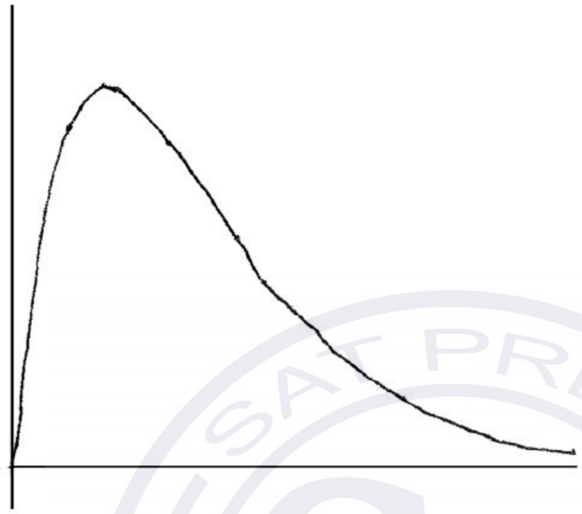
Subject – Math (Standard Level)
Topic - Calculus
Year - Nov 2011 – Nov 2019
Paper -2

Question1

(a)



A1A1A1 *N3*

Note: Award *A1* for approximately correct shape with inflexion/ change of curvature, *A1* for maximum skewed to the left, *A1* for asymptotic behaviour to the right.

[3 marks]

(b) (i) $x = 3.33$ *A1* *N1*

(ii) correct interval, with right end point $3\frac{1}{3}$ *A1A1* *N2*

e.g. $0 < x \le 3.33,\ 0 \le x < 3\frac{1}{3}$

Note: Accept any inequalities in the right direction.

[3 marks]

(c) valid approach *(M1)*

e.g. quotient rule, product rule

2 correct derivatives (must be seen in product or quotient rule) *(A1)(A1)*

e.g. $20,\ 0.3e^{0.3x}$ or $-0.3e^{-0.3x}$

correct substitution into product or quotient rule *A1*

e.g. $\frac{20e^{0.3x} - 20x(0.3)e^{0.3x}}{(e^{0.3x})^2},\ 20e^{-0.3x} + 20x(-0.3)e^{-0.3x}$

correct working *A1*

e.g. $\frac{20e^{0.3x} - 6xe^{0.3x}}{e^{0.6x}},\ \frac{e^{0.3x}(20 - 20x(0.3))}{(e^{0.3x})^2},\ e^{-0.3x}(20 + 20x(-0.3))$

$f'(x) = \frac{20 - 6x}{e^{0.3x}}$ *AG* *N0*

[5 marks]

(d) consideration of f' or f'' *(M1)*

valid reasoning **R1**
e.g. sketch of f', f'' is positive, $f''=0$, reference to minimum of f'

correct value $6.6666666\ldots\ \left(6\frac{2}{3}\right)$ *(A1)*

correct interval, with **both** end points **A1** **N3**
e.g $6.67 < x \le 20$, $6\frac{2}{3} \le x < 20$

[4 marks]

Total [15 marks]

Question 2

(a) evidence of valid approach *(M1)*
e.g. $y = 0$, $\sin x = 0$
$2\pi = 6.283185\ldots$
$k = 6.28$ **A1** **N2**
[2 marks]

(b) attempt to substitute either limits or the function into formula *(M1)*
(accept absence of dx)
e.g. $V = \pi\int_{\pi}^{k}(f(x))^2\,dx$, $\pi\int((x-1)\sin x)^2$, $\pi\int_{\pi}^{6.28\ldots} y^2\,dx$

correct expression **A2** **N3**
e.g. $\pi\int_{\pi}^{6.28}(x-1)^2\sin^2 x\,dx$, $\pi\int_{\pi}^{2\pi}((x-1)\sin x)^2\,dx$

[3 marks]

(c) $V = 69.60192562\ldots$
$V = 69.6$ **A2** **N2**

[2 marks]

Total [7 marks]

Question 3

(a) $f'(x) = -e^x \sin(e^x)$ *A1A1* *N2*

[2 marks]

(b)

A1A1A1A1 *N4*

Note:	Award ***A1*** for shape that must have the correct domain (from -2 to $+2$) and correct range (from -6 to 4), ***A1*** for minimum in circle, ***A1*** for maximum in circle and ***A1*** for intercepts in circles.

[4 marks]
Total [6 marks]

Question 4

(a) recognizing that acceleration is the derivative of velocity (seen anywhere) *(R1)*

e.g. $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$, v', $12 - 6t^2$

correctly substituting 2.7 into their expression for a (not into v) *(A1)*

e.g. $s''(2.7)$

acceleration = −31.74 (exact), −31.7 ***A1*** ***N3***

[3 marks]

(b) recognizing that displacement is the integral of velocity ***R1***

e.g. $s = \int v$

Correctly substituting 1.3 *(A1)*

e.g. $\int_0^{1.3} v \, \mathrm{d}t$

displacement = 7.41195 (exact), 7.41 (cm) ***A1*** ***N2***

[3 marks]

Total [6 marks]

Question 5

(a)	attempt to substitute coordinates in f *e.g.* $f(2)=9$	*(M1)*	
	correct substitution *e.g.* $a\times 2^3+b\times 2^2+c=9$	*A1*	
	$8a+4b+c=9$	***AG***	***N0*** ***[2 marks]***
(b)	recognizing that $(1, 4)$ is on the graph of f *e.g.* $f(1)=4$	*(M1)*	
	correct equation *e.g.* $a+b+c=4$	*A1*	
	recognizing that $f'=0$ at minimum (seen anywhere) *e.g.* $f'(1)=0$	*(M1)*	
	$f'(x)=3ax^2+2bx$ (seen anywhere)	*A1A1*	
	correct substitution into derivative *e.g.* $3a\times 1^2+2b\times 1=0$	*(A1)*	
	correct simplified equation *e.g.* $3a+2b=0$	*A1*	
			[7 marks]
(c)	valid method for solving system of equations *e.g.* inverse of a matrix, substitution	*(M1)*	
	$a=2, b=-3, c=5$	*A1A1A1*	***N4*** ***[4 marks]***

Total [13 marks]

Question 6

(a) $x = 2$ $\left(\text{accept } (2, 0)\right)$ *A1* *N1*

[1 mark]

(b) evidence of finding gradient of f at $x = 2$ *(M1)*

e.g. $f'(2)$

the gradient is 10 *A1* *N2*

[2 marks]

(c) evidence of negative reciprocal of gradient *(M1)*

e.g. $\frac{-1}{f'(x)}$, $-\frac{1}{10}$

evidence of correct substitution into equation of a line *(A1)*

e.g. $y - 0 = \frac{-1}{10}(x - 2)$, $0 = -0.1(2) + b$

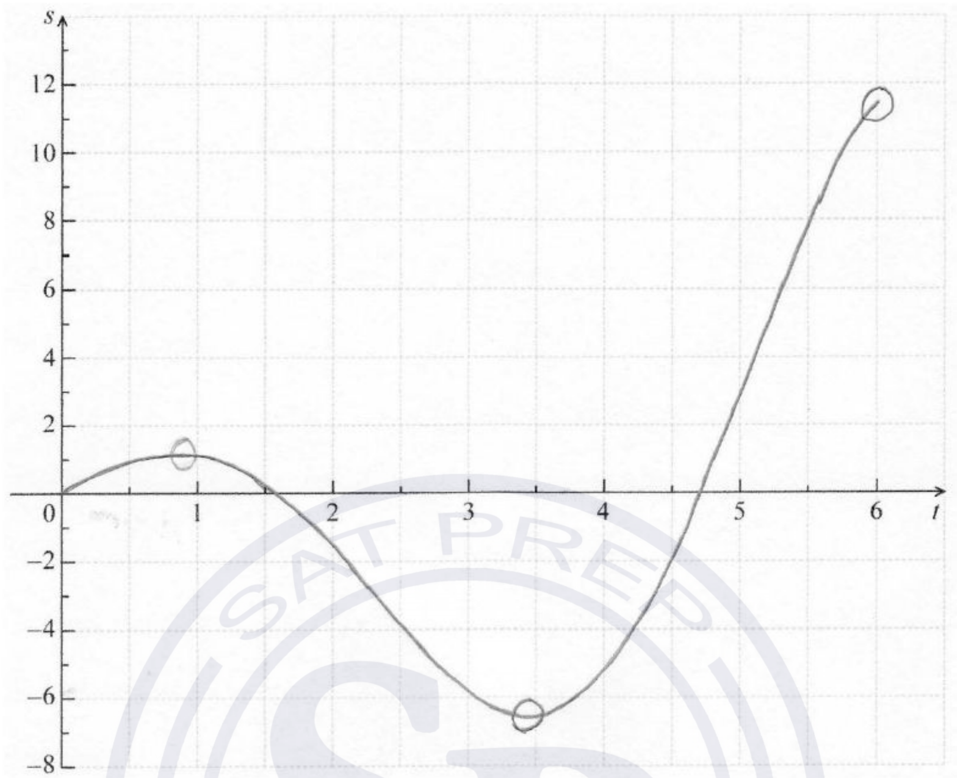
$y = -\frac{1}{10}x + \frac{2}{10}$ (accept $a = -0.1$, $b = 0.2$) *A1* *N2*

[3 marks]

Total [6 marks]

Question 7

(a)



A1A1A1A1 *N4*

> **Note:** Award ***A1*** for approximately correct shape (do not accept line segments).
> **Only** if this ***A1*** is awarded, award the following:
> ***A1*** for maximum and minimum within circles,
> ***A1*** for x-intercepts between 1 and 2 **and** between 4 and 5,
> ***A1*** for left endpoint at $(0, 0)$ and right endpoint within circle.

[4 marks]

(b) appropriate approach ***(M1)***

e.g. recognizing that $v = s'$, finding derivative, $a = s''$

valid method to find maximum ***(M1)***

e.g. sketch of v, $v'(t) = 0$, $t = 5.08698\ldots$

$v = 10.20025\ldots$

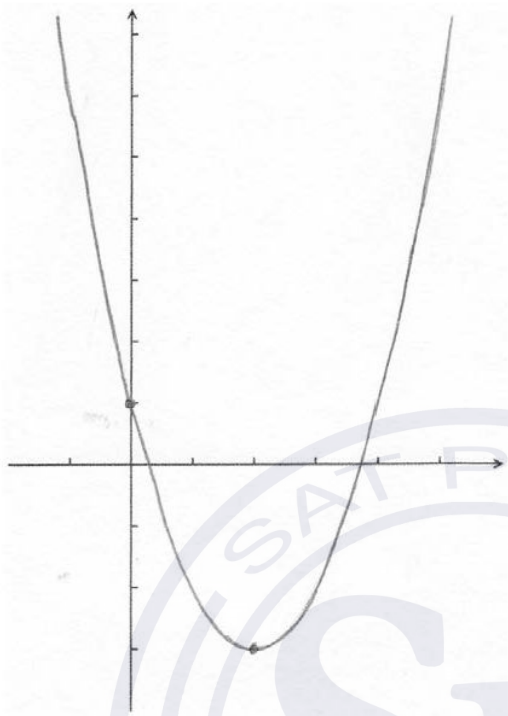
$v = 10.2$ [10.2,10.3] ***A1*** ***N2***

[3 marks]

[7 marks]

Question 8

(a)



A1A1A1A1 *N4*

Note: The shape **must** be an approximately correct upwards parabola.
Only if the shape is approximately correct, award the following:
A1 for vertex $x \approx 2$, *A1* for x-intercepts between 0 and 1, and 3 and 4,
A1 for correct y-intercept $(0, 1)$, *A1* for correct domain. $[-1, 5]$.
Scale not required on the axes, but approximate positions need to be clear.

[4 marks]

(b) $p = 2$ *A1* *N1*

[1 mark]

(c) correct vertical reflection, correct vertical translation *(A1)(A1)*

e.g. $-f(x)$, $-\left((x-2)^2-3\right)$, $-y$, $-f(x)+6$, $y+6$

transformations in correct order *(A1)*

e.g. $-(x^2-4x+1)+6$, $-\left((x-2)^2-3\right)+6$

simplification which clearly leads to given answer *A1*

e.g. $-x^2+4x-1+6$, $-(x^2-4x+4-3)+6$

$g(x) = -x^2+4x+5$ *AG* *N0*

Note: If working shown, award *A1A1A0A0* if transformations correct, but done in reverse order, *e.g.* $-(x^2-4x+1+6)$.

[4 marks]

(d) valid approach *(M1)*

e.g. sketch, $f = g$

$-0.449489\ldots, 4.449489\ldots$

$\left(2 \pm \sqrt{6}\right)$ (exact), $-0.449[-0.450, -0.449]$; $4.45[4.44, 4.45]$ *A1A1* *N3*

[3 marks]

(e) attempt to substitute limits or functions into area formula (accept absence of $\mathrm{d}x$) *(M1)*

e.g. $\int_a^b \left((-x^2 + 4x + 5) - (x^2 - 4x + 1)\right)\mathrm{d}x$, $\int_{4.45}^{-0.449} (f - g)$,

$\int (-2x^2 + 8x + 4)\,\mathrm{d}x$

approach involving subtraction of integrals/areas (accept absence of $\mathrm{d}x$) *(M1)*

e.g. $\int_a^b (-x^2 + 4x + 5) - \int_a^b (x^2 - 4x + 1)$, $\int (f - g)\,\mathrm{d}x$

area $= 39.19183\ldots$

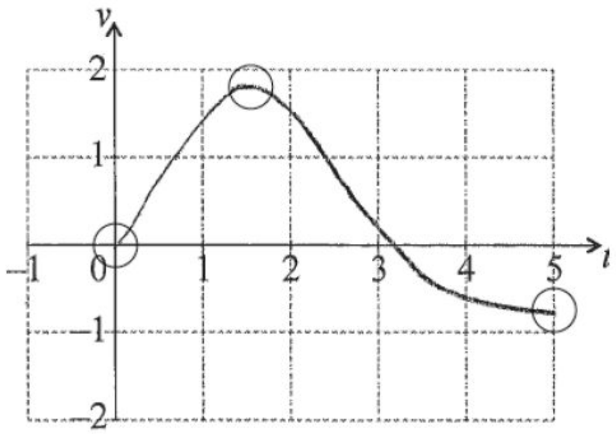
area $= 39.2$ [39.1, 39.2] *A1* *N3*

[3 marks]

Total [15 marks]

Question 9

(a)



A1A1A1 *N3*

Note: Award *A1* for approximately correct shape crossing x-axis with $3 < x < 3.5$.
Only if this *A1* is awarded, award the following:
A1 for maximum in circle, *A1* for endpoints in circle.

[3 marks]

(b) (i) $t = \pi$ (exact), 3.14 *A1* *N1*

(ii) recognizing distance is area under velocity curve ***(M1)***

eg $s = \int v$, shading on diagram, attempt to integrate v

valid approach to find the total area ***(M1)***

eg area A + area B, $\int v\mathrm{d}t - \int v\mathrm{d}t$, $\int_0^{3.14} v\mathrm{d}t + \int_{3.14}^{5} v\mathrm{d}t$, $\int |v|$

correct working with integration and limits (accept dx or missing dt) ***(A1)***

eg $\int_0^{3.14} v\mathrm{d}t + \int_5^{3.14} v\mathrm{d}t$, $3.067\ldots + 0.878\ldots$, $\int_0^5 \left|\mathrm{e}^{\sin t} - 1\right|$

distance $= 3.95$ (m) *A1* *N3*

[5 marks]

Total [8 marks]

Question 10

(a) $f(0) = \frac{100}{51}$ (exact), 1.96 *A1* *N1*

[1 mark]

(b) setting up equation *(M1)*

eg $95 = \frac{100}{1+50e^{-0.2x}}$, sketch of graph with horizontal line at $y = 95$

$x = 34.3$ *A1* *N2*

[2 marks]

(c) upper bound of y is 100 *(A1)*

lower bound of y is 0 *(A1)*

range is $0 < y < 100$ *A1* *N3*

[3 marks]

(d) **METHOD 1**

setting function ready to apply the chain rule *(M1)*

eg $100\left(1+50e^{-0.2x}\right)^{-1}$

evidence of correct differentiation (must be substituted into chain rule) *(A1)(A1)*

eg $u' = -100\left(1+50e^{-0.2x}\right)^{-2}$, $v' = \left(50e^{-0.2x}\right)(-0.2)$

correct chain rule derivative *A1*

eg $f'(x) = -100\left(1+50e^{-0.2x}\right)^{-2}\left(50e^{-0.2x}\right)(-0.2)$

correct working clearly leading to the required answer *A1*

eg $f'(x) = 1000e^{-0.2x}\left(1+50e^{-0.2x}\right)^{-2}$

$f'(x) = \frac{1000e^{-0.2x}}{\left(1+50e^{-0.2x}\right)^2}$ *AG* *N0*

METHOD 2

attempt to apply the quotient rule (accept reversed numerator terms) *(M1)*

eg $\frac{vu' - uv'}{v^2}$, $\frac{uv' - vu'}{v^2}$

evidence of correct differentiation inside the quotient rule *(A1)(A1)*

eg $f'(x) = \frac{\left(1+50e^{-0.2x}\right)(0) - 100\left(50e^{-0.2x} \times -0.2\right)}{\left(1+50e^{-0.2x}\right)^2}$, $\frac{100(-10)e^{-0.2x} - 0}{\left(1+50e^{-0.2x}\right)^2}$

any correct expression for derivative (0 may not be explicitly seen) *(A1)*

eg $\frac{-100\left(50e^{-0.2x} \times -0.2\right)}{\left(1+50e^{-0.2x}\right)^2}$

correct working clearly leading to the required answer *A1*

eg $f'(x) = \frac{0 - 100(-10)e^{-0.2x}}{\left(1+50e^{-0.2x}\right)^2}$, $\frac{-100(-10)e^{-0.2x}}{\left(1+50e^{-0.2x}\right)^2}$

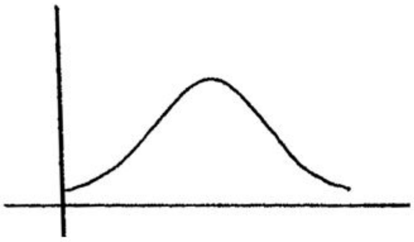
$f'(x) = \frac{1000e^{-0.2x}}{\left(1+50e^{-0.2x}\right)^2}$ *AG* *N0*

[5 marks]

(e) **METHOD 1**

sketch of $f'(x)$ *(A1)*

eg



recognizing maximum on $f'(x)$ *(M1)*

eg dot on max of sketch

finding maximum on graph of $f'(x)$ ***A1***

eg $(19.6, 5)$, $x = 19.560\ldots$

maximum rate of increase is 5 ***A1*** ***N2***

[4 marks]

METHOD 2

recognizing $f''(x) = 0$ *(M1)*

finding any correct expression for $f''(x)$ *(A1)*

eg $$\frac{\left(1+50\mathrm{e}^{-0.2x}\right)^2\left(-200\mathrm{e}^{-0.2x}\right)-\left(1000\mathrm{e}^{-0.2x}\right)\left(2\left(1+50\mathrm{e}^{-0.2x}\right)\left(-10\mathrm{e}^{-0.2x}\right)\right)}{\left(1+50\mathrm{e}^{-0.2x}\right)^4}$$

finding $x = 19.560\ldots$ ***A1***

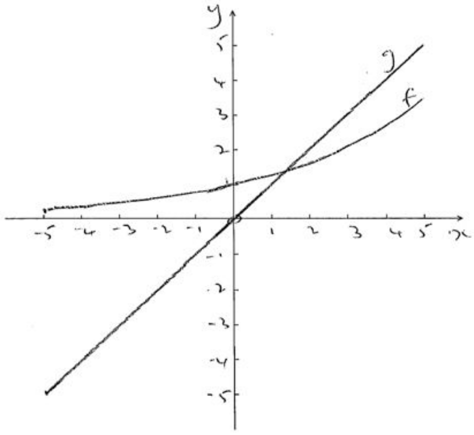
maximum rate of increase is 5 ***A1*** ***N2***

[4 marks]

Total [15 marks]

Question 11

(a) (i)



A1A1 *N2*

Notes:	Award ***A1*** for the graph of f positive, increasing and concave up. Award ***A1*** for graph of g increasing and linear with y-intercept of 0. Penalize one mark if domain is not [–5, 5] and/or if f and g do not intersect in the first quadrant.

(ii) attempt to find intersection of the graphs of f and g ***(M1)***

eg $e^{\frac{x}{4}} = x$

$x = 1.42961\ldots$ ***A1***

valid attempt to find area of R ***(M1)***

eg $\int (x - e^{\frac{x}{4}})\,dx$, $\int_0^1 (g - f)$, $\int (f - g)$

Area = 0.697 ***A2*** ***N3***

[7 marks]

(b) recognize that area of R is a maximum at point of tangency ***(R1)***

eg $m = f'(x)$

equating functions ***(M1)***

eg $f(x) = g(x)$, $e^{\frac{x}{4}} = mx$

$f'(x) = \frac{1}{4}e^{\frac{x}{4}}$ ***(A1)***

equating gradients ***(A1)***

eg $f'(x) = g'(x)$, $\frac{1}{4}e^{\frac{x}{4}} = m$

attempt to solve system of two equations for x ***(M1)***

eg $\frac{1}{4}e^{\frac{x}{4}} \times x = e^{\frac{x}{4}}$

$x = 4$ ***(A1)***

attempt to find m ***(M1)***

eg $f'(4)$, $\frac{1}{4}e^{\frac{4}{4}}$

$m = \frac{1}{4}e$ (exact), 0.680 ***A1*** ***N3***

[8 marks]

Total [15 marks]

Question 12

(a) valid approach *(M1)*

eg $f(x)=0$, sketch of parabola showing two x-intercepts

$x=1,\ x=4$ (accept $(1,0),\ (4,0)$) *A1A1* *N3*

[3 marks]

(b) attempt to substitute either limits or the function into formula involving f^2 *(M1)*

eg $\int_1^4 (f(x))^2\,dx,\ \pi\int((x-1)(x-4))^2$

volume $=8.1\pi$ (exact), 25.4 *A2* *N3*

[3 marks]

Total [6 marks]

Question 13

(a) expressing f as $x^{\frac{4}{3}}$ *(M1)*

$f'(x)=\frac{4}{3}x^{\frac{1}{3}}\quad\left(=\frac{4}{3}\sqrt[3]{x}\right)$ *A1* *N2*

[2 marks]

(b) attempt to integrate $\sqrt[3]{x^4}$ *(M1)*

eg $\dfrac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1}$

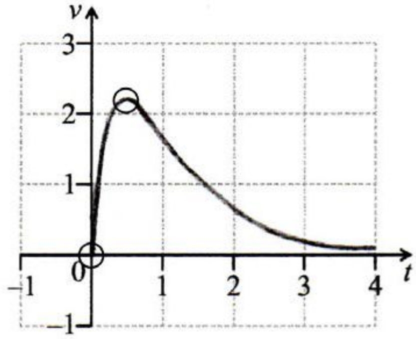
$\int f(x)\,dx=\frac{3}{7}x^{\frac{7}{3}}-\frac{x}{2}+c$ *A1A1A1* *N4*

[4 marks]

Total [6 marks]

Question 14

(a)



A1A2 *N3*

Notes: Award ***A1*** for approximately correct domain $0 \le t \le 4$.

The shape must be approximately correct, with maximum skewed left. **Only** if the shape is approximately correct, award ***A2*** for all the following approximately correct features, in circle of tolerance where drawn (accept seeing correct coordinates for the maximum, even if point outside circle):

Maximum point, passes through origin, asymptotic to t-axis (but must not touch the axis).

If only two of these features are correct, award ***A1***.

[3 marks]

(b) valid approach (including 0 and 3) ***(M1)***

eg $\int_0^3 10te^{-1.7t}dt$, $\int_0^3 f(x)$, area from 0 to 3 (may be shaded in diagram)

distance $= 3.33$ (m) ***A1*** ***N2***

[2 marks]

(c) recognizing acceleration is derivative of velocity ***(R1)***

eg $a = \frac{dv}{dt}$, attempt to find $\frac{dv}{dt}$, reference to maximum on the graph of v

valid approach to find v when $a = 0$ (may be seen on graph) ***(M1)***

eg $\frac{dv}{dt} = 0$, $10e^{-1.7t} - 17te^{-1.7t} = 0$, $t = 0.588$

velocity $= 2.16$ (m s^{-1}) ***A1*** ***N3***

Note: Award ***R1M1A0*** for $(0.588, 2.16)$ if velocity is not identified as final answer

[3 marks]

Total [8 marks]

Question 15

(a) (i) valid approach (may be seen on diagram) *(M1)*

eg Q to 6 is x

$PQ = 6 - 2x$ *A1* *N2*

(ii) $A = (6-2x)\sqrt{6x - x^2}$ *A1* *N1*

[3 marks]

(b) (i) recognising $\frac{dA}{dx}$ at $x = 2$ needed (must be the derivative of area) *(M1)*

$\frac{dA}{dx} = -\frac{7\sqrt{2}}{2}, \quad -4.95$ *A1* *N2*

(ii) $a = 0.879 \quad b = 3$ *A1A1* *N2*

[4 marks]

Total [7 marks]

Question 16

METHOD 1

$S_L(0) = 60$ (seen anywhere) *(A1)*

recognizing need to integrate V_R *(M1)*

eg $S_R(t) = \int V_R \, dt$

correct expression

eg $40t - \frac{1}{3}t^3 + C$ *A1A1*

Note: Award *A1* for $40t$, and *A1* for $-\frac{1}{3}t^3$.

equate displacements to find C *(R1)*

eg $40(0) - \frac{1}{3}(0)^3 + C = 60$, $S_L(0) = S_R(0)$

$C = 60$ *A1*

attempt to find displacement *(M1)*

eg $S_R(10)$, $40(10) - \frac{1}{3}(10)^3 + 60$

126.666

$126\frac{2}{3}$ (exact), 127 (m) *A1* *N5*

METHOD 2

recognizing need to integrate V_R *(M1)*

eg $S_R(t)=\int V_R \mathrm{d}t$

valid approach involving a definite integral *(M1)*

eg $\int_a^b V_R \,\mathrm{d}t$

correct expression with limits *(A1)*

eg $\int_0^{10}\left(40-t^2\right)\mathrm{d}t$, $\int_0^{10} V_R \,\mathrm{d}t$, $\left[40t-\frac{1}{3}t^3\right]_0^{10}$

66.6666 ***A2***

$S_L\left(0\right)=60$ (seen anywhere) *(A1)*

valid approach to find total displacement *(M1)*

eg $60+66.666$

126.666

$126\frac{2}{3}$ (exact), 127 (m) ***A1*** ***N5***

METHOD 3

$S_L(0)=60$ (seen anywhere) *(A1)*

recognizing need to integrate V_R *(M1)*

eg $S_R(t)=\int V_R \,\mathrm{d}t$

correct expression ***A1A1***

eg $40t-\frac{1}{3}t^3+C$

> **Note:** Award ***A1*** for $40t$, and ***A1*** for $-\frac{1}{3}t^3$.

correct expression for Ramiro displacement ***A1***

eg $S_R\left(10\right)-S_R\left(0\right)$, $\left[40t-\frac{1}{3}t^3+C\right]_0^{10}$

66.6666 ***A1***

valid approach to find total displacement *(M1)*

eg $60+66.6666$

126.666

$126\frac{2}{3}$ (exact), 127 (m) ***A1*** ***N5***

[8 marks]

Question 17

recognizing need to find $f(2)$ or $f'(2)$ *(R1)*

$f(2)=\frac{18}{6}$ (seen anywhere) *(A1)*

correct substitution into the quotient rule *(A1)*

eg $\frac{6(5)-18(2)}{6^2}$

$f'(2)=-\frac{6}{36}$ *A1*

gradient of normal is 6 *(A1)*

attempt to use the point and gradient to find equation of straight line *(M1)*

eg $y-f(2)=-\frac{1}{f'(2)}(x-2)$

correct equation in any form *A1* *N4*

eg $y-3=6(x-2)$, $y=6x-9$

[7 marks]

Question 18

(a) recognizing $f(x)=0$ *(M1)*

eg $f=0$, $x^2=5$

$x=\pm 2.23606$

$x=\pm\sqrt{5}$ (exact), $x=\pm 2.24$ *A1A1* *N3*

[3 marks]

(b) attempt to substitute either limits or the function into formula involving f^2 *(M1)*

eg $\pi\int\left(5-x^2\right)^2 \mathrm{d}x$, $\pi\int_{-2.24}^{2.24}\left(x^4-10x^2+25\right)$, $2\pi\int_0^{\sqrt{5}} f^2$

187.328

volume $=$ 187 *A2* *N3*

[3 marks]

Total [6 marks]

Question 19

(a) substituting $t=1$ into v — *(M1)*

eg $v(1), \left(1^2-4\right)^3$

velocity $=-27$ $\left(\text{ms}^{-1}\right)$ — *A1* *N2*

[2 marks]

(b) valid reasoning — *(R1)*

eg $v=0$, $\left(t^2-4\right)^3=0$

correct working — *(A1)*

eg $t^2-4=0$, $t=\pm 2$, sketch

$t=2$ — *A1* *N2*

[3 marks]

(c) correct integral expression for distance — *(A1)*

eg $\int_0^3 |v|$, $\int \left|\left(t^2-4\right)^3\right|$, $-\int_0^2 v\,dt+\int_2^3 v\,dt$,

$\int_0^2 \left(4-t^2\right)^3 \mathrm{d}t+\int_2^3 \left(t^2-4\right)^3 \mathrm{d}t$ (do not accept $\int_0^3 v\,\mathrm{d}t$)

86.2571

distance $=86.3$ (m) — *A2* *N3*

[3 marks]

(d) evidence of differentiating velocity — *(M1)*

eg $v'(t)$

$a=3\left(t^2-4\right)^2(2t)$ — *A2*

$a=6t\left(t^2-4\right)^2$ — *AG* *N0*

[3 marks]

(e) **METHOD 1**

valid approach — *M1*

eg graphs of v and a

correct working — *(A1)*

eg areas of same sign indicated on graph

$2<t\le 3$ (accept $t>2$) — *A2* *N2*

METHOD 2

recognizing that $a\ge 0$ (accept a *is* always positive) (seen anywhere) — *R1*

recognizing that v is positive when $t>2$ (seen anywhere) — *(R1)*

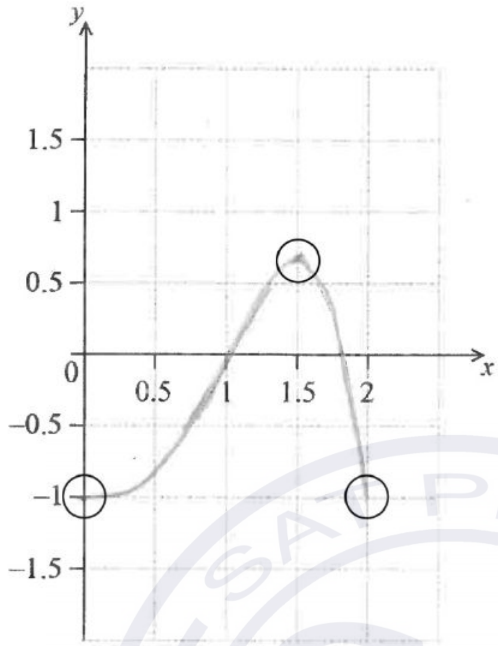
$2<t\le 3$ (accept $t>2$) — *A2* *N2*

[4 marks]

Total [15 marks]

Question 20

(a)



A1A1A1 *N3*

[3 marks]

Note: Award *A1* for both endpoints in circles,
A1 for approximately correct shape (concave up to concave down).
Only if this *A1* for shape is awarded, award *A1* for maximum point in circle.

(b) $x=1$ $x=1.83928$

$x=1$ (exact) $x=1.84$ [1.83, 1.84] *A1A1* *N2*

[2 marks]

(c) attempt to substitute either (***FT***) limits or function into formula with f^2 *(M1)*

(accept absence of π or dx, but do not accept any errors, including extra bits)

eg $V=\pi\int_1^{1.84} f^2$, $\int\left(-x^4+2x^3-1\right)^2 \mathrm{d}x$

0.636581

$V=0.637$ [0.636, 0.637] *A2* *N3*

[3 marks]

Total [8 marks]

Question 21

(a) correct substitution of function and/or limits into formula **(A1)**
(accept absence of d*t*, but do not accept any errors)

eg $\int_0^{\frac{\pi}{2}} v$, $\int \left| e^{\frac{1}{2}\cos t} - 1 \right| dt$, $\int \left(e^{\frac{1}{2}\cos t} - 1 \right)$

0.613747
distance is 0.614 [0.613, 0.614] (m) **A1** **N2**
[2 marks]

(b) **METHOD 1**

valid attempt to find the distance travelled between $t = \frac{\pi}{2}$ and $t = 4$ **(M1)**

eg $\int_{\frac{\pi}{2}}^{4} \left(e^{\frac{1}{2}\cos t} - 1 \right)$, $\int_0^4 \left| e^{\frac{1}{2}\cos t} - 1 \right| dt - 0.614$

distance is 0.719565 **A1**

valid reason, referring to change of direction (may be seen in explanation) **R1**

valid explanation comparing **their** distances **R1**
eg $0.719565 > 0.614$, distance moving back is more than distance moving forward

Note: Do not award the final ***R1*** unless the ***A1*** is awarded.

particle passes through A again **AG** **N0**

METHOD 2

valid attempt to find displacement **(M1)**

eg $\int_{\frac{\pi}{2}}^{4} \left(e^{\frac{1}{2}\cos t} - 1 \right)$, $\int_0^4 \left(e^{\frac{1}{2}\cos t} - 1 \right)$

correct displacement **A1**
eg −0.719565, −0.105817

recognising that displacement from 0 to $\frac{\pi}{2}$ is positive **R1**

eg displacement = distance from 0 to $\frac{\pi}{2}$

valid explanation referring to positive and negative displacement **R1**
eg $0.719565 > 0.614$, overall displacement is negative, since displacement after $\frac{\pi}{2}$ is negative, then particle gone backwards more than forwards

Note: Do not award the final ***R1*** unless the ***A1*** and the first ***R1*** are awarded.

particle passes through A again **AG** **N0**
[4 marks]

Note: Special Case. If all working shown, and candidates seem to have misread the question, using $v = e^{\frac{1}{2}\cos t}$, award marks as follows:

(a) correct substitution of function and/or limits into formula *A0MR*
(accept absence of d*t*, but do not accept any errors)

eg $\int_0^{\frac{\pi}{2}} \left(e^{\frac{1}{2}\cos t}\right)$, $\int \left|e^{\frac{1}{2}\cos t}\right| dt$, $\int \left(e^{\frac{1}{2}\cos t}\right)$

2.184544
distance is 2.18 [2.18, 2.19] (m) *A1* *N0*

(b) **METHOD 1**

valid attempt to find the distance travelled between $t = \frac{\pi}{2}$ and $t = 4$ *M1*

eg $\int_{\frac{\pi}{2}}^{4} \left(e^{\frac{1}{2}\cos t}\right)$, $\int_0^4 \left|e^{\frac{1}{2}\cos t}\right| dt - 2.18$

distance is 1.709638 *A1*

reference to change of direction (may be seen in explanation) *R1*

reasoning/stating particle passes/does not pass through A again *R0*

METHOD 2

valid attempt to find displacement *M1*

eg $\int_{\frac{\pi}{2}}^{4} \left(e^{\frac{1}{2}\cos t}\right)$, $\int_0^4 \left(e^{\frac{1}{2}\cos t}\right)$

correct displacement *A1*
eg 1.709638, 3.894182

recognising that displacement from 0 to $\frac{\pi}{2}$ is positive *R0*

reasoning/stating particle passes/does not pass through A again *R0*

With method 2, there is no valid reasoning about whether the particle passes through A again or not, so they cannot gain the ***R*** marks.

Total [6 marks]

Question 22

recognizing that the gradient of tangent is the derivative **(M1)**

eg f'

finding the gradient of f at P **(A1)**

eg $f'(0.25)=16$

evidence of taking negative reciprocal of **their** gradient at P **(M1)**

eg $\frac{-1}{m}, -\frac{1}{f'(0.25)}$

equating derivatives **M1**

eg $f'(x)=\frac{-1}{16}, f'=-\frac{1}{m}, \frac{x\left(\frac{1}{x}\right)-\ln(4x)}{x^2}=16$

finding the x-coordinate of Q, $x=0.700750$

$x=0.701$ **A1** **N3**

attempt to substitute **their** x into f to find the y-coordinate of Q **(M1)**

eg $f(0.7)$

$y=1.47083$

$y=1.47$ **A1** **N2**

[7 marks]

Question 23

(a)	$p = 6$	***A1***	***N1***
	recognising that turning points occur when $f'(x) = 0$ eg correct sign diagram	***R1***	***N1***
	f' changes from positive to negative at $x = 6$	***R1***	***N1***
			[3 marks]
(b)	$f'(2) = -2$	***A1***	***N1***
			[1 mark]
(c)	attempt to apply chain rule *eg* $\ln(x)' \times f'(x)$	***(M1)***	
	correct expression for $g'(x)$ *eg* $g'(x) = \frac{1}{f(x)} \times f'(x)$	***(A1)***	
	substituting $x = 2$ into **their** g' *eg* $\frac{f'(2)}{f(2)}$	***(M1)***	
	-0.666667 $g'(2) = -\frac{2}{3}$ (exact), -0.667	***A1***	***N3***
			[4 marks]
(d)	evidence of integrating $g'(x)$ eg $g(x)\big\vert_2^a$, $g(x)\big\vert_a^2$	***(M1)***	
	applying the fundamental theorem of calculus (seen anywhere) eg $\int_2^a g'(x) = g(a) - g(2)$	***R1***	
	correct substitution into integral eg $\ln 3 + g(a) - g(2)$, $\ln 3 + g(a) - \ln(f(2))$	***(A1)***	
	$\ln 3 + g(a) - \ln 3$	***A1***	
	$\ln 3 + \int_2^a g'(x) = g(a)$	***AG***	***N0***
			[4 marks]

(e) **METHOD 1**

substituting $a = 5$ into the formula for $g(a)$ ***(M1)***

eg $\int_2^5 g'(x)\,dx$, $g(5) = \ln 3 + \int_2^5 g'(x)\,dx$ (do not accept only $g(5)$)

attempt to substitute areas ***(M1)***

eg $\ln 3 + 0.66 - 0.21$, $\ln 3 + 0.66 + 0.21$

correct working

eg $g(5) = \ln 3 + (-0.66 + 0.21)$ ***(A1)***

0.648612

$g(5) = \ln 3 - 0.45$ (exact), 0.649 ***A1*** ***N3***

METHOD 2

attempt to set up an equation for one shaded region ***(M1)***

eg $\int_4^5 g'(x)\,dx = 0.21$, $\int_2^4 g'(x)\,dx = -0.66$, $\int_2^5 g'(x)\,dx = -0.45$

two correct equations ***(A1)***

eg $g(5) - g(4) = 0.21$, $g(2) - g(4) = 0.66$

combining equations to eliminate $g(4)$ ***(M1)***

eg $g(5) - [\ln 3 - 0.66] = 0.21$

0.648612

$g(5) = \ln 3 - 0.45$ (exact), 0.649 ***A1*** ***N3***

METHOD 3

attempt to set up a definite integral ***(M1)***

eg $\int_2^5 g'(x)\,dx = -0.66 + 0.21$, $\int_2^5 g'(x)\,dx = -0.45$

correct working ***(A1)***

eg $g(5) - g(2) = -0.45$

correct substitution ***(A1)***

eg $g(5) - \ln 3 = -0.45$

0.648612

$g(5) = \ln 3 - 0.45$ (exact), 0.649 ***A1*** ***N3***

[4 marks]

Total [16 marks]

Question 24

(a) attempt to find intersection *(M1)*

eg $f = g$

$p = 1,\ q = 3$ **A1A1** **N3**

[3 marks]

(b) $f'(p) = -1$ **A2** **N2**

[2 marks]

(c) (i) correct approach to find the gradient of the normal *(A1)*

eg $m_1 m_2 = -1,\ -\dfrac{1}{f'(p)}$, correct value of 1

EITHER

attempt to substitute coordinates (in any order) and correct normal gradient to find c *(M1)*

eg $3 = -\dfrac{1}{f'(p)} \times 1 + c,\ 1 = 1 \times 3 + c$

$c = 2$ *(A1)*

$y = x + 2$ **A1** **N2**

OR

attempt to substitute coordinates (in any order) and correct normal gradient into equation of a straight line *(M1)*

eg $y - 3 = -\dfrac{1}{f'(p)}(x - 1),\ y - 1 = 1 \times (x - 3)$

correct working *(A1)*

eg $y = (x - 1) + 3$

$y = x + 2$ **A1** **N2**

(ii) $(0,\ 2)$ **A1** **N1**

[5 marks]

(d) appropriate approach involving subtraction *(M1)*

eg $\int_a^b (L - g)\,dx,\ \int \left(3x^2 - (x + 2)\right)$

substitution of **their** limits or function *(A1)*

eg $\int_0^p (L - g)\,dx,\ \int \left((x + 2) - 3x^2\right)$

area $= 1.5$ **A1** **N2**

[3 marks]

Total [13 marks]

Question 25

(a) area of $\text{ABCD} = \text{AB}^2$ (seen anywhere) ***(A1)***

choose cosine rule to find a side of the square ***(M1)***

eg $a^2 = b^2 + c^2 - 2bc\cos\theta$

correct substitution (for triangle AOB) ***A1***

eg $r^2 + r^2 - 2 \times r \times r\cos\theta$, $\text{OA}^2 + \text{OB}^2 - 2 \times \text{OA} \times \text{OB}\cos\theta$

correct working for AB^2 ***A1***

eg $2r^2 - 2r^2\cos\theta$

area $= 2r^2(1 - \cos\theta)$ ***AG*** ***N0***

Note: Award no marks if the only working is $2r^2 - 2r^2\cos\theta$.

[4 marks]

(b) (i) $\frac{1}{2}\alpha r^2$ (accept $2r^2(1 - \cos\alpha)$) ***A1*** ***N1***

(ii) correct equation in one variable ***(A1)***

eg $2(1 - \cos\alpha) = \frac{1}{2}\alpha$

$\alpha = 0.511024$

$\alpha = 0.511$ (accept $\theta = 0.511$) ***A2*** ***N2***

Note: Award ***A1*** for $\alpha = 0.511$ and additional answers.

[4 marks]

(c)

Note: In this part, accept θ instead of β, and the use of equations instead of inequalities in the working.

attempt to find R *(M1)*

eg subtraction of areas, $\text{square} - \text{segment}$

correct expression for segment area *(A1)*

eg $\frac{1}{2}\beta r^2 - \frac{1}{2}r^2 \sin\beta$

correct expression for R *(A1)*

eg $2r^2(1-\cos\beta) - \left(\frac{1}{2}\beta r^2 - \frac{1}{2}r^2 \sin\beta\right)$

correct inequality *(A1)*

eg $2r^2(1-\cos\beta) - \left(\frac{1}{2}\beta r^2 - \frac{1}{2}r^2 \sin\beta\right) > 2\left(\frac{1}{2}\beta r^2\right)$

correct inequality in terms of angle only ***A1***

eg $2(1-\cos\beta) - \left(\frac{1}{2}\beta - \frac{1}{2}\sin\beta\right) > \beta$

attempt to solve their inequality, must represent R> twice sector *(M1)*

eg sketch, one correct value

ote: Do not award the second ***(M1)*** unless the first ***(M1)*** for attempting to find R has been awarded.

both correct values 1.30573 and 2.67369 *(A1)*

correct inequality $1.31 < \beta < 2.67$ ***A1*** ***N3***

[8 marks]

Total [16 marks]

Question 26

(a) valid approach *(M1)*
eg horizontal translation 3 units to the right

$x = 3$ (must be an equation) ***A1*** ***N2***
[2 marks]

(b) valid approach *(M1)*
eg $f(x) = 0$, $e^0 = x - 3$

4, $x = 4$, $(4, 0)$ ***A1*** ***N2***
[2 marks]

(c) attempt to substitute either **their correct** limits or the function into formula involving f^2 *(M1)*
eg $\int_4^{10} f^2$, $\pi \int \left(2\ln(x-3)\right)^2 \mathrm{d}x$

141.537
volume = 142 ***A2*** ***N3***
[3 marks]
Total [7 marks]

Question 27

(a) recognizing particle at rest when $v = 0$ *(M1)*
eg $(0.3t + 0.1)^t - 4 = 0$, x-intercept on graph of v

$t = 4.27631$
$t = 4.28$ (seconds) ***A2*** ***N3***
[3 marks]

(b) valid approach to find t when a is 0 *(M1)*
eg $v'(t) = 0$, v minimum,

$t = 1.19236$
$t = 1.19$ (seconds) ***A2*** ***N3***
[3 marks]

Total [6 marks]

Question 28

7. (a) correct substitution into chain rule **A2**

eg $f'(x) = \frac{1}{x^2} \times 2x$

$f'(x) = \frac{2}{x}$ **AG** **N0**

[2 marks]

There are many approaches to this question, especially the steps to set up the correct equation, for the two M marks. There are a few processes they may need to apply at some stage, for the ***M1M1***. These include substituting $f'(d)$ and points P and/or Q into the gradient of PQ or equation of the tangent line PQ. There may be other approaches, please check working and award marks in line with markscheme.

(b) at P, $y = \ln(d^2)$ (seen anywhere) **A1**

gradient of tangent at P is $\frac{2}{d}$ (seen anywhere) **A1**

substituting $(1, -3)$, $(d, \ln d^2)$ or gradient $\frac{2}{d}$ into equation of tangent at P ***(M1)***

eg $y - (-3) = m(x-1)$, $y = \left(\frac{2}{d}\right)x + b$, $y - \ln d^2 = m(x-d)$

second substitution ***(M1)***

eg $y + 3 = \left(\frac{2}{d}\right)(x-1)$, $-3 = \left(\frac{2}{d}\right)1 + b$, $m = \frac{\ln d^2 + 3}{d-1}$

any correct equation (in d or x) **A1**

eg $-3 - \ln(d^2) = \left(\frac{2}{d}\right)(1-d)$, $\ln(x^2) + 1 + \left(\frac{2}{x}\right) = 0$

-1.30505

$d = -1.31$ (accept $x = -1.31$) **A1** **N2**

[6 marks]

Total [8 marks]

Question 29

(a) **METHOD 1**

recognizing $s = \int v$ *(M1)*

recognizing displacement of P in first 5 seconds (seen anywhere) (accept missing $\mathrm{d}t$) *A1*

eg $\int_0^5 v\,\mathrm{d}t$, -3.71591

valid approach to find total displacement *(M1)*

eg $4 + (-3.7159)$, $s = 4 + \int_0^5 v$

0.284086

0.284 (m) ***A2*** ***N3***

METHOD 2

recognizing $s = \int v$ *(M1)*

correct integration *A1*

eg $\frac{1}{3}\sin 3t + 2\cos t - \frac{t}{2} + c$ (do not penalize missing "c")

attempt to find c *(M1)*

eg $4 = \frac{1}{3}\sin(0) + 2\cos(0) - \frac{0}{2} + c$, $4 = \frac{1}{3}\sin 3t + 2\cos t - \frac{t}{2} + c$, $2 + c = 4$

attempt to substitute $t = 5$ into their expression with c *(M1)*

eg $s(5)$, $\frac{1}{3}\sin(15) + 2\cos(5) - \frac{5}{2} + 2$

0.284086

0.284 (m) ***A1*** ***N3***

[5 marks]

(b) recognizing that at rest, $v = 0$ *(M1)*

$t = 0.179900$

$t = 0.180$ (secs) ***A1*** ***N2***

[2 marks]

(c) recognizing when change of direction occurs *(M1)*

eg v crosses t axis

2 (times) ***A1*** ***N2***

[2 marks]

(d) acceleration is v' (seen anywhere) *(M1)*

eg $v'(3)$

0.743631

0.744 (ms^{-2}) *A1* *N2*

[2 marks]

(e) valid approach involving max or min of v *(M1)*

eg $v' = 0$, $a = 0$, graph

one correct co-ordinate for min *(A1)*

eg 1.14102, –3.27876

3.28 (ms^{-1}) *A1* *N2*

[3 marks]

Total [14 marks]

Question 30

correct approach *(A1)*

eg $s = \int v$, $\int_0^p 6t - 6\,dt$

correct integration *(A1)*

eg $\int 6t - 6\,dt = 3t^2 - 6t + C$, $\left[3t^2 - 6t\right]_0^p$

recognizing that there are two possibilities *(M1)*

eg 2 correct answers, $s = \pm 2$, $c \pm 2$

two correct equations in p *A1A1*

eg $3p^2 - 6p = 2$, $3p^2 - 6p = -2$

0.42265, 1.57735

$p = 0.423$ or $p = 1.58$ *A1A1* *N3*

[7 marks]

Question 31

(a) $y = 2$ (correct equation only) **A2** **N2**

[2 marks]

(b) valid approach **(M1)**

eg $(x-1)^{-1}+2$, $f'(x)=\dfrac{0(x-1)-1}{(x-1)^2}$

$-(x-1)^{-2}$, $f'(x)=\dfrac{-1}{(x-1)^2}$ **A1** **N2**

[2 marks]

(c) correct equation for the asymptote of g

eg $y = b$ **(A1)**

$b = 2$ **A1** **N2**

[2 marks]

(d) correct derivative of g (seen anywhere) **(A2)**

eg $g'(x) = -a\mathrm{e}^{-x}$

correct equation **(A1)**

eg $-\mathrm{e} = -a\mathrm{e}^{-1}$

7.38905

$a = \mathrm{e}^2$ (exact), 7.39 **A1** **N2**

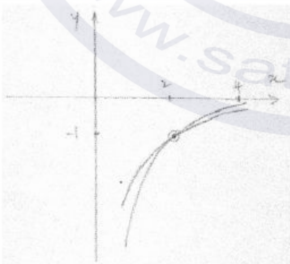
[4 marks]

(e) attempt to equate **their** derivatives **(M1)**

eg $f'(x) = g'(x)$, $\dfrac{-1}{(x-1)^2} = -a\mathrm{e}^{-x}$

valid attempt to solve **their** equation **(M1)**

eg correct value outside the domain of f such as 0.522 or 4.51,



correct solution (may be seen in sketch) **(A1)**

eg $x = 2$, $(2, -1)$

gradient is -1 **A1** **N3**

[4 marks]

Total [14 marks]

Question 32

(a) valid attempt to find the intersection *(M1)*

eg $f = g$, sketch, one correct answer

$p = 0.357402$, $q = 2.15329$

$p = 0.357$, $q = 2.15$ **A1A1** **N3**

[3 marks]

(b) attempt to set up an integral involving subtraction (in any order) *(M1)*

eg $\int_p^q [f(x) - g(x)]\,dx$, $\int_p^q f(x)\,dx - \int_p^q g(x)\,dx$

0.537667

area $= 0.538$ **A2** **N3**

[3 marks]

[Total 6 marks]

Question 33

(a) attempt to substitute correct limits or the function into the formula involving y^2 *(M1)*

eg $\pi\int_{-0.5}^{0.5} y^2\,dx$, $\pi\int\left(-0.8x^2 + 0.5\right)^2 dx$

0.601091

volume $= 0.601$ (m^3) **A2** **N3**

[3 marks]

(b) attempt to equate half **their** volume to V *(M1)*

eg $0.30055 = 0.8\left(1 - e^{-0.1t}\right)$, graph

4.71104

4.71 (minutes) **A2** **N3**

[3 marks]

[Total 6 marks]

Question 34

(a) valid attempt to substitute $t=0$ into the correct function **(M1)**

eg $-2(0)+2$

2 **A1** **N2**

[2 marks]

(b) recognizing $v=0$ when P is at rest **(M1)**

5.21834

$p=5.22$ (seconds) **A1** **N2**

[2 marks]

(c) (i) recognizing that $a=v'$ **(M1)**

eg $v'=0$, minimum on graph

1.95343

$q=1.95$ **A1** **N2**

(ii) valid approach to find **their** minimum **(M1)**

eg $v(q)$, -1.75879, reference to min on graph

1.75879

speed $=1.76$ $\left(\mathrm{cm\,s^{-1}}\right)$ **A1** **N2**

[4 marks]

(d) (i) substitution of **correct** $v(t)$ into distance formula, **(A1)**

eg $\int_1^p \left|3\sqrt{t}+\frac{4}{t^2}-7\right| \mathrm{d}t$, $\left|\int 3\sqrt{t}+\frac{4}{t^2}-7\,\mathrm{d}t\right|$,

4.45368

distance $=4.45$ (cm) **A1** **N2**

(ii) displacement from $t=1$ to $t=p$ (seen anywhere) **(A1)**

eg -4.45368, $\int_1^p \left(3\sqrt{t}+\frac{4}{t^2}-7\right)\mathrm{d}t$

displacement from $t=0$ to $t=1$ **(A1)**

eg $\int_0^1 (-2t+2)\,\mathrm{d}t$, $0.5\times1\times2$, 1

valid approach to find displacement for $0\le t\le p$ **M1**

eg $\int_0^1 (-2t+2)\,\mathrm{d}t+\int_1^p \left(3\sqrt{t}+\frac{4}{t^2}-7\right)\mathrm{d}t$, $\int_0^1 (-2t+2)\,\mathrm{d}t-4.45$

-3.45368

displacement $=-3.45$ (cm) **A1** **N2**

[6 marks]

[Total 14 marks]

Question 35

METHOD 1

derivative of $f(x)$ **A2**

$7(x^2+3)^6(2x)$

recognizing need to find x^4 term in $(x^2+3)^6$ (seen anywhere) **R1**

eg $14x$ $(\text{term in } x^4)$

valid approach to find the terms in $(x^2+3)^6$ ***(M1)***

eg $\binom{6}{r}(x^2)^{6-r}(3)^r$, $(x^2)^6(3)^0+(x^2)^5(3)^1+\ldots$, Pascal's triangle to 6th row

identifying correct term (may be indicated in expansion) ***(A1)***

eg 5th term, $r=2$, $\binom{6}{4}$, $(x^2)^2(3)^4$

correct working (may be seen in expansion) ***(A1)***

eg $\binom{6}{4}(x^2)^2(3)^4$, 15×3^4, $14x\times15\times81(x^2)^2$

$17010x^5$ **A1** **N3**

METHOD 2

recognition of need to find x^6 in $(x^2+3)^7$ (seen anywhere) **R1**

valid approach to find the terms in $(x^2+3)^7$ ***(M1)***

eg $\binom{7}{r}(x^2)^{7-r}(3)^r$, $(x^2)^7(3)^0+(x^2)^6(3)^1+\ldots$, Pascal's triangle to 7th row

identifying correct term (may be indicated in expansion) ***(A1)***

eg 6th term, $r=3$, $\binom{7}{3}$, $(x^2)^3(3)^4$

correct working (may be seen in expansion) ***(A1)***

eg $\binom{7}{4}(x^2)^3(3)^4$, 35×3^4

correct term ***(A1)***

$2835x^6$

differentiating their term in x^6 ***(M1)***

eg $(2835x^6)'$, $(6)(2835x^5)$

$17010x^5$ **A1** **N3**

[7 marks]

Question 36

(a)	(i)	$t = 2$	*A1*	*N1*
	(ii)	substitution of limits or function into formula or correct sum	*(A1)*	
		eg $\int_0^8 \lvert v \rvert \mathrm{d}t$, $\int \lvert v_Q \rvert \mathrm{d}t$, $\int_0^2 v\mathrm{d}t - \int_2^4 v\mathrm{d}t + \int_4^6 v\mathrm{d}t - \int_6^8 v\mathrm{d}t$		
		9.64782		
		distance $= 9.65$ (metres)	*A1*	*N2*
				[3 marks]
(b)		correct approach	*(A1)*	
		eg $s = \int \sqrt{t}$, $\int_0^k \sqrt{t}\,\mathrm{d}t$, $\int_0^k \lvert v_Q \rvert \mathrm{d}t$		
		correct integration	*(A1)*	
		eg $\int \sqrt{t} = \frac{2}{3}t^{\frac{3}{2}} + c$, $\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^k$, $\frac{2}{3}k^{\frac{3}{2}}$		
		equating their expression to the distance travelled by their P	*(M1)*	
		eg $\frac{2}{3}k^{\frac{3}{2}} = 9.65$, $\int_0^k \sqrt{t}\,\mathrm{d}t = 9.65$		
		5.93855		
		5.94 (seconds)	*A1*	*N3*
				[4 marks]

Total [7 marks]

Question 37

(a) (i) $q = 2$ **_A1_** **_N1_**

(ii) $h = 0$ **_A1_** **_N1_**

(iii) $k = 3$ **_A1_** **_N1_**

> **Note:** Accept $q = 1$, $h = 0$, and $k = 3 - \ln(2)$, 2.31 as candidate may have rewritten $g(x)$ as equal to $3 + \ln(x) - \ln(2)$.

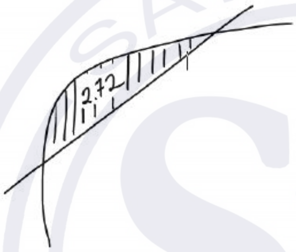
[3 marks]

(b) (i) 2.72409

2.72 **_A2_** **_N2_**

(ii) recognizing area between $y = x$ and h equals 2.72 **_(M1)_**

eg



recognizing graphs of h and h^{-1} are reflections of each other in $y = x$ **_(M1)_**

eg area between $y = x$ and h equals between $y = x$ and h^{-1}

2×2.72, $\int_{0.111}^{3.31} \left(x - h^{-1}(x)\right) dx = 2.72$

5.44819

5.45 **_A1_** **_N3_**

[5 marks]

(c) valid attempt to find d *(M1)*

eg difference in y-coordinates, $d = h(x) - x$

correct expression for d *(A1)*

eg $\left(\ln\frac{1}{2}x + 3\right)(\cos 0.1x) - x$

valid approach to find when d is a maximum *(M1)*

eg max on sketch of d, attempt to solve $d' = 0$

0.973679

$x = 0.974$ ***A2*** ***N4***

substituting **their** x value into $h(x)$ *(M1)*

2.26938

$y = 2.27$ ***A1*** ***N2***

[7 marks]

[15 marks]

Question 38

METHOD 1 (displacement)

recognizing $s = \int v\,\mathrm{d}t$ ***(M1)***

consideration of displacement at $t = 2$ **and** $t = 5$ (seen anywhere) ***M1***

eg $\int_0^2 v$ and $\int_0^5 v$

Note: Must have both for any further marks.

correct displacement at $t = 2$ and $t = 5$ (seen anywhere) ***A1A1***

-2.28318 (accept 2.28318), 1.55513

valid reasoning comparing correct displacements ***R1***

eg $|-2.28| > |1.56|$, more left than right

2.28 (m) ***A1*** ***N1***

Note: Do not award the final ***A1*** without the ***R1***.

METHOD 2 (distance travelled)

recognizing distance $= \int |v|\,\mathrm{d}t$ ***(M1)***

consideration of distance travelled from $t = 0$ to 2 **and** $t = 2$ to 5 (seen anywhere) ***M1***

eg $\int_0^2 v$ and $\int_2^5 v$

Note: Must have both for any further marks.

correct distances travelled (seen anywhere) ***A1A1***

2.28318, (accept -2.28318), 3.83832

valid reasoning comparing correct distance values ***R1***

eg $3.84 - 2.28 < 2.28$, $3.84 < 2 \times 2.28$

2.28 (m) ***A1*** ***N1***

Note: Do not award the final ***A1*** without the ***R1***.

[6 marks]

Question 39

(a)	evidence of valid approach *eg* $f(x)=0$, $y=0$		*(M1)*	
	2.73205 $p=2.73$		***A1***	***N2***
				[2 marks]
(b)	(i)	1.87938, 8.11721 (1.88, 8.12)	***A2***	***N2***
	(ii)	rate of change is 0 (do not accept decimals)	***A1***	***N1***
				[3 marks]
(c)	(i)	**METHOD 1 (using GDC)**		
		valid approach *eg* $f''=0$, max/min on f', $x=-1$	***M1***	
		sketch of either f' or f'', with max/min or root (respectively)	*(A1)*	
		$x=1$	***A1***	***N1***
		Substituting **their** x value into f *eg* $f(1)$	*(M1)*	
		$y=4.5$	***A1***	***N1***
		METHOD 2 (analytical)		
		$f''=-6x^2+6$	***A1***	
		setting $f''=0$	*(M1)*	
		$x=1$	***A1***	***N1***
		substituting **their** x value into f *eg* $f(1)$	*(M1)*	
		$y=4.5$	***A1***	***N1***

(ii) recognizing rate of change is f' *(M1)*

eg y', $f'(1)$

rate of change is 6 ***A1*** ***N2***

[7 marks]

(d) attempt to substitute either limits or the function into formula involving f^2 (accept absence of π and/or $\mathrm{d}x$) *(M1)*

eg $\pi\int\left(-0.5x^4+3x^2+2x\right)^2\mathrm{d}x$, $\int_1^{1.88} f^2$

128.890

volume $= 129$ ***A2*** ***N3***

[3 marks]

[Total 15 marks]

Question 40

(a) valid approach *(M1)*

eg $f(p)=4$, intersection with $y=4$, ± 2.32

2.32143

$p=\sqrt{\mathrm{e}^2-2}$ (exact), 2.32 ***A1*** ***N2***

[2 marks]

(b) attempt to substitute **either their** limits **or** the function into volume formula (must involve f^2, accept reversed limits and absence of π and/or $\mathrm{d}x$, but do not accept any other errors) *(M1)*

eg $\int_{-2.32}^{2.32} f^2$, $\pi\int\left(6-\ln(x^2+2)\right)^2\mathrm{d}x$, 105.675

331.989

volume $= 332$ ***A2*** ***N3***

[3 marks]

Total [5 marks]

Question 41

(a) $t=\frac{2}{3}$(exact), 0.667, $t=4$ **A1A1** **N2**

[2 marks]

(b) recognizing that v is decreasing when a is negative **(M1)**

eg $a<0$, $3t^2-14t+8\le 0$, sketch of a

correct interval **A1** **N2**

eg $\frac{2}{3}<t<4$

[2 marks]

(c) valid approach (do not accept a definite integral) **(M1)**

eg $v=\int a$

correct integration (accept missing c) **(A1)(A1)(A1)**

t^3-7t^2+8t+c

substituting $t=0$, $v=3$ (must have c) **(M1)**

eg $3=0^3-7(0^2)+8(0)+c$, $c=3$

$v=t^3-7t^2+8t+3$ **A1** **N6**

[6 marks]

(d) recognizing that v increases outside the interval found in part (b) **(M1)**

eg $0<t<\frac{2}{3}$, $4<t<5$, diagram

one correct substitution into distance formula **(A1)**

eg $\int_0^{\frac{2}{3}}|v|$, $\int_4^5|v|$, $\int_{\frac{2}{3}}^4|v|$, $\int_0^5|v|$

one correct pair **(A1)**

eg 3.13580 and 11.0833, 20.9906 and 35.2097

14.2191 **A1** **N2**

$d=14.2$ (m)

[4 marks]

Total [14 marks]

Question 42

(a) initial velocity when $t = 0$ *(M1)*

eg $v(0)$

$v = 7\ (\text{m s}^{-1})$ **A1** **N2**

[2 marks]

(b) recognizing maximum speed when $|v|$ is greatest *(M1)*

eg minimum, maximum, $v' = 0$

one correct coordinate for minimum *(A1)*

eg $6.37896, -24.6571$

$24.7\ \ (\text{m s}^{-1})$ **A1** **N2**

[3 marks]

(c) recognizing $a = v'$ *(M1)*

eg $a = \frac{\mathrm{d}v}{\mathrm{d}t}$, correct derivative of first term

identifying when $a = 0$ *(M1)*

eg turning points of v, t-intercepts of v'

3 **A1** **N3**

[3 marks]

(d) recognizing P changes direction when $v = 0$ *(M1)*

$t = 0.863851$ *(A1)*

-9.24689

$a = -9.25\ (\text{m s}^{-2})$ **A2** **N3**

[4 marks]

(e) correct substitution of limits or function into formula *(A1)*

eg $\int_0^7 |v|,\ \int_0^{0.8638} v\,\mathrm{d}t - \int_{0.8638}^{7} v\,\mathrm{d}t,\ \int \left| 7\cos x - 5x^{\cos x} \right| \mathrm{d}x,\ 3.32 + 60.6$

63.8874

63.9 (metres) **A2** **N3**

[3 marks]

[Total: 15 Marks]

Question 43

(a) valid approach (M1)

eg $f(x)=0$, $e^x = 180$ or $0 \ldots$

1.14472

$x = \ln \pi$ (exact), 1.14 **A1** **N2**

[2 marks]

(b) attempt to substitute either **their** limits or the function into formula involving f^2 (M1)

eg $\int_0^{1.14} f^2$, $\pi\int\left(\sin\left(e^x\right)\right)^2 dx$, 0.795135

2.49799

volume $= 2.50$ **A2** **N3**

[3 marks]

[Total: 5 marks]

Question 44

(a) $-0.394791, 13$

$A(-0.395, 13)$ **A1A1** **N2**

[2 marks]

(b) (i) 13 **A1** **N1**

(ii) 2π, 6.28 **A1** **N1**

[2 marks]

(c) valid approach (M1)

eg recognizing that amplitude is p or shift is r

$f(x) = 13\cos(x + 0.395)$ (accept $p = 13$, $r = 0.395$) **A1A1** **N3**

> ***Note:*** Accept any value of r of the form $0.395 + 2\pi k$, $k \in \mathbb{Z}$

[3 marks]

(d) recognizing need for $d'(t)$ *(M1)*

eg $-12\sin(t)-5\cos(t)$

correct approach (accept any variable for t) *(A1)*

eg $-13\sin(t+0.395)$, sketch of d', $(1.18,-13)$, $t=4.32$

maximum speed $=13\ (\text{cm s}^{-1})$ ***A1*** ***N2***

[3 marks]

(e) recognizing that acceleration is needed *(M1)*

eg $a(t)$, $d''(t)$

correct equation (accept any variable for t) *(A1)*

eg $a(t)=-2$, $\left|\frac{\mathrm{d}}{\mathrm{d}t}(d'(t))\right|=2$, $-12\cos(t)+5\sin(t)=-2$

valid attempt to solve **their** equation *(M1)*

eg sketch, 1.33

1.02154

1.02 ***A2*** ***N3***

[5 marks]

Total [15 marks]

Question 45

. (a) attempt to substitute correct limits or the function into formula involving f^2 ***(M1)***

eg $\pi\int_{-2}^{2} y^2\,\mathrm{d}y$, $\pi\int\left(\sqrt{\frac{4-x^2}{8}}\right)^2 \mathrm{d}x$

4.18879

volume $= 4.19$, $\frac{4}{3}\pi$ (exact) (m^3) ***A2*** ***N3***

Note: If candidates have their GDC incorrectly set in degrees, award ***M*** marks where appropriate, but no ***A*** marks may be awarded. Answers from degrees are $p = 13.1243$ and $q = 26.9768$ in (b)(i) and 12.3130 or 28.3505 in (b)(ii).

[3 marks]

(b) (i) recognizing the volume increases when g' is positive ***(M1)***

eg $g'(t) > 0$, sketch of graph of g' indicating correct interval

1.73387, 3.56393

$p = 1.73$, $q = 3.56$ ***A1A1*** ***N3***

(ii) valid approach to find change in volume ***(M1)***

eg $g(q) - g(p)$, $\int_p^q g'(t)\,\mathrm{d}t$

3.74541

total amount $= 3.75$ (m^3) ***A2*** ***N3***

[6 marks]

continued…

(c)

> **Note:** There may be slight differences in the final answer, depending on which values candidates carry through from previous parts. Accept answers that are consistent with correct working.

recognizing when the volume of water is a maximum *(M1)*

eg maximum when $t = q$, $\int_0^q g'(t)\,dt$

valid approach to find maximum volume of water *(M1)*

eg $2.3 + \int_0^q g'(t)\,dt$, $2.3 + \int_0^p g'(t)\,dt + 3.74541$, 3.85745

correct expression for the difference between volume of container and maximum value *(A1)*

eg $4.18879 - \left(2.3 + \int_0^q g'(t)\,dt\right)$, $4.19 - 3.85745$

0.331334

0.331 (m^3) **A2** **N3**

[5 marks]

Total [14 marks]

Question 46

(a) valid approach *(M1)*

eg $v(t) = 0$, sketch of graph

2.95195

$t = \log_{1.4} 2.7$ (exact), $t = 2.95$ (s) **A1** **N2**

[2 marks]

(b) valid approach *(M1)*

eg $a(t) = v'(t)$, $v'(2)$

0.659485

$a(2) = 1.96 \ln 1.4$ (exact), $a(2) = 0.659$ $(\mathrm{m\,s^{-2}})$ **A1** **N2**

[2 marks]

(c) correct approach *(A1)*

eg $\int_0^5 |v(t)|\,dt$, $\int_0^{2.95} (-v(t))\,dt + \int_{2.95}^5 v(t)\,dt$

5.3479

distance = 5.35 (m) **A2** **N3**

[3 marks]

Total [7 marks]

Question 47

(a) valid approach *(M1)*
eg $s_A(0)$, $s(0)$, $t=0$

15 (cm) ***A1*** ***N2***
[2 marks]

(b) valid approach *(M1)*
eg $s_A = 0$, $s = 0$, 6.79321, 14.8651

2.46941
$t = 2.47$ (seconds) ***A1*** ***N2***
[2 marks]

(c) recognizing when change in direction occurs *(M1)*
eg slope of s changes sign, $s' = 0$, minimum point, 10.0144, (4.08, -4.66)

4.07702
$t = 4.08$ (seconds) ***A1*** ***N2***
[2 marks]

(d) **METHOD 1 (using displacement)**

correct displacement or distance from P at $t = 3$ (seen anywhere) *(A1)*
eg -2.69630, 2.69630

valid approach *(M1)*
eg $15 + 2.69630$, $s(3) - s(0)$, -17.6963

17.6963
17.7 (cm) ***A1*** ***N2***

METHOD 2 (using velocity)

attempt to substitute either limits or the velocity function into distance formula involving $|v|$ *(M1)*

eg $\int_0^3 |v|\, dt$, $\int \left| -1 - 18t^2 e^{-0.8t} + 4.8t^3 e^{-0.8t} \right|$

17.6963
17.7 (cm) ***A2*** ***N2***
[3 marks]

(e) (i) recognize the need to integrate velocity *(M1)*

eg $\int v(t)$

$8t - \frac{2t^2}{2} + c$ (accept x instead of t and missing c) *(A2)*

substituting initial condition into their integrated expression (must have c) *(M1)*

eg $15 = 8(0) - \frac{2(0)^2}{2} + c$, $c = 15$

$s_B(t) = 8t - t^2 + 15$ ***A1*** ***N3***

(ii) valid approach *(M1)*

eg $s_A = s_B$, sketch, $(9.30404, 2.86710)$

9.30404

$t = 9.30$ (seconds) ***A1*** ***N2***

> **Note:** If candidates obtain $s_B(t) = 8t - t^2$ in part (e)(i), there are 2 solutions for part (e)(ii), 1.32463 and 7.79009. Award the last ***A1*** in part (e)(ii) only if both solutions are given.

[7 marks]

Total [16 marks]

Question 48

(a) valid approach *(M1)*

eg $f(10)$

235.402

235 (fish) (must be an integer) ***A1*** ***N2***

[2 marks]

(b) recognizing rate of change is derivative *(M1)*

eg rate $= f'$, $f'(10)$, sketch of f', 35 (fish per month)

35.9976

36.0 (fish per month) ***A1*** ***N2***

[2 marks]

(c) valid approach *(M1)*

eg maximum of f', $f'' = 0$

15.890

15.9 (months) ***A1*** ***N2***

[2 marks]

Total [6 marks]

Question 49

(a) valid approach *(M1)*

eg $f(x)=0$, $4-2e^{x}=0$

0.693147

$x=\ln 2$ (exact), 0.693 ***A1*** ***N2***

[2 marks]

(b) attempt to substitute either their correct limits or the function into formula involving f^2 *(M1)*

eg $\int_0^{0.693} f^2$, $\pi\int\left(4-2e^{x}\right)^2 dx$, $\int_0^{\ln 2}\left(4-2e^{x}\right)^2$

3.42545

volume = 3.43 ***A2*** ***N3***

[3 marks]

Total [5 marks]

Question 50

(a) attempt to find $f'(8)$ *(M1)*

eg $f'(x)$, y', $-16x^{-2}$

-0.25 (exact) ***A1*** ***N2***

[2 marks]

(b) $\boldsymbol{u}=\begin{pmatrix}4\\-1\end{pmatrix}$ or any scalar multiple ***A2*** ***N2***

[2 marks]

(c) correct scalar product and magnitudes *(A1)(A1)(A1)*

scalar product $=1\times 4+1\times -1\ (=3)$

magnitudes = $\sqrt{1^2+1^2}, \sqrt{4^2+(-1)^2}\ \left(=\sqrt{2}, \sqrt{17}\right)$

substitution of **their** values into correct formula *(M1)*

eg $\dfrac{4-1}{\sqrt{1^2+1^2}\sqrt{4^2+(-1)^2}}$, $\dfrac{-3}{\sqrt{2}\sqrt{17}}$, 2.1112, 120.96°

1.03037, 59.0362°

angle = 1.03, 59.0° ***A1*** ***N4***

[5 marks]

(d) (i) attempt to form composite $(f \circ f)(x)$ ***(M1)***

eg $f(f(x))$, $f\left(\dfrac{16}{x}\right)$, $\dfrac{16}{f(x)}$

correct working ***(A1)***

eg $\dfrac{16}{16/x}$, $16 \times \dfrac{x}{16}$

$(f \circ f)(x) = x$ ***A1*** ***N2***

(ii) $f^{-1}(x) = \dfrac{16}{x}$ (accept $y = \dfrac{16}{x}$, $\dfrac{16}{x}$) ***A1*** ***N1***

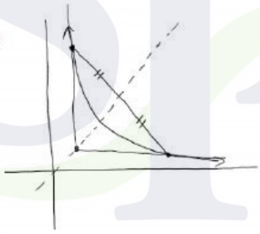
> **Note:** Award ***A0*** in part (ii) if part (i) is incorrect.
>
> Award ***A0*** in part (ii) if the candidate has found $f^{-1}(x) = \dfrac{16}{x}$ by interchanging x and y.

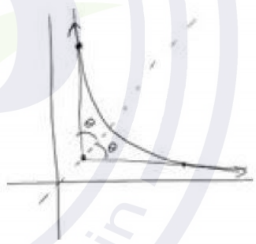
(iii) **METHOD 1**

recognition of symmetry about $y = x$ ***(M1)***

eg ,

$(2, 8) \Leftrightarrow (8, 2)$





evidence of doubling **their** angle ***(M1)***

eg 2×1.03, 2×59.0

2.06075, $118.072°$

2.06 (radians) (118 degrees) ***A1*** ***N2***

METHOD 2

finding direction vector for tangent line at $x = 2$ ***(A1)***

eg $\begin{pmatrix} -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

substitution of **their** values into correct formula (must be from vectors) ***(M1)***

eg $\frac{-4-4}{\sqrt{1^2+4^2}\sqrt{4^2+(-1)^2}}, \frac{8}{\sqrt{17}\sqrt{17}}$

$2.06075, 118.072^\circ$

2.06 (radians) (118 degrees) ***A1*** ***N2***

METHOD 3

using trigonometry to find an angle with the horizontal ***(M1)***

eg $\tan\theta = -\frac{1}{4}, \tan\theta = -4$

finding both angles of rotation ***(A1)***

eg $\theta_1 = 0.244978, 14.0362^\circ, \theta_2 = 1.81577, 104.036^\circ$

$2.06075, 118.072^\circ$

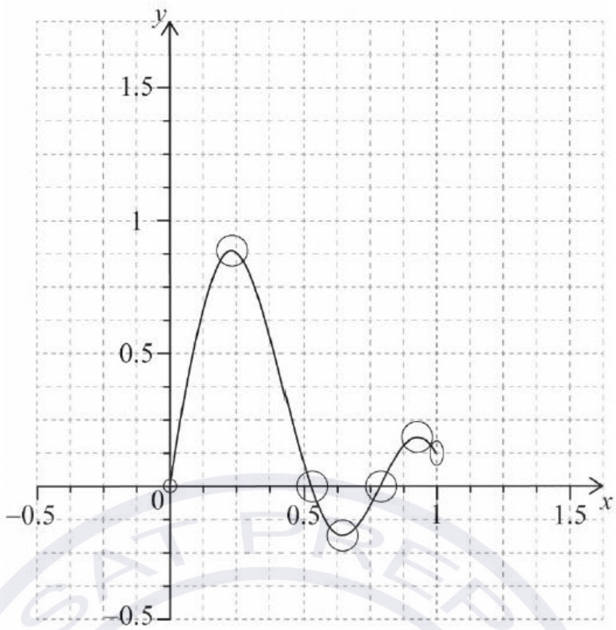
2.06 (radians) (118 degrees) ***A1*** ***N2***

[7 marks]

Total [16 marks]

Question 51

(a)



A1A1A1 ***N3***

> **Note:** Only if the shape is approximately correct with exactly 2 maximums and 1 minimum on the interval $0 \le x \le 1$, award the following:
> ***A1*** for correct domain with both endpoints within circle and oval.
> ***A1*** for passing through the other x-intercepts within the circles.
> ***A1*** for passing through the three turning points within circles (ignore x-intercepts and extrema outside of the domain).

[3 marks]

(b) evidence of reasoning (may be seen on graph) ***(M1)***

eg $f'' = 0$, $(0.524, 0)$, $(0.785, 0)$

0.523598, 0.785398

$x = 0.524 \left(= \frac{\pi}{6}\right)$, $x = 0.785 \left(= \frac{\pi}{4}\right)$ ***A1A1*** ***N3***

> **Note:** Award ***M1A1A0*** if any solution outside domain (eg $x = 0$) is also included.

[3 marks]

(c) $0.524 < x < 0.785$ $\left(\frac{\pi}{6} < x < \frac{\pi}{4}\right)$ ***A2*** ***N2***

> **Note:** Award ***A1*** if any correct interval outside domain also included, unless additional solutions already penalized in (b).
> Award ***A0*** if any incorrect intervals are also included.

[2 marks]

Total [8 marks]

Question 52

(a) choosing product rule **(M1)**

eg $uv' + vu'$, $\left(x^2\right)'(e^{3x}) + \left(e^{3x}\right)' x^2$

correct derivatives (must be seen in the rule) **A1A1**

eg $2x$, $3e^{3x}$

$f'(x) = 2xe^{3x} + 3x^2e^{3x}$ **A1** **N4**

[4 marks]

(b) valid method **(M1)**

eg $f'(x) = 0$,

$a = -0.667\left(= -\frac{2}{3}\right)$ (accept $x = -0.667$) **A1** **N2**

[2 marks]

Total [6 marks]

Question 53

(a) recognizing that $v = \int a$ **(M1)**

correct integration **A1**

eg $-120\cos(2t) + c$

attempt to find c using their $v(t)$ **(M1)**

eg $-120\cos(0) + c = 140$

$v(t) = -120\cos(2t) + 260$ **A1** **N3**

[4 marks]

(b) evidence of valid approach to find time taken in first stage **(M1)**

eg graph, $-120\cos(2t) + 260 = 375$

$k = 1.42595$ **A1**

attempt to substitute **their** v and/or **their** limits into distance formula **(M1)**

eg $\int_0^{1.42595} |v|$, $\int 260 - 120\cos(2t)$, $\int_0^k (260 - 120\cos(2t))\,dt$

353.608

distance is 354 (m) **A1** **N3**

[4 marks]

(c) recognizing velocity of second stage is linear (seen anywhere) **R1**

eg graph, $s = \frac{1}{2}h(a+b)$, $v = mt + c$

valid approach ***(M1)***

eg $\int v = 353.608$

correct equation ***(A1)***

eg $\frac{1}{2}h(375+500) = 353.608$

time for stage two $= 0.808248$ (0.809142 from 3 sf) **A2**

2.23420 (2.23914 from 3 sf)

2.23 (seconds) (2.24 from 3 sf) **A1** **N3**

[6 marks]

Total [14 marks]

Question 54

(a) evidence of valid approach ***(M1)***

eg $f(x) = 0$, $y = 0$

1.13843

$p = 1.14$ **A1** **N2**

[2 marks]

(b) (i) $0.562134, 16.7641$

$(0.562, 16.8)$ **A2** **N2**

(ii) valid approach ***(M1)***

eg tangent at maximum point is horizontal, $f' = 0$

$y = 16.8$ (must be an equation) **A1** **N2**

[4 marks]

(c) (i) **METHOD 1 (using GDC)**

valid approach **M1**

eg $f'' = 0$, max/min on f', $x = -3$

sketch of either f' or f'', with max/min or root (respectively) ***(A1)***

$x = 3$ **A1** **N1**

substituting **their** x value into f ***(M1)***

eg $f(3)$

$y = -225$ (exact) (accept $(3, -225)$) **A1** **N1**

METHOD 2 (analytical)

$f'' = 12x^2 - 108$ *A1*

valid approach *(M1)*

eg $f'' = 0$, $x = \pm 3$

$x = 3$ *A1* *N1*

substituting **their** x value into f *(M1)*

eg $f(3)$

$y = -225$ (exact) (accept $(3, -225)$) *A1* *N1*

(ii) recognizing rate of change is f' *(M1)*

eg y', $f'(3)$

rate of change is -156 (exact) *A1* *N2*

[7 marks]

(d) attempt to substitute **either their** limits **or** the function into volume formula *(M1)*

eg $\int_{1.14}^{3} f^2$, $\pi\int (x^4 - 54x^2 + 60x)^2 \,dx$, $25\,752.0$

$80\,902.3$

volume $= 80\,900$ *A2* *N3*

[3 marks]

Total [16 marks]

Question 55

(a) attempt to form composite (in any order) *(M1)*

eg $f(x^4 - 3)$, $(x-8)^4 - 3$

$h(x) = x^4 - 11$ *A1* *N2*

[2 marks]

(b) recognizing that the gradient of the tangent is the derivative *(M1)*

eg h'

correct derivative (seen anywhere) *(A1)*

$h'(x) = 4x^3$

correct value for gradient of f (seen anywhere) *(A1)*

$f'(x) = 1$, $m = 1$

setting **their** derivative equal to 1 *(M1)*

$4x^3 = 1$

0.629960

$x = \sqrt[3]{\frac{1}{4}}$ (exact), 0.630 *A1* *N3*

[5 marks]

Total [7 marks]