Subject – Math(Higher Level)
Topic - Vector
Year - Nov 2011 – Nov 2019

Question 1

(a) (i) $|\boldsymbol{a}-\boldsymbol{b}| = |\boldsymbol{a}+\boldsymbol{b}|$

$\Rightarrow (\boldsymbol{a}-\boldsymbol{b})\bullet(\boldsymbol{a}-\boldsymbol{b}) = (\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{a}+\boldsymbol{b})$ ***(M1)***

$\Rightarrow |\boldsymbol{a}|^2 - 2\boldsymbol{a}\bullet\boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a}\bullet\boldsymbol{b} + |\boldsymbol{b}|^2$ ***A1***

$\Rightarrow 4\boldsymbol{a}\bullet\boldsymbol{b} = 0 \Rightarrow \boldsymbol{a}\bullet\boldsymbol{b} = 0$ ***A1***

therefore ***a*** and ***b*** are perpendicular ***R1***

Note: Allow use of 2-d components.

Note: Do not condone sloppy vector notation, so we must see something to the effect that $|\boldsymbol{c}|^2 = \boldsymbol{c.c}$ is clearly being used for the ***M1***.

Note: Allow a correct geometric argument, for example that the diagonals of a parallelogram have the same length only if it is a rectangle.

(ii) $|\boldsymbol{a}\times\boldsymbol{b}|^2 = (|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***M1A1***

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\bullet\boldsymbol{b})^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***M1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***A1***

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\bullet\boldsymbol{b})^2$ ***AG***

[8 marks]

(b) (i) area of triangle $= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ ***(M1)***

$= \frac{1}{2}|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})|$ ***A1***

$= \frac{1}{2}|\boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{b}\times-\boldsymbol{a} + -\boldsymbol{a}\times\boldsymbol{c} + -\boldsymbol{a}\times-\boldsymbol{a}|$ ***A1***

$\boldsymbol{b}\times-\boldsymbol{a} = \boldsymbol{a}\times\boldsymbol{b}$; $\boldsymbol{c}\times\boldsymbol{a} = -\boldsymbol{a}\times\boldsymbol{c}$; $-\boldsymbol{a}\times-\boldsymbol{a} = 0$ ***M1***

hence, area of triangle is $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{c}\times\boldsymbol{a}|$ ***AG***

(ii) D is the foot of the perpendicular from B to AC

area of triangle ABC $= \frac{1}{2}|\overrightarrow{AC}||\overrightarrow{BD}|$ ***A1***

therefore

$\frac{1}{2}|\overrightarrow{AC}||\overrightarrow{BD}| = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ ***M1***

hence, $|\overrightarrow{BD}| = \frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{|\overrightarrow{AC}|}$ ***A1***

$= \frac{|\boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{c}\times\boldsymbol{a}|}{|\boldsymbol{c}-\boldsymbol{a}|}$ ***AG***

[7 marks]

Total [15 marks]

Question 2

perpendicular when $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix} = 0$ ***(M1)***

$\Rightarrow -1 + 4\sin x \cos x = 0$ ***A1***

$\Rightarrow \sin 2x = \frac{1}{2}$ ***M1***

$\Rightarrow 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$ ***A1A1***

ote: Accept answers in degrees.

[5 marks]

Question 3

METHOD 1

(a) $9t_A = 7 - 4t_B$ and
$3 - 6t_A = -6 + 7t_B$ ***M1A1***

solve simultaneously

$t_A = \frac{1}{3},\ t_B = 1$ ***A1***

Note: Only need to see one time for the ***A1***.

therefore meet at $(3, 1)$ ***A1***

[4 marks]

(b) boats do not collide because the two times $\left(t_A = \frac{1}{3},\ t_B = 1\right)$ ***(A1)***
are different ***R1***

[2 marks]

Total [6 marks]

METHOD 2

(a) path of A is a straight line: $y = -\frac{2}{3}x + 3$ ***M1A1***

Note: Award ***M1*** for an attempt at simultaneous equations.

path of B is a straight line: $y = -\frac{7}{4}x + \frac{25}{4}$ ***A1***

$-\frac{2}{3}x + 3 = -\frac{7}{4}x + \frac{25}{4}\ (\Rightarrow x = 3)$
so the common point is $(3, 1)$ ***A1***

[4 marks]

(b) for boat A, $9t = 3 \Rightarrow t = \frac{1}{3}$ and for boat B, $7 - 4t = 3 \Rightarrow t = 1$ ***A1***

times are different so boats do not collide ***R1AG***

[2 marks]

Total [6 marks]

Question 4

(a) (i) $\vec{AB} = \vec{OB} - \vec{OA} = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ (or in column vector form) *(A1)*

Note: Award ***A1*** if any one of the vectors, or its negative, representing the sides of the triangle is seen.

$|\vec{AB}| = |5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}| = \sqrt{30}$

$|\vec{BC}| = |-\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}| = \sqrt{11}$

$|\vec{CA}| = |-4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}| = \sqrt{33}$ ***A2***

Note: Award ***A1*** for two correct and ***A0*** for one correct.

(ii) **METHOD 1**

$\cos BAC = \frac{20+4+2}{\sqrt{30}\sqrt{33}}$ ***M1A1***

Note: Award ***M1*** for an attempt at the use of the scalar product for two vectors representing the sides AB and AC, or their negatives, ***A1*** for the correct computation using their vectors.

$= \frac{26}{\sqrt{990}} \left(= \frac{26}{3\sqrt{110}} \right)$ ***A1***

Note: Candidates who use the modulus need to justify it – the angle is not stated in the question to be acute.

METHOD 2

using the cosine rule

$\cos BAC = \frac{30+33-11}{2\sqrt{30}\sqrt{33}}$ ***M1A1***

$= \frac{26}{\sqrt{990}} \left(= \frac{26}{3\sqrt{110}} \right)$ ***A1***

[6 marks]

(b) (i) $\vec{BC} \times \vec{CA} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ -4 & 4 & 1 \end{vmatrix}$ ***A1***

$= ((-3)\times 1 - 1\times 4)\boldsymbol{i} + (1\times(-4) - (-1)\times 1)\boldsymbol{j} + ((-1)\times 4 - (-3)\times(-4))\boldsymbol{k}$ ***M1A1***

$= -7\boldsymbol{i} - 3\boldsymbol{j} - 16\boldsymbol{k}$ ***AG***

(ii) the area of $\Delta ABC = \frac{1}{2}\left|\vec{BC} \times \vec{CA}\right|$ *(M1)*

$\frac{1}{2}\sqrt{(-7)^2 + (-3)^2 + (-16)^2}$ ***A1***

$= \frac{1}{2}\sqrt{314}$ ***AG***

[5 marks]

(c) attempt at the use of "$(\boldsymbol{r}-\boldsymbol{a})\bullet\boldsymbol{n}=0$" ***(M1)***

using $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{a}=\overrightarrow{\text{OA}}$ and $\boldsymbol{n}=-7\boldsymbol{i}-3\boldsymbol{j}-16\boldsymbol{k}$ ***(A1)***

$7x+3y+16z=47$ ***A1***

Note:	Candidates who adopt a 2-parameter approach should be awarded, ***A1*** for correct 2-parameter equations for x, y and z; ***M1*** for a serious attempt at elimination of the parameters; ***A1*** for the final Cartesian equation.

[3 marks]

(d) $\boldsymbol{r}=\overrightarrow{\text{OA}}+t\overrightarrow{\text{AB}}$ (or equivalent) ***M1***

$\boldsymbol{r}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$ ***A1***

Note:	Award ***M1A0*** if "$\boldsymbol{r}=$" is missing.

Note:	Accept forms of the equation starting with B or with the direction reversed.

[2 marks]

(e) (i) $\overrightarrow{\text{OD}}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$

statement that $\overrightarrow{\text{OD}}\bullet\overrightarrow{\text{BC}}=0$ ***(M1)***

$$\begin{pmatrix}-1+5t\\2-t\\3-2t\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\1\end{pmatrix}=0$$ ***A1***

$-2-4t=0$ or $t=-\dfrac{1}{2}$ ***A1***

coordinates of D are $\left(-\dfrac{7}{2},\dfrac{5}{2},4\right)$ ***A1***

Note:	Different forms of $\overrightarrow{\text{OD}}$ give different values of t, but the same final answer.

(ii) $t<0 \Rightarrow$ D is not between A and B ***R1***

[5 marks]

Total [21 marks]

Question 5

(a) $\vec{CA} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ *(A1)*

$\vec{CB} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ *(A1)*

Note: If $\vec{AC}$ and $\vec{BC}$ found correctly award ***(A1) (A0)***.

$\vec{CA} \times \vec{CB} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$ ***(M1)***

$\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$ ***A1***

[4 marks]

(b) **METHOD 1**

$\frac{1}{2}\left|\vec{CA} \times \vec{CB}\right|$ ***(M1)***

$= \frac{1}{2}\sqrt{(-2)^2 + (-3)^2 + 4^2}$ ***(A1)***

$= \frac{\sqrt{29}}{2}$ ***A1***

METHOD 2

attempt to apply $\frac{1}{2}|CA||CB|\sin C$ ***(M1)***

$CA.CB = \sqrt{5}.\sqrt{6}\cos C \Rightarrow \cos C = \frac{1}{\sqrt{30}} \Rightarrow \sin C = \frac{\sqrt{29}}{\sqrt{30}}$ ***(A1)***

area $= \frac{\sqrt{29}}{2}$ ***A1***

[3 marks]

(c) **METHOD 1**

$$\mathbf{r}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}$$ ***M1A1***

$\Rightarrow -2x-3y+4z=-2$ ***A1***

$\Rightarrow 2x+3y-4z=2$ ***AG***

METHOD 2

$-2x-3y+4z=\mathrm{d}$

substituting a point in the plane ***M1A1***

$\mathrm{d}=-2$ ***A1***

$\Rightarrow -2x-3y+4z=-2$

$\Rightarrow 2x+3y-4z=2$ ***AG***

Note: Accept verification that all 3 vertices of the triangle lie on the given plane.

[3 marks]

(d) **METHOD 1**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -7 \\ -14 \\ -14 \end{pmatrix}$$ ***M1A1***

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$z = 0 \Rightarrow y = 0,\ x = 1$ ***(M1)(A1)***

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 2

eliminate 1 of the variables, *eg* x ***M1***
$-7y + 7z = 0$ ***(A1)***
introduce a parameter ***M1***
$\Rightarrow z = \lambda,$

$y = \lambda,\ x = 1 + \frac{\lambda}{2}$ ***(A1)***

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 3

$z = t$ ***M1***
write x and y in terms of $t \Rightarrow 4x - y = 4 + t,\ 2x + 3y = 2 + 4t$ or equivalent ***A1***
attempt to eliminate x or y ***M1***
x, y, z expressed in parameters
$\Rightarrow z = t,$

$y = t,\ x = 1 + \frac{t}{2}$ ***A1***

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

[5 marks]

(e) **METHOD 1**

direction of the line is perpendicular to the normal of the plane

$$\begin{pmatrix}16\\ \alpha\\ -3\end{pmatrix}\cdot\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=0$$ ***M1A1***

$16+2\alpha-6=0\Rightarrow\alpha=-5$ ***A1***

METHOD 2

solving line/plane simultaneously

$16(1+\lambda)+2\alpha\lambda-6\lambda=\beta$ ***M1A1***

$16+(10+2\alpha)\lambda=\beta$

$\Rightarrow\alpha=-5$ ***A1***

METHOD 3

$$\begin{vmatrix}2 & 3 & -4\\ 4 & -1 & -1\\ 16 & \alpha & -3\end{vmatrix}=0$$ ***M1***

$2(3+\alpha)-3(-12+16)-4(4\alpha+16)=0$ ***A1***

$\Rightarrow\alpha=-5$ ***A1***

METHOD 4

attempt to use row reduction on augmented matrix ***M1***

to obtain $\left(\begin{array}{ccc|c}2 & 3 & -4 & 2\\ 0 & -1 & 1 & 0\\ 0 & 0 & \alpha+5 & \beta-16\end{array}\right)$ ***A1***

$\Rightarrow\alpha=-5$ ***A1***

[3 marks]

(f) $\alpha=-5$ ***A1***

$\beta\neq16$ ***A1***

[2 marks]

Total [20 marks]

Question 6

(a) $|\vec{OA}| = |\vec{CB}| = |\vec{OC}| = |\vec{AB}| = 6$ (therefore a rhombus) ***A1A1***

> **Note:** Award ***A1*** for two correct lengths, ***A2*** for all four.

> **Note:** Award ***A1A0*** for $\vec{OA} = \vec{CB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ or $\vec{OC} = \vec{AB} = \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix}$ if no magnitudes are shown.

$\vec{OA} \cdot \vec{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = 0$ (therefore a square) ***A1***

> **Note:** Other arguments are possible with a minimum of three conditions.

[3 marks]

(b) $M\left(3, -\frac{\sqrt{24}}{2}, \frac{\sqrt{12}}{2}\right)\left(= \left(3, -\sqrt{6}, \sqrt{3}\right)\right)$ ***A1***

[1 mark]

(c) **METHOD 1**

$\vec{OA} \times \vec{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -6\sqrt{12} \\ -6\sqrt{24} \end{pmatrix} \left(= \begin{pmatrix} 0 \\ -12\sqrt{3} \\ -12\sqrt{6} \end{pmatrix} \right)$ ***M1A1***

> **Note:** Candidates may use other pairs of vectors.

equation of plane is $-6\sqrt{12}y - 6\sqrt{24}z = d$

any valid method showing that $d = 0$ ***M1***

$\Pi : y + \sqrt{2}z = 0$ ***AG***

METHOD 2

equation of plane is $ax + by + cz = d$

substituting O to find $d = 0$ ***(M1)***

substituting two points (A, B, C or M) ***M1***

eg

$6a = 0$, $-\sqrt{24}b + \sqrt{12}c = 0$ ***A1***

$\Pi : y + \sqrt{2}z = 0$ ***AG***

(d) $\boldsymbol{r}=\begin{pmatrix}3\\-\sqrt{6}\\\sqrt{3}\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\\sqrt{2}\end{pmatrix}$ *A1A1A1*

Note: Award ***A1*** for $\boldsymbol{r}=$, ***A1A1*** for two correct vectors.

[3 marks]

(e) Using $y=0$ to find λ ***M1***

Substitute their λ into their equation from part (d) ***M1***

D has coordinates $\left(3,0,3\sqrt{3}\right)$ ***A1***

[3 marks]

(f) λ for point E is the negative of the λ for point D ***(M1)***

Note: Other possible methods may be seen.

E has coordinates $\left(3,-2\sqrt{6},-\sqrt{3}\right)$ ***A1A1***

Note: Award ***A1*** for each of the y and z coordinates.

[3 marks]

(g) (i) $\overrightarrow{\mathrm{DA}}\text{g}\overrightarrow{\mathrm{DO}}=\begin{pmatrix}3\\0\\-3\sqrt{3}\end{pmatrix}\text{g}\begin{pmatrix}-3\\0\\-3\sqrt{3}\end{pmatrix}=18$ ***M1A1***

$\cos\mathrm{O\hat{D}A}=\dfrac{18}{\sqrt{36}\sqrt{36}}=\dfrac{1}{2}$ ***M1***

hence $\mathrm{O\hat{D}A}=60°$ ***A1***

Note: Accept method showing OAD is equilateral.

(ii) OABCDE is a regular octahedron (accept equivalent description) ***A2***

Note: ***A2*** for saying it is made up of 8 equilateral triangles Award ***A1*** for two pyramids, ***A1*** for equilateral triangles. (can be either stated or shown in a sketch – but there must be clear indication the triangles are equilateral)

[6 marks]

Question 7

(a) $\overrightarrow{PR} = \boldsymbol{a} + \boldsymbol{b}$ ***A1***

$\overrightarrow{QS} = \boldsymbol{b} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) $\overrightarrow{PR} \cdot \overrightarrow{QS} = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{b} - \boldsymbol{a})$ ***M1***

$= |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$ ***A1***

for a rhombus $|\boldsymbol{a}| = |\boldsymbol{b}|$ ***R1***

hence $|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 = 0$ ***A1***

Note: Do not award the final ***A1*** unless ***R1*** is awarded.

hence the diagonals intersect at right angles ***AG***

[4 marks]

Question 8

(a) direction vector $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\overrightarrow{BA} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$ ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or equivalent ***A1***

Note: Do not award final ***A1*** unless '$r = \ldots$' (or equivalent) seen. Allow FT on direction vector for final ***A1***.

[2 marks]

(b) both lines expressed in parametric form:

L_1:

$x = 1 + t$

$y = 3t$

$z = 4 - 5t$

L_2:

$x = 1 + 3s$

$y = -2 + s$ ***M1A1***

$z = -2s + 1$

Notes: Award ***M1*** for an attempt to convert L_2 from Cartesian to parametric form. Award ***A1*** for correct parametric equations for L_1 and L_2. Allow ***M1A1*** at this stage if same parameter is used in both lines.

attempt to solve simultaneously for x and y: ***M1***

$1 + t = 1 + 3s$

$3t = -2 + s$

$t = -\frac{3}{4}$, $s = -\frac{1}{4}$ ***A1***

substituting both values back into z values respectively gives $z = \frac{31}{4}$

and $z = \frac{3}{2}$ so a contradiction ***R1***

therefore L_1 and L_2 are skew lines ***AG***

[5 marks]

(c) finding the cross product:

$$\begin{pmatrix}1\\3\\-5\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}$$ ***(M1)***

$= -\boldsymbol{i} - 13\boldsymbol{j} - 8\boldsymbol{k}$ ***A1***

Note: Accept $\boldsymbol{i} + 13\boldsymbol{j} + 8\boldsymbol{k}$

$-1(0) - 13(1) - 8(-2) = 3$ ***(M1)***

$\Rightarrow -x - 13y - 8z = 3$ or equivalent ***A1***

[4 marks]

(d) (i) $$(\cos\theta =)\frac{\begin{pmatrix}k\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\sqrt{k^2+1+1}\times\sqrt{1+1}}$$ ***M1***

Note: Award ***M1*** for an attempt to use angle between two vectors formula.

$$\frac{\sqrt{3}}{2} = \frac{k+1}{\sqrt{2(k^2+2)}}$$ ***A1***

obtaining the quadratic equation

$4(k+1)^2 = 6(k^2+2)$ ***M1***

$k^2 - 4k + 4 = 0$

$(k-2)^2 = 0$

$k = 2$ ***A1***

Note: Award ***M1A0M1A0*** if $\cos 60^{\circ}$ is used $(k = 0$ or $k = -4)$.

(a) direction vector $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\overrightarrow{BA} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$ ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or equivalent ***A1***

Note: Do not award final ***A1*** unless '$r = \text{K}$' (or equivalent) seen.
Allow FT on direction vector for final ***A1***.

[2 marks]

(b) both lines expressed in parametric form:

L_1:
$x = 1 + t$
$y = 3t$
$z = 4 - 5t$

L_2:
$x = 1 + 3s$
$y = -2 + s$ ***M1A1***
$z = -2s + 1$

Notes: Award ***M1*** for an attempt to convert L_2 from Cartesian to parametric form.
Award ***A1*** for correct parametric equations for L_1 and L_2.
Allow ***M1A1*** at this stage if same parameter is used in both lines.

attempt to solve simultaneously for x and y: ***M1***

$1 + t = 1 + 3s$
$3t = -2 + s$
$t = -\frac{3}{4}$, $s = -\frac{1}{4}$ ***A1***

substituting both values back into z values respectively gives $z = \frac{31}{4}$ and $z = \frac{3}{2}$ so a contradiction ***R1***

therefore L_1 and L_2 are skew lines ***AG***

[5 marks]

(ii) $\boldsymbol{r}=\begin{pmatrix}3\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$

substituting into the equation of the plane Π_2:

$3+2\lambda+\lambda=12$ ***M1***

$\lambda=3$ ***A1***

point P has the coordinates:

$(9, 3, -2)$ ***A1***

Notes:	Accept $9\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$ and $\begin{pmatrix}9\\3\\-2\end{pmatrix}$.
	Do not allow FT if two values found for *k*.

[7 marks]

Total [18 marks]

Question 9

METHOD 1

$\left|\vec{OP}\right| = \sqrt{(1+s)^2 + (3+2s)^2 + (1-s)^2} \quad \left(= \sqrt{6s^2 + 12s + 11}\right)$ ***A1***

te: Award ***A1*** if the square of the distance is found.

EITHER

attempt to differentiate: $\frac{d}{ds}\left|\vec{OP}\right|^2 \left(= 12s + 12\right)$ ***M1***

attempting to solve $\frac{d}{ds}\left|\vec{OP}\right|^2 = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt to differentiate: $\frac{d}{ds}\left|\vec{OP}\right| \left(= \frac{6s + 6}{\sqrt{6s^2 + 12s + 11}}\right)$ ***M1***

attempting to solve $\frac{d}{ds}\left|\vec{OP}\right| = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt at completing the square: $\left(\left|\vec{OP}\right|^2 = 6(s + 1)^2 + 5\right)$ ***M1***

minimum value ***(M1)***
occurs at $s = -1$ ***(A1)***

THEN

the minimum length of $\vec{OP}$ is $\sqrt{5}$ ***A1***

METHOD 2

the length of $\vec{OP}$ is a minimum when $\vec{OP}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***(R1)***

Question 10

(a) (i) $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$ *(M1)*

$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ *A1*

(ii) $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$ *M1*

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ *A1*

$\overrightarrow{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ *AG*

[4 marks]

(b) (i) $\overrightarrow{RA} = \frac{1}{3}\overrightarrow{BA}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ *A1*

(ii) $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$

$= \frac{2}{3}\left(\overrightarrow{RA} + \overrightarrow{AS}\right)$ *(M1)*

$= \frac{2}{3}\left(\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})\right)$ or equivalent. *A1A1*

$= \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$ *A1*

$\overrightarrow{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ *AG*

[5 marks]

(c) $\overrightarrow{BT} = \overrightarrow{BR} + \overrightarrow{RT}$

$= \frac{2}{3}\overrightarrow{BA} + \overrightarrow{RT}$ *(M1)*

$= \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} - \frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ *A1*

$\overrightarrow{BT} = \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$ *A1*

point B is common to $\overrightarrow{BT}$ and $\overrightarrow{BM}$ and $\overrightarrow{BT} = \frac{8}{9}\overrightarrow{BM}$ *R1R1*

so T lies on [BM] *AG*

[5 marks]

Total [14 marks]

Question 11

(a) $\vec{OP} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$\vec{OQ} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\vec{PQ} = \vec{OQ} - \vec{OP}$ ***(M1)***

$\vec{PQ} = \boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k} - \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$= (1 - \lambda + \mu)\boldsymbol{i} + (-1 - \lambda - \mu)\boldsymbol{j} + (-4 - \lambda + 2\mu)\boldsymbol{k}$ ***A1***

[2 marks]

(b) **METHOD 1**

use of scalar product ***M1***

perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ gives

$(1 - \lambda + \mu) + (-1 - \lambda - \mu) + (-4 - \lambda + 2\mu) = 0$

$\Rightarrow -3\lambda + 2\mu = 4$ ***A1***

perpendicular to $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ gives

$(1 - \lambda + \mu) - (-1 - \lambda - \mu) + 2(-4 - \lambda + 2\mu) = 0$

$\Rightarrow -2\lambda + 6\mu = 6$ ***A1***

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$ ***A1A1***

METHOD 2

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***M1A1***

$\vec{PQ} = a(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$1 - \lambda + \mu = 3a$

$-1 - \lambda - \mu = -a$ ***A1***

$-4 - \lambda + 2\mu = -2a$

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$ ***A1A1***

[5 marks]

(c) $\vec{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ ***A1***

shortest distance $= \left|\vec{PQ}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[3 marks]

(d) **METHOD 1**

vector perpendicular to Π is given by vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ ***(R1)***

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***(M1)A1***

so equation of Π is $3x - y - 2z + d = 0$

through $(1, 2, 3) \Rightarrow d = 5$ ***M1***

so equation is $3x - y - 2z + 5 = 0$ ***A1***

METHOD 2

from part (b) $\overrightarrow{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ is a vector perpendicular to Π ***R1A2***

so equation of Π is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + c = 0$

through $(1, 2, 3) \Rightarrow c = \frac{30}{7}$ ***M1***

so equation is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + \frac{30}{7} = 0 \quad (3x - y - 2z + 5 = 0)$ ***A1***

Note: Allow other methods *ie* via vector parametric equation.

[5 marks]

(e) $\overrightarrow{OT} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \eta(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$T = (2 + 3\eta, 1 - \eta, -1 - 2\eta)$ lies on Π implies

$3(2 + 3\eta) - (1 - \eta) - 2(-1 - 2\eta) + 5 = 0$ ***M1***

$\Rightarrow 12 + 14\eta = 0 \Rightarrow \eta = -\frac{6}{7}$ ***A1***

Note: If no marks awarded in (d) but correct vector product calculated in (e) award ***M1A1*** in (d).

[2 marks]

(f) $\left|\overrightarrow{BT}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[2 marks]

(g) they agree ***A1***

Note: FT is inappropriate here.

$\overrightarrow{BT}$ is perpendicular to both Π and l_2

so its length is the shortest distance between Π and l_2 which is the shortest distance between l_1 and l_2 ***R1***

[2 marks]

Total [21 marks]

Question 12

(a) $\vec{BR} = \vec{BA} + \vec{AR}$ $\left(= \vec{BA} + \frac{1}{2}\vec{AC}\right)$ ***(M1)***

$= (\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***A1***

[2 marks]

(b) (i) $\mathrm{r}_{\mathrm{BR}} = \boldsymbol{b} + \lambda\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= \frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c}\right)$ ***A1A1***

Note: Award ***A1A0*** if the $\mathrm{r} =$ is omitted in an otherwise correct expression/equation.

(ii) $\vec{AQ} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***(A1)***

$\mathrm{r}_{\mathrm{AQ}} = \boldsymbol{a} + \mu\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}\right)$ ***A1***

(iii) when $\vec{AQ}$ and $\vec{BP}$ intersect we will have $\mathrm{r}_{\mathrm{BR}} = \mathrm{r}_{\mathrm{AQ}}$ ***(M1)***

$\frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c} = (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}$

attempt to equate the coefficients of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ ***M1***

$$\left.\begin{aligned} \frac{\lambda}{2} &= 1-\mu \\ 1-\lambda &= \frac{\mu}{2} \\ \frac{\lambda}{2} &= \frac{\mu}{2} \end{aligned}\right\}$$ ***(A1)***

$\lambda = \frac{2}{3}$ or $\mu = \frac{2}{3}$ ***A1***

substituting parameters back into one of the equations ***M1***

$\vec{OG} = \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{a} + \left(1 - \frac{2}{3}\right)\boldsymbol{b} + \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{c} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ ***AG***

[9 marks]

(c) $\vec{CP} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}$ ***(M1)A1***

so we have that $\mathrm{r}_{\mathrm{CP}} = \boldsymbol{c} + \beta\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}\right)$ and when $\beta = \frac{2}{3}$ the line passes through

the point G (*ie*, with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$) ***R1***

hence [AQ], [BR] and [CP] all intersect in G ***AG***

[3 marks]

(d) $\vec{OG} = \frac{1}{3}\left(\begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}3\\7\\-5\end{pmatrix} + \begin{pmatrix}2\\2\\1\end{pmatrix}\right) = \begin{pmatrix}2\\4\\-1\end{pmatrix}$ ***A1***

Note: This independent mark for the vector may be awarded wherever the vector is calculated.

$\vec{AB} \times \vec{AC} = \begin{pmatrix}2\\4\\-6\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-6\\-6\\-6\end{pmatrix}$ ***M1A1***

$\vec{GX} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$ ***(M1)***

volume of Tetrahedron given by $\frac{1}{3} \times \text{Area ABC} \times \text{GX}$

$= \frac{1}{3}\left(\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|\right) \times \text{GX} = 12$ ***(M1)(A1)***

Note: Accept alternative methods, for example the use of a scalar triple product.

$= \frac{1}{6}\sqrt{(-6)^2 + (-6)^2 + (-6)^2} \times \sqrt{\alpha^2 + \alpha^2 + \alpha^2} = 12$ ***(A1)***

$= \frac{1}{6} 6\sqrt{3}\,|\alpha|\sqrt{3} = 12$

$\Rightarrow |\alpha| = 4$ ***A1***

Note: Condone absence of absolute value.

this gives us the position of X as $\begin{pmatrix}2\\4\\-1\end{pmatrix} \pm \begin{pmatrix}4\\4\\4\end{pmatrix}$

$X(6, 8, 3)$ or $(-2, 0, -5)$ ***A1***

Note: Award ***A1*** for either result.

[9 marks]

Total [23 marks]

Question 13

(a) angle between planes is equal to the angles between the normal to the planes **(M1)**

$\begin{pmatrix}4\\1\\1\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-1\end{pmatrix}=18$ **(A1)**

let θ be the angle between the normal to the planes

$\cos\theta=\frac{18}{\sqrt{18}\sqrt{26}}=\sqrt{\frac{18}{26}}\left(\text{or equivalent, for example }\sqrt{\frac{324}{468}}\text{ or }\sqrt{\frac{9}{13}}\right)$ ***M1A1***

[4 marks]

(b) (i) **METHOD 1**

$\begin{pmatrix}4\\1\\1\end{pmatrix}\times\begin{pmatrix}4\\3\\-1\end{pmatrix}=\begin{pmatrix}-4\\8\\8\end{pmatrix}$ ***M1A1***

which is a multiple of $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***R1AG***

Note: Allow any equivalent wording or $\begin{pmatrix}-4\\8\\8\end{pmatrix}=4\begin{pmatrix}-1\\2\\2\end{pmatrix}$, do not allow $\begin{pmatrix}-4\\8\\8\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$.

METHOD 2

let $z=t$ (or equivalent)

solve simultaneously to get ***M1***

$y=t-4,\ x=3-0.5t$ ***A1***

hence direction vector is $\begin{pmatrix}-0.5\\1\\1\end{pmatrix}$

which is a multiple of $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***R1AG***

METHOD 3

$\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = -4 + 2 + 2 = 0$ **M1A1**

$\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = -4 + 6 - 2 = 0$ **A1**

Note: If only one scalar product is found award ***M0A0A0***.

(ii) $\Pi_1 : 4 + 0 + 4 = 8$ and $\Pi_2 : 4 + 0 - 4 = 0$ **R1**

(iii) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ **A1A1**

Note: ***A1*** for "$\boldsymbol{r} =$" and a correct point on the line, ***A1*** for a parameter and a correct direction vector.

[6 marks]

(c) $\overrightarrow{AB} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} a-1 \\ b \\ -3 \end{pmatrix}$ ***(A1)***

$\begin{pmatrix} a-1 \\ b \\ -3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ **M1**

Note: Award ***M0*** for $\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$.

$-a + 1 + 2b - 6 = 0 \Rightarrow a - 2b = -5$ **A1**

lies on Π_1 so $4a + b + 1 = 8 \Rightarrow 4a + b = 7$ **M1**

$a = 1, b = 3$ **A1**

[5 marks]

(d) $AB = \sqrt{0^2 + 3^2 + (-3)^2} = 3\sqrt{2}$ ***M1AG***

(e) **METHOD 1**

$\left|\vec{AB}\right| = \left|\vec{AP}\right| = 3\sqrt{2}$ ***(M1)***

$\vec{AP} = t\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***(A1)***

$|3t| = 3\sqrt{2} \Rightarrow t = \pm\sqrt{2}$ ***(M1)A1***

$P\left(1-\sqrt{2},\ 2\sqrt{2},\ 4+2\sqrt{2}\right)$ and $\left(1+\sqrt{2},\ -2\sqrt{2},\ 4-2\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 2

let P have coordinates $(1-\lambda,\ 2\lambda,\ 4+2\lambda)$ ***M1***

$\vec{BA} = \begin{pmatrix}0\\-3\\3\end{pmatrix},\quad \vec{BP} = \begin{pmatrix}-\lambda\\2\lambda-3\\3+2\lambda\end{pmatrix}$ ***A1***

$\cos 45^0 = \dfrac{\vec{BA}\bullet\vec{BP}}{|BA||BP|}$ ***M1***

Note: Award ***M1*** even if AB rather than BA is used in the scalar product.

$\vec{BA}\bullet\vec{BP} = 18$

$\dfrac{1}{\sqrt{2}} = \dfrac{18}{\sqrt{18}\sqrt{9\lambda^2+18}}$

$\lambda = \pm\sqrt{2}$ ***A1***

$P\left(1-\sqrt{2},\ 2\sqrt{2},\ 4+2\sqrt{2}\right)$ and $\left(1+\sqrt{2},\ -2\sqrt{2},\ 4-2\sqrt{2}\right)$ ***A1***

Note: Accept answers given as position vectors.

[5 marks]

Total [21 marks]

Question 14

(a) $\boldsymbol{a} \times \boldsymbol{b} = -12\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ ***(M1)A1***

[2 marks]

(b) **METHOD 1**

$-12x - 2y - 3z = d$ ***M1***

$-12 \times 1 - 2 \times 0 - 3(-1) = d$ ***(M1)***

$\Rightarrow d = -9$ ***A1***

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$)

METHOD 2

$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-12\\-2\\-3\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}-12\\-2\\-3\end{pmatrix}$ ***M1A1***

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$) ***A1***

[3 marks]

Total [5 marks]

Question 15

(a) **METHOD 1**

$$l_1 : r = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = -3 + \beta \\ y = -2 + 4\beta \\ z = a + 2\beta \end{cases}$$ ***M1***

$$\frac{6-(-3+\beta)}{3} = \frac{(-2+4\beta)-2}{4} \Rightarrow 4 = \frac{4\beta}{3} \Rightarrow \beta = 3$$ ***M1A1***

$$\frac{6-(-3+\beta)}{3} = 1-(a+2\beta) \Rightarrow 2 = -5-a \Rightarrow a = -7$$ ***A1***

METHOD 2

$$\begin{cases} -3 + \beta = 6 - 3\lambda \\ -2 + 4\beta = 4\lambda + 2 \\ a + 2\beta = 1 - \lambda \end{cases}$$ ***M1***

attempt to solve ***M1***

$\lambda = 2,\ \beta = 3$ ***A1***

$a = 1 - \lambda - 2\beta = -7$ ***A1***

[4 marks]

(b) $$\vec{OP} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$$ ***A1***

$\therefore P(0, 10, -1)$

[2 marks]

Total [6 marks]

Question 16

(a) $\vec{AB} \times \vec{AD} = -\boldsymbol{i} + 10\boldsymbol{j} - 7\boldsymbol{k}$ ***M1A1***

area $= \left|\vec{AB} \times \vec{AD}\right| = \sqrt{1^2 + 10^2 + 7^2}$

$= 5\sqrt{6}\left(\sqrt{150}\right)$ ***A1***

[3 marks]

(b) **METHOD 1**

$\vec{AB} \cdot \vec{AD} = -4 - 2 - 6$ ***M1A1***

$= -12$

considering the sign of the answer

$\vec{AB} \cdot \vec{AD} < 0$, therefore angle $D\hat{A}B$ is obtuse ***M1***

(as it is a parallelogram), $A\hat{B}C$ is acute ***A1***

[4 marks]

METHOD 2

$\vec{BA} \cdot \vec{BC} = +4 + 2 + 6$ ***M1A1***

$= 12$

considering the sign of the answer ***M1***

$\vec{BA} \cdot \vec{BC} > 0 \Rightarrow A\hat{B}C$ is acute ***A1***

[4 marks]

Total [7 marks]

Question 17

(a) $\vec{AB} = \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 6 \\ -5 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ is not seen (or equivalent).

[3 marks]

(b) substitute line L in $\Pi : 4(6\lambda) - 3(3 - 8\lambda) + 2(-6 + 17\lambda) = 20$ ***M1***

$82\lambda = 41$

$\lambda = \frac{1}{2}$ ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$

so coordinate is $\left(3, -1, \frac{5}{2}\right)$ ***A1***

Note: Accept coordinate expressed as position vector $\begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$.

[3 marks]

Total [6 marks]

Question 18

(a) **EITHER**

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ ***M1***

row of zeroes implies infinite solutions, (or equivalent). ***R1***

Note: Award ***M1*** for any attempt at row reduction.

OR

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0$$ ***M1***

$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0$ with one valid point ***R1***

OR

$$x + y + 2z = -2$$
$$3x - y + 14z = 6$$
$$x + 2y = -5 \quad \Rightarrow x = -5 - 2y$$

substitute $x = -5 - 2y$ into the first two equations:

$$-5 - 2y + y + 2z = -2$$
$$3(-5 - 2y) - y + 14z = 6$$ ***M1***
$$-y + 2z = 3$$
$$-7y + 14z = 21$$

the latter two equations are equivalent (by multiplying by 7) therefore an infinite number of solutions. ***R1***

OR

for example, $7 \times R_1 - R_2$ gives $4x + 8y = -20$ ***M1***

this equation is a multiple of the third equation, therefore an infinite number of solutions. ***R1***

(b) let $y = t$ *M1*

then $x = -5 - 2t$ *A1*

$z = \frac{t+3}{2}$ *A1*

OR

let $x = t$ *M1*

then $y = \frac{-5-t}{2}$ *A1*

$z = \frac{1-t}{4}$ *A1*

OR

let $z = t$ *M1*
then $x = 1 - 4t$ *A1*
$y = -3 + 2t$ *A1*

OR

attempt to find cross product of two normal vectors:

eg: $\begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4i + 2j + k$ *M1A1*

$x = 1 - 4t$
$y = -3 + 2t$
$z = t$ *A1*
(or equivalent)

Total [5 marks]

Question 19

EITHER

eliminating a variable, x, for example to obtain $y + 3z = -16$ and $-5y - 3z = 8$ ***M1A1***
attempting to find the value of one variable ***M1***
point of intersection is $(-1, 2, -6)$ ***A1A1A1***

OR

attempting row reduction of relevant matrix, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & 3 & 1 & -1 \\ 1 & 2 & -2 & 15 \end{array}\right)$ ***M1***

correct matrix with two zeroes in a column, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & 5 & 3 & -8 \\ 0 & 1 & 3 & -16 \end{array}\right)$ ***A1***

further attempt at reduction ***M1***
point of intersection is $(-1, 2, -6)$ ***A1A1A1***

:e: Allow solution expressed as $x = -1$, $y = 2$, $z = -6$ for final ***A*** marks.

[6 marks]

Question 20

$\boldsymbol{c} \cdot (\boldsymbol{b} - \boldsymbol{a}) = 0$ ***M1***

te: Allow $c \cdot \overrightarrow{AB} = 0$ or similar for ***M1***.

$\boldsymbol{c} \cdot \boldsymbol{b} = \boldsymbol{c} \cdot \boldsymbol{a}$ ***A1***
$\boldsymbol{b} \cdot (\boldsymbol{c} - \boldsymbol{a}) = 0$
$\boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{a}$ ***A1***
$\boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a}$ ***M1***
$(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ ***A1***

hence $\boldsymbol{a}$ is perpendicular to $\overrightarrow{BC}$ ***AG***

te: Only award the final ***A1*** if a dot is used throughout to indicate scalar product.
Condone any lack of specific indication that the letters represent vectors.

[5 marks]

Question 21

(a) **EITHER**

$\boldsymbol{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix}$ ***A1A1***

and $\boldsymbol{n} \neq k\boldsymbol{d}$ ***R1***

OR

$\boldsymbol{n} \times \boldsymbol{d} = \begin{pmatrix} -5 \\ 3p-1 \\ 2-p \end{pmatrix}$ ***M1A1***

the vector product is non-zero for $p \in \mathbb{R}$ ***R1***

THEN

L is not perpendicular to Π ***AG***

[3 marks]

(b) **METHOD 1**

$(2 + p\lambda) + (q + 2\lambda) + 3(1 + \lambda) = 9$ ***M1***

$(q + 5) + (p + 5)\lambda = 9$ ***(A1)***

$p = -5$ and $q = 4$ ***A1A1***

METHOD 2

direction vector of line is perpendicular to plane, so

$\begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$ ***M1***

$p = -5$ ***A1***

$(2, q, 1)$ is common to both L and Π

either $\begin{pmatrix} 2 \\ q \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 9$ or by substituting into $x + y + 3z = 9$ ***M1***

$q = 4$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

α is the acute angle between $\boldsymbol{n}$ and L

if $\sin\theta = \frac{1}{\sqrt{11}}$ then $\cos\alpha = \frac{1}{\sqrt{11}}$ ***(M1)(A1)***

attempting to use $\cos\alpha = \frac{\boldsymbol{n}\cdot\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}$ or $\sin\theta = \frac{\boldsymbol{n}\cdot\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}$ ***M1***

$\frac{p+5}{\sqrt{11}\times\sqrt{p^2+5}} = \frac{1}{\sqrt{11}}$ ***A1A1***

$(p+5)^2 = p^2+5$ ***M1***

$10p = -20$ (or equivalent) ***A1***

$p = -2$ ***AG***

METHOD 2

α is the angle between $\boldsymbol{n}$ and L

if $\sin\theta = \frac{1}{\sqrt{11}}$ then $\sin\alpha = \frac{\sqrt{10}}{\sqrt{11}}$ ***(M1)A1***

attempting to use $\sin\alpha = \frac{|\boldsymbol{n}\times\boldsymbol{d}|}{|\boldsymbol{n}||\boldsymbol{d}|}$ ***M1***

$\frac{\sqrt{(-5)^2+(3p-1)^2+(2-p)^2}}{\sqrt{11}\times\sqrt{p^2+5}} = \frac{\sqrt{10}}{\sqrt{11}}$ ***A1A1***

$p^2 - p + 3 = p^2 + 5$ ***M1***

$-p + 3 = 5$ (or equivalent) ***A1***

$p = -2$ ***AG***

(ii) $p = -2$ and $z = -1 \Rightarrow \frac{x-2}{-2} = \frac{y-q}{2} = -2$ ***(A1)***

$x = 6$ and $y = q - 4$ ***(A1)***

this satisfies Π so $6 + q - 4 - 3 = 9$ ***M1***

$q = 10$ ***A1***

[11 marks]

Total [18 marks]

Question 22

METHOD 1

for eliminating one variable from two equations ***(M1)***

eg, $\begin{cases}(x+y+z=3)\\ 2x+2z=8\\ 2x+3z=11\end{cases}$ ***A1A1***

for finding correctly one coordinate

eg, $\Rightarrow \begin{cases}(x+y+z=3)\\ (2x+2z=8)\\ z=3\end{cases}$ ***A1***

for finding correctly the other two coordinates ***A1***

$\Rightarrow \begin{cases}x=1\\ y=-1\\ z=3\end{cases}$

the intersection point has coordinates $(1,\ -1,\ 3)$

METHOD 2

for eliminating two variables from two equations or using row reduction ***(M1)***

eg, $\begin{cases}(x+y+z=3)\\ -2y=2\\ z=3\end{cases}$ **or** $\left(\begin{array}{ccc|c}1 & 1 & 1 & 3\\ 0 & -2 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right)$ ***A1A1***

for finding correctly the other coordinates ***A1A1***

$\Rightarrow \begin{cases}x=1\\ y=-1\\ (z=3)\end{cases}$ **or** $\left(\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 3\end{array}\right)$

the intersection point has coordinates $(1,\ -1,\ 3)$

Question 23

(a) **METHOD 1**

$$l_1 : \mathbf{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = -3 + \beta \\ y = -2 + 4\beta \\ z = a + 2\beta \end{cases}$$ ***M1***

$$\frac{6-(-3+\beta)}{3} = \frac{(-2+4\beta)-2}{4} \Rightarrow 4 = \frac{4\beta}{3} \Rightarrow \beta = 3$$ ***M1A1***

$$\frac{6-(-3+\beta)}{3} = 1-(a+2\beta) \Rightarrow 2 = -5-a \Rightarrow a = -7$$ ***A1***

METHOD 2

$$\begin{cases} -3 + \beta = 6 - 3\lambda \\ -2 + 4\beta = 4\lambda + 2 \\ a + 2\beta = 1 - \lambda \end{cases}$$ ***M1***

attempt to solve ***M1***

$\lambda = 2,\ \beta = 3$ ***A1***

$a = 1 - \lambda - 2\beta = -7$ ***A1***

[4 marks]

(b) $$\vec{OP} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$$ ***A1***

$\therefore P(0, 10, -1)$

[2 marks]

Total [6 marks]

Question 24

METHOD 1

(a) $\det\begin{pmatrix}1 & 3 & a-1\\ 2 & 2 & a-2\\ 3 & 1 & a-3\end{pmatrix}$ ***M1***

$=1\left(2(a-3)-(a-2)\right)-3\left(2(a-3)-3(a-2)\right)+(a-1)(2-6)$

(or equivalent) ***A1***

$=0$ (therefore there is no unique solution) ***A1***

[3 marks]

(b) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right) : \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ ***M1A1***

$: \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ ***A1***

$b=1$ ***A1*** ***N2***

Note: Award ***M1*** for an attempt to use row operations.

[4 marks]

METHOD 2

(a) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right) : \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ ***M1A1***

$: \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ (and 3 zeros imply no unique solution) ***A1***

[3 marks]

(b) $b=1$ ***A4***

Note: Award ***A4*** only if "$b-1$" seen in (a).

[4 marks]

Total [7 marks]

Question 25

(a) $\vec{AB} = \begin{pmatrix}1\\0\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\-2\\2\end{pmatrix}$, $\vec{AC} = \begin{pmatrix}2\\-1\\4\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-3\\1\end{pmatrix}$ *A1A1*

Note: Award the above marks if the components are seen in the line below.

$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & -2 & 2 \\ 1 & -3 & 1 \end{vmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix}$ *(M1)A1*

[4 marks]

(b) area $= \frac{1}{2}\left|\left(\vec{AB} \times \vec{AC}\right)\right|$ *(M1)*

$= \frac{1}{2}\sqrt{4^2 + 2^2 + 2^2} = \frac{1}{2}\sqrt{24}\left(= \sqrt{6}\right)$ *A1*

Note: Award ***M0A0*** for attempts that do not involve the answer to (a).

[2 marks]

Total [6 marks]

Question 26

(a) (i) a pair of opposite sides have equal length and are parallel ***R1***
hence ABCD is a parallelogram ***AG***

(ii) attempt to rewrite the given information in vector form ***M1***
$\boldsymbol{b} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{d}$ ***A1***
rearranging $\boldsymbol{d} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{b}$ ***M1***
hence $\vec{AD} = \vec{BC}$ ***AG***

:e: Candidates may correctly answer part i) by answering part ii) correctly and then deducing there are two pairs of parallel sides.

[4 marks]

(b) **EITHER**

use of $\vec{AB} = \vec{DC}$ ***(M1)***

$\begin{pmatrix}2\\-3\\p+3\end{pmatrix} = \begin{pmatrix}q+1\\1-r\\4\end{pmatrix}$ ***A1A1***

OR

use of $\vec{AD} = \vec{BC}$ ***(M1)***

$\begin{pmatrix}-2\\r-2\\1\end{pmatrix} = \begin{pmatrix}q-3\\2\\2-p\end{pmatrix}$ ***A1A1***

THEN

attempt to compare coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ in their equation or statement to that effect ***M1***
clear demonstration that the given values satisfy their equation ***A1***
$p = 1,\ q = 1,\ r = 4$ ***AG***

[5 marks]

(c) attempt at computing $\vec{AB} \times \vec{AD}$ (or equivalent) ***M1***

$\begin{pmatrix} -11 \\ -10 \\ -2 \end{pmatrix}$ ***A1***

area $= \left|\vec{AB} \times \vec{AD}\right| \left(= \sqrt{225}\right)$ ***(M1)***

$= 15$ ***A1***

[4 marks]

(d) valid attempt to find $\vec{OM}$ $\left(= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})\right)$ ***(M1)***

$\begin{pmatrix} 1 \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$ ***A1***

the equation is

$\boldsymbol{r} = \begin{pmatrix} 1 \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix}$ or equivalent ***M1A1***

Note: Award maximum ***M1A0*** if '$\mathbf{r} = \ldots$' (or equivalent) is not seen.

[4 marks]

(e) attempt to obtain the equation of the plane in the form $ax + by + cz = d$ ***M1***

$11x + 10y + 2z = 25$ ***A1A1***

Note: ***A1*** for right hand side, ***A1*** for left hand side.

[3 marks]

(f) (i) putting two coordinates equal to zero ***(M1)***

$X\left(\frac{25}{11}, 0, 0\right), Y\left(0, \frac{5}{2}, 0\right), Z\left(0, 0, \frac{25}{2}\right)$ ***A1***

(ii) $YZ = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{25}{2}\right)^2}$ ***M1***

$= \sqrt{\frac{325}{2}} \left(= \frac{5\sqrt{104}}{4} = \frac{5\sqrt{26}}{2}\right)$ ***A1***

[4 marks]

Total [24 marks]

Question 27

$\cos\theta = \frac{(3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}) \bullet (5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k})}{|3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}||5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}|}$ ***(M1)***

$= \frac{16}{\sqrt{50}\sqrt{50}}$ ***A1A1***

e: ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{8}{25} \left(= \frac{16}{50} = 0.32\right)$ ***A1***

[4 marks]

Question 28

(a) recognising normal to plane or attempting to find cross product of two vectors lying in the plane ***(M1)***

for example, $\vec{AB} \times \vec{AD} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_1 : x + z = 1$ ***A1***

[3 marks]

(b) **EITHER**

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 = \sqrt{2}\sqrt{2}\cos\theta$ ***M1A1***

OR

$\left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right| = \sqrt{3} = \sqrt{2}\sqrt{2}\sin\theta$ ***M1A1***

Note: ***M1*** is for an attempt to find the scalar or vector product of the two normal vectors.

$\Rightarrow \theta = 60^\circ \left(= \frac{\pi}{3} \right)$ ***A1***

angle between faces is $120^\circ \left(= \frac{2\pi}{3} \right)$ ***A1***

[4 marks]

(c) $\vec{DB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ or $\vec{BD} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_3 : x + y - z = k$ ***(M1)***

$\Pi_3 : x + y - z = 0$ ***A1***

[3 marks]

Continue...

(d) **METHOD 1**

line $\text{AD}:(\mathbf{r}=)\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$ *M1A1*

intersects Π_3 when $\lambda-(1-\lambda)=0$ *M1*

so $\lambda=\frac{1}{2}$ *A1*

hence P is the midpoint of AD *AG*

METHOD 2

midpoint of AD is $(0.5, 0, 0.5)$ *(M1)A1*

substitute into $x+y-z=0$ *M1*

$0.5+0-0.5=0$ *A1*

hence P is the midpoint of AD *AG*

[4 marks]

(e) **METHOD 1**

$\text{OP}=\frac{1}{\sqrt{2}}$, $\text{O}\hat{\text{P}}\text{Q}=90^\circ$, $\text{O}\hat{\text{Q}}\text{P}=60^\circ$ *A1A1A1*

$\text{PQ}=\frac{1}{\sqrt{6}}$ *A1*

area $=\frac{1}{2\sqrt{12}}=\frac{1}{4\sqrt{3}}=\frac{\sqrt{3}}{12}$ *A1*

Continue...

METHOD 2

line $\text{BD}:(\mathbf{r}=)\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\Rightarrow \lambda=\frac{2}{3}$ ***(A1)***

$\vec{\text{OQ}}=\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$ ***A1***

area $=\frac{1}{2}\left|\vec{\text{OP}}\times\vec{\text{OQ}}\right|$ ***M1***

$\vec{\text{OP}}=\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}$ ***A1***

ote: This ***A1*** is dependent on ***M1***.

area $=\frac{\sqrt{3}}{12}$ ***A1***

[5 marks]

Total [19 marks]

Question 29

(a) **METHOD 1**

$$n = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2b \\ 0 \\ b-1 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$$ ***(M1)A1***

$(0, 0, 0)$ on Π so $(b-1)x + 4by - 2bz = 0$ ***(M1)A1***

METHOD 2

using equation of the form $px + qy + rz = 0$ ***(M1)***

$(0, 1, 2)$ on $\Pi \Rightarrow q + 2r = 0$

$(2b, 0, b-1)$ on $\Pi \Rightarrow 2bp + r(b-1) = 0$ ***(M1)A1***

Note: Award ***(M1)A1*** for both equations seen.

solve for p, q, and r ***(M1)***

$(b-1)x + 4by - 2bz = 0$ ***A1***

[5 marks]

(b) M has coordinates $\left(b, 0, \frac{b-1}{2}\right)$ ***(A1)***

$$r = \begin{pmatrix} b \\ 0 \\ \frac{b-1}{2} \end{pmatrix} + \lambda \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$$ ***M1A1***

Note: Award ***M1A0*** if $r =$ (or equivalent) is not seen.

Note: Allow equivalent forms such as $\frac{x-b}{b-1} = \frac{y}{4b} = \frac{2z-b+1}{-4b}$.

[3 marks]

(c) **METHOD 1**

$x = z = 0$ ***(M1)***

Note: Award ***M1*** for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ ***A1***

attempt to eliminate λ ***M1***

$\Rightarrow -\frac{b}{b-1} = \frac{b-1}{4b}$ ***(A1)***

$-4b^2 = (b-1)^2$ ***A1***

EITHER

consideration of the signs of LHS and RHS ***(M1)***
the LHS is negative and the RHS must be positive (or equivalent statement) ***R1***

OR

$-4b^2 = b^2 - 2b + 1$

$\Rightarrow 5b^2 - 2b + 1 = 0$

$\Delta = (-2)^2 - 4 \times 5 \times 1 = -16\ (< 0)$ ***M1***

$\therefore$ no real solutions ***R1***

THEN

so no point of intersection ***AG***

METHOD 2

$x = z = 0$ ***(M1)***

Note: Award ***M1*** for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ ***A1***

attempt to eliminate b ***M1***

$\Rightarrow \frac{\lambda}{1+\lambda} = \frac{1}{1-4\lambda}$ ***(A1)***

$-4\lambda^2 = 1\left(\Rightarrow \lambda^2 = -\frac{1}{4}\right)$ ***A1***

consideration of the signs of LHS and RHS ***(M1)***
there are no real solutions (or equivalent statement) ***R1***
so no point of intersection ***AG***

[7 marks]

Total [15 marks]

Question 30

(a) $\boldsymbol{a}\bullet\boldsymbol{b} = (1 \times 0) + (1 \times -t) + (t \times 4t)$ *(M1)*

$= -t + 4t^2$ ***A1***

[2 marks]

(b) recognition that $\boldsymbol{a}\bullet\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ *(M1)*

$\boldsymbol{a}\bullet\boldsymbol{b} < 0$ or $-t + 4t^2 < 0$ or $\cos\theta < 0$ ***R1***

Note: Allow $\le$ for ***R1***.

attempt to solve using sketch or sign diagram *(M1)*

$0 < t < \frac{1}{4}$ ***A1***

[4 marks]

Total [6 marks]

Question 31

(a) (i) $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ ***A1***

(ii) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ ***A1***

Note: Accept row vectors or equivalent.

[2 marks]

(b) **METHOD 1**

attempt at vector product using $\overrightarrow{AB}$ and $\overrightarrow{AC}$. *(M1)*

$\pm(2\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k})$ ***A1***

attempt to use area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ ***M1***

$= \frac{\sqrt{76}}{2}\left(= \sqrt{19}\right)$ ***A1***

[4 marks]

Continue….

METHOD 2

attempt to use $\vec{AB} \cdot \vec{AC} = \left|\vec{AB}\right|\left|\vec{AC}\right| \cos\theta$ ***M1***

$\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \sqrt{0^2 + 2^2 + (-2)^2}\sqrt{3^2 + 1^2 + (-2)^2} \cos\theta$

$6 = \sqrt{8}\sqrt{14} \cos\theta$ ***A1***

$\cos\theta = \dfrac{6}{\sqrt{8}\sqrt{14}} = \dfrac{6}{\sqrt{112}}$

attempt to use area $= \dfrac{1}{2}\left|\vec{AB}\right|\left|\vec{AC}\right| \sin\theta$ ***M1***

$= \dfrac{1}{2}\sqrt{8}\sqrt{14}\sqrt{1 - \dfrac{36}{112}} \left(= \dfrac{1}{2}\sqrt{8}\sqrt{14}\sqrt{\dfrac{76}{112}} \right)$

$= \dfrac{\sqrt{76}}{2} \left(= \sqrt{19} \right)$ ***A1***

[4 marks]

Total [6 marks]

Question 32

(a) (i) appreciation that two points distinct from P need to be chosen from each line ***M1***

$^{4}C_2 \times {}^{3}C_2$

$= 18$ ***A1***

(ii) **EITHER**

consider cases for triangles including P **or** triangles not including P ***M1***

$3 \times 4 + 4 \times {}^{3}C_2 + 3 \times {}^{4}C_2$ ***(A1)(A1)***

Note: Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

OR

consider total number of ways to select 3 points and subtract those with 3 points on the same line ***M1***

$^{8}C_3 - {}^{5}C_3 - {}^{4}C_3$ ***(A1)(A1)***

Note: Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

$56 - 10 - 4$

THEN

$= 42$ ***A1***

[6 marks]

continue…

(b) **METHOD 1**

substitution of $(4, 6, 4)$ into both equations *(M1)*

$\lambda = 3$ and $\mu = 1$ ***A1A1***

$(4, 6, 4)$ ***AG***

METHOD 2

attempting to solve two of the three parametric equations ***M1***

$\lambda = 3$ or $\mu = 1$ ***A1***

check both of the above give $(4, 6, 4)$ ***M1AG***

Note: If they have shown the curve intersects for all three coordinates they only need to check (4,6,4) with one of "λ" or "μ".

[3 marks]

(c) $\lambda = 2$ ***A1***

[1 mark]

(d) $\vec{PA} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, $\vec{PB} = \begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix}$ ***A1A1***

e: Award ***A1A0*** if both are given as coordinates.

[2 marks]

continued…

(e) **METHOD 1**

area triangle ABP $= \frac{1}{2}\left|\vec{PB} \times \vec{PA}\right|$ ***M1***

$\left(= \frac{1}{2}\left|\begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right|\right) = \frac{1}{2}\left|\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right|$ ***A1***

$= \frac{\sqrt{29}}{2}$ ***A1***

EITHER

$\vec{PC} = 3\vec{PA}$, $\vec{PD} = 3\vec{PB}$ ***(M1)***

area triangle PCD $= 9 \times$ area triangle ABP ***(M1)A1***

$= \frac{9\sqrt{29}}{2}$ ***A1***

Continue…

OR

D has coordinates $(-11, -12, -2)$ ***A1***

area triangle $\text{PCD} = \frac{1}{2}\left|\overrightarrow{\text{PD}} \times \overrightarrow{\text{PC}}\right| = \frac{1}{2}\left|\begin{pmatrix}-15\\-18\\-6\end{pmatrix} \times \begin{pmatrix}-3\\-6\\-3\end{pmatrix}\right|$ ***M1A1***

Note: A1 is for the correct vectors in the correct formula.

$= \frac{9\sqrt{29}}{2}$ ***A1***

THEN

area of $\text{CDBA} = \frac{9\sqrt{29}}{2} - \frac{\sqrt{29}}{2}$

$= 4\sqrt{29}$ ***A1***

[8 marks]

continued…

METHOD 2

D has coordinates $(-11, -12, -2)$ ***A1***

area $= \frac{1}{2}\left|\overrightarrow{\text{CB}} \times \overrightarrow{\text{CA}}\right| + \frac{1}{2}\left|\overrightarrow{\text{BC}} \times \overrightarrow{\text{BD}}\right|$ ***M1***

Note: Award ***M1*** for use of correct formula on appropriate non-overlapping triangles.

Note: Different triangles or vectors could be used.

$\overrightarrow{\text{CB}} = \begin{pmatrix}-2\\0\\1\end{pmatrix}, \overrightarrow{\text{CA}} = \begin{pmatrix}2\\4\\2\end{pmatrix}$ ***A1***

$\overrightarrow{\text{CB}} \times \overrightarrow{\text{CA}} = \begin{pmatrix}-4\\6\\-8\end{pmatrix}$ ***A1***

$\overrightarrow{\text{BC}} = \begin{pmatrix}2\\0\\-1\end{pmatrix}, \overrightarrow{\text{BD}} = \begin{pmatrix}-10\\-12\\-4\end{pmatrix}$ ***A1***

$\overrightarrow{\text{BC}} \times \overrightarrow{\text{BD}} = \begin{pmatrix}-12\\18\\-24\end{pmatrix}$ ***A1***

Note: Other vectors which might be used are $\overrightarrow{DA} = \begin{pmatrix} 14 \\ 16 \\ 5 \end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}$.

Note: Previous ***A1A1A1A1*** are all dependent on the first ***M1***.

valid attempt to find a value of $\frac{1}{2}|a \times b|$ ***M1***

Note: ***M1*** independent of triangle chosen.

area $= \frac{1}{2} \times 2 \times \sqrt{29} + \frac{1}{2} \times 6 \times \sqrt{29}$

$= 4\sqrt{29}$ ***A1***

Note: Accept $\frac{1}{2}\sqrt{116} + \frac{1}{2}\sqrt{1044}$ or equivalent.

[8 marks]

Total [20 marks]

Question 33

$a \bullet b = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ k+2 \\ k \end{pmatrix}$

$= -6 + k(k+2) - k$ ***A1***

$a \bullet b = 0$ ***(M1)***

$k^2 + k - 6 = 0$

attempt at solving their quadratic equation ***(M1)***

$(k+3)(k-2) = 0$

$k = -3, 2$ ***A1***

te: Attempt at solving using $|a||b| = |a \times b|$ will be ***M1A0A0A0*** if neither answer found ***M1(A1)A1A0*** for one correct answer and ***M1(A1)A1A1*** for two correct answers.

Total [4 marks]

Question 34

(a) (i) $\overrightarrow{AV} = \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AV} = \begin{pmatrix} 0 \\ 10 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10(p-10)+10p \\ -10p \\ -10p \end{pmatrix}$ ***A1***

$= \begin{pmatrix} 20p-100 \\ -10p \\ -10p \end{pmatrix} = -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$ ***AG***

$\overrightarrow{AC} \times \overrightarrow{AV} = \begin{pmatrix} 10 \\ 0 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10p \\ 100-20p \\ 10p \end{pmatrix} \left(= 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} \right)$ ***A1***

(ii) attempt to find a scalar product **M1**

$$-10\begin{pmatrix}10-2p\\p\\p\end{pmatrix}\bullet 10\begin{pmatrix}p\\10-2p\\p\end{pmatrix}=100\left(3p^2-20p\right)$$

OR $-\begin{pmatrix}10-2p\\p\\p\end{pmatrix}\bullet\begin{pmatrix}p\\10-2p\\p\end{pmatrix}=3p^2-20p$ **A1**

attempt to find magnitude of either $\overrightarrow{AB}\times\overrightarrow{AV}$ or $\overrightarrow{AC}\times\overrightarrow{AV}$ **M1**

$$\left|-10\begin{pmatrix}10-2p\\p\\p\end{pmatrix}\right|=\left|10\begin{pmatrix}p\\10-2p\\p\end{pmatrix}\right|=10\sqrt{(10-2p)^2+2p^2}$$ **A1**

$$100\left(3p^2-20p\right)=100\left(\sqrt{(10-2p)^2+2p^2}\right)^2\cos\theta$$

$$\cos\theta=\frac{3p^2-20p}{(10-2p)^2+2p^2}$$ **A1**

Note: Award ***A1*** for any intermediate step leading to the correct answer.

$$=\frac{p(3p-20)}{6p^2-40p+100}$$ **AG**

Note: Do not allow FT marks from part (a)(i).

[8 marks]

(b) (i) $p(3p-20)=0\Rightarrow p=0$ or $p=\frac{20}{3}$ **M1A1**

coordinates are $(0,0,0)$ and $\left(\frac{20}{3},\frac{20}{3},\frac{20}{3}\right)$ **A1**

Note: Do not allow column vectors for the final ***A*** mark.

(ii) two points are mirror images in the plane
or opposite sides of the plane
or equidistant from the plane
or the line connecting the two Vs is perpendicular to the plane **R1**

[4 marks]

(c) (i) geometrical consideration or attempt to solve $-1=\frac{p(3p-20)}{6p^2-40p+100}$ ***(M1)***

$p=\frac{10}{3}$, $\theta=\pi$ or $\theta=180°$ **A1A1**

(ii) $p\to\infty\Rightarrow\cos\theta\to\frac{1}{2}$ **M1**

hence the asymptote has equation $\theta=\frac{\pi}{3}$ **A1**

[5 marks]

Question 35

a vector normal to Π_p is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ***(A1)***

te: Allow any scalar multiple of $\begin{pmatrix}1\\0\\0\end{pmatrix}$, including $\begin{pmatrix}p\\0\\0\end{pmatrix}$

attempt to find scalar product (or vector product) of direction vector of line with any scalar multiple of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ***M1***

$\begin{pmatrix}1\\0\\0\end{pmatrix}\bullet\begin{pmatrix}5\\\sin\theta\\\cos\theta\end{pmatrix}=5$ (or $\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}5\\\sin\theta\\\cos\theta\end{pmatrix}=\begin{pmatrix}0\\-\cos\theta\\\sin\theta\end{pmatrix}$) ***A1***

(if α is the angle between the line and the normal to the plane)

$\cos\alpha=\dfrac{5}{1\times\sqrt{25+\sin^2\theta+\cos^2\theta}}$ (or $\sin\alpha=\dfrac{1}{1\times\sqrt{25+\sin^2\theta+\cos^2\theta}}$) ***A1***

$\Rightarrow\cos\alpha=\dfrac{5}{\sqrt{26}}$ or $\sin\alpha=\dfrac{1}{\sqrt{26}}$ ***A1***

this is independent of p and θ, hence the angle between the line and the plane, $(90-\alpha)$, is also independent of p and θ ***R1***

[6 marks]