A-level

Topic :Complex Number

May 2013-May 2025

Questions

Question 1

(a) Without using a calculator, solve the equation

$$3w + 2\mathrm{i}w^* = 17 + 8\mathrm{i},$$

where w^* denotes the complex conjugate of w. Give your answer in the form $a + b\mathrm{i}$. [4]

(b) In an Argand diagram, the loci

$$\arg(z - 2\mathrm{i}) = \tfrac{1}{6}\pi \quad \text{and} \quad |z - 3| = |z - 3\mathrm{i}|$$

intersect at the point P. Express the complex number represented by P in the form $r\mathrm{e}^{\mathrm{i}\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. [5]

Question 2

(a) The complex number w is such that $\operatorname{Re} w > 0$ and $w + 3w^* = \mathrm{i}w^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2\mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

Question 3

The complex number z is defined by $z = a + \mathrm{i}b$, where a and b are real. The complex conjugate of z is denoted by z^*.

(i) Show that $|z|^2 = zz^*$ and that $(z - k\mathrm{i})^* = z^* + k\mathrm{i}$, where k is real. [2]

In an Argand diagram a set of points representing complex numbers z is defined by the equation $|z - 10\mathrm{i}| = 2|z - 4\mathrm{i}|$.

(ii) Show, by squaring both sides, that

$$zz^* - 2\mathrm{i}z^* + 2\mathrm{i}z - 12 = 0.$$

Hence show that $|z - 2\mathrm{i}| = 4$. [5]

(iii) Describe the set of points geometrically. [1]

Question 4

(a) The complex numbers u and v satisfy the equations

$$u + 2v = 2\text{i} \quad \text{and} \quad \text{i}u + v = 3.$$

Solve the equations for u and v, giving both answers in the form $x + \text{i}y$, where x and y are real. [5]

(b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + \text{i}| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$. Find the least value of $|z - w|$ for points on these loci. [5]

Question 5

(a) Without using a calculator, use the formula for the solution of a quadratic equation to solve

$$(2 - \text{i})z^2 + 2z + 2 + \text{i} = 0.$$

Give your answers in the form $a + b\text{i}$. [5]

(b) The complex number w is defined by $w = 2\text{e}^{\frac{1}{4}\pi\text{i}}$. In an Argand diagram, the points A, B and C represent the complex numbers w, w^3 and w^* respectively (where w^* denotes the complex conjugate of w). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. [5]

Question 6

The complex number z is defined by $z = \dfrac{9\sqrt{3} + 9\text{i}}{\sqrt{3} - \text{i}}$. Find, showing all your working,

(i) an expression for z in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

Question 7

(a) It is given that $-1 + (\sqrt{5})\text{i}$ is a root of the equation $z^3 + 2z + a = 0$, where a is real. Showing your working, find the value of a, and write down the other complex root of this equation. [4]

(b) The complex number w has modulus 1 and argument 2θ radians. Show that $\dfrac{w - 1}{w + 1} = \text{i}\tan\theta$. [4]

Question 8

(a) The complex number $\frac{3-5\text{i}}{1+4\text{i}}$ is denoted by u. Showing your working, express u in the form $x+\text{i}y$, where x and y are real. [3]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z-2-\text{i}| \leqslant 1$ and $|z-\text{i}| \leqslant |z-2|$. [4]

(ii) Calculate the maximum value of $\arg z$ for points lying in the shaded region. [2]

Question 9

The complex numbers w and z satisfy the relation

$$w = \frac{z+\text{i}}{\text{i}z+2}.$$

(i) Given that $z = 1+\text{i}$, find w, giving your answer in the form $x+\text{i}y$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x+\text{i}y$, where x and y are real. [4]

Question 10

The complex numbers w and z are defined by $w = 5+3\text{i}$ and $z = 4+\text{i}$.

(i) Express $\frac{\text{i}w}{z}$ in the form $x+\text{i}y$, showing all your working and giving the exact values of x and y. [3]

(ii) Find wz and hence, by considering arguments, show that

$$\tan^{-1}\left(\tfrac{3}{5}\right) + \tan^{-1}\left(\tfrac{1}{4}\right) = \tfrac{1}{4}\pi.$$ [4]

Question 11

The complex number w is defined by $w = \frac{22+4\text{i}}{(2-\text{i})^2}$.

(i) Without using a calculator, show that $w = 2+4\text{i}$. [3]

(ii) It is given that p is a real number such that $\frac{1}{4}\pi \leqslant \arg(w+p) \leqslant \frac{3}{4}\pi$. Find the set of possible values of p. [3]

(iii) The complex conjugate of w is denoted by w^*. The complex numbers w and w^* are represented in an Argand diagram by the points S and T respectively. Find, in the form $|z-a| = k$, the equation of the circle passing through S, T and the origin. [3]

Question 12

The complex number u is given by $u = -1 + (4\sqrt{3})\text{i}$.

(i) Without using a calculator and showing all your working, find the two square roots of u. Give your answers in the form $a + \text{i}b$, where the real numbers a and b are exact. [5]

(ii) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the relation $|z - u| = 1$. Determine the greatest value of $\arg z$ for points on this locus. [4]

Question 13

The complex number $1 - \text{i}$ is denoted by u.

(i) Showing your working and without using a calculator, express

$$\frac{\text{i}}{u}$$

in the form $x + \text{i}y$, where x and y are real. [2]

(ii) On an Argand diagram, sketch the loci representing complex numbers z satisfying the equations $|z - u| = |z|$ and $|z - \text{i}| = 2$. [4]

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part **(ii)**. [3]

Question 14

The complex number $3 - \text{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively. What type of quadrilateral is $OABC$? [4]

(ii) Showing your working and without using a calculator, express $\dfrac{u^*}{u}$ in the form $x + \text{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\dfrac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]

Question 15

(a) It is given that $(1 + 3\text{i})w = 2 + 4\text{i}$. Showing all necessary working, prove that the exact value of $|w^2|$ is 2 and find $\arg(w^2)$ correct to 3 significant figures. [6]

(b) On a single Argand diagram sketch the loci $|z| = 5$ and $|z - 5| = |z|$. Hence determine the complex numbers represented by points common to both loci, giving each answer in the form $re^{\text{i}\theta}$. [4]

Question 16

(a) Find the complex number z satisfying the equation $z^* + 1 = 2iz$, where z^* denotes the complex conjugate of z. Give your answer in the form $x + iy$, where x and y are real. [5]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z + 1 - 3i| \leqslant 1$ and $\operatorname{Im} z \geqslant 3$, where $\operatorname{Im} z$ denotes the imaginary part of z. [4]

(ii) Determine the difference between the greatest and least values of $\arg z$ for points lying in this region. [2]

Question 17

(a) Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})i$. Give your answers in the form $x + iy$, where x and y are real and exact. [5]

(b) **(i)** On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2i| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

Question 18

(a) Showing all necessary working, solve the equation $iz^2 + 2z - 3i = 0$, giving your answers in the form $x + iy$, where x and y are real and exact. [5]

(b) **(i)** On a sketch of an Argand diagram, show the locus representing complex numbers satisfying the equation $|z| = |z - 4 - 3i|$. [2]

(ii) Find the complex number represented by the point on the locus where $|z|$ is least. Find the modulus and argument of this complex number, giving the argument correct to 2 decimal places. [3]

Question 19

The complex numbers $-1 + 3i$ and $2 - i$ are denoted by u and v respectively. In an Argand diagram with origin O, the points A, B and C represent the numbers u, v and $u + v$ respectively.

(i) Sketch this diagram and state fully the geometrical relationship between OB and AC. [4]

(ii) Find, in the form $x + iy$, where x and y are real, the complex number $\frac{u}{v}$. [3]

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

Question 20

(a) Solve the equation $(1 + 2i)w^2 + 4w - (1 - 2i) = 0$, giving your answers in the form $x + iy$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 1 - i| \leqslant 2$ and $-\frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{4}\pi$. [5]

Question 21

The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})i$. The complex conjugate of z is denoted by z^*.

(i) Find the modulus and argument of z. [2]

(ii) Express each of the following in the form $x + iy$, where x and y are real and exact:

(a) $z + 2z^*$;

(b) $\dfrac{z^*}{iz}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and iz respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

Question 22

The polynomial $z^4 + 3z^2 + 6z + 10$ is denoted by $p(z)$. The complex number $-1 + i$ is denoted by u.

(i) Showing all your working, verify that u is a root of the equation $p(z) = 0$. [3]

(ii) Find the other three roots of the equation $p(z) = 0$. [7]

Question 23

The complex numbers u and w are defined by $u = -1 + 7i$ and $w = 3 + 4i$.

(i) Showing all your working, find in the form $x + iy$, where x and y are real, the complex numbers $u - 2w$ and $\dfrac{u}{w}$. [4]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, w and $u - 2w$ respectively.

(ii) Prove that angle $AOB = \frac{1}{4}\pi$. [2]

(iii) State fully the geometrical relation between the line segments OB and CA. [2]

Question 24

The complex number $2 - \mathrm{i}$ is denoted by u.

(i) It is given that u is a root of the equation $x^3 + ax^2 - 3x + b = 0$, where the constants a and b are real. Find the values of a and b. [4]

(ii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| < 1$ and $|z| < |z + \mathrm{i}|$. [4]

Question 25

(a) The complex numbers z and w satisfy the equations

$$z + (1 + \mathrm{i})w = \mathrm{i} \qquad \text{and} \qquad (1 - \mathrm{i})z + \mathrm{i}w = 1.$$

Solve the equations for z and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) The complex numbers u and v are given by $u = 1 + (2\sqrt{3})\mathrm{i}$ and $v = 3 + 2\mathrm{i}$. In an Argand diagram, u and v are represented by the points A and B. A third point C lies in the first quadrant and is such that $BC = 2AB$ and angle $ABC = 90°$. Find the complex number z represented by C, giving your answer in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

Question 26

(a) The complex number u is given by $u = 8 - 15\mathrm{i}$. Showing all necessary working, find the two square roots of u. Give answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5]

(b) On an Argand diagram, shade the region whose points represent complex numbers satisfying both the inequalities $|z - 2 - \mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z - \mathrm{i}) \leqslant \frac{1}{4}\pi$. [4]

Question 27

The complex number $1 - (\sqrt{3})\mathrm{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Show that $u^3 + 8 = 0$. [2]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| \leqslant 2$ and $\operatorname{Re} z \geqslant 2$, where $\operatorname{Re} z$ denotes the real part of z. [4]

Question 28

The complex number $1 + 2i$ is denoted by u.

(i) It is given that u is a root of the equation $2x^3 - x^2 + 4x + k = 0$, where k is a constant.

(a) Showing all working and without using a calculator, find the value of k. [3]

(b) Showing all working and without using a calculator, find the other two roots of this equation. [4]

(ii) On an Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 1$. Determine the least value of $\arg z$ for points on this locus. Give your answer in radians correct to 2 decimal places. [4]

Question 29

(i) Showing all working and without using a calculator, solve the equation $z^2 + (2\sqrt{6})z + 8 = 0$, giving your answers in the form $x + iy$, where x and y are real and exact. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) The points representing the roots are A and B, and O is the origin. Find angle AOB. [3]

(iv) Prove that triangle AOB is equilateral. [1]

Question 30

The complex numbers $-3\sqrt{3} + i$ and $\sqrt{3} + 2i$ are denoted by u and v respectively.

(i) Find, in the form $x + iy$, where x and y are real and exact, the complex numbers uv and $\frac{u}{v}$. [5]

(ii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers u and v respectively. Prove that angle $AOB = \frac{2}{3}\pi$. [3]

Question 31

(a) Find the complex number z satisfying the equation

$$3z - iz^* = 1 + 5i,$$

where z^* denotes the complex conjugate of z. [4]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| \leqslant 3$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of z. Calculate the greatest value of $\arg z$ for points in this region. Give your answer in radians correct to 2 decimal places. [5]

Question 32

(a) Showing all necessary working, express the complex number $\frac{2+3\mathrm{i}}{1-2\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the values of r and θ correct to 3 significant figures. [5]

(b) On an Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z - 3 + 2\mathrm{i}| = 1$. Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [4]

Question 33

(a) **(i)** Without using a calculator, express the complex number $\frac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Hence, without using a calculator, express $\frac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - 3\mathrm{i}| \leqslant 1$ and $\mathrm{Re}\, z \leqslant 0$, where $\mathrm{Re}\, z$ denotes the real part of z. Find the greatest value of $\arg z$ for points in this region, giving your answer in radians correct to 2 decimal places. [5]

Question 34

(a) Showing all working and without using a calculator, solve the equation

$$(1+\mathrm{i})z^2 - (4+3\mathrm{i})z + 5 + \mathrm{i} = 0.$$

Give your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) The complex number u is given by

$$u = -1 - \mathrm{i}.$$

On a sketch of an Argand diagram show the point representing u. Shade the region whose points represent complex numbers satisfying the inequalities $|z| < |z - 2\mathrm{i}|$ and $\frac{1}{4}\pi < \arg(z - u) < \frac{1}{2}\pi$. [4]

Question 35

The complex number $(\sqrt{3}) + \mathrm{i}$ is denoted by u.

(i) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. Hence or otherwise state the exact values of the modulus and argument of u^4. [5]

(ii) Verify that u is a root of the equation $z^3 - 8z + 8\sqrt{3} = 0$ and state the other complex root of this equation. [3]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - u| \leqslant 2$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of z. [5]

Question 36

It is given that the complex number $-1 + (\sqrt{3})\mathrm{i}$ is a root of the equation

$$kx^3 + 5x^2 + 10x + 4 = 0,$$

where k is a real constant.

(i) Write down another root of the equation. [1]

(ii) Find the value of k and the third root of the equation. [6]

Question 37

The complex number u is defined by

$$u = \frac{4\mathrm{i}}{1 - (\sqrt{3})\mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(ii) Find the exact modulus and argument of u. [2]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| < 2$ and $|z - u| < |z|$. [4]

Question 38

(a) The complex number u is given by $u = -3 - (2\sqrt{10})\mathrm{i}$. Showing all necessary working and without using a calculator, find the square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5]

(b) On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$, $\arg z \geqslant \frac{1}{4}\pi$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of the complex number z. [5]

Question 39

(a) Find the complex number z satisfying the equation

$$z + \frac{\mathrm{i}z}{z^*} - 2 = 0,$$

where z^* denotes the complex conjugate of z. Give your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) **(i)** On a single Argand diagram sketch the loci given by the equations $|z - 2\mathrm{i}| = 2$ and $\operatorname{Im} z = 3$, where $\operatorname{Im} z$ denotes the imaginary part of z. [2]

(ii) In the first quadrant the two loci intersect at the point P. Find the exact argument of the complex number represented by P. [2]

Question 40

The complex number with modulus 1 and argument $\frac{1}{3}\pi$ is denoted by w.

(i) Express w in the form $x + \mathrm{i}y$, where x and y are real and exact. [1]

The complex number $1 + 2\mathrm{i}$ is denoted by u. The complex number v is such that $|v| = 2|u|$ and $\arg v = \arg u + \frac{1}{3}\pi$.

(ii) Sketch an Argand diagram showing the points representing u and v. [2]

(iii) Explain why v can be expressed as $2uw$. Hence find v, giving your answer in the form $a + \mathrm{i}b$, where a and b are real and exact. [4]

Question 41

(a) The complex numbers v and w satisfy the equations

$$v + \mathrm{i}w = 5 \qquad \text{and} \qquad (1 + 2\mathrm{i})v - w = 3\mathrm{i}.$$

Solve the equations for v and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) **(i)** On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z - 2 - 3\mathrm{i}| = 1$. [2]

(ii) Calculate the least value of $\arg z$ for points on this locus. [2]

Question 42

(a) The complex number u is defined by $u = \frac{3\mathrm{i}}{a + 2\mathrm{i}}$, where a is real.

(i) Express u in the Cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [3]

(ii) Find the exact value of a for which $\arg u^* = \frac{1}{3}\pi$. [3]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2\mathrm{i}| \leqslant |z - 1 - \mathrm{i}|$ and $|z - 2 - \mathrm{i}| \leqslant 2$. [4]

(ii) Calculate the least value of $\arg z$ for points in this region. [2]

Question 43

(a) Solve the equation $(1 + 2\mathrm{i})w + \mathrm{i}w^* = 3 + 5\mathrm{i}$. Give your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2 - 2\mathrm{i}| \leqslant 1$ and $\arg(z - 4\mathrm{i}) \geqslant -\frac{1}{4}\pi$. [4]

(ii) Find the least value of $\mathrm{Im}\, z$ for points in this region, giving your answer in an exact form. [2]

Question 44

(a) The complex numbers u and w are such that

$$u - w = 2\mathrm{i} \quad \text{and} \quad uw = 6.$$

Find u and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities

$$|z - 2 - 2\mathrm{i}| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad \mathrm{Re}\, z \leqslant 3.$$ [5]

Question 45

(a) Verify that $-1 + \sqrt{5}\,\mathrm{i}$ is a root of the equation $2x^3 + x^2 + 6x - 18 = 0$. [3]

(b) Find the other roots of this equation. [4]

Question 46

The complex number u is defined by

$$u = \frac{7 + \mathrm{i}}{1 - \mathrm{i}}.$$

(a) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + \mathrm{i}$ and $1 - \mathrm{i}$ respectively. [2]

(c) By considering the arguments of $7 + \mathrm{i}$ and $1 - \mathrm{i}$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$ [3]

Question 47

(a) Verify that $-1 + \sqrt{5}\mathrm{i}$ is a root of the equation $2x^3 + x^2 + 6x - 18 = 0$. [3]

(b) Find the other roots of this equation. [4]

Question 48

The complex numbers u and v are defined by $u = -4 + 2\mathrm{i}$ and $v = 3 + \mathrm{i}$.

(a) Find $\frac{u}{v}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) Hence express $\frac{u}{v}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ are exact. [2]

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC. [2]

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

Question 49

(a) Verify that $-1 + \sqrt{2}\mathrm{i}$ is a root of the equation $z^4 + 3z^2 + 2z + 12 = 0$. [3]

(b) Find the other roots of this equation. [7]

Question 50

The complex number u is given by $u = 10 - 4\sqrt{6}\mathrm{i}$.

Find the two square roots of u, giving your answers in the form $a + \mathrm{i}b$, where a and b are real and exact. [5]

Question 51

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 1 - \mathrm{i}| \leqslant 1$ and $\arg(z - 1) \leqslant \frac{3}{4}\pi$. [4]

Question 52

(a) Solve the equation $z^2 - 2p\mathrm{i}z - q = 0$, where p and q are real constants. [2]

In an Argand diagram with origin O, the roots of this equation are represented by the distinct points A and B.

(b) Given that A and B lie on the imaginary axis, find a relation between p and q. [2]

(c) Given instead that triangle OAB is equilateral, express q in terms of p. [3]

Question 53

The complex number $-\sqrt{3} + \mathrm{i}$ is denoted by u.

(a) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

(b) Hence show that u^6 is real and state its value. [2]

(c) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $0 \leqslant \arg(z - u) \leqslant \frac{1}{4}\pi$ and $\operatorname{Re} z \leqslant 2$. [4]

(ii) Find the greatest value of $|z|$ for points in the shaded region. Give your answer correct to 3 significant figures. [2]

Question 54

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - 2\mathrm{i}| \leqslant 1$ and $\operatorname{Im} z \geqslant 2$. [4]

(b) Find the greatest value of $\arg z$ for points in the shaded region, giving your answer in degrees. [3]

Question 55

(a) Given the complex numbers $u = a + ib$ and $w = c + id$, where a, b, c and d are real, prove that $(u + w)^* = u^* + w^*$. [2]

(b) Solve the equation $(z + 2 + i)^* + (2 + i)z = 0$, giving your answer in the form $x + iy$ where x and y are real. [4]

Question 56

The complex number $1 + 2i$ is denoted by u. The polynomial $2x^3 + ax^2 + 4x + b$, where a and b are real constants, is denoted by $p(x)$. It is given that u is a root of the equation $p(x) = 0$.

(a) Find the values of a and b. [4]

(b) State a second complex root of this equation. [1]

(c) Find the real factors of $p(x)$. [2]

(d) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - u| \leqslant \sqrt{5}$ and $\arg z \leqslant \frac{1}{4}\pi$. [4]

(ii) Find the least value of $\operatorname{Im} z$ for points in the shaded region. Give your answer in an exact form. [1]

Question 57

Find the complex numbers w which satisfy the equation $w^2 + 2iw^* = 1$ and are such that $\operatorname{Re} w \leqslant 0$. Give your answers in the form $x + iy$, where x and y are real. [6]

Question 58

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2 - 3i| \leqslant 2$ and $\arg z \leqslant \frac{3}{4}\pi$. [4]

Question 59

The complex number $3 - i$ is denoted by u.

(a) Show, on an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively.

State the type of quadrilateral formed by the points O, A, B and C. [3]

(b) Express $\dfrac{u^*}{u}$ in the form $x + iy$, where x and y are real. [3]

(c) By considering the argument of $\dfrac{u^*}{u}$, or otherwise, prove that $\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$. [2]

Question 60

The complex number $-1 + \sqrt{7}\mathrm{i}$ is denoted by u. It is given that u is a root of the equation

$$2x^3 + 3x^2 + 14x + k = 0,$$

where k is a real constant.

(a) Find the value of k. [3]

(b) Find the other two roots of the equation. [4]

(c) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 2$. [2]

(d) Determine the greatest value of $\arg z$ for points on this locus, giving your answer in radians. [2]

Question 61

The complex number u is defined by $u = \dfrac{\sqrt{2} - a\sqrt{2}\mathrm{i}}{1 + 2\mathrm{i}}$, where a is a positive integer.

(a) Express u in terms of a, in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

It is now given that $a = 3$.

(b) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

(c) Using your answer to part **(b)**, find the two square roots of u. Give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [3]

Question 62

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2| \leqslant 2$ and $\operatorname{Im} z \geqslant 1$. [4]

(b) Find the greatest value of $\arg z$ for points in the shaded region. [2]

Question 63

Solve the quadratic equation $(1 - 3\mathrm{i})z^2 - (2 + \mathrm{i})z + \mathrm{i} = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

Question 64

(a) Solve the equation $z^2 - 6\mathrm{i}z - 12 = 0$, giving the answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(b) On a sketch of an Argand diagram with origin O, show points A and B representing the roots of the equation in part **(a)**. [1]

(c) Find the exact modulus and argument of each root. [3]

(d) Hence show that the triangle OAB is equilateral. [1]

Question 65

On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z| \leqslant 3$, $\operatorname{Re} z \geqslant -2$ and $\frac{1}{4}\pi \leqslant \arg z \leqslant \pi$. [4]

Question 66

The complex numbers u and w are defined by $u = 2e^{\frac{1}{4}\pi i}$ and $w = 3e^{\frac{1}{3}\pi i}$.

(a) Find $\frac{u^2}{w}$, giving your answer in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [3]

(b) State the least positive integer n such that both $\operatorname{Im} w^n = 0$ and $\operatorname{Re} w^n > 0$. [1]

Question 67

(a) On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leqslant \arg(z - 1 - 2i) \leqslant \frac{1}{3}\pi$ and $\operatorname{Re} z \leqslant 3$. [3]

(b) Calculate the least value of $\arg z$ for points in the region from **(a)**. Give your answer in radians correct to 3 decimal places. [2]

Question 68

Solve the equation

$$\frac{5z}{1+2i} - zz^* + 30 + 10i = 0,$$

giving your answers in the form $x + iy$, where x and y are real. [5]

Question 69

The complex number z is defined by $z = \frac{5a - 2i}{3 + ai}$, where a is an integer. It is given that $\arg z = -\frac{1}{4}\pi$.

(a) Find the value of a and hence express z in the form $x + iy$, where x and y are real. [6]

(b) Express z^3 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the simplified exact values of r and θ. [3]

Question 70

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - i| \leqslant 3$ and $|z| \geqslant |z - 4i|$. [4]

Question 71

The complex number $2 + y\mathrm{i}$ is denoted by a, where y is a real number and $y < 0$. It is given that $\mathrm{f}(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $\mathrm{f}(a)$ in terms of y. [3]

The complex number $2 + y\mathrm{i}$ is denoted by a, where y is a real number and $y < 0$. It is given that $\mathrm{f}(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $\mathrm{f}(a)$ in terms of y. [3]

Question 72

(a) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z + 3 - 2\mathrm{i}| = 2$. [2]

(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [2]

Question 73

The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $\mathrm{p}(x)$.

(a) Show that $(x + 3)$ is a factor of $\mathrm{p}(x)$. [2]

(b) Show that $z = -1 + 2\sqrt{6}\mathrm{i}$ is a root of $\mathrm{p}(z) = 0$. [3]

(c) Hence find the complex numbers z which are roots of $\mathrm{p}(z^2) = 0$. [7]

Question 74

Solve the quadratic equation $(3 + \mathrm{i})w^2 - 2w + 3 - \mathrm{i} = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

Question 75

On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 1 + 2\mathrm{i}| \leqslant |z|$ and $|z - 2| \leqslant 1$. [5]

Question 76

It is given that $\dfrac{2 + 3a\mathrm{i}}{a + 2\mathrm{i}} = \lambda(2 - \mathrm{i})$, where a and λ are real constants.

(a) Show that $3a^2 + 4a - 4 = 0$. [4]

(b) Hence find the possible values of a and the corresponding values of λ. [3]

Question 77

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 4 - 3\mathrm{i}| \leqslant 2$ and $\mathrm{Re}\, z \leqslant 3$. [4]

(b) Find the greatest value of $\arg z$ for points in this region. [2]

Question 78

The complex number u is defined by $u = \dfrac{3 + 2\mathrm{i}}{a - 5\mathrm{i}}$, where a is real.

(a) Express u in the Cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [3]

(b) Given that $\arg u = \frac{1}{4}\pi$, find the value of a. [2]

Question 79

On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2\mathrm{i}| \leqslant |z + 2 - \mathrm{i}|$ and $0 \leqslant \arg(z + 1) \leqslant \frac{1}{4}\pi$. [4]

Question 80

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 4 - 2\mathrm{i}| \leqslant 3$ and $|z| \geqslant |10 - z|$. [4]

(b) Find the greatest value of $\arg z$ for points in this region. [2]

Question 81

It is given that $z = -\sqrt{3} + \mathrm{i}$.

(a) Express z^2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

(b) The complex number ω is such that $z^2\omega$ is real and $\left|\dfrac{z^2}{\omega}\right| = 12$.

Find the two possible values of ω, giving your answers in the form $R\mathrm{e}^{\mathrm{i}\alpha}$, where $R > 0$ and $-\pi < \alpha \leqslant \pi$. [3]

Question 82

(a) On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 4 - 3\mathrm{i}| \leqslant 2$ and $\arg(z - 2 - \mathrm{i}) \geqslant \frac{1}{3}\pi$. [5]

(b) Calculate the greatest value of $\arg z$ for points in this region. [2]

Question 83

The square roots of $24 - 7\mathrm{i}$ can be expressed in the Cartesian form $x + \mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $24 - 7\mathrm{i}$ in exact Cartesian form. [5]

Question 84

The complex numbers z and ω are defined by $z = 1 - \mathrm{i}$ and $\omega = -3 + 3\sqrt{3}\mathrm{i}$.

(a) Express $z\omega$ in the form $a + b\mathrm{i}$, where a and b are real and in exact surd form. [1]

(b) Express z and ω in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ in each case. [4]

(c) On an Argand diagram, the points representing ω and $z\omega$ are A and B respectively.

Prove that OAB is an isosceles right-angled triangle, where O is the origin. [2]

(d) Using your answers to part **(b)**, prove that $\tan\frac{5}{12}\pi = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$. [3]

Question 85

(a) On a single Argand diagram sketch the loci given by the equations $|z - 3 + 2\mathrm{i}| = 2$ and $|w - 3 + 2\mathrm{i}| = |w + 3 - 4\mathrm{i}|$ where z and w are complex numbers. [4]

(b) Hence find the least value of $|z - w|$ for points on these loci. Give your answer in an exact form. [2]

Question 86

The complex number u is given by $u = -1 - \mathrm{i}\sqrt{3}$.

(a) Express u in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [2]

The complex number v is given by $v = 5\left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)$.

(b) Express the complex number $\dfrac{v}{u}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

Question 87

Find the complex number z satisfying the equation

$$\frac{z - 3\mathrm{i}}{z + 3\mathrm{i}} = \frac{2 - 9\mathrm{i}}{5}.$$

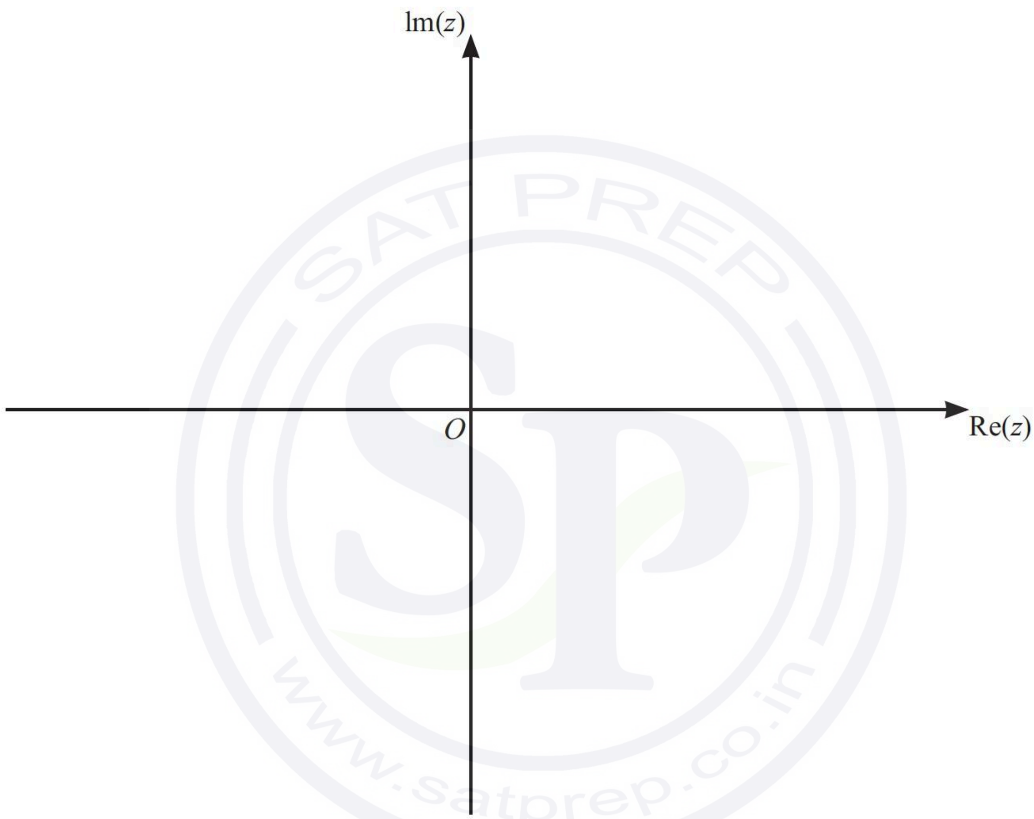
Give your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

Question 88

The complex number z satisfies $|z| = 2$ and $0 \leqslant \arg z \leqslant \frac{1}{4}\pi$.

(a) On the Argand diagram below, sketch the locus of the points representing z. [2]

(b) On the **same diagram**, sketch the locus of the points representing z^2. [2]



Question 89

(a) The complex number u is given by

$$u = \frac{(\cos\frac{1}{7}\pi + \mathrm{i}\sin\frac{1}{7}\pi)^4}{\cos\frac{1}{7}\pi - \mathrm{i}\sin\frac{1}{7}\pi}.$$

Find the exact value of $\arg u$. [2]

(b) The complex numbers u and u^* are plotted on an Argand diagram.

Describe the single geometrical transformation that maps u onto u^* and state the exact value of $\arg u^*$. [2]

Question 90

The square roots of $6 - 8\mathrm{i}$ can be expressed in the Cartesian form $x + \mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $6 - 8\mathrm{i}$ in exact Cartesian form. [5]

Question 91

(a) Given that $z = 1 + y\mathrm{i}$ and that y is a real number, express $\frac{1}{z}$ in the form $a + b\mathrm{i}$, where a and b are functions of y. [2]

(b) Show that $\left(a - \frac{1}{2}\right)^2 + b^2 = \frac{1}{4}$, where a and b are the functions of y found in part **(a)**. [3]

(c) On a single Argand diagram, sketch the loci given by the equations $\mathrm{Re}(z) = 1$ and $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, where z is a complex number. [3]

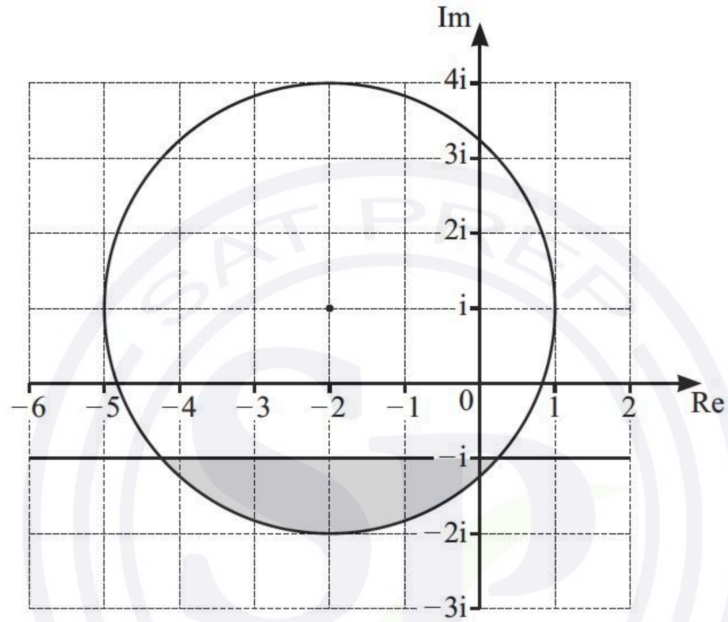
(d) The complex number z is such that $\mathrm{Re}(z) = 1$. Use your answer to part **(b)** to give a geometrical description of the locus of $\frac{1}{z}$. [1]

Question 92

The square roots of $-4+6\sqrt{5}\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-4+6\sqrt{5}\mathrm{i}$ in exact Cartesian form. [5]

Question 93



The shaded region on the Argand diagram shows points representing complex numbers z defined by two inequalities. The shaded region is bounded by a circle and a line parallel to the real axis. The boundaries of the region are included in the shaded region.

(a) Find two inequalities in terms of z that define the shaded region. [3]

(b) Find the greatest value of $|z|$ for points in this region. [3]

Question 94

Find the complex numbers z for which $\dfrac{z+4}{z+4\mathrm{i}}$ is real and $|z|=\sqrt{10}$. Give your answers in the form $z=x+\mathrm{i}y$, where x and y are real. [6]

Question 95

(a) It is given that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

Show that $(z_1 z_2)^* = z_1^* z_2^*$. [3]

(b) $z = 3e^{\frac{1}{4}\pi i}$ is a root of the equation $z^2 + bz + c = 0$, where b and c are real.

State the other root and hence find the values of b and c. [3]

Question 96

The square roots of $-1 - 4\sqrt{5}i$ can be expressed in the Cartesian form $x + iy$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-1 - 4\sqrt{5}i$ in exact Cartesian form. [5]

Question 97

On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 3i| \leqslant 2$ and $\frac{1}{4}\pi \leqslant \arg(z - 1 - 2i) \leqslant \frac{3}{4}\pi$. [5]

Question 98

It is given that $z_1 = 3e^{\frac{1}{4}\pi i}$, $z_2 = \frac{3}{2}e^{\frac{1}{6}\pi i}$ and $\omega = 2e^{\frac{1}{2}\pi i}$.

(a) State the values of ωz_1 and ωz_2. Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

(b) On a sketch of an Argand diagram with origin O, show the points A, B, C and D representing the complex numbers z_1, z_2, ωz_1 and ωz_2 respectively. [2]

(c) State the geometric effects of multiplying z_1 and z_2 by ω. [2]

Question 99

Find the complex numbers z for which $\dfrac{z + 5i}{z - 5}$ is real and $|z| = \sqrt{17}$. Give your answers in the form $z = x + iy$, where x and y are real. [6]