet. *[6 marks]*

Question 14

[Maximum mark: 6]

(a) Find $\int x \sec^2 x \, dx$. *[4 marks]*

(b) Determine the value of m if $\int_0^m x \sec^2 x \, dx = 0.5$, where $m > 0$. *[2 marks]*

Question 15

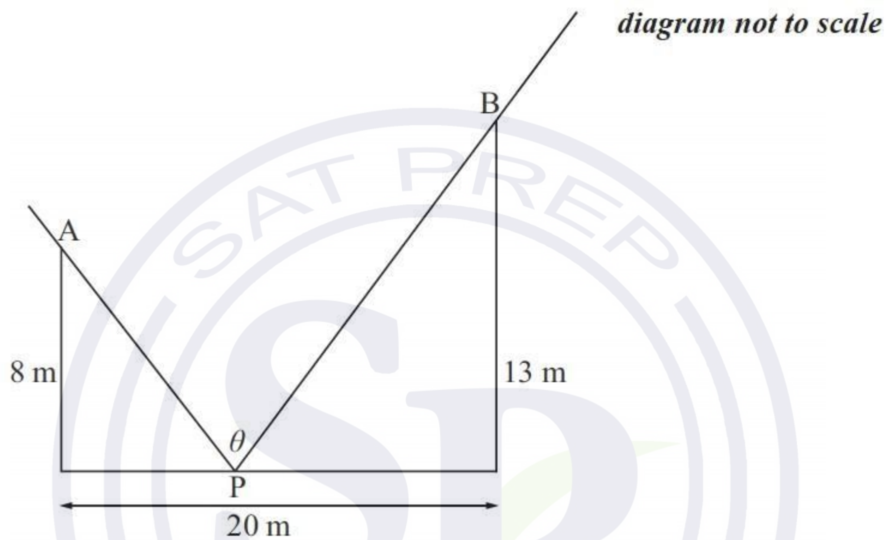
[Maximum mark: 6]

The acceleration of a car is $\frac{1}{40}(60 - v)$ ms^{-2}, when its velocity is v ms^{-1}. Given the car starts from rest, find the velocity of the car after 30 seconds.

Question 16

[Maximum mark: 19]

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \mathrm{A\hat{P}B}$, as shown in the diagram.



(a) Find an expression for θ in terms of x, where x is the distance of P from the base of the wall of height 8m. *[2 marks]*

(b) (i) Calculate the value of θ when $x = 0$.

(ii) Calculate the value of θ when $x = 20$. *[2 marks]*

(c) Sketch the graph of θ, for $0 \le x \le 20$. *[2 marks]*

(d) Show that $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{5(744 - 64x - x^2)}{(x^2 + 64)(x^2 - 40x + 569)}$. *[6 marks]*

(e) Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. *[3 marks]*

(f) The point P moves across the street with speed $0.5\ \mathrm{m\,s^{-1}}$. Determine the rate of change of θ with respect to time when P is at the midpoint of the street. *[4 marks]*

Question 17

[Maximum mark: 7]

Consider $f(x) = \ln x - e^{\cos x}$, $0 < x \leq 10$.

(a) Sketch the graph of $y = f(x)$, stating the coordinates of any maximum and minimum points and points of intersection with the x-axis. *[5]*

(b) Solve the inequality $\ln x \leq e^{\cos x}$, $0 < x \leq 10$. *[2]*

Question 18

[Maximum mark: 7]

By using the substitution $x = 2\tan u$, show that $\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{4x} + C$.

Question 19

[Maximum mark: 20]

(a) A particle P moves in a straight line with velocity $v\,\text{ms}^{-1}$. At time $t = 0$, P is at the point O and has velocity $12\,\text{ms}^{-1}$. Its acceleration at time t seconds is given by $\frac{dv}{dt} = 3\cos\frac{t}{4}\,\text{ms}^{-2}$, $(t \geq 0)$.

(i) Find an expression for the particle's velocity v, in terms of t.

(ii) Sketch a velocity/time graph for the particle for $0 \leq t \leq 8\pi$, showing clearly where the curve meets the axes and any maximum or minimum points.

(iii) Find the distance travelled by the particle before first coming to rest. *[8]*

(b) Another particle Q moves in a straight line with displacement s metres and velocity $v\,\text{ms}^{-1}$. Its acceleration is given by $a = -(v^2+4)\,\text{ms}^{-2}$, $(0 \leq t \leq 1)$. At time $t = 0$, Q is at the point O and has velocity $2\,\text{ms}^{-1}$.

(i) Show that the velocity v at time t is given by $v = 2\tan\left(\frac{\pi - 8t}{4}\right)$.

(ii) Show that $\frac{dv}{ds} = -\frac{(v^2+4)}{v}$.

(iii) Find the distance travelled by the particle before coming to rest. *[12]*

Question 20

[Maximum mark: 22]

A function f is defined by $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$, $x \in \mathbb{R}$.

(a) (i) Explain why the inverse function f^{-1} does not exist.

(ii) Show that the equation of the normal to the curve at the point P where $x = \ln 3$ is given by $9x + 12y - 9\ln 3 - 20 = 0$.

(iii) Find the x-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$. *[14]*

(b) The domain of f is now restricted to $x \geq 0$.

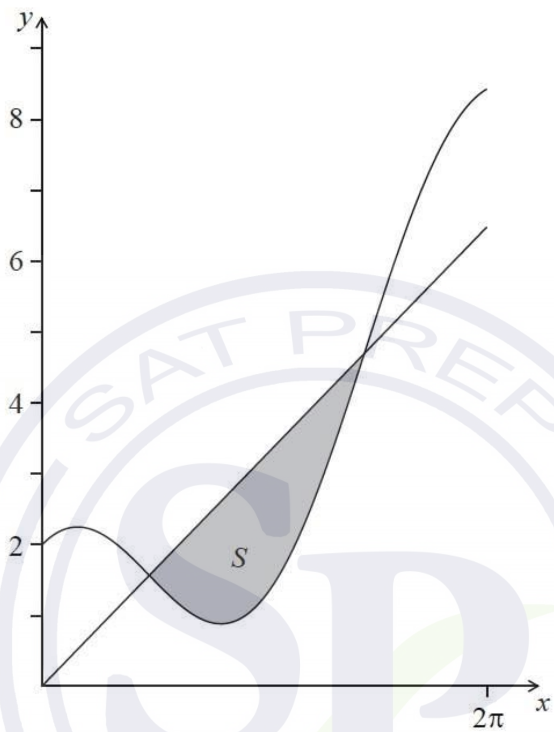
(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y = f(x)$ and the lines $x = 0$ and $y = 5$ is rotated through an angle of 2π radians about the y-axis. *[8]*

Question 21

[Maximum mark: 8]

The shaded region S is enclosed between the curve $y = x + 2\cos x$, for $0 \le x \le 2\pi$, and the line $y = x$, as shown in the diagram below.



(a) Find the coordinates of the points where the line meets the curve. *[3]*

The region S is rotated by 2π about the x-axis to generate a solid.

(b) (i) Write down an integral that represents the volume V of the solid.

(ii) Find the volume V. *[5]*

Question 22

[Maximum mark: 5]

Sand is being poured to form a cone of height h cm and base radius r cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5\,\text{cm}\,\text{min}^{-1}$.

Find the rate at which sand is being poured, in $\text{cm}^3\,\text{min}^{-1}$, when the height is 4 cm.

Question 23

[Maximum mark: 8]

Consider the curve with equation $\left(x^2+y^2\right)^2 = 4xy^2$.

(a) Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[5]*

(b) Find the equation of the normal to the curve at the point $(1, 1)$. *[3]*

Question 24

[Maximum mark: 12]

Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geq 0$.

(a) Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2]*

(b) Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4}\mathrm{d}t$. *[4]*

(c) Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds. Give your answer in the form $k\arctan(b)$, $k, b \in \mathbb{R}$. *[3]*

Particle B moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

(d) Find the acceleration of particle B when $s = 0.1\,\mathrm{m}$. *[3]*

Question 25

[Maximum mark: 5]

Two cyclists are at the same road intersection. One cyclist travels north at $20\,\mathrm{km\,h^{-1}}$. The other cyclist travels west at $15\,\mathrm{km\,h^{-1}}$.

Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

Question 26

[Maximum mark: 7]

A particle moves in a straight line such that its velocity, $v\,\text{m}\,\text{s}^{-1}$, at time t seconds, is given by

$$v(t)=\begin{cases}5-(t-2)^2, & 0\le t\le 4\\ 3-\dfrac{t}{2}, & t>4\end{cases}.$$

(a) Find the value of t when the particle is instantaneously at rest. *[2]*

The particle returns to its initial position at $t=T$.

(b) Find the value of T. *[5]*

Question 27

[Maximum mark: 12]

Consider the triangle PQR where $\text{Q}\hat{\text{P}}\text{R}=30^\circ$, $\text{PQ}=(x+2)\text{cm}$ and $\text{PR}=(5-x)^2\text{cm}$, where $-2<x<5$.

(a) Show that the area, $A\ \text{cm}^2$, of the triangle is given by $A=\dfrac{1}{4}\left(x^3-8x^2+5x+50\right)$. *[2]*

(b) (i) State $\dfrac{\text{d}A}{\text{d}x}$.

(ii) Verify that $\dfrac{\text{d}A}{\text{d}x}=0$ when $x=\dfrac{1}{3}$. *[3]*

(c) (i) Find $\dfrac{\text{d}^2A}{\text{d}x^2}$ and hence justify that $x=\dfrac{1}{3}$ gives the maximum area of triangle PQR.

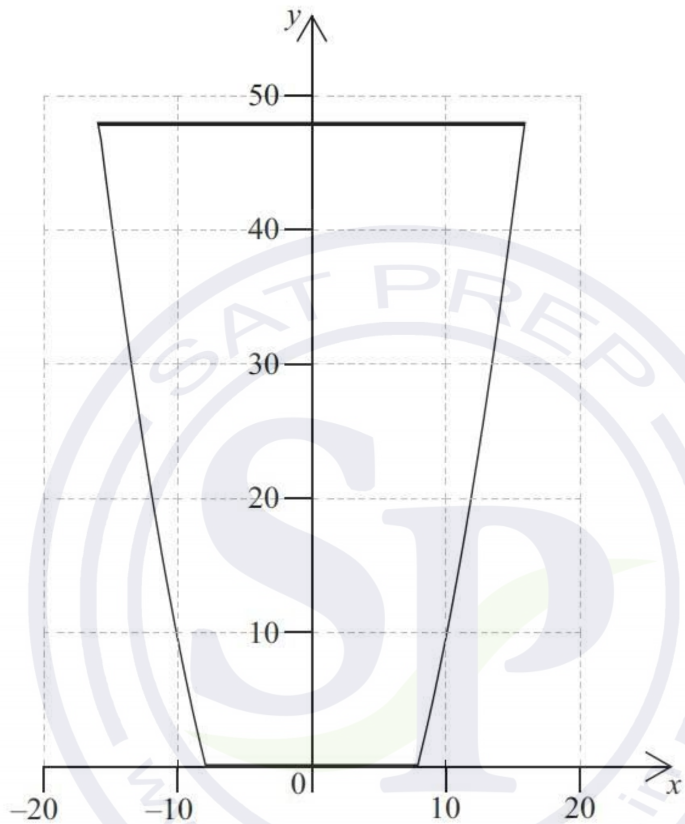
(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum. *[7]*

Question 28

[Maximum mark: 16]

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is $48\,\text{cm}$.

(a) If the container is filled with water to a depth of h cm, show that the volume, $V\,\text{cm}^3$, of the water is given by $V = 4\pi\left(\dfrac{h^2}{2} + 16h\right)$. *[3]*

(Question 13 continued)

The container, initially full of water, begins leaking from a small hole at a rate given by $\frac{dV}{dt} = -\frac{250\sqrt{h}}{\pi(h+16)}$ where t is measured in seconds.

(b) (i) Show that $\frac{dh}{dt} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$.

(ii) State $\frac{dt}{dh}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)dh$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute) *[10]*

Once empty, water is pumped back into the container at a rate of $8.5\,\text{cm}^3\,\text{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\text{cm}^3\,\text{s}^{-1}$.

(c) Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container. *[3]*

Question 29

[Maximum mark: 4]

The region R is enclosed by the graph of $y = e^{-x^2}$, the x-axis and the lines $x = -1$ and $x = 1$. Find the volume of the solid of revolution that is formed when R is rotated through 2π about the x-axis.

Question 30

[Maximum mark: 5]

A bicycle inner tube can be considered as a joined up cylinder of fixed length $200\,\text{cm}$ and radius $r\,\text{cm}$. The radius r increases as the inner tube is pumped up. Air is being pumped into the inner tube so that the volume of air in the tube increases at a constant rate of $30\,\text{cm}^3\,\text{s}^{-1}$. Find the rate at which the radius of the inner tube is increasing when $r = 2\,\text{cm}$.

Question 31

[Maximum mark: 4]

A function f is defined by $f(x) = x^3 + e^x + 1$, $x \in \mathbb{R}$. By considering $f'(x)$ determine whether f is a one-to-one or a many-to-one function.

Question 32

[Maximum mark: 7]

Find the equation of the normal to the curve $y = \frac{e^x \cos x \ln(x+e)}{(x^{17}+1)^5}$ at the point where $x = 0$.

In your answer give the value of the gradient, of the normal, to three decimal places.

Question 33

[Maximum mark: 21]

Let $f(x) = \frac{e^{2x}+1}{e^x-2}$.

(a) Find the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. *[4]*

(b) (i) Find $f'(x)$.

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point. *[8]*

(c) Find the equation of L_1, the normal to the curve at the point where it crosses the y-axis. *[4]*

The line L_2 is parallel to L_1 and tangent to the curve $y = f(x)$.

(d) Find the equation of the line L_2. *[5]*

Question 34

[Maximum mark: 10]

Farmer Bill owns a rectangular field, $10\,\text{m}$ by $4\,\text{m}$. Bill attaches a rope to a wooden post at one corner of his field, and attaches the other end to his goat Gruff.

(a) Given that the rope is $5\,\text{m}$ long, calculate the percentage of Bill's field that Gruff is able to graze. Give your answer correct to the nearest integer. [4]

(b) Bill replaces Gruff's rope with another, this time of length a, $4 < a < 10$, so that Gruff can now graze exactly one half of Bill's field.

Show that a satisfies the equation

$$a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40\,.$$ [4]

(c) Find the value of a. [2]

Question 35

[Maximum mark: 15]

A curve is defined by $x^2 - 5xy + y^2 = 7$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y - 2x}{2y - 5x}$. [3]

(b) Find the equation of the normal to the curve at the point $(6, 1)$. [4]

(c) Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

Question 36

[Maximum mark: 15]

A particle moves in a straight line, its velocity $v\ \mathrm{ms^{-1}}$ at time t seconds is given by $v = 9t - 3t^2$, $0 \le t \le 5$.

At time $t = 0$, the displacement s of the particle from an origin O is $3\ \mathrm{m}$.

(a) Find the displacement of the particle when $t = 4$. [3]

(b) Sketch a displacement/time graph for the particle, $0 \le t \le 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. [5]

For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \ge 0$.

(c) Given further that $s = 16.5$ when $t = 7.5$, find the values of a and b. [3]

(d) Find the times t_1 and t_2 $\left(0 < t_1 < t_2 < 8\right)$ when the particle returns to its starting point. [4]

Question 37

[Maximum mark: 21]

Richard, a marine soldier, steps out of a stationary helicopter, $1000\,\mathrm{m}$ above the ground, at time $t = 0$. Let his height, in metres, above the ground be given by $s(t)$. For the first 10 seconds his velocity, $v(t)\,\mathrm{ms}^{-1}$, is given by $v(t) = -10t$.

(a) (i) Find his acceleration $a(t)$ for $t < 10$.

(ii) Calculate $v(10)$.

(iii) Show that $s(10) = 500$. [6]

At $t = 10$ his parachute opens and his acceleration $a(t)$ is subsequently given by $a(t) = -10 - 5v$, $t \geq 10$.

(b) Given that $\dfrac{\mathrm{d}t}{\mathrm{d}v} = \dfrac{1}{\frac{\mathrm{d}v}{\mathrm{d}t}}$, write down $\dfrac{\mathrm{d}t}{\mathrm{d}v}$ in terms of v. [1]

You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$.

(c) Hence show that $t = 10 + \dfrac{1}{5}\ln\left(\dfrac{98}{-2-v}\right)$. [5]

(d) Hence find an expression for the velocity, v, for $t \geq 10$. [2]

(e) Find an expression for his height, s, above the ground for $t \geq 10$. [5]

(f) Find the value of t when Richard lands on the ground. [2]

Question 38

[Maximum mark: 6]

A function is defined by $f(x) = x^2 + 2$, $x \geq 0$. A region R is enclosed by $y = f(x)$, the y-axis and the line $y = 4$.

(a) (i) Express the area of the region R as an integral with respect to y.

(ii) Determine the area of R, giving your answer correct to four significant figures. [3]

(b) Find the exact volume generated when the region R is rotated through 2π radians about the y-axis. [3]

Question 39

[Maximum mark: 8]

A particle can move along a straight line from a point O. The velocity v, in m s^{-1}, is given by the function $v(t) = 1 - e^{-\sin t^2}$ where time $t \geq 0$ is measured in seconds.

(a) Write down the first two times $t_1, t_2 > 0$, when the particle changes direction. [2]

(b) (i) Find the time $t < t_2$ when the particle has a maximum velocity.

(ii) Find the time $t < t_2$ when the particle has a minimum velocity. [4]

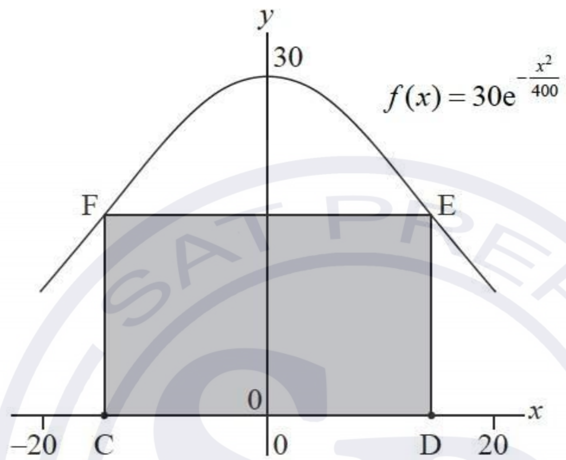
(c) Find the distance travelled by the particle between times $t = t_1$ and $t = t_2$. [2]

Question 40

[Maximum mark: 21]

The following diagram shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $f(x) = 30e^{-\frac{x^2}{400}}$, where $-20 \le x \le 20$.

Ground level is represented by the x-axis.



(a) Find $f''(x)$. [4]

(b) Show that the gradient of the roof function is greatest when $x = -\sqrt{200}$. [3]

The cross section of the living space under the roof can be modelled by a rectangle CDEF with points $C(-a, 0)$ and $D(a, 0)$, where $0 < a \le 20$.

(c) Show that the maximum area A of the rectangle CDEF is $600\sqrt{2}e^{-\frac{1}{2}}$. [5]

(d) A function I is known as the Insulation Factor of CDEF. The function is defined as $I(a) = \dfrac{P(a)}{A(a)}$ where P = Perimeter and A = Area of the rectangle.

(i) Find an expression for P in terms of a.

(ii) Find the value of a which minimizes I.

(iii) Using the value of a found in part (ii) calculate the percentage of the cross sectional area under the whole roof that is not included in the cross section of the living space. [9]

Question 41

[Maximum mark: 5]

The displacement, s, in metres, of a particle t seconds after it passes through the origin is given by the expression $s = \ln\left(2 - e^{-t}\right)$, $t \geq 0$.

(a) Find an expression for the velocity, v, of the particle at time t. [2]

(b) Find an expression for the acceleration, a, of the particle at time t. [2]

(c) Find the acceleration of the particle at time $t = 0$. [1]

Question 42

[Maximum mark: 22]

Let $f(x) = x^4 + 0.2x^3 - 5.8x^2 - x + 4$, $x \in \mathbb{R}$.

(a) Find the solutions of $f(x) > 0$. [3]

(b) For the curve $y = f(x)$.

(i) Find the coordinates of both local minimum points.

(ii) Find the x-coordinates of the points of inflexion. [5]

The domain of f is now restricted to $[0, a]$.

(c) (i) Write down the largest value of a for which f has an inverse. Give your answer correct to 3 significant figures.

(ii) For this value of a sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing clearly the coordinates of the end points of each curve.

(iii) Solve $f^{-1}(x) = 1$. [6]

Let $g(x) = 2\sin(x - 1) - 3$, $-\frac{\pi}{2} + 1 \leq x \leq \frac{\pi}{2} + 1$.

(d) (i) Find an expression for $g^{-1}(x)$, stating the domain.

(ii) Solve $\left(f^{-1} \circ g\right)(x) < 1$. [8]

Question 43

[Maximum mark: 16]

Consider the curve, C defined by the equation $y^2 - 2xy = 5 - e^x$. The point A lies on C and has coordinates $(0, a)$, $a > 0$.

(a) Find the value of a. [2]

(b) Show that $\frac{dy}{dx} = \frac{2y - e^x}{2(y - x)}$. [4]

(c) Find the equation of the normal to C at the point A. [3]

(d) Find the coordinates of the second point at which the normal found in part (c) intersects C. [4]

(e) Given that $v = y^3$, $y > 0$, find $\frac{dv}{dx}$ at $x = 0$. [3]

Question 44

[Maximum mark: 8]

Consider the curve with equation $x^3 + y^3 = 4xy$.

(a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$. [3]

The tangent to this curve is parallel to the x-axis at the point where $x = k$, $k > 0$.

(b) Find the value of k. [5]

Question 45

[Maximum mark: 6]

A particle moves such that its velocity $v\,\mathrm{ms}^{-1}$ is related to its displacement s m, by the equation $v(s) = \arctan(\sin s)$, $0 \le s \le 1$. The particle's acceleration is $a\,\mathrm{ms}^{-2}$.

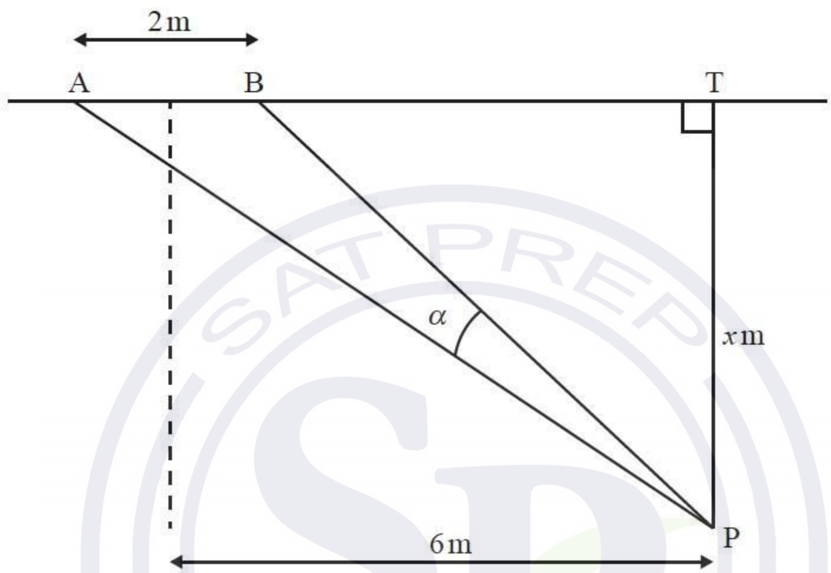
(a) Find the particle's acceleration in terms of s. [4]

(b) Using an appropriate sketch graph, find the particle's displacement when its acceleration is $0.25\,\mathrm{ms}^{-2}$. [2]

Question 46

[Maximum mark: 22]

Points A, B and T lie on a line on an indoor soccer field. The goal, [AB], is 2 metres wide. A player situated at point P kicks a ball at the goal. [PT] is perpendicular to (AB) and is 6 metres from a parallel line through the centre of [AB]. Let PT be x metres and let $\alpha = \mathrm{A\hat{P}B}$ measured in degrees. Assume that the ball travels along the floor.



(a) Find the value of α when $x = 10$. [4]

(b) Show that $\tan\alpha = \dfrac{2x}{x^2 + 35}$. [4]

The maximum for $\tan\alpha$ gives the maximum for α.

(c) (i) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha)$.

(ii) Hence or otherwise find the value of α such that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0$.

(iii) Find $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ and hence show that the value of α never exceeds 10°. [11]

(d) Find the set of values of x for which $\alpha \geq 7^\circ$. [3]

Question 47

[Maximum mark: 23]

The functions f and g are defined by

$$f(x) = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$

$$g(x) = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

(a) (i) Show that $\frac{1}{4f(x) - 2g(x)} = \frac{e^x}{e^{2x} + 3}$.

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x) - 2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a, b \in \mathbb{Z}^+$. [9]

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R}, n > 1$.

(b) (i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$.

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$. [8]

Let $t(x) = \frac{g(x)}{f(x)}$.

(c) (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$.

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$. [6]

Question 48

[Maximum mark: 6]

An earth satellite moves in a path that can be described by the curve $72.5x^2 + 71.5y^2 = 1$ where $x = x(t)$ and $y = y(t)$ are in thousands of kilometres and t is time in seconds.

Given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 7.75 \times 10^{-5}$ when $x = 3.2 \times 10^{-3}$, find the possible values of $\frac{\mathrm{d}y}{\mathrm{d}t}$.

Give your answers in standard form.

Question 49

[Maximum mark: 22]

Let the function f be defined by $f(x) = \frac{2 - \mathrm{e}^x}{2\mathrm{e}^x - 1}$, $x \in D$.

(a) Determine D, the largest possible domain of f. [2]

(b) Show that the graph of f has three asymptotes and state their equations. [5]

(c) Show that $f'(x) = -\frac{3\mathrm{e}^x}{\left(2\mathrm{e}^x - 1\right)^2}$. [3]

(d) Use your answers from parts (b) and (c) to justify that f has an inverse and state its domain. [4]

(e) Find an expression for $f^{-1}(x)$. [4]

(f) Consider the region R enclosed by the graph of $y = f(x)$ and the axes.
Find the volume of the solid obtained when R is rotated through 2π about the y-axis. [4]

Question 50

[Maximum mark: 7]

The curve C is defined by equation $xy - \ln y = 1$, $y > 0$.

(a) Find $\frac{dy}{dx}$ in terms of x and y . [4]

(b) Determine the equation of the tangent to C at the point $\left(\frac{2}{e}, e\right)$. [3]

Question 51

[Maximum mark: 6]

The region A is enclosed by the graph of $y = 2\arcsin(x-1) - \frac{\pi}{4}$, the y-axis and the line $y = \frac{\pi}{4}$.

(a) Write down a definite integral to represent the area of A . [4]

(b) Calculate the area of A . [2]

Question 52

[Maximum mark: 7]

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

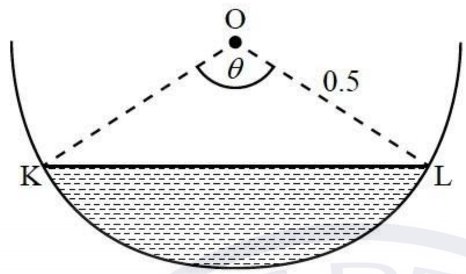


diagram not to scale

(a) Find an expression for the volume of water $V\left(m^3\right)$ in the trough in terms of θ. [3]

The volume of water is increasing at a constant rate of $0.0008\,\mathrm{m^3\,s^{-1}}$.

(b) Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. [4]

Question 53

[Maximum mark: 18]

Consider $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$.

(a) Find the largest possible domain D for f to be a function. [2]

The function f is defined by $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$, $x \in D$.

(b) Sketch the graph of $y = f(x)$ showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]

(c) Explain why f is an even function. [1]

(d) Explain why the inverse function f^{-1} does not exist. [1]

The function g is defined by $g(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$, $x \in]1, \infty[$.

(e) Find the inverse function g^{-1} and state its domain. [4]

(f) Find $g'(x)$. [3]

(g) Hence, show that there are no solutions to

(i) $g'(x) = 0$;

(ii) $\left(g^{-1}\right)'(x) = 0$. [4]

Question 54

[Maximum mark: 9]

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

(a) Find the equation of the normal to the curve at the point $\left(1, \sqrt{3}\right)$. [6]

(b) Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geq 0$ and the y-axis for $y \geq 0$ is rotated through 2π about the x-axis. [3]

Question 55

[Maximum mark: 7]

By using the substitution $x^2 = 2\sec\theta$, show that $\int \frac{dx}{x\sqrt{x^4 - 4}} = \frac{1}{4}\arccos\left(\frac{2}{x^2}\right) + c$.

Question 56

[Maximum mark: 17]

Consider the function $f(x) = \frac{\sqrt{x}}{\sin x}$, $0 < x < \pi$.

(a) (i) Show that the x-coordinate of the minimum point on the curve $y = f(x)$ satisfies the equation $\tan x = 2x$.

(ii) Determine the values of x for which $f(x)$ is a decreasing function. [7]

(b) Sketch the graph of $y = f(x)$ showing clearly the minimum point and any asymptotic behaviour. [3]

(c) Find the coordinates of the point on the graph of f where the normal to the graph is parallel to the line $y = -x$. [4]

Consider the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$.

(d) This region is now rotated through 2π radians about the x-axis. Find the volume of revolution. [3]

Question 57

[Maximum mark: 18]

Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9,\ 0 \leq x < \frac{\pi}{2}$.

(a) (i) Determine an expression for $f'(x)$ in terms of x.

(ii) Sketch a graph of $y = f'(x)$ for $0 \leq x < \frac{\pi}{2}$.

(iii) Find the x-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. [8]

(b) Let $u = \tan x$.

(i) Express $\sin x$ in terms of u.

(ii) Express $\sin 2x$ in terms of u.

(iii) Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. [7]

(c) Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. [3]

Question 58

[Maximum mark: 5]

A point P moves in a straight line with velocity v ms^{-1} given by $v(t) = e^{-t} - 8t^2e^{-2t}$ at time t seconds, where $t \geq 0$.

(a) Determine the first time t_1 at which P has zero velocity. [2]

(b) (i) Find an expression for the acceleration of P at time t.

(ii) Find the value of the acceleration of P at time t_1. [3]

Question 59

[Maximum mark: 19]

A curve C is given by the implicit equation $x + y - \cos(xy) = 0$.

(a) Show that $\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$. [5]

(b) The curve $xy = -\frac{\pi}{2}$ intersects C at P and Q.

(i) Find the coordinates of P and Q.

(ii) Given that the gradients of the tangents to C at P and Q are m_1 and m_2 respectively, show that $m_1 \times m_2 = 1$. [7]

(c) Find the coordinates of the three points on C, nearest the origin, where the tangent is parallel to the line $y = -x$. [7]

Question 60

[Maximum mark: 7]

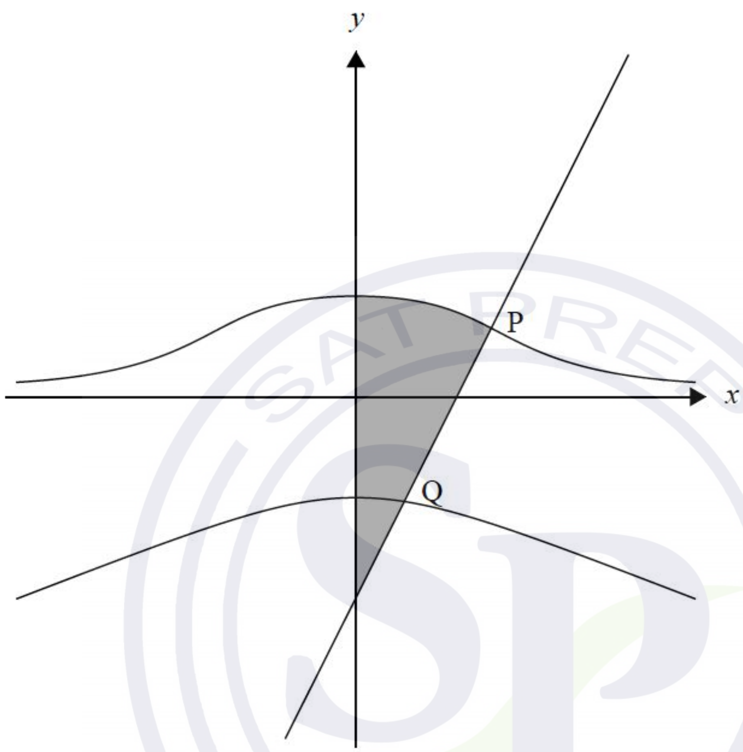
(a) Given that $2x^3 - 3x + 1$ can be expressed in the form $Ax(x^2 + 1) + Bx + C$, find the values of the constants A, B and C. [2]

(b) Hence find $\int \frac{2x^3 - 3x + 1}{x^2 + 1} dx$. [5]

Question 61

[Maximum mark: 22]

The following graph shows the two parts of the curve defined by the equation $x^2y = 5 - y^4$, and the normal to the curve at the point $P(2, 1)$.



(a) Show that there are exactly two points on the curve where the gradient is zero. [7]

(b) Find the equation of the normal to the curve at the point P. [5]

(c) The normal at P cuts the curve again at the point Q. Find the x-coordinate of Q. [3]

(d) The shaded region is rotated by 2π about the y-axis. Find the volume of the solid formed. [7]

Question 62

[Maximum mark: 7]

A function f satisfies the conditions $f(0) = -4$, $f(1) = 0$ and its second derivative is $f''(x) = 15\sqrt{x} + \frac{1}{(x+1)^2}$, $x \geq 0$.

Find $f(x)$.

Question 63

[Maximum mark: 5]

Differentiate from first principles the function $f(x) = 3x^3 - x$.

Question 64

[Maximum mark: 6]

A particle moves along a horizontal line such that at time t seconds, $t \geq 0$, its acceleration a is given by $a = 2t - 1$. When $t = 6$, its displacement s from a fixed origin O is 18.25 m. When $t = 15$, its displacement from O is 922.75 m. Find an expression for s in terms of t.

Question 65

[Maximum mark: 4]

Let l be the tangent to the curve $y = xe^{2x}$ at the point $(1, e^2)$.

Find the coordinates of the point where l meets the x-axis.

Question 66

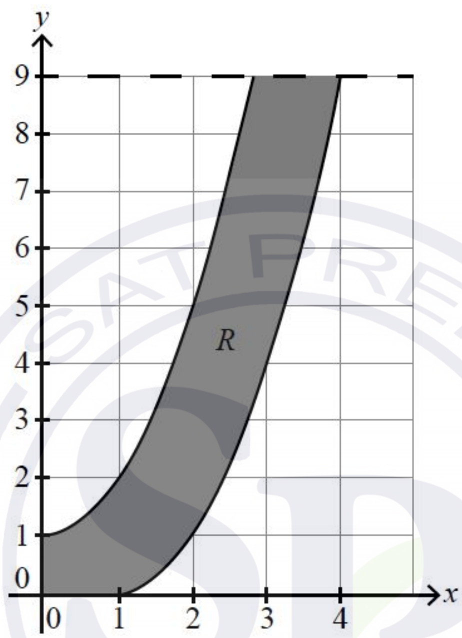
[Maximum mark: 6]

Use integration by parts to find $\int (\ln x)^2 \, dx$.

Question 67

[Maximum mark: 7]

The function f is defined by $f(x) = (x-1)^2$, $x \geq 1$ and the function g is defined by $g(x) = x^2 + 1$, $x \geq 0$.
The region R is bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $y = 0$, $x = 0$ and $y = 9$ as shown on the following diagram.



The shape of a clay vase can be modelled by rotating the region R through 360° about the y-axis.

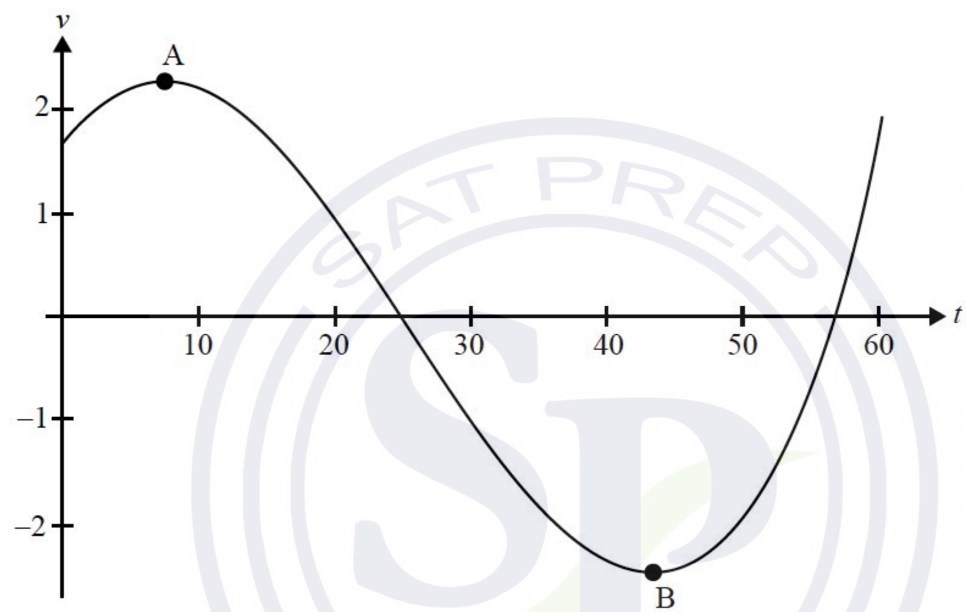
Find the volume of clay used to make the vase.

Question 68

[Maximum mark: 14]

A body moves in a straight line such that its velocity, $v\ \mathrm{m\,s^{-1}}$, after t seconds is given by $v = 2\sin\left(\frac{t}{10} + \frac{\pi}{5}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right)$ for $0 \le t \le 60$.

The following diagram shows the graph of v against t. Point A is a local maximum and point B is a local minimum.



(a) (i) Determine the coordinates of point A and the coordinates of point B.

(ii) Hence, write down the maximum speed of the body. [5]

(b) The body first comes to rest at time $t = t_1$. Find

(i) the value of t_1;

(ii) the distance travelled between $t = 0$ and $t = t_1$;

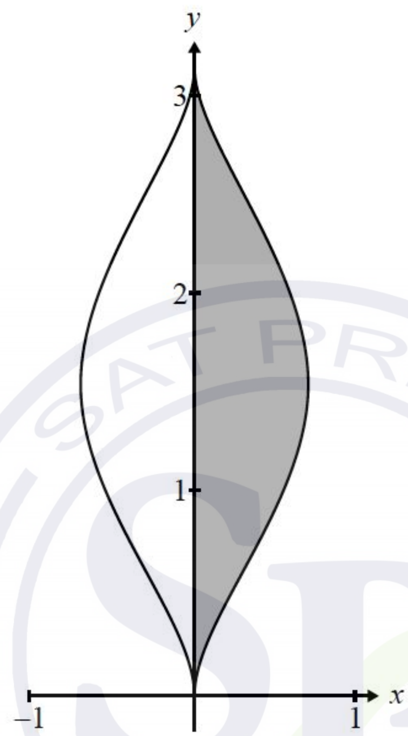
(iii) the acceleration when $t = t_1$. [6]

(c) Find the distance travelled in the first 30 seconds. [3]

Question 69

[Maximum mark: 17]

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \le y \le \pi$.



(a) (i) Using implicit differentiation, find an expression for $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. [8]

The shaded region R is the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = \pi$.

(b) Find the area of R. [3]

The region R is now rotated about the y-axis, through 2π radians, to form a solid.

(c) By writing $\sin^3 y$ as $(1 - \cos^2 y)\sin y$, show that the volume of the solid formed is $\frac{2\pi}{3}$. [6]