Subject – Math (Standard Level)
Topic - Functions and Equations
Year - Nov 2011 – Nov 2019
Paper -2

Question 1

[Maximum mark: 7]

Let $f(x) = 2x + 4$ and $g(x) = 7x^2$.

(a) Find $f^{-1}(x)$. *[3 marks]*

(b) Find $(f \circ g)(x)$. *[2 marks]*

(c) Find $(f \circ g)(3.5)$. *[2 marks]*

Question 2

[Maximum mark: 8]

Jose takes medication. After t minutes, the concentration of medication left in his bloodstream is given by $A(t) = 10(0.5)^{0.014t}$, where A is in milligrams per litre.

(a) Write down $A(0)$. *[1 mark]*

(b) Find the concentration of medication left in his bloodstream after 50 minutes. *[2 marks]*

(c) At 13:00, when there is no medication in Jose's bloodstream, he takes his first dose of medication. He can take his medication again when the concentration of medication reaches 0.395 milligrams per litre. What time will Jose be able to take his medication again? *[5 marks]*

Question 3

[Maximum mark: 7]

Let $f(t) = 2t^2 + 7$, where $t > 0$. The function v is obtained when the graph of f is transformed by

a stretch by a scale factor of $\frac{1}{3}$ parallel to the y-axis,

followed by a translation by the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

(a) Find $v(t)$, giving your answer in the form $a(t-b)^2 + c$. *[4 marks]*

(b) A particle moves along a straight line so that its velocity in m s^{-1}, at time t seconds, is given by v. Find the distance the particle travels between $t = 5.0$ and $t = 6.8$. *[3 marks]*

Question 4

[Maximum mark: 7]

Let $f(x) = 2x^2 - 8x - 9$.

(a) (i) Write down the coordinates of the vertex.

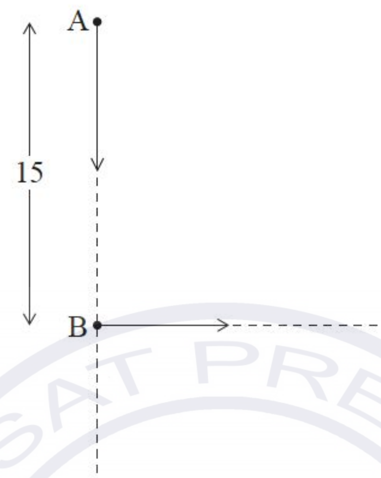
(ii) Hence or otherwise, express the function in the form $f(x) = 2(x-h)^2 + k$. *[4 marks]*

(b) Solve the equation $f(x) = 0$. *[3 marks]*

Question 5

[Maximum mark: 17]

The following diagram shows two ships A and B. At noon, ship A was 15 km due north of ship B. Ship A was moving south at 15 km h^{-1} and ship B was moving east at 11 km h^{-1}.



(a) Find the distance between the ships

(i) at 13:00;

(ii) at 14:00. *[5 marks]*

Let $s(t)$ be the distance between the ships t hours after noon, for $0 \le t \le 4$.

(b) Show that $s(t) = \sqrt{346t^2 - 450t + 225}$. *[6 marks]*

(c) Sketch the graph of $s(t)$. *[3 marks]*

(d) Due to poor weather, the captain of ship A can only see another ship if they are less than 8 km apart. Explain why the captain cannot see ship B between noon and 16:00. *[3 marks]*

Question 6

[Maximum mark: 6]

Let f and g be functions such that $g(x) = 2f(x+1) + 5$.

(a) The graph of f is mapped to the graph of g under the following transformations:

vertical stretch by a factor of k, followed by a translation $\begin{pmatrix} p \\ q \end{pmatrix}$.

Write down the value of

(i) k;

(ii) p;

(iii) q. *[3 marks]*

(b) Let $h(x) = -g(3x)$. The point $\mathrm{A}(6, 5)$ on the graph of g is mapped to the point A' on the graph of h. Find A'. *[3 marks]*

Question 7

[Maximum mark: 14]

Let $f(x) = \dfrac{3x}{x-q}$, where $x \neq q$.

(a) Write down the equations of the vertical and horizontal asymptotes of the graph of f. *[2]*

The vertical and horizontal asymptotes to the graph of f intersect at the point $\mathrm{Q}(1, 3)$.

(b) Find the value of q. *[2]*

(c) The point $\mathrm{P}(x, y)$ lies on the graph of f. Show that $\mathrm{PQ} = \sqrt{(x-1)^2 + \left(\dfrac{3}{x-1}\right)^2}$. *[4]*

(d) Hence find the coordinates of the points on the graph of f that are closest to $(1, 3)$. *[6]*

Question 8

[Maximum mark: 15]

The number of bacteria in two colonies, A and B, starts increasing at the same time.

The number of bacteria in colony A after t hours is modelled by the function $A(t) = 12e^{0.4t}$.

(a) Find the initial number of bacteria in colony A. *[2]*

(b) Find the number of bacteria in colony A after four hours. *[3]*

(c) How long does it take for the number of bacteria in colony A to reach 400? *[3]*

The number of bacteria in colony B after t hours is modelled by the function $B(t) = 24e^{kt}$.

(d) After four hours, there are 60 bacteria in colony B. Find the value of k. *[3]*

(e) The number of bacteria in colony A first exceeds the number of bacteria in colony B after n hours, where $n \in \mathbb{Z}$. Find the value of n. *[4]*

Question 9

[Maximum mark: 5]

Let $f(x) = 2x + 3$ and $g(x) = x^3$.

(a) Find $(f \circ g)(x)$. *[2]*

(b) Solve the equation $(f \circ g)(x) = 0$. *[3]*

Question 10

[Maximum mark: 8]

Let $f(x) = -x^4 + 2x^3 - 1$, for $0 \le x \le 2$.

(a) Sketch the graph of f on the following grid. *[3]*

(b) Solve $f(x) = 0$. *[2]*

(c) The region enclosed by the graph of f and the x-axis is rotated $360°$ about the x-axis. Find the volume of the solid formed. *[3]*

Question 11

[Maximum mark: 7]

Let $f(x) = \dfrac{2x-6}{1-x}$, for $x \neq 1$.

(a) For the graph of f

(i) find the x-intercept;

(ii) write down the equation of the vertical asymptote;

(iii) find the equation of the horizontal asymptote. [5]

(b) Find $\lim_{x \to \infty} f(x)$. [2]

Question 12

[Maximum mark: 6]

Let $G(x) = 95e^{(-0.02x)} + 40$, for $20 \leq x \leq 200$.

(a) On the following grid, sketch the graph of G. [3]

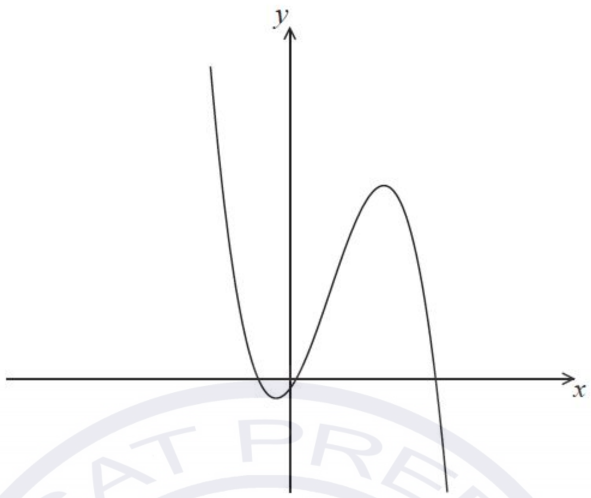
(b) Robin and Pat are planning a wedding banquet. The cost per guest, G dollars, is modelled by the function $G(n) = 95e^{(-0.02n)} + 40$, for $20 \leq n \leq 200$, where n is the number of guests.

Calculate the **total** cost for 45 guests. [3]

Question 13

[Maximum mark: 7]

The following diagram shows part of the graph of $f(x) = -2x^3 + 5.1x^2 + 3.6x - 0.4$.



(a) Find the coordinates of the local minimum point. [2]

(b) The graph of f is translated to the graph of g by the vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$. Find all values of k so that $g(x) = 0$ has exactly one solution. [5]

Question 14

[Maximum mark: 6]

Let $f(x) = e^{x+1} + 2$, for $-4 \le x \le 1$.

(a) On the following grid, sketch the graph of f. [3]

(b) The graph of f is translated by the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ to obtain the graph of a function g.

Find an expression for $g(x)$. [3]

Question 15

[Maximum mark: 8]

Let $f(x) = kx^2 + kx$ and $g(x) = x - 0.8$. The graphs of f and g intersect at two distinct points. Find the possible values of k.

Question 16

[Maximum mark: 6]

Let $f(x) = x^2$ and $g(x) = 3\ln(x+1)$, for $x > -1$.

(a) Solve $f(x) = g(x)$. [3]

(b) Find the area of the region enclosed by the graphs of f and g. [3]

Question 17

[Maximum mark: 8]

Note: One decade is 10 years

A population of rare birds, P_t, can be modelled by the equation $P_t = P_0 e^{kt}$, where P_0 is the initial population, and t is measured in decades. After one decade, it is estimated that $\frac{P_1}{P_0} = 0.9$.

(a) (i) Find the value of k.

(ii) Interpret the meaning of the value of k. [3]

(b) Find the least number of **whole** years for which $\frac{P_t}{P_0} < 0.75$. [5]

Question 18

[Maximum mark: 7]

Let $f(x) = e^{0.5x} - 2$.

(a) For the graph of f

(i) write down the y-intercept;

(ii) find the x-intercept;

(iii) write down the equation of the horizontal asymptote. [4]

(b) On the following grid, sketch the graph of f, for $-4 \leq x \leq 4$. [3]

Question 19

[Maximum mark: 7]

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x - 5$, for $x \in \mathbb{R}$.

(a) Find $f(8)$. [2]

(b) Find $(g \circ f)(x)$. [2]

(c) Solve $(g \circ f)(x) = 0$. [3]

Question 20

[Maximum mark: 7]

Let $f(x) = 0.225x^3 - 2.7x$, for $-3 \le x \le 3$. There is a local minimum point at A.

(a) Find the coordinates of A. [2]

(b) On the following grid,

(i) sketch the graph of f, clearly indicating the point A;

(ii) sketch the tangent to the graph of f at A. [5]

Question 21

[Maximum mark: 6]

Consider the graph of $f(x) = \dfrac{e^x}{5x - 10} + 3$, for $x \ne 2$.

(a) Find the y-intercept. [2]

(b) Find the equation of the vertical asymptote. [2]

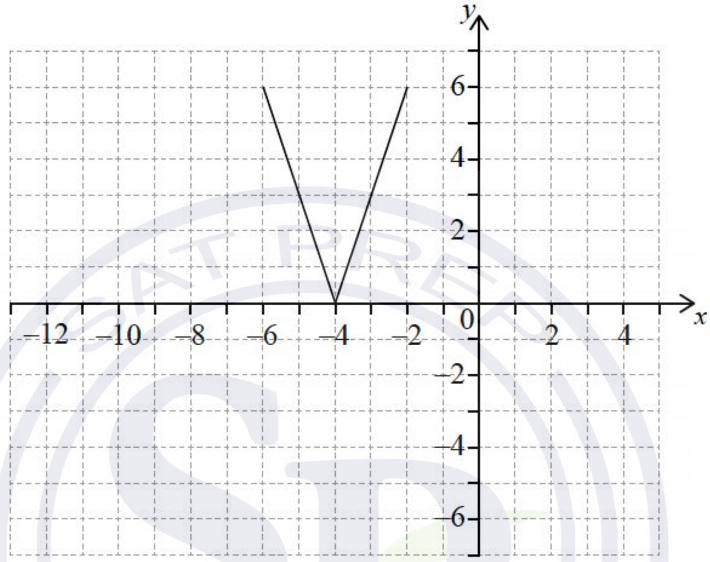
(c) Find the minimum value of $f(x)$ for $x > 2$. [2]

Question 22

[Maximum mark: 6]

The following diagram shows the graph of a function $y = f(x)$, for $-6 \le x \le -2$.
The points $(-6, 6)$ and $(-2, 6)$ lie on the graph of f. There is a minimum point at $(-4, 0)$.

(a) Write down the range of f. [2]



Let $g(x) = f(x-5)$.

(b) On the grid above, sketch the graph of g. [2]

(c) Write down the domain of g. [2]

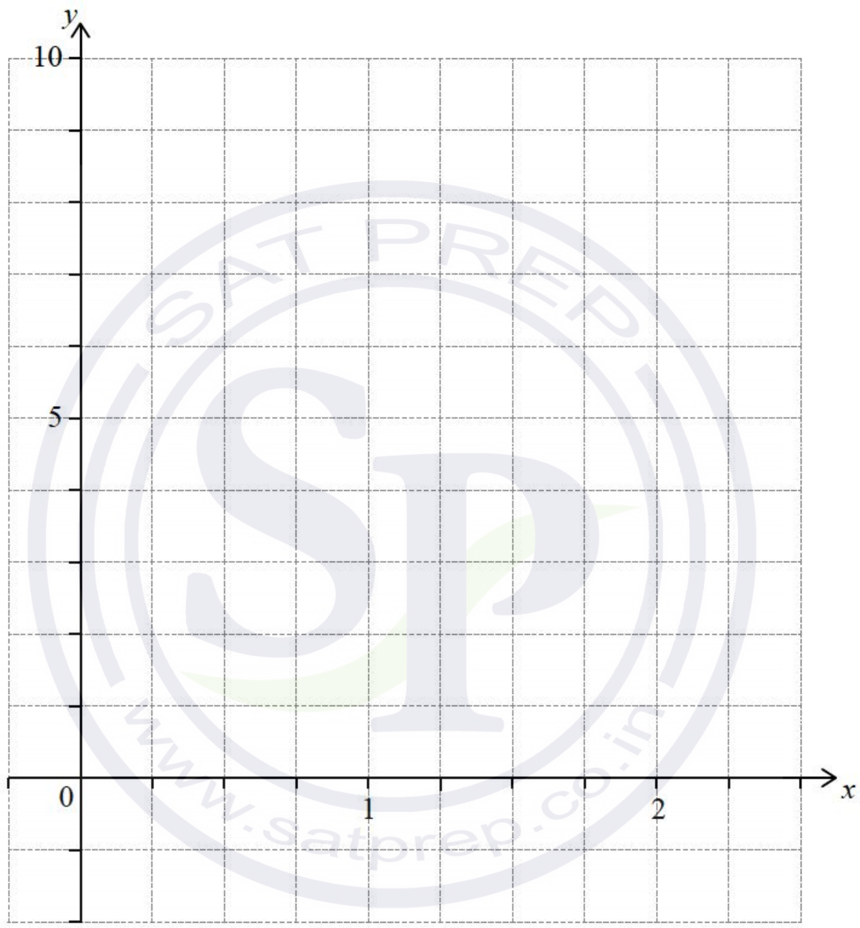
Question 23

[Maximum mark: 8]

Let $f(x) = x^2 - 1$ and $g(x) = x^2 - 2$, for $x \in \mathbb{R}$.

(a) Show that $(f \circ g)(x) = x^4 - 4x^2 + 3$. [2]

(b) On the following grid, sketch the graph of $(f \circ g)(x)$, for $0 \leq x \leq 2.25$. [3]



(c) The equation $(f \circ g)(x) = k$ has exactly two solutions, for $0 \leq x \leq 2.25$. Find the possible values of k. [3]

Question 24

[Maximum mark: 7]

Let $f(x)=\frac{6x^2-4}{e^x}$, for $0 \le x \le 7$.

(a) Find the x-intercept of the graph of f. [2]

(b) The graph of f has a maximum at the point A. Write down the coordinates of A. [2]

(c) On the following grid, sketch the graph of f. [3]

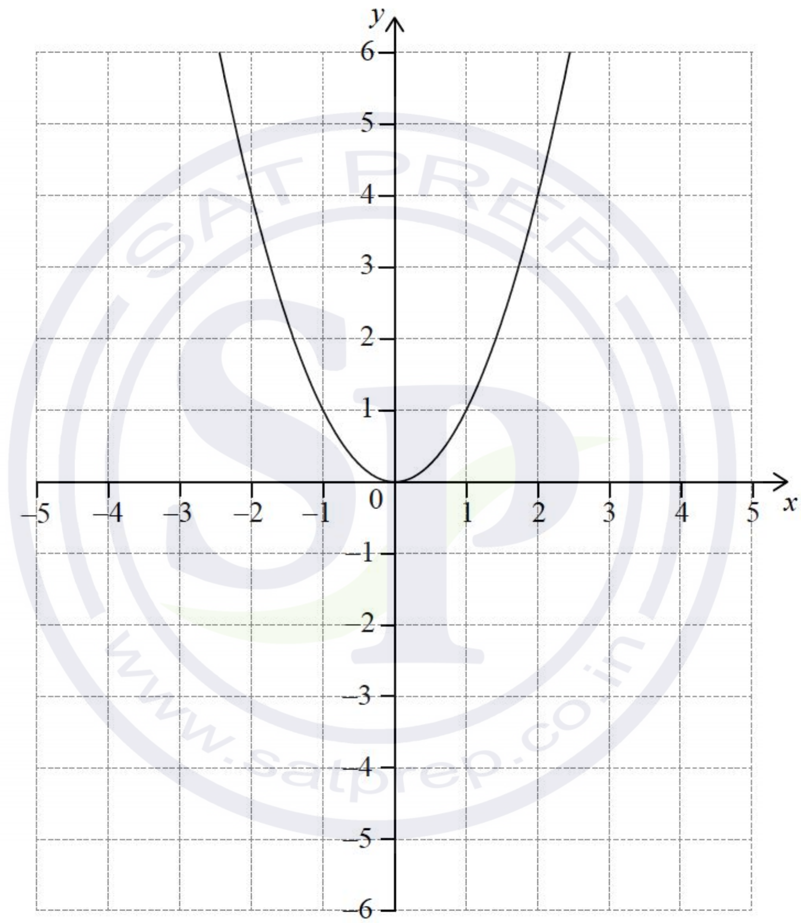
Question 25

[Maximum mark: 7]

Let $g(x) = -(x-1)^2 + 5$.

(a) Write down the coordinates of the vertex of the graph of g. [1]

Let $f(x) = x^2$. The following diagram shows part of the graph of f.



The graph of g intersects the graph of f at $x = -1$ and $x = 2$.

(b) On the grid above, sketch the graph of g for $-2 \le x \le 4$. [3]

(c) Find the area of the region enclosed by the graphs of f and g. [3]

Question 26

[Maximum mark: 8]

Let $f(x) = e^{2\sin\left(\frac{\pi x}{2}\right)}$, for $x > 0$.

The kth maximum point on the graph of f has x-coordinate x_k where $k \in \mathbb{Z}^+$.

(a) Given that $x_{k+1} = x_k + a$, find a. [4]

(b) Hence find the value of n such that $\sum_{k=1}^{n} x_k = 861$. [4]

Question 27

[Maximum mark: 7]

Let $f(x) = \frac{8x-5}{cx+6}$ for $x \neq -\frac{6}{c}$, $c \neq 0$.

(a) The line $x = 3$ is a vertical asymptote to the graph of f. Find the value of c. [2]

(b) Write down the equation of the horizontal asymptote to the graph of f. [2]

(c) The line $y = k$, where $k \in \mathbb{R}$ intersects the graph of $|f(x)|$ at exactly one point. Find the possible values of k. [3]

Question 28

[Maximum mark: 7]

Let $f(x) = \frac{6x-1}{2x+3}$, for $x \neq -\frac{3}{2}$.

(a) For the graph of f,

(i) find the y-intercept;

(ii) find the equation of the vertical asymptote;

(iii) find the equation of the horizontal asymptote. [5]

(b) Hence or otherwise, write down $\lim_{x \to \infty}\left(\frac{6x-1}{2x+3}\right)$. [2]

Question 29

[Maximum mark: 5]

Let $f(x) = \ln x - 5x$, for $x > 0$.

(a) Find $f'(x)$. [2]

(b) Find $f''(x)$. [1]

(c) Solve $f'(x) = f''(x)$. [2]

Question 30

[Maximum mark: 6]

The population of fish in a lake is modelled by the function

$$f(t) = \frac{1000}{1 + 24e^{-0.2t}}, \ 0 \le t \le 30, \text{ where } t \text{ is measured in months.}$$

(a) Find the population of fish at $t = 10$. [2]

(b) Find the rate at which the population of fish is increasing at $t = 10$. [2]

(c) Find the value of t for which the population of fish is increasing most rapidly. [2]