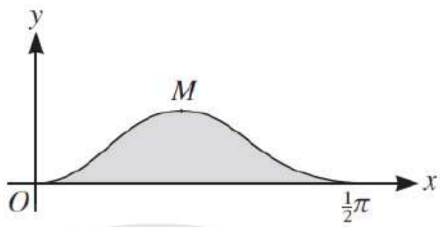
A-Level

Calculus

2013-2018

Question 1



The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [6]

(ii) Using the substitution $u = \sin x$, find by integration the area of the shaded region bounded by the curve and the x-axis. [4]

Question 2

(i) Express $(\sqrt{3})\cos x + \sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{1}{((\sqrt{3})\cos x + \sin x)^2}\,dx = \frac{1}{4}\sqrt{3}.$$ [4]

Question 3

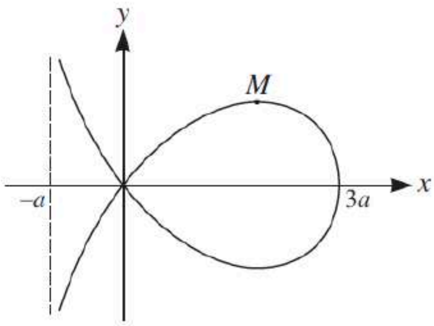
(i) By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3})\tan\theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3 + x^2)}}\,dx,$$

expressing your answer as a single logarithm. [4]

Question 4



The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$

where a is a positive constant. The maximum point on the curve is M. Find the x-coordinate of M in terms of a. [6]

Question 5

(a) Show that $\int_2^4 4x \ln x \, dx = 56 \ln 2 - 12$. [5]

(b) Use the substitution $u = \sin 4x$ to find the exact value of $\int_0^{\frac{1}{24}\pi} \cos^3 4x \, dx$. [5]

Question 6

For each of the following curves, find the gradient at the point where the curve crosses the y-axis:

(i) $y = \dfrac{1 + x^2}{1 + e^{2x}}$; [3]

(ii) $2x^3 + 5xy + y^3 = 8$. [4]

Question 7

A curve has equation $3e^{2x}y + e^x y^3 = 14$. Find the gradient of the curve at the point $(0, 2)$. [5]

Question 8

Use the substitution $u = 3x + 1$ to find $\displaystyle\int \frac{3x}{3x + 1} \, dx$. [4]

Question 9

Find the exact value of $\displaystyle\int_1^4 \frac{\ln x}{\sqrt{x}} \, dx$. [5]

Question 10

The parametric equations of a curve are

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t.$$

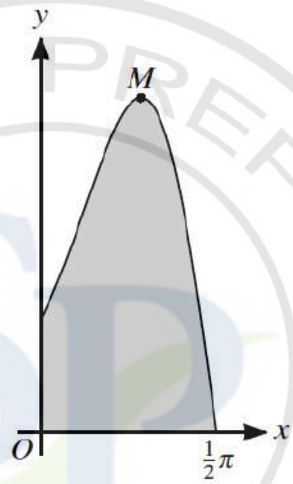
Show that $\frac{dy}{dx} = \tan\left(t - \frac{1}{4}\pi\right)$. [6]

Question 11

(i) Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. [3]

(ii) Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta \, d\theta = \frac{1}{2}\ln 3$. [4]

Question 12

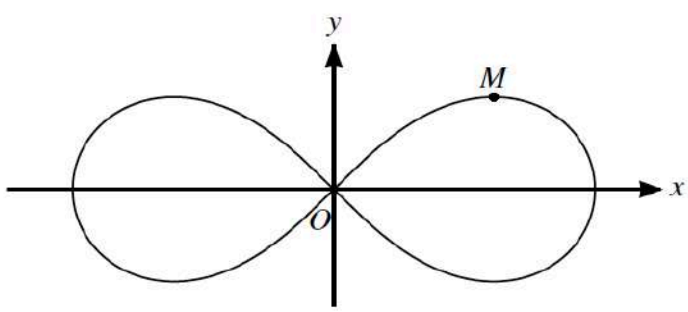


The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \sin x$, find the exact value of the area of the shaded region bounded by the curve and the axes. [5]

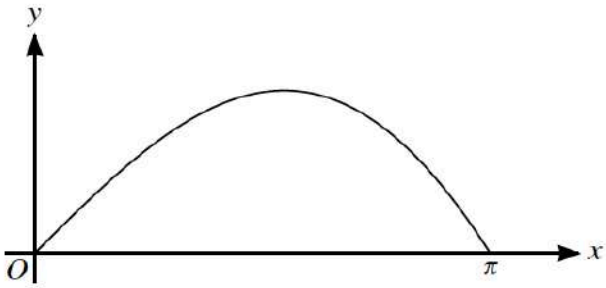
(ii) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

Question 13



The diagram shows the curve $(x^2 + y^2)^2 = 2(x^2 - y^2)$ and one of its maximum points M. Find the coordinates of M. [7]

Question 14



The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.

(i) Find $\frac{dy}{dx}$ and show that $4\frac{d^2y}{dx^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

(ii) Find the exact value of the area of the region enclosed by this part of the curve and the x-axis. [5]

Question 15

The parametric equations of a curve are

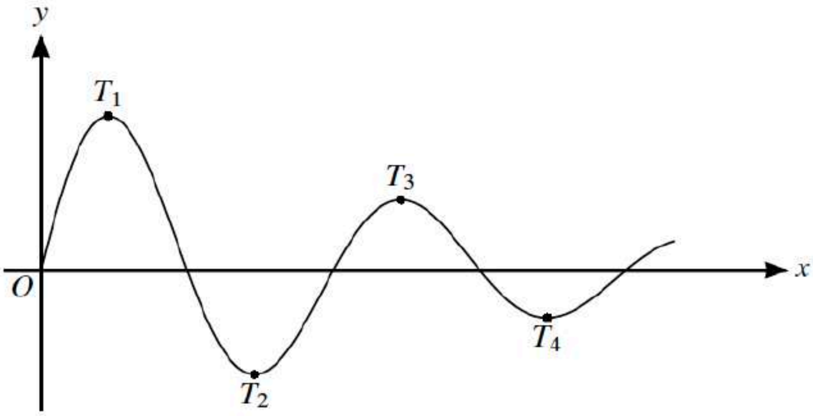
$$x = t - \tan t, \quad y = \ln(\cos t),$$

for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \cot t$. [5]

(ii) Hence find the x-coordinate of the point on the curve at which the gradient is equal to 2. Give your answer correct to 3 significant figures. [2]

Question 16



The diagram shows the curve $y = 10e^{-\frac{1}{2}x}\sin 4x$ for $x \geqslant 0$. The stationary points are labelled T_1, T_2, T_3, ... as shown.

(i) Find the x-coordinates of T_1 and T_2, giving each x-coordinate correct to 3 decimal places. [6]

(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. [4]

Question 17

The parametric equations of a curve are

$$x = \ln(2t + 3), \quad y = \frac{3t + 2}{2t + 3}.$$

Find the gradient of the curve at the point where it crosses the y-axis. [

Question 18

Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1 + 3\tan x)}}{\cos^2 x}\,dx.$$ [5]

Question 19

By first using the substitution $u = e^x$, show that

$$\int_0^{\ln 4} \frac{e^{2x}}{e^{2x} + 3e^x + 2}\,dx = \ln\left(\frac{8}{5}\right).$$ [10]

Question 20

A curve is defined for $0 < \theta < \frac{1}{2}\pi$ by the parametric equations

$$x = \tan\theta, \quad y = 2\cos^2\theta\sin\theta.$$

Show that $\dfrac{dy}{dx} = 6\cos^5\theta - 4\cos^3\theta$. [5]

Question 21

It is given that $I = \int_0^{0.3} (1 + 3x^2)^{-2}\, dx$.

(i) Use the trapezium rule with 3 intervals to find an approximation to I, giving the answer correct to 3 decimal places. [3]

(ii) For small values of x, $(1 + 3x^2)^{-2} \approx 1 + ax^2 + bx^4$. Find the values of the constants a and b. Hence, by evaluating $\int_0^{0.3} (1 + ax^2 + bx^4)\, dx$, find a second approximation to I, giving the answer correct to 3 decimal places. [5]

Question 22

The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x \sin t - \tan t$. [3]

Question 23

It is given that $\int_1^a \ln(2x)\, dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

Question 24

Let $f(x) = \frac{11x + 7}{(2x - 1)(x + 2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\int_1^2 f(x)\, dx = \frac{1}{4} + \ln\left(\frac{9}{4}\right)$. [5]

Question 25

The curve with equation $y = \frac{e^{2x}}{4 + e^{3x}}$ has one stationary point. Find the exact values of the coordinates of this point. [6]

Question 26

The parametric equations of a curve are

$$x = a\cos^4 t, \quad y = a\sin^4 t,$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin^2 t + y\cos^2 t = a\sin^2 t\cos^2 t.$$ [3]

(iii) Hence show that if the tangent meets the x-axis at P and the y-axis at Q, then

$$OP + OQ = a,$$

where O is the origin. [2]

Question 27

Let $I = \int_0^1 \frac{\sqrt{x}}{2-\sqrt{x}}\,dx$.

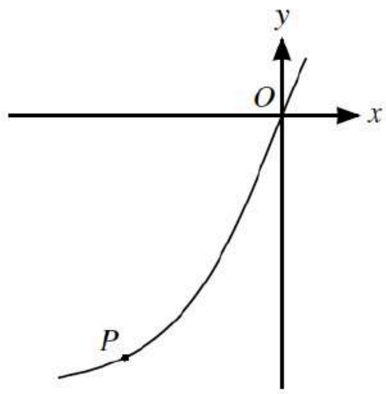
(i) Using the substitution $u = 2 - \sqrt{x}$, show that $I = \int_1^2 \frac{2(2-u)^2}{u}\,du$. [4]

(ii) Hence show that $I = 8\ln 2 - 5$. [4]

Question 28

A curve has equation $y = \cos x\cos 2x$. Find the x-coordinate of the stationary point on the curve in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

Question 29

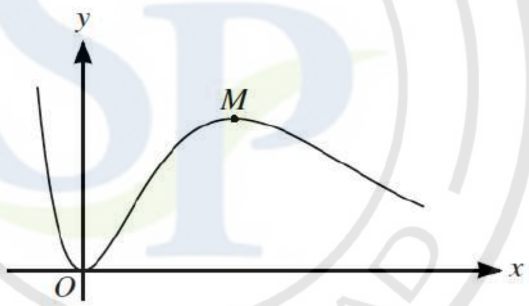


The diagram shows part of the curve with parametric equations

$$x = 2\ln(t+2), \qquad y = t^3 + 2t + 3.$$

(i) Find the gradient of the curve at the origin. [5]

Question 30



The diagram shows the curve $y = x^2e^{2-x}$ and its maximum point M.

(i) Show that the x-coordinate of M is 2. [3]

(ii) Find the exact value of $\int_0^2 x^2e^{2-x}\,dx$. [6]

Question 31

(a) Find $\int (4 + \tan^2 2x)\,dx$. [3]

(b) Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin(x + \frac{1}{6}\pi)}{\sin x}\,dx$. [5]

Question 32

The equation of a curve is

$$y = 3\cos 2x + 7\sin x + 2.$$

Find the x-coordinates of the stationary points in the interval $0 \leqslant x \leqslant \pi$. Give each answer correct to 3 significant figures. [7]

Question 33

Use the substitution $u = 4 - 3\cos x$ to find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4 - 3\cos x)}}\,dx$. [8]

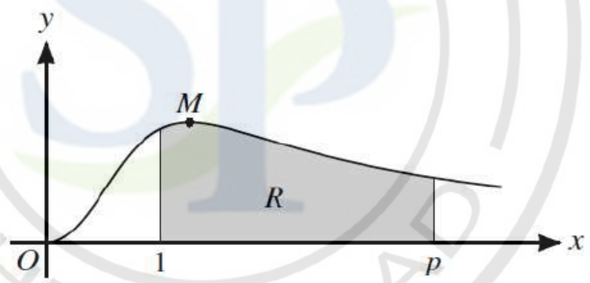
Question 34

A curve has equation

$$y = \frac{2 - \tan x}{1 + \tan x}.$$

Find the equation of the tangent to the curve at the point for which $x = \frac{1}{4}\pi$, giving the answer in the form $y = mx + c$ where c is correct to 3 significant figures. [6]

Question 35



The diagram shows the curve $y = \frac{x^2}{1 + x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

Question 36

The equation of a curve is $y = e^{-2x}\tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where a and b are constants. [5]

(ii) Explain why the gradient of the curve is never negative. [1]

(iii) Find the value of x for which the gradient is least. [1]

Question 37

A curve has equation

$$\sin y \ln x = x - 2\sin y,$$

for $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [5]

(ii) Hence find the exact x-coordinate of the point on the curve at which the tangent is parallel to the x-axis. [3]

Question 38

Let $I = \int_0^1 \frac{9}{(3 + x^2)^2}\,dx$.

(i) Using the substitution $x = (\sqrt{3})\tan\theta$, show that $I = \sqrt{3}\int_0^{\frac{1}{6}\pi} \cos^2\theta\,d\theta$. [3]

(ii) Hence find the exact value of I. [4]

Question 39

Let $I = \int_0^1 \frac{x^5}{(1 + x^2)^3}\,dx$.

(i) Using the substitution $u = 1 + x^2$, show that $I = \int_1^2 \frac{(u-1)^2}{2u^3}\,du$. [3]

(ii) Hence find the exact value of I. [5]

Question 40

The parametric equations of a curve are

$$x = t + \cos t, \qquad y = \ln(1 + \sin t),$$

where $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \sec t$. [5]

(ii) Hence find the x-coordinates of the points on the curve at which the gradient is equal to 3. Give your answers correct to 3 significant figures. [3]

Question 41

The curve with equation $y = \frac{(\ln x)^2}{x}$ has two stationary points. Find the exact values of the coordinates of these points. [6]

Question 42

Find the exact value of $\int_0^{\frac{1}{2}\pi} x^2 \sin 2x \, dx$. [5]

Question 43

The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - y^2}$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

Question 44

The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

Question 45

Find the exact value of $\int_0^{\frac{1}{2}} xe^{-2x} \, dx$. [5]

Question 46

Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})} \, dx$.

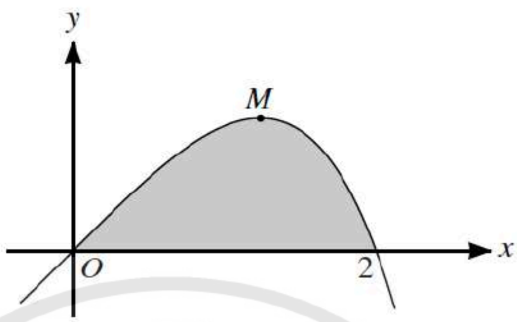
(i) Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u - 1}{u + 1} \, du$. [3]

(ii) Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

Question 47

The equation of a curve is $y = \dfrac{\sin x}{1 + \cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all x in the given interval. [4]

Question 48



The diagram shows part of the curve $y = (2x - x^2)e^{\frac{1}{2}x}$ and its maximum point M.

(i) Find the exact x-coordinate of M. [4]

(ii) Find the exact value of the area of the shaded region bounded by the curve and the positive x-axis. [5]

Question 49

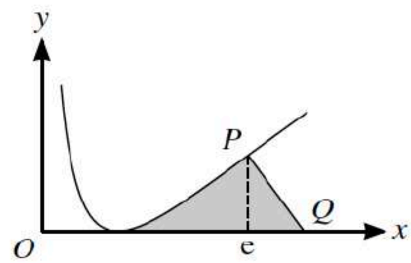
(i) Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. [4]

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$. [4]

Question 50

The equation of a curve is $xy(x - 6y) = 9a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [7]

Question 51



The diagram shows the curve $y = (\ln x)^2$. The x-coordinate of the point P is equal to e, and the normal to the curve at P meets the x-axis at Q.

(i) Find the x-coordinate of Q. [4]

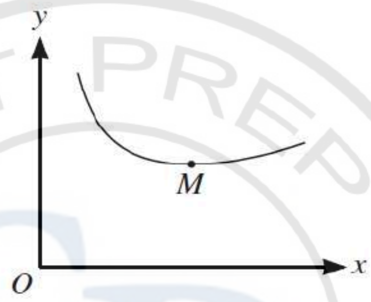
(ii) Show that $\int \ln x \, dx = x \ln x - x + c$, where c is a constant. [1]

(iii) Using integration by parts, or otherwise, find the exact value of the area of the shaded region between the curve, the x-axis and the normal PQ. [5]

Question 52

The curve with equation $y = e^{-ax} \tan x$, where a is a positive constant, has only one point in the interval $0 < x < \frac{1}{2}\pi$ at which the tangent is parallel to the x-axis. Find the value of a and state the exact value of the x-coordinate of this point. [7]

Question 53



The diagram shows a sketch of the curve $y = \dfrac{e^{\frac{1}{2}x}}{x}$ for $x > 0$, and its minimum point M.

(i) Find the x-coordinate of M. [4]

Question 54

A curve has equation $y = \frac{2}{3}\ln(1 + 3\cos^2 x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Express $\dfrac{dy}{dx}$ in terms of $\tan x$. [4]

(ii) Hence find the x-coordinate of the point on the curve where the gradient is -1. Give your answer correct to 3 significant figures. [2]

Question 55

Using the substitution $u = e^x$, solve the equation $4e^{-x} = 3e^x + 4$. Give your answer correct to 3 significant figures. [4]

Question 56

Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \theta \sin \tfrac{1}{2}\theta \, d\theta$. [4]

Question 57

(i) Prove that if $y = \frac{1}{\cos\theta}$ then $\frac{dy}{d\theta} = \sec\theta\tan\theta$. [2]

(ii) Prove the identity $\frac{1+\sin\theta}{1-\sin\theta} \equiv 2\sec^2\theta + 2\sec\theta\tan\theta - 1$. [3]

(iii) Hence find the exact value of $\int_0^{\frac{1}{4}\pi} \frac{1+\sin\theta}{1-\sin\theta}\,d\theta$. [4]

Question 58

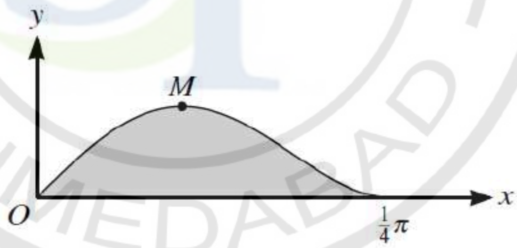
The parametric equations of a curve are

$$x = t^2 + 1, \quad y = 4t + \ln(2t - 1).$$

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Find the equation of the normal to the curve at the point where $t = 1$. Give your answer in the form $ax + by + c = 0$. [3]

Question 59



The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Find the x-coordinate of M. Give your answer correct to 2 decimal places. [6]

Question 60

The parametric equations of a curve are

$$x = \ln \cos\theta, \quad y = 3\theta - \tan\theta,$$

where $0 \leqslant \theta < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $\tan\theta$. [5]

(ii) Find the exact y-coordinate of the point on the curve at which the gradient of the normal is equal to 1. [3]

Question 61

It is given that $x = \ln(1 - y) - \ln y$, where $0 < y < 1$.

(i) Show that $y = \frac{e^{-x}}{1 + e^{-x}}$. [2]

(ii) Hence show that $\int_0^1 y\,dx = \ln\left(\frac{2e}{e+1}\right)$. [4]

Question 62

The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(i) Find the exact values of the x-coordinates of the stationary points of the curve. [4]

(ii) Show that the exact value of the area of R is $18 - \frac{42}{e}$. [5]

Question 63

The equation of a curve is $2x^4 + xy^3 + y^4 = 10$.

(i) Show that $\frac{dy}{dx} = -\frac{8x^3 + y^3}{3xy^2 + 4y^3}$. [4]

(ii) Hence show that there are two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

Question 64

The equation of a curve is $x^3y - 3xy^3 = 2a^4$, where a is a non-zero constant.

(i) Show that $\frac{dy}{dx} = \frac{3x^2y - 3y^3}{9xy^2 - x^3}$. [4]

(ii) Hence show that there are only two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

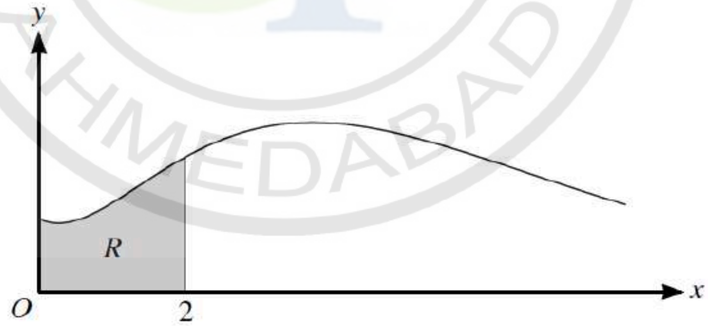
Question 65

The curve with equation $y = \frac{2 - \sin x}{\cos x}$ has one stationary point in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

(i) Find the exact coordinates of this point. [5]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Question 66



The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(i) Find the exact values of the x-coordinates of the stationary points of the curve. [4]

(ii) Show that the exact value of the area of R is $18 - \frac{42}{e}$. [5]

Question 67

The equation of a curve is $2x^4 + xy^3 + y^4 = 10$.

(i) Show that $\frac{dy}{dx} = -\frac{8x^3 + y^3}{3xy^2 + 4y^3}$. [4]

(ii) Hence show that there are two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

Question 68

The equation of a curve is $2x^3 - y^3 - 3xy^2 = 2a^3$, where a is a non-zero constant.

(i) Show that $\frac{dy}{dx} = \frac{2x^2 - y^2}{y^2 + 2xy}$. [4]

(ii) Find the coordinates of the two points on the curve at which the tangent is parallel to the y-axis. [5]

Question 69

(i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and $\tan\alpha$. [3]

(ii) Hence, showing all necessary working, show that $\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2}\, d\theta = 5$. [5]

Question 70

The curve with equation $y = \frac{\ln x}{3 + x}$ has a stationary point at $x = p$.

(i) Show that p satisfies the equation $\ln x = 1 + \frac{3}{x}$. [3]

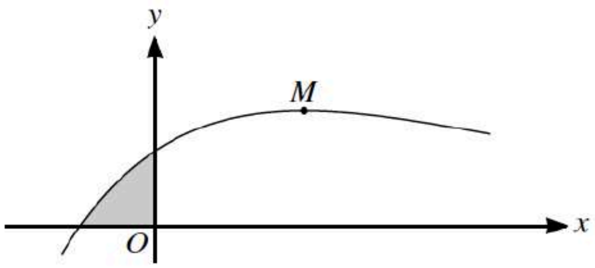
(ii) By sketching suitable graphs, show that the equation in part **(i)** has only one root. [2]

(iii) It is given that the equation in part **(i)** can be written in the form $x = \frac{3 + x}{\ln x}$. Use an iterative formula based on this rearrangement to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 71

Showing all necessary working, find the value of $\int_0^{\frac{1}{6}\pi} x\cos 3x\, dx$, giving your answer in terms of π. [5]

Question 72



The diagram shows the curve $y = (x+1)e^{-\frac{1}{3}x}$ and its maximum point M.

(i) Find the x-coordinate of M. [4]

(ii) Find the area of the shaded region enclosed by the curve and the axes, giving your answer in terms of e. [5]

Question 73

The equation of a curve is $x^2(x+3y) - y^3 = 3$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 + 2xy}{y^2 - x^2}$. [4]

(ii) Hence find the exact coordinates of the two points on the curve at which the gradient of the normal is 1. [4]

Question 74

(i) Show that $\dfrac{2\sin x - \sin 2x}{1 - \cos 2x} \equiv \dfrac{\sin x}{1 + \cos x}$. [4]

(ii) Hence, showing all necessary working, find $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{2\sin x - \sin 2x}{1 - \cos 2x}\, dx$, giving your answer in the form $\ln k$. [4]

Question 75

Let $I = \displaystyle\int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\, dx$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\, d\theta$. [4]

(ii) Hence find the exact value of I. [4]

Question 76

A curve has equation $y = \dfrac{e^{3x}}{\tan \frac{1}{2}x}$. Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. Give your answers correct to 3 decimal places. [6]

Question 77

The parametric equations of a curve are

$$x = 2t + \sin 2t, \quad y = 1 - 2\cos 2t,$$

for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = 2\tan t$. [5]

(ii) Hence find the x-coordinate of the point on the curve at which the gradient of the normal is 2. Give your answer correct to 3 significant figures. [2]

Question 78

(i) Using the expansions of $\cos(3x + x)$ and $\cos(3x - x)$, show that

$$\frac{1}{2}(\cos 4x + \cos 2x) \equiv \cos 3x \cos x.$$ [3]

(ii) Hence show that $\displaystyle\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x \cos x \, dx = \frac{3}{8}\sqrt{3}$. [3]