Subject – Math(Higher Level)
Topic - Vector
Year - Nov 2011 – Nov 2019
Paper 2

Question 1

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}$$ *(M1)*

$$= a\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix}$$

$= a(a^2 - 1) - (a - 1) + (1 - a)$ *(A1)*

$= a^3 - 3a + 2$ ***A1***

set $a^3 - 3a + 2 = 0$ ***M1***

$\Rightarrow a = -2$; $a = 1$ ***A1A1***

hence the system has a unique solution for all reals such that

$a \neq -2$; $a \neq 1$ ***R1***

ote: Award ***R1*** for their values of a.

[7 marks]

Question 2

(a) **METHOD 1**

solving simultaneously (gdc) ***(M1)***

$x = 1 + 2z$; $y = -1 - 5z$ ***A1A1***

$$L : \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$ ***A1A1A1***

Note: 1st ***A1*** is for $\boldsymbol{r} =$.

[6 marks]

METHOD 2

direction of line $= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$ (last two rows swapped) ***M1***

$= 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ ***A1***

putting $z = 0$, a point on the line satisfies $2x + y = 1$, $3x + y = 2$ ***M1***

i.e. $(1, -1, 0)$ ***A1***

the equation of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$ ***A1A1***

Note: Award ***A0A1*** if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is missing.

[6 marks]

(b) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ ***M1***

$= 6\boldsymbol{i} - 12\boldsymbol{k}$ ***A1***

hence, $\boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{k}$

$\boldsymbol{n} \bullet \boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1$ ***M1A1***

therefore $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{a} \bullet \boldsymbol{n} \Rightarrow x - 2z = 1$ ***AG***

[4 marks]

(c) **METHOD 1**

$P=(-2, 4, 1)$, $Q=(x, y, z)$

$\vec{PQ}=\begin{pmatrix} x+2 \\ y-4 \\ z-1 \end{pmatrix}$ ***A1***

$\vec{PQ}$ is perpendicular to $3x+y-z=2$

$\Rightarrow \vec{PQ}$ is parallel to $3\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ ***R1***

$\Rightarrow x+2=3t$; $y-4=t$; $z-1=-t$ ***A1***

$1-z=t \Rightarrow x+2=3-3z \Rightarrow x+3z=1$ ***A1***

solving simultaneously $x+3z=1$; $x-2z=1$ ***M1***

$5z=0 \Rightarrow z=0$; $x=1$, $y=5$ ***A1***

hence, $Q=(1, 5, 0)$

[6 marks]

METHOD 2

Line passing through PQ has equation

$\mathbf{r}=\begin{matrix} -2 \\ 4 \\ 1 \end{matrix} + t \begin{matrix} 3 \\ 1 \\ -1 \end{matrix}$ ***M1A1***

Meets π_3 when:

$-2+3t-2(1-t)=1$ ***M1A1***

$t=1$ ***A1***

Q has coordinates (1, 5, 0) ***A1***

[6 marks]

Total [16 marks]

Question 3

point on line is $x=\dfrac{-1-5\lambda}{5}$, $y=\dfrac{9+5\lambda}{5}$, $z=\lambda$ or similar ***M1A1***

:e: Accept use of point on the line or elimination of one of the variables using the equations of the planes

$\dfrac{-1-5\lambda}{5}-\dfrac{9+5\lambda}{5}+2\lambda=k$ ***M1A1***

:e: Award ***M1A1*** if coordinates of point and equation of a plane is used to obtain linear equation in *k* or equations of the line are used in combination with equation obtained by elimination to get linear equation in *k*.

$k=-2$ ***A1***

[5 marks]

Question 4

(a) $\vec{AB} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \Rightarrow AB = \sqrt{72}$ ***A1***

$\vec{AC} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \Rightarrow AC = \sqrt{72}$ ***A1***

so they are the same ***AG***

$\vec{AB} \cdot \vec{AC} = 36 = (\sqrt{72})(\sqrt{72})\cos\theta$ ***(M1)***

$\cos\theta = \frac{36}{(\sqrt{72})(\sqrt{72})} = \frac{1}{2} \Rightarrow \theta = 60^\circ$ ***A1AG***

Note: Award ***M1A1*** if candidates find BC and claim that triangle ABC is equilateral.

[4 marks]

(b) **METHOD 1**

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 6 & -6 \\ -6 & 0 & -6 \end{vmatrix} = -36\boldsymbol{i} + 36\boldsymbol{j} + 36\boldsymbol{k}$ ***(M1)A1***

equation of plane is $x - y - z = k$ ***(M1)***

goes through A, B or C $\Rightarrow x - y - z = 2$ ***A1***

[4 marks]

METHOD 2

$x + by + cz = d$ (or similar) ***M1***

$5 - 2b + 5c = d$

$5 + 4b - c = d$ ***A1***

$-1 - 2b - c = d$

solving simultaneously ***M1***

$b = -1,\ c = -1,\ d = 2$

so $x - y - z = 2$ ***A1***

[4 marks]

(c) (i) midpoint is $(5, 1, 2)$, so equation of Π_1 is $y - z = -1$ ***A1A1***

(ii) midpoint is $(2, -2, 2)$, so equation of Π_2 is $x + z = 4$ ***A1A1***

Note: In each part, award ***A1*** for midpoint and ***A1*** for the equation of the plane.

[4 marks]

(d) **EITHER**

solving the two equations above **M1**

$$L: \boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$ **A1**

OR

L has the direction of the vector product of the normal vectors to the planes Π_1 and Π_2 **(M1)**

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$ (or its opposite) **A1**

THEN

direction is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ as required **R1**

[3 marks]

(e) D is of the form $(4-\lambda, -1+\lambda, \lambda)$ **M1**

$(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = 72$ **M1**

$3\lambda^2 - 6\lambda - 45 = 0$

$\lambda = 5$ or $\lambda = -3$ **A1**

$D(-1, 4, 5)$ **AG**

Note:	Award ***M0M0A0*** if candidates just show that $D(-1, 4, 5)$ satisfies AB=AD; Award ***M1M1A0*** if candidates also show that D is of the form $(4-\lambda, -1+\lambda, \lambda)$

[3 marks]

(f) **EITHER**

G is of the form $(4-\lambda, -1+\lambda, \lambda)$ and $DG = AG$, BG or CG ***M1***

e.g. $(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = (5-\lambda)^2 + (5-\lambda)^2 + (5-\lambda)^2$ ***M1***

$(1+\lambda)^2 = (5-\lambda)^2$

$\lambda = 2$ ***A1***

$G(2, 1, 2)$ ***AG***

OR

G is the centre of mass (barycentre) of the regular tetrahedron ABCD ***(M1)***

$$G\left(\frac{5+5+(-1)+(-1)}{4}, \frac{-2+4+(-2)+4}{4}, \frac{5+(-1)+(-1)+5}{4}\right)$$ ***M1A1***

THEN

Note: the following part is independent of previous work and candidates may use ***AG*** to answer it (here it is possible to award ***M0M0A0A1M1A1)***

$$\vec{GD} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \text{ and } \vec{GA} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$ ***A1***

$$\cos\theta = \frac{-9}{(3\sqrt{3})(3\sqrt{3})} = -\frac{1}{3} \Rightarrow \theta = 109^\circ \text{ (or 1.91 radians)}$$ ***M1A1***

[6 marks]

Total [24 marks]

Question 5

(a) in augmented matrix form $\left|\begin{array}{cccc} 1 & -3 & 1 & 3 \\ 1 & 5 & -2 & 1 \\ 0 & 16 & -6 & k \end{array}\right|$

attempt to find a line of zeros *(M1)*

$r_2 - r_1 \left|\begin{array}{cccc} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 16 & -6 & k \end{array}\right|$ *(A1)*

$r_3 - 2r_2 \left|\begin{array}{cccc} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{array}\right|$ *(A1)*

there is an infinite number of solutions when $k = -4$ ***R1***

there is no solution when

$k \neq -4, (k \in \mathbb{R})$ ***R1***

Note: Approaches other than using the augmented matrix are acceptable.

[5 marks]

(b) using $\left|\begin{array}{cccc} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{array}\right|$ and letting $z = \lambda$ *(M1)*

$8y - 3\lambda = -2$

$\Rightarrow y = \frac{3\lambda - 2}{8}$ *(A1)*

$x - 3y + z = 3$

$\Rightarrow x - \left(\frac{9\lambda - 6}{8}\right) + \lambda = 3$ *(M1)*

$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$

$\Rightarrow x = \frac{18 + \lambda}{8}$ *(A1)*

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{18}{8} \\ -\frac{2}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{pmatrix}$ *(M1)(A1)*

$\boldsymbol{r} = \begin{pmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$ ***A1***

Note: Accept equivalent answers.

[7 marks]

(c) recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane *(A1)*

direction normal of the plane is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$ *(M1)*

$= 16\boldsymbol{i} + 24\boldsymbol{j} - 11\boldsymbol{k}$ ***A1***

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is $(1, 2, 0)$ *(M1)*

$\Rightarrow 16 + 48 = d$

$d = 64$ ***A1***

hence Cartesian equation of plane is $16x + 24y - 11z = 64$ ***AG***

Note:	Accept alternative methods using dot product.

[5 marks]

(d) the plane crosses the z-axis when $x = y = 0$ *(M1)*

coordinates of P are $\left(0, 0, -\frac{64}{11}\right)$ ***A1***

Note:	Award ***A1*** for stating $z = -\frac{64}{11}$.

Note:	Accept. $\begin{pmatrix} 0 \\ 0 \\ -\frac{64}{11} \end{pmatrix}$

[2 marks]

(e) recognition that the angle between the line and the direction normal is given by:

$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta$ where θ is the angle between the line and the normal vector ***M1A1***

$\Rightarrow 122 = \sqrt{29}\sqrt{953}\cos\theta$ *(A1)*

$\Rightarrow \theta = 42.8^\circ$ (0.747 radians) *(A1)*

hence the angle between the line and the plane is $90^\circ - 42.8^\circ = 47.2^\circ$ (0.824 radians) ***A1***

[5 marks]

Note:	Accept use of the formula $\boldsymbol{a}.\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta$.

Total [24 marks]

Question 6

(a) $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ and $\boldsymbol{m} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ ***(A1)***

$\cos\theta = \dfrac{\boldsymbol{n} \bullet \boldsymbol{m}}{|\boldsymbol{n}||\boldsymbol{m}|}$ ***(M1)***

$\cos\theta = \dfrac{2+2+3}{\sqrt{1+4+9}\sqrt{4+1+1}} = \dfrac{7}{\sqrt{14}\sqrt{6}}$ ***A1***

$\theta = 40.2^{\circ}$ (0.702 rad) ***A1***

[4 marks]

(b) **METHOD 1**

eliminate z from $x-2y-3z=2$ and $2x-y-z=k$

$5x - y = 3k - 2 \Rightarrow x = \dfrac{y-(2-3k)}{5}$ ***M1A1***

eliminate y from $x-2y-3z=2$ and $2x-y-z=k$

$3x + z = 2k - 2 \Rightarrow x = \dfrac{z-(2k-2)}{-3}$ ***A1***

$x = t$, $y = (2-3k) + 5t$ and $z = (2k-2) - 3t$ ***A1A1***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***AG***

[5 marks]

METHOD 2

$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} \Rightarrow$ direction is $\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***M1A1***

Let $x = 0$

$0 - 2y - 3z = 2$ and $2 \times 0 - y - z = k$ ***(M1)***

solve simultaneously ***(M1)***

$y = 2 - 3k$ and $z = 2k - 2$ ***A1***

therefore $\mathbf{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***AG***

[5 marks]

METHOD 3

substitute $x = t$, $y = (2-3k)+5t$ and $z = (2k-2)-3t$ into π_1 and π_2 ***M1***

for $\pi_1 : t - 2(2-3k+5t) \ - \ 3(2k-2-3t) = 2$ ***A1***

for $\pi_2 : 2t - (2-3k+5t) - (2k-2-3t) = k$ ***A1***

the planes have a unique line of intersection ***R2***

therefore the line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***AG***

[5 marks]

(c) $5 - t = (2-3k+5t)+3 = 2-2(2k-2-3t)$ ***M1A1***

Note: Award ***M1A1*** if candidates use vector or parametric equations of L_2

eg $\begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$ or $\Rightarrow \begin{cases} t = 5-2s \\ 2-3k+5t = -3+2s \\ 2k-2-3t = 1+s \end{cases}$

solve simultaneously ***M1***

$k = 2, t = 1 \ (s = 2)$ ***A1***

intersection point $(1, 1, -1)$ ***A1***

[5 marks]

(d) $\vec{l_2} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***A1***

$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 5 & -3 \\ 2 & -2 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -7 \\ -12 \end{pmatrix}$ ***(M1)A1***

$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix}$ ***(M1)***

$x + 7y + 12z = -4$ ***A1***

[5 marks]

(e) Let θ be the angle between the lines $\vec{l}_1 = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ and $\vec{l}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

$\cos\theta = \frac{|2-10-3|}{\sqrt{35}\sqrt{9}} \Rightarrow \theta = 0.902334\ldots\ (51.699\ldots°)$ ***(M1)***

as the triangle XYZ has a right angle at Y,

$XZ = a \Rightarrow YZ = a\sin\theta$ and $XY = a\cos\theta$ ***(M1)***

$\text{area} = 3 \Rightarrow \frac{a^2 \sin\theta\cos\theta}{2} = 3$ ***(M1)***

$a = 3.5122\ldots$ ***(A1)***

$\text{perimeter} = a + a\sin\theta + a\cos\theta = 8.44537\ldots = 8.45$ ***A1***

Note: If candidates attempt to find coordinates of Y and Z award ***M1*** for expression of vector YZ in terms of two parameters, ***M1*** for attempt to use perpendicular condition to determine relation between parameters, ***M1*** for attempt to use the area to find the parameters and ***A2*** for final answer.

[5 marks]

Total [24 marks]

Question 7

METHOD 1

determinant $= 0$ ***M1***

$k(-2-16)-(0-12)+2(0+3)=0$ ***(M1)(A1)***

$-18k+18=0$ ***(A1)***

$k=1$ ***A1***

METHOD 2

writes in the form

$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ (or attempts to solve simultaneous equations) ***(M1)***

Having two 0's in first column (obtaining two equations in the same two variables) ***M1***

$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 0 & 0 & 18k-18 & 21k-27 \end{pmatrix}$ (or isolating one variable) ***A1***

te: The ***A1*** is to be awarded for the $18k-18$. The final column may not be seen.

$k=1$ ***(M1)A1***

[5 marks]

Question 8

(a) $\vec{PQ} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ *(A1)*

equation of line: $\boldsymbol{r} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ (or equivalent) ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ is omitted.

[3 marks]

(b) **METHOD 1**

$x: \quad -4 + 5s = -3 + 8t$
$y: \quad 2s = -1 + 6t$
$z: \quad 4 = 2 + 4t$ ***M1***

solving any two simultaneously ***M1***

$t = 0.5,\ s = 1$ (or equivalent) ***A1***

verification that these values give R when substituted into **both** equations (or that the three equations are consistent and that one gives R) ***R1***

METHOD 2

$(1, 2, 4)$ is given by $t = 0.5$ for L_1 and $s = 1$ for L_2 ***M1A1A1***

because $(1, 2, 4)$ is on both lines it is the point of intersection of the two lines ***R1***

[4 marks]

(c) $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 26 = \sqrt{29} \times \sqrt{29} \cos\theta$ ***M1***

$\cos\theta = \frac{26}{29}$ *(A1)*

$\theta = 0.459$ or 26.3° ***A1***

[3 marks]

(e) **EITHER**

midpoint of $[PS_1]$ is $M(-3.5, -0.5, 3)$ ***M1A1***

$$\vec{RM} = \begin{pmatrix} -4.5 \\ -2.5 \\ -1 \end{pmatrix}$$ ***A1***

OR

$$\vec{RS_1} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}$$ ***M1***

the direction of the line is $\vec{RS_1} + \vec{RP}$

$$\begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ -5 \\ -2 \end{pmatrix}$$ ***M1A1***

THEN

the equation of the line is:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note:	Marks cannot be awarded for methods involving halving the angle, unless it is clear that the candidate considers also the equation of the plane of L_1 and L_2 to reduce the number of parameters involved to one (to obtain the vector equation of the required line).

[4 marks]

Total [20 marks]

Question 9

(a) attempting to express the system in matrix form *M1*

$$\begin{pmatrix} 0.1 & -1.7 & 0.9 \\ -2.4 & 0.3 & 3.2 \\ 2.5 & 0.6 & -3.7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4.4 \\ 1.2 \\ 0.8 \end{pmatrix}$$ *A1*

Note: Award ***M1A1*** for a correct augmented matrix.

[2 marks]

(b) either direct GDC use, attempting elimination or using an inverse matrix. ***(M1)***

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2.4 \\ 1.6 \\ -1.6 \end{pmatrix} \text{ (correct to 2sf) or } \begin{pmatrix} -2.40 \\ 1.61 \\ -1.57 \end{pmatrix} \text{ (correct to 3sf) or } \begin{pmatrix} -\frac{932}{389} \\ \frac{628}{389} \\ -\frac{612}{389} \end{pmatrix} \text{ (exact)}$$ *A2*

[3 marks]

Total [5 marks]

Question 10

(a) attempting to form $(3\cos\theta + 6)(\cos\theta - 2) + 7(1 + \sin\theta) = 0$ *M1*

$3\cos^2\theta - 12 + 7\sin\theta + 7 = 0$ *A1*

$3\left(1 - \sin^2\theta\right) + 7\sin\theta - 5 = 0$ *M1*

$3\sin^2\theta - 7\sin\theta + 2 = 0$ *AG*

[3 marks]

(b) attempting to solve algebraically (including substitution) or graphically for $\sin\theta$ ***(M1)***

$\sin\theta = \frac{1}{3}$ ***(A1)***

$\theta = 0.340 \quad (= 19.5^\circ)$ *A1*

[3 marks]

Total [6 marks]

Question 11

(a) **METHOD 1**

for P on L_1, $\vec{OP} = \begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix}$

require $\begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ ***M1***

$5+\lambda-6+4\lambda+4+4\lambda=0$ (or equivalent) ***A1***

$\lambda = -\frac{1}{3}$ ***A1***

$\therefore \vec{OP} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen.

[5 marks]

METHOD 2

Calculating either $\left|\vec{OP}\right|$ or $\left|\vec{OP}\right|^2$ *eg*

$\left|\vec{OP}\right| = \sqrt{(-5-\lambda)^2+(-3+2\lambda)^2+(2+2\lambda)^2}$ ***A1***

$= \sqrt{9\lambda^2+6\lambda+38}$

Solving either $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\vec{OP}\right|\right)=0$ or $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\vec{OP}\right|^2\right)=0$ for λ. ***M1***

$\lambda = -\frac{1}{3}$ ***A1***

$\vec{OP} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

e: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen. ***[5 marks]***

(b) **METHOD 1**

$\left|\vec{OP}\right| = \sqrt{\left(-\frac{14}{3}\right)^2 + \left(-\frac{11}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$ ***(M1)***

$= 6.08 \quad \left(= \sqrt{37}\right)$ ***A1***

METHOD 2

$$\text{shortest distance} = \frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix} \times \begin{pmatrix}-5\\-3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\right|}$$ ***(M1)***

$$= \frac{\left|10\boldsymbol{i} + 8\boldsymbol{j} + 13\boldsymbol{k}\right|}{\sqrt{1+4+4}}$$

$= 6.08 \left(= \sqrt{37}\right)$ ***A1***

[2 marks]

Total [7 marks]

Question 12

(a) $\begin{cases} x+2y-z=2 \\ 2x+y+z=1 \\ -x+4y+az=4 \end{cases}$

$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ 6y+(a-1)z=6 \end{cases}$ ***M1A1***

$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ (a+5)z=0 \end{cases}$ ***A1***

(or equivalent)

if not a unique solution then $a=-5$ ***A1***

Note: The first ***M1*** is for attempting to eliminate a variable, the first ***A1*** for obtaining two expression in just two variables (plus *a*), and the second ***A1*** for obtaining an expression in just *a* and one other variable

[4 marks]

(b) if $a=-5$ there are an infinite number of solutions as last equation always true ***R1***

and if $a \neq -5$ there is a unique solution ***R1***

hence always a solution ***AG***

[2 marks]

Total [6 marks]

Question 13

$n_1 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and $n_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ ***(A1)(A1)***

use of $\cos\theta = \dfrac{n_1 \bullet n_2}{|n_1||n_2|}$ ***(M1)***

$\cos\theta = \dfrac{7}{\sqrt{21}\sqrt{19}} \left(= \dfrac{7}{\sqrt{399}}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$\theta = 69^{\circ}$ ***A1***

Note: Award ***A1*** for 111°.

Total [6 marks]

Question 14

(a) attempting to find a normal to π *eg* $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}$ ***(M1)***

$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = 17\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***(A1)***

$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***M1***

$2x - 2y + z = 4$ (or equivalent) ***A1***

[4 marks]

(b) $l_3 : \boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, t \in \mathbb{R}$ ***(A1)***

attempting to solve $\begin{pmatrix} 4+2t \\ -2t \\ 8+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$ for t *ie* $9t + 16 = 4$ for t ***M1***

$t = -\frac{4}{3}$ ***A1***

$\left(\frac{4}{3}, \frac{8}{3}, \frac{20}{3}\right)$ ***A1***

[4 marks]

Total [8 marks]

Question 15

(a) require $\begin{pmatrix}4\\\lambda\\10\end{pmatrix}=s\begin{pmatrix}2\\3\\5\end{pmatrix}$ ***(M1)***

$\Rightarrow 4=2s\Rightarrow s=2\Rightarrow \lambda=6$ ***A1***

Note: Accept cross product solution.

[2 marks]

(b) require $\boldsymbol{v}\bullet\boldsymbol{w}=2\times4+3\times\lambda+5\times10=0\Rightarrow 3\lambda=-58\Rightarrow\lambda=\frac{-58}{3}(-19.3)$ ***M1A1***

[2 marks]

(c) $\boldsymbol{v}\bullet\boldsymbol{w}=2\times4+3\times\lambda+5\times10=\sqrt{2^2+3^2+5^2}\times\sqrt{4^2+\lambda^2+10^2}\times\cos10^\circ$ ***(M1)(A1)***

$58+3\lambda=\sqrt{38}\times\sqrt{116+\lambda^2}\times\cos10^\circ$

$\lambda=3.73$ or 8.76 ***A1A1***

[4 marks]

Total [8 marks]

Question 16

(a) $\boldsymbol{u}\times\boldsymbol{v}=\begin{pmatrix}2(0)+2b\\-2a-1(0)\\b-2a\end{pmatrix}=\begin{pmatrix}2b\\-2a\\b-2a\end{pmatrix}$ ***(M1)(A1)***

$\begin{pmatrix}2b\\-2a\\b-2a\end{pmatrix}=\begin{pmatrix}4\\b\\c\end{pmatrix}$ ***(M1)***

$\Rightarrow a=-1,\ b=2,\ c=4$ ***A2***

Note: Award ***A1*** for two correct.

[5 marks]

(b) $\boldsymbol{n}=\begin{pmatrix}4\\2\\4\end{pmatrix}$ ***(A1)***

$\Rightarrow 4x+2y+4z=0\ (2x+y+2z=0)$ ***A1***

[2 marks]

Total [7 marks]

Question 17

(a)
$$2x + y + 6z = 0$$
$$4x + 3y + 14z = 4$$
$$2x - 2y + (\alpha - 2)z = \beta - 12$$

attempt at row reduction *M1*

eg $R_2 - 2R_1$ and $R_3 - R_1$

$$2x + y + 6z = 0$$
$$y + 2z = 4$$
$$-3y + (\alpha - 8)z = \beta - 12$$
A1

eg $R_3 + 3R_2$

$$2x + y + 6z = 0$$
$$y + 2z = 4$$
$$(\alpha - 2)z = \beta$$
A1

(i) no solutions if $\alpha = 2$, $\beta \neq 0$ *A1*

(ii) one solution if $\alpha \neq 2$ *A1*

(iii) infinite solutions if $\alpha = 2$, $\beta = 0$ *A1*

Note: Accept alternative methods e.g. determinant of a matrix

Note: Award ***A1A1A0*** if all three consistent with their reduced form, ***A1A0A0*** if two or one answer consistent with their reduced form.

[6 marks]

(b) $y + 2z = 4 \Rightarrow y = 4 - 2z$ *A1*

$2x = -y - 6z = 2z - 4 - 6z = -4z - 4 \Rightarrow x = -2z - 2$ *A1*

therefore Cartesian equation is $\frac{x+2}{-2} = \frac{y-4}{-2} = \frac{z}{1}$ or equivalent *A1*

[3 marks]

Total [9 marks]

Question 18

(a) L_1 and L_2 are not parallel, since $\begin{pmatrix}-1\\1\\2\end{pmatrix} \neq k\begin{pmatrix}2\\1\\6\end{pmatrix}$ ***R1***

if they meet, then $1-\lambda = 1+2\mu$ and $2+\lambda = 2+\mu$ ***M1***

solving simultaneously $\Rightarrow \lambda = \mu = 0$ ***A1***

$2+2\lambda = 4+6\mu \Rightarrow 2 \neq 4$ contradiction, ***R1***

so lines are skew ***AG***

Note: Do not award the second ***R1*** if their values of parameters are incorrect.

[4 marks]

(b) $\begin{pmatrix}-1\\1\\2\end{pmatrix} \bullet \begin{pmatrix}2\\1\\6\end{pmatrix} (=11) = \sqrt{6}\sqrt{41}\cos\theta$ ***M1A1***

$\cos\theta = \dfrac{11}{\sqrt{246}}$ ***(A1)***

$\theta = 45.5^{\circ}$ (0.794 radians) ***A1***

[4 marks]

(c) (i) $\begin{pmatrix}-1\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\6\end{pmatrix} = \begin{pmatrix}6-2\\4+6\\-1-2\end{pmatrix}$ ***(M1)***

$= \begin{pmatrix}4\\10\\-3\end{pmatrix} = 4\boldsymbol{i} + 10\boldsymbol{j} - 3\boldsymbol{k}$ ***A1***

continued…

(ii) **METHOD 1**

let P be the intersection of L_1 and L_3
let Q be the intersection of L_2 and L_3

$\vec{OP} = \begin{pmatrix} 1-\lambda \\ 2+\lambda \\ 2+2\lambda \end{pmatrix}$ $\vec{OQ} = \begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix}$ ***M1***

therefore $\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix} 2\mu+\lambda \\ \mu-\lambda \\ 2+6\mu-2\lambda \end{pmatrix}$ ***M1A1***

$\begin{pmatrix} 2\mu+\lambda \\ \mu-\lambda \\ 2+6\mu-2\lambda \end{pmatrix} = t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***M1***

$2\mu+\lambda-4t=0$
$\mu-\lambda-10t=0$
$6\mu-2\lambda+3t=-2$

solving simultaneously ***(M1)***

$\lambda = \frac{32}{125}(0.256),\ \mu = -\frac{28}{125}(-0.224)$ ***A1***

Note: Award ***A1*** for either correct λ or μ.

EITHER

therefore $\vec{OP} = \begin{pmatrix} 1-\lambda \\ 2+\lambda \\ 2+2\lambda \end{pmatrix} = \begin{pmatrix} \frac{93}{125} \\ \frac{282}{125} \\ \frac{314}{125} \end{pmatrix} = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix} + \alpha\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1***

continued…

OR

therefore $\vec{OQ} = \begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix} = \begin{pmatrix} \frac{69}{125} \\ \frac{222}{125} \\ \frac{332}{125} \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

[10 marks]

METHOD 2

$L_3 : r_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$

forming two equations as intersections with L_1 and L_2

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t_2 \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$ ***M1A1A1***

Note: Only award ***M1A1A1*** if two different parameters t_1, t_2 used.

attempting to solve simultaneously ***M1***

$\lambda = \frac{32}{125}(0.256),\ \mu = -\frac{28}{125}(-0.224)$ ***A1***

Note: Award ***A1*** for either correct λ or μ.

continued…

EITHER

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1A1***

OR

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

Total [18 marks]

Question 19

using technology and/or by elimination (*eg* ref on GDC) ***(M1)***

$x = 1.89\left(= \frac{17}{9}\right),\ y = 1.67\left(= \frac{5}{3}\right),\ z = -2.22\left(= \frac{-20}{9}\right)$ ***A1A1A1***

[4 marks]

Question 20

(a) $\boldsymbol{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ ***A1***

$\boldsymbol{b} = \frac{1}{3}\left(\begin{pmatrix} 4 \\ 16 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} \frac{5}{3} \\ 4 \end{pmatrix}$ ***(M1)A1***

[3 marks]

(b) **METHOD 1**

Roderick must signal in a direction vector perpendicular to Ed's path. ***(M1)***

the equation of the signal is $\boldsymbol{s} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} -12 \\ 5 \end{pmatrix}$ (or equivalent) ***A1***

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \frac{t}{3}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} -12 \\ 5 \end{pmatrix}$ ***M1***

$\frac{5}{3}t + 12\lambda = 12$ and $4t - 5\lambda = 5$ ***M1***

$t = 2.13\left(= \frac{360}{169}\right)$ ***A1***

[5 marks]

METHOD 2

$\begin{pmatrix} 5 \\ 12 \end{pmatrix} \cdot \left(\begin{pmatrix} 11 \\ 9 \end{pmatrix} - \begin{pmatrix} -1 + \frac{5}{3}t \\ 4 + 4t \end{pmatrix}\right) = 0$ (or equivalent) ***M1A1A1***

Note: Award the ***M1*** for an attempt at a scalar product equated to zero, ***A1*** for the first factor and ***A1*** for the complete second factor.

attempting to solve for t ***(M1)***

$t = 2.13\left(= \frac{360}{169}\right)$ ***A1***

[5 marks]

continued…

METHOD 3

$x=\sqrt{\left(12-\frac{5t}{3}\right)^2+(5-4t)^2}$ (or equivalent) $\left(x^2=\left(12-\frac{5t}{3}\right)^2+(5-4t)^2\right)$ ***M1A1A1***

Note: Award ***M1*** for use of Pythagoras' theorem, ***A1*** for $\left(12-\frac{5t}{3}\right)^2$ and ***A1*** for $(5-4t)^2$.

attempting (graphically or analytically) to find t such that $\frac{dx}{dt}=0\left(\frac{d(x^2)}{dt}=0\right)$ ***(M1)***

$t=2.13\left(=\frac{360}{169}\right)$ ***A1***

[5 marks]

METHOD 4

$\cos\theta=\frac{\begin{pmatrix}12\\5\end{pmatrix}\cdot\begin{pmatrix}5\\12\end{pmatrix}}{\left|\begin{pmatrix}12\\5\end{pmatrix}\right|\left|\begin{pmatrix}5\\12\end{pmatrix}\right|}=\frac{120}{169}$ ***M1A1***

Note: Award ***M1*** for attempting to calculate the scalar product.

$\frac{120}{13}=\frac{t}{3}\left|\begin{pmatrix}5\\12\end{pmatrix}\right|$ (or equivalent) ***(A1)***

attempting to solve for t ***(M1)***

$t=2.13\left(=\frac{360}{169}\right)$ ***A1***

[5 marks]

Total [8 marks]

Question 21

(a) $\vec{OA}\times\vec{OB}=\begin{pmatrix}4\\-4\\-2\end{pmatrix}$ ***(M1)A1***

Note: ***M1A0*** can be awarded for attempt at a correct method **shown**, or correct method implied by the digits 4, 4, 2 found in the correct order.

[2 marks]

(b) area $=\frac{1}{2}\sqrt{4^2+4^2+2^2}=3$ ***M1A1***

[2 marks]

Total [4 marks]

Question 22

Total [6 marks]

(a) (i) **METHOD 1**

$|\vec{OC}|^2 = \vec{OC} \cdot \vec{OC}$

$= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$ ***A1***

$= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b}$ ***A1***

$= |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

continued..

METHOD 2

$|\vec{OC}|^2 = |\vec{OA}|^2 + |\vec{OB}|^2 - 2|\vec{OA}||\vec{OB}|\cos\left(\text{O}\hat{\text{A}}\text{C}\right)$ ***A1***

$|\vec{OA}||\vec{OB}|\cos\left(\text{O}\hat{\text{A}}\text{C}\right) = -(\boldsymbol{a} \cdot \boldsymbol{b})$ ***A1***

$|\vec{OC}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

(ii) **METHOD 1**

$|\vec{AB}|^2 = \vec{AB} \cdot \vec{AB}$

$= (\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{b} - \boldsymbol{a})$ ***A1***

$= \boldsymbol{b} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a}$ ***A1***

$= |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

METHOD 2

$|\vec{AB}|^2 = |\vec{AC}|^2 + |\vec{BC}|^2 - 2|\vec{AC}||\vec{BC}|\cos\left(\text{A}\hat{\text{C}}\text{B}\right)$ ***A1***

$|\vec{AC}||\vec{BC}|\cos\left(\text{A}\hat{\text{C}}\text{B}\right) = \boldsymbol{a} \cdot \boldsymbol{b}$ ***A1***

$|\vec{AB}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

[4 marks]

(b) $|\vec{OC}| = |\vec{AB}| \Rightarrow |\vec{OC}|^2 = |\vec{AB}|^2 \Rightarrow |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$ ***R1(M1)***

Note: Award ***R1*** for $|\vec{OC}| = |\vec{AB}| \Rightarrow |\vec{OC}|^2 = |\vec{AB}|^2$ and ***(M1)*** for $|\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2$.

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ***A1***

hence OACB is a rectangle ($\boldsymbol{a}$ and $\boldsymbol{b}$ both non-zero)
with adjacent sides at right angles ***R1***

Note: Award ***R1(M1)A0R1*** if the dot product has not been used.

[4 marks]

Total [8 marks]

Question 23

$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

EITHER

$\theta = \arccos\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \left(\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$ ***(M1)***

$= \arccos\left(\frac{2+0-1}{\sqrt{3}\sqrt{5}}\right) \left(\cos\theta = \frac{2+0-1}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arccos\left(\frac{1}{\sqrt{15}}\right) \left(\cos\theta = \frac{1}{\sqrt{15}}\right)$

OR

$\theta = \arcsin\left(\frac{|\boldsymbol{n}_1 \times \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \left(\sin\theta = \frac{|\boldsymbol{n}_1 \times \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$ ***(M1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right) \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{15}}\right) \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{15}}\right)$

THEN

$= 75.0^\circ$ (or 1.31) ***A1***

[5 marks]

Question 24

METHOD 1

$\vec{AB} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$ ***(A1)***

$\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***M1A1***

$= \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ ***A1***

$x - y - z = k$ ***M1***

$k = 3$

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

METHOD 2

let plane Π be $ax + by + cz = d$

attempt to form one or more simultaneous equations: ***M1***

$a + 2b - c = 0$ (1) ***A1***

$6a + 2b + c = d$ (2)

$3a - b + c = d$ (3) ***A1***

Note: Award second ***A1*** for equations (2) and (3).

attempt to solve ***M1***

EITHER

using GDC gives $a = \frac{d}{3},\ b = -\frac{d}{3},\ c = -\frac{d}{3}$ ***(A1)***

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

OR

row reduction ***M1***

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

[6 marks]

Question 25

METHOD 1

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$

$(\boldsymbol{a} \times \boldsymbol{b}) - (\boldsymbol{b} \times \boldsymbol{c}) = 0$

$(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \boldsymbol{b}) = 0$ ***M1A1***

$(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b} = 0$ ***A1***

$(\boldsymbol{a} + \boldsymbol{c})$ is parallel to $\boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***R1AG***

Note:	Condone absence of arrows, underlining, or other otherwise “correct” vector notation throughout this question.

Note:	Allow “is in the same direction to”, for the final ***R*** mark.

METHOD 2

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \Rightarrow \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{pmatrix}$$ ***M1A1***

$a_2 b_3 - a_3 b_2 = b_2 c_3 - b_3 c_2 \Rightarrow b_3 (a_2 + c_2) = b_2 (a_3 + c_3)$

$a_3 b_1 - a_1 b_3 = b_3 c_1 - b_1 c_3 \Rightarrow b_1 (a_3 + c_3) = b_3 (a_1 + c_1)$

$a_1 b_2 - a_2 b_1 = b_1 c_2 - b_2 c_1 \Rightarrow b_2 (a_1 + c_1) = b_1 (a_2 + c_2)$

$$\frac{(a_1 + c_1)}{b_1} = \frac{(a_2 + c_2)}{b_2} = \frac{(a_3 + c_3)}{b_3} = s$$ ***A1***

$\Rightarrow a_1 + c_1 = s b_1$

$\Rightarrow a_2 + c_2 = s b_2$

$\Rightarrow a_3 + c_3 = s b_3$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = s \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$ ***A1***

$\Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***AG***

[4 marks]

Question 26

(a) $\vec{\mathrm{BC}} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ ***(A1)***

$\boldsymbol{r} = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

(or $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$ ***(M1)A1***

Note: Do not award ***A1*** unless $\boldsymbol{r} =$ or equivalent correct notation seen.

[3 marks]

(b) attempt to write in parametric form using two different parameters **AND** equate ***M1***

$2\mu = 2 - \lambda$

$\mu = -1 + 4\lambda$

$-2\mu = 2 + \lambda$ ***A1***

attempt to solve first pair of simultaneous equations for two parameters ***M1***

solving first two equations gives $\lambda = \frac{4}{9}, \mu = \frac{7}{9}$ ***(A1)***

substitution of these two values in third equation ***(M1)***

since the values do not fit, the lines do not intersect ***R1***

Note: Candidates may note that adding the first and third equations immediately leads to a contradiction and hence they can immediately deduce that the lines do not intersect.

[6 marks]

(c) **METHOD 1**

plane is of the form $\boldsymbol{r} \cdot (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) =$ [redacted] ***(A1)***

[redacted] $\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) \cdot (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -1$ ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

METHOD 2

plane is of the form $2x + y - 2z = d$ ***(A1)***

substituting $(1, 3, 3)$ (to find gives $2 + 3 - 6 = -1$) ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

[3 marks]

continued…

(d) **METHOD 1**

attempt scalar product of direction vector BC with normal to plane **M1**

$(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})\bullet(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -2 + 4 - 2$

$= 0$ **A1**

hence BC lies in Π_1 **AG**

METHOD 2

substitute eqn of line into plane **M1**

line $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$. Plane $\pi_1 : 2x + y - 2z = -1$

$2(2-\lambda) + (-1+4\lambda) - 2(2+\lambda)$

$= -1$ **A1**

hence BC lies in Π_1 **AG**

Note: Candidates may also just substitute $2i - j + 2k$ into the plane since they are told C lies on π_1.

Note: Do not award ***A1FT***.

[2 marks]

(e) **METHOD 1**

applying scalar product to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ **M1**

$(2\boldsymbol{j} + \boldsymbol{k})\bullet(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 0$ **A1**

$(2\boldsymbol{j} + \boldsymbol{k})\bullet(2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$ **A1**

METHOD 2

attempt to find cross product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ **M1**

plane Π_2 has normal $\overrightarrow{OA} \times \overrightarrow{OB} = -8\boldsymbol{j} - 4\boldsymbol{k}$ **A1**

since $-8\boldsymbol{j} - 4\boldsymbol{k} = -4(2\boldsymbol{j} + \boldsymbol{k})$, $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2 **R1**

[3 marks]

(f) plane Π_3 has normal $\overrightarrow{OA} \times \overrightarrow{OC} = 9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$ **A1**

[1 mark]

continued…

(g) attempt to use dot product of normal vectors ***(M1)***

$\cos\theta = \dfrac{(2\boldsymbol{j} + \boldsymbol{k})\bullet(9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k})}{|2\boldsymbol{j} + \boldsymbol{k}||9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}|}$ ***(M1)***

$= \dfrac{-11}{\sqrt{5}\sqrt{170}} (= -0.377...)$ ***(A1)***

Note: Accept $\dfrac{11}{\sqrt{5}\sqrt{170}}$.

acute angle between planes $= 67.8^\circ (= 1.18^c)$ **A1**

[4 marks]

Total [22 marks]

Question 27

(a) $\boldsymbol{r}_A = \boldsymbol{r}_B$ ***(M1)***

$2 - t = -0.5t \Rightarrow t = 4$ ***A1***

checking $t = 4$ satisfies $4 + t = 3.2 + 1.2t$ and $-1 - 0.15t = -2 + 0.1t$ ***R1***

$\mathrm{P}(-2, 8, -1.6)$ ***A1***

Note: Do not award final ***A1*** if answer given as column vector.

[4 marks]

(b) (i) $0.9 \times \begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}$ ***A1***

Note: Accept use of cross product equalling zero.

hence in the same direction ***AG***

(ii) $\begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ -1.6 \end{pmatrix}$ ***M1***

Note: The ***M1*** can be awarded for any one of the resultant equations.

$\Rightarrow t = \frac{40}{9} = 4.44\ldots$ ***A1***

[3 marks]

(c) (i) $\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 2 - t \\ 4 + t \\ -1 - 0.15t \end{pmatrix} - \begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix}$ ***(M1)(A1)***

$= \begin{pmatrix} 2 - 0.55t \\ 0.8 - 0.08t \\ 1 - 0.24t \end{pmatrix}$ ***(A1)***

Note: Accept $\boldsymbol{r}_B - \boldsymbol{r}_A$.

distance $D = \sqrt{(2 - 0.55t)^2 + (0.8 - 0.08t)^2 + (1 - 0.24t)^2}$ ***M1A1***

$(= \sqrt{8.64 - 2.688t + 0.317t^2})$

(ii) minimum when $\frac{\mathrm{d}D}{\mathrm{d}t} = 0$ ***(M1)***

$t = 3.83$ ***A1***

(iii) 0.511 (km) ***A1***

[8 marks]

Total [15 marks]

Question 28

(d) (i) substituting $\begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix}$ into Π_3: *M1*

$\frac{5a}{4} - \frac{7c}{4} = 1$ *A1*

$5a - 7c = 4$ *AG*

(ii) attempt to find scalar products for Π_1 and Π_3, Π_2 and Π_3 and equating *M1*

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = \frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ *M1*

Note: Accept $3a+b+c = a-3b-c$.

$\Rightarrow a + 2b + c = 0$ *A1*

attempt to solve $a+2b+c=0$, $a+2b-5c=0$, $5a-7c=4$ *M1*

$\Rightarrow a = \frac{4}{5}, b = -\frac{2}{5}, c = 0$ *A1*

hence equation is $\frac{4x}{5} - \frac{2y}{5} = 1$

for second equation:

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = -\frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ *(M1)*

$\Rightarrow 2a - b = 0$

attempt to solve $2a-b=0$, $a+2b-5c=0$, $5a-7c=4$

$\Rightarrow a = -2, b = -4, c = -2$ *A1*

hence equation is $-2x - 4y - 2z = 1$

[9 marks]

Total [19 marks]