Subject – Math(Higher Level)
Topic - Vector
Year - Nov 2011 – Nov 2019
Paper -2

Question 1

[Maximum mark: 7]

Given the following system of linear equations,

$$ax + y + z = 1$$
$$x + ay + z = a$$
$$x + y + az = a^2$$

find the values of the real constant, a, for which the system has a unique solution.

Question 2

[Maximum mark: 16]

Two planes Π_1 and Π_2 have equations $2x + y + z = 1$ and $3x + y - z = 2$ respectively.

(a) Find the vector equation of L, the line of intersection of Π_1 and Π_2. *[6 marks]*

(b) Show that the plane Π_3 which is perpendicular to Π_1 and contains L, has equation $x - 2z = 1$. *[4 marks]*

(c) The point P has coordinates $(-2, 4, 1)$, the point Q lies on Π_3 and PQ is perpendicular to Π_2. Find the coordinates of Q. *[6 marks]*

Question 3

[Maximum mark: 5]

The planes $2x + 3y - z = 5$ and $x - y + 2z = k$ intersect in the line $5x + 1 = 9 - 5y = -5z$.
Find the value of k.

Question 4

[Maximum mark: 24]

The coordinates of points A, B and C are given as $(5, -2, 5)$, $(5, 4, -1)$ and $(-1, -2, -1)$ respectively.

(a) Show that $AB = AC$ and that $B\hat{A}C = 60^\circ$. *[4 marks]*

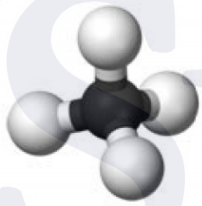
(b) Find the Cartesian equation of Π, the plane passing through A, B, and C. *[4 marks]*

(c) (i) Find the Cartesian equation of Π_1, the plane perpendicular to (AB) passing through the midpoint of [AB].

(ii) Find the Cartesian equation of Π_2, the plane perpendicular to (AC) passing through the midpoint of [AC]. *[4 marks]*

(d) Find the vector equation of L, the line of intersection of Π_1 and Π_2, and show that it is perpendicular to Π. *[3 marks]*

A methane molecule consists of a carbon atom with four hydrogen atoms symmetrically placed around it in three dimensions.



The positions of the centres of three of the hydrogen atoms are A, B and C as given. The position of the centre of the fourth hydrogen atom is D.

(e) Using the fact that $AB = AD$, show that the coordinates of one of the possible positions of the fourth hydrogen atom is $(-1, 4, 5)$. *[3 marks]*

(f) Letting D be $(-1, 4, 5)$, show that the coordinates of G, the position of the centre of the carbon atom, are $(2, 1, 2)$. Hence calculate $D\hat{G}A$, the bonding angle of carbon. *[6 marks]*

Question 5

[Maximum mark: 24]

(a) Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions.

$$x - 3y + z = 3$$
$$x + 5y - 2z = 1$$
$$16y - 6z = k$$

[5 marks]

(b) Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where the components of $\boldsymbol{b}$ are integers. *[7 marks]*

(c) The plane ÷ is parallel to both the line in part (b) and the line $\dfrac{x-4}{3} = \dfrac{y-6}{-2} = \dfrac{z-2}{0}$.

Given that ÷ contains the point $(1, 2, 0)$, show that the Cartesian equation of ÷ is $16x + 24y - 11z = 64$. *[5 marks]*

(d) The z-axis meets the plane ÷ at the point P. Find the coordinates of P. *[2 marks]*

(e) Find the angle between the line $\dfrac{x-2}{3} = \dfrac{y+5}{4} = \dfrac{z}{2}$ and the plane ÷ . *[5 marks]*

Question 6

[Maximum mark: 24]

Consider the planes $\pi_1 : x - 2y - 3z = 2$ and $\pi_2 : 2x - y - z = k$.

(a) Find the angle between the planes π_1 and π_2. *[4 marks]*

(b) The planes π_1 and π_2 intersect in the line L_1. Show that the vector equation of L_1 is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$. *[5 marks]*

(c) The line L_2 has Cartesian equation $5 - x = y + 3 = 2 - 2z$. The lines L_1 and L_2 intersect at a point X. Find the coordinates of X. *[5 marks]*

(d) Determine a Cartesian equation of the plane π_3 containing both lines L_1 and L_2. *[5 marks]*

(e) Let Y be a point on L_1 and Z be a point on L_2 such that XY is perpendicular to YZ and the area of the triangle XYZ is 3. Find the perimeter of the triangle XYZ. *[5 marks]*

Question 7

[Maximum mark: 5]

Find the value of k such that the following system of equations does not have a unique solution.

$$\begin{aligned} kx + y + 2z &= 4 \\ -y + 4z &= 5 \\ 3x + 4y + 2z &= 1 \end{aligned}$$

Question 8

[Maximum mark: 20]

Consider the points $\mathrm{P}(-3, -1, 2)$ and $\mathrm{Q}(5, 5, 6)$.

(a) Find a vector equation for the line, L_1, which passes through the points P and Q. *[3 marks]*

The line L_2 has equation

$$\boldsymbol{r} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}.$$

(b) Show that L_1 and L_2 intersect at the point $\mathrm{R}(1, 2, 4)$. *[4 marks]*

(c) Find the acute angle between L_1 and L_2. *[3 marks]*

Let S be a point on L_2 such that $|\overrightarrow{\mathrm{RP}}| = |\overrightarrow{\mathrm{RS}}|$.

(d) Show that one of the possible positions for S is $\mathrm{S}_1(-4, 0, 4)$ and find the coordinates of the other possible position, S_2. *[6 marks]*

(e) Find a vector equation of the line which passes through R and bisects $\mathrm{P\widehat{R}S_1}$. *[4 marks]*

Question 9

[Maximum mark: 5]

Consider the system of equations

$$\begin{aligned} 0.1x - 1.7y + 0.9z &= -4.4 \\ -2.4x + 0.3y + 3.2z &= 1.2 \\ 2.5x + 0.6y - 3.7z &= 0.8\,. \end{aligned}$$

(a) Express the system of equations in matrix form. *[2 marks]*

(b) Find the solution to the system of equations. *[3 marks]*

Question 10

[Maximum mark: 6]

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $\boldsymbol{a} = (3\cos\theta + 6)\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = (\cos\theta - 2)\boldsymbol{i} + (1 + \sin\theta)\boldsymbol{j}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular,

(a) show that $3\sin^2\theta - 7\sin\theta + 2 = 0$; *[3]*

(b) find the smallest possible positive value of θ. *[3]*

Question 11

[Maximum mark: 7]

A line L_1 has equation $\boldsymbol{r} = \begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

A line L_2 passing through the origin intersects L_1 and is perpendicular to L_1.

(a) Find a vector equation of L_2. *[5]*

(b) Determine the shortest distance from the origin to L_1. *[2]*

Question 12

[Maximum mark: 6]

A system of equations is given below.

$$\begin{aligned} x+2y-z&=2 \\ 2x+y+z&=1 \\ -x+4y+az&=4 \end{aligned}$$

(a) Find the value of a so that the system does not have a unique solution. *[4]*

(b) Show that the system has a solution for any value of a. *[2]*

Question 13

[Maximum mark: 6]

Consider the two planes

$$\pi_1 : 4x+2y-z=8$$

$$\pi_2 : x+3y+3z=3.$$

Find the angle between π_1 and π_2, giving your answer correct to the nearest degree.

Question 14

[Maximum mark: 8]

The lines l_1 and l_2 are defined as

$$l_1 : \frac{x-1}{3}=\frac{y-5}{2}=\frac{z-12}{-2}$$

$$l_2 : \frac{x-1}{8}=\frac{y-5}{11}=\frac{z-12}{6}.$$

The plane π contains both l_1 and l_2.

(a) Find the Cartesian equation of π. *[4]*

The line l_3 passing through the point $(4, 0, 8)$ is perpendicular to π.

(b) Find the coordinates of the point where l_3 meets π. *[4]*

Question 15

[Maximum mark: 8]

Let $\boldsymbol{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 4 \\ \lambda \\ 10 \end{pmatrix}$.

(a) Find the value of λ for $\boldsymbol{v}$ and $\boldsymbol{w}$ to be parallel. [2]

(b) Find the value of λ for $\boldsymbol{v}$ and $\boldsymbol{w}$ to be perpendicular. [2]

(c) Find the two values of λ if the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 10°. [4]

Question 16

[Maximum mark: 7]

Consider the vectors given by $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, where a and b are constants.

It is given that $\boldsymbol{u} \times \boldsymbol{v} = 4\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, where c is a constant.

(a) Find the value of each of the constants a, b and c. [5]

(b) Hence find the Cartesian equation of the plane containing the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and passing through the point $(0, 0, 0)$. [2]

Question 17

[Maximum mark: 9]

Consider the following system of equations

$$2x + y + 6z = 0$$
$$4x + 3y + 14z = 4$$
$$2x - 2y + (\alpha - 2)z = \beta - 12 .$$

(a) Find conditions on α and β for which

(i) the system has no solutions;

(ii) the system has only one solution;

(iii) the system has an infinite number of solutions. [6]

(b) In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. [3]

Question 18

[Maximum mark: 18]

The equations of the lines L_1 and L_2 are

$$L_1 : \boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$L_2 : \boldsymbol{r}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$$

(a) Show that the lines L_1 and L_2 are skew. [4]

(b) Find the acute angle between the lines L_1 and L_2. [4]

(c) (i) Find a vector perpendicular to both lines.

(ii) Hence determine an equation of the line L_3 that is perpendicular to both L_1 and L_2 and intersects both lines. [10]

Question 19

[Maximum mark: 4]

The three planes having Cartesian equations $2x + 3y - z = 11$, $x + 2y + z = 3$ and $5x - y - z = 10$ meet at a point P. Find the coordinates of P.

Question 20

[Maximum mark: 8]

Ed walks in a straight line from point $\mathrm{P}(-1, 4)$ to point $\mathrm{Q}(4, 16)$ with constant speed.
Ed starts from point P at time $t = 0$ and arrives at point Q at time $t = 3$, where t is measured in hours.
Given that, at time t, Ed's position vector, relative to the origin, can be given in the form, $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$,

(a) find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. [3]

Roderick is at a point $\mathrm{C}(11, 9)$. During Ed's walk from P to Q Roderick wishes to signal to Ed. He decides to signal when Ed is at the closest point to C.

(b) Find the time when Roderick signals to Ed. [5]

Question 21

[Maximum mark: 4]

The points A and B have position vectors $\overrightarrow{\mathrm{OA}} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\overrightarrow{\mathrm{OB}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(a) Find $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}}$. [2]

(b) Hence find the area of the triangle OAB. [2]

Question 22

[Maximum mark: 8]

OACB is a parallelogram with $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$ and $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors.

(a) Show that

(i) $|\overrightarrow{\mathrm{OC}}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$;

(ii) $|\overrightarrow{\mathrm{AB}}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$. [4]

(b) Given that $|\overrightarrow{\mathrm{OC}}| = |\overrightarrow{\mathrm{AB}}|$, prove that OACB is a rectangle. [4]

Question 23

[Maximum mark: 5]

Find the acute angle between the planes with equations $x + y + z = 3$ and $2x - z = 2$.

Question 24

[Maximum mark: 6]

Find the Cartesian equation of plane Π containing the points $\mathrm{A}(6, 2, 1)$ and $\mathrm{B}(3, -1, 1)$ and perpendicular to the plane $x + 2y - z - 6 = 0$.

Question 25

[Maximum mark: 4]

Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \neq \mathbf{0}$ prove that $\boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ where s is a scalar.

Question 26

[Maximum mark: 22]

The points A, B and C have the following position vectors with respect to an origin O.

$$\vec{\mathrm{OA}} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

$$\vec{\mathrm{OB}} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

$$\vec{\mathrm{OC}} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$$

(a) Find the vector equation of the line (BC). [3]

(b) Determine whether or not the lines (OA) and (BC) intersect. [6]

(c) Find the Cartesian equation of the plane Π_1, which passes through C and is perpendicular to $\vec{\mathrm{OA}}$. [3]

(d) Show that the line (BC) lies in the plane Π_1. [2]

The plane Π_2 contains the points O, A and B and the plane Π_3 contains the points O, A and C.

(e) Verify that $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2. [3]

(f) Find a vector perpendicular to the plane Π_3. [1]

(g) Find the acute angle between the planes Π_2 and Π_3. [4]

Question 27

[Maximum mark: 15]

Two submarines A and B have their routes planned so that their positions at time t hours, $0 \le t < 20$, would be defined by the position vectors $\boldsymbol{r}_A = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -0.15 \end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t \begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix}$ relative to a fixed point on the surface of the ocean (all lengths are in kilometres).

(a) Show that the two submarines would collide at a point P and write down the coordinates of P. [4]

To avoid the collision submarine B adjusts its velocity so that its position vector is now given by

$$\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t \begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}.$$

(b) (i) Show that submarine B travels in the same direction as originally planned.

(ii) Find the value of t when submarine B passes through P. [3]

(c) (i) Find an expression for the distance between the two submarines in terms of t.

(ii) Find the value of t when the two submarines are closest together.

(iii) Find the distance between the two submarines at this time. [8]

Question 28

[Maximum mark: 19]

The plane Π_1 contains the points P$(1, 6, -7)$, Q$(0, 1, 1)$ and R$(2, 0, -4)$.

(a) Find the Cartesian equation of the plane containing P, Q and R. [6]

The Cartesian equation of the plane Π_2 is given by $x - 3y - z = 3$.

(b) Given that Π_1 and Π_2 meet in a line L, verify that the vector equation of L can be given by $\boldsymbol{r} = \begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$. [3]

Continue..

The Cartesian equation of the plane Π_3 is given by $ax + by + cz = 1$.

(c) Given that Π_3 is parallel to the line L, show that $a + 2b - 5c = 0$. [1]

Consider the case that Π_3 contains L.

(d) (i) Show that $5a - 7c = 4$.

(ii) Given that Π_3 is equally inclined to both Π_1 and Π_2, determine two distinct possible Cartesian equations for Π_3. [9]