A-level

Topic :Vector

May 2013-May 2023

Questions

Question 1

The points P and Q have position vectors, relative to the origin O, given by

$$\overrightarrow{OP} = 7\mathbf{i} + 7\mathbf{j} - 5\mathbf{k} \quad \text{and} \quad \overrightarrow{OQ} = -5\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

The mid-point of PQ is the point A. The plane Π is perpendicular to the line PQ and passes through A.

(i) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(ii) The straight line through P parallel to the x-axis meets Π at the point B. Find the distance AB, correct to 3 significant figures. [5]

Question 2

The points A and B have position vectors $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ respectively. The plane p has equation $x + y = 5$.

(i) Find the position vector of the point of intersection of the line through A and B and the plane p. [4]

(ii) A second plane q has an equation of the form $x + by + cz = d$, where b, c and d are constants. The plane q contains the line AB, and the acute angle between the planes p and q is $60°$. Find the equation of q. [7]

Question 3

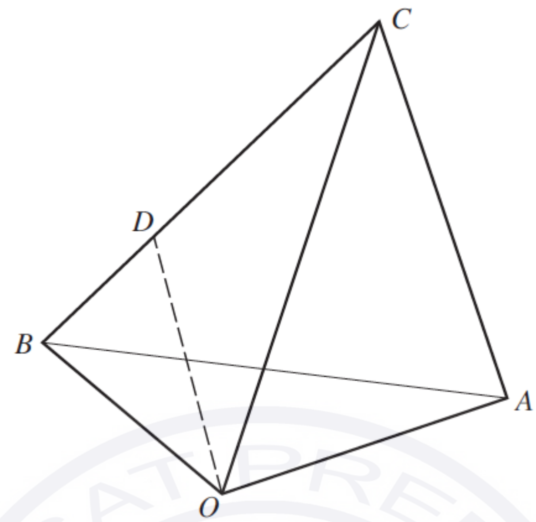
The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, where a is a constant. The plane p has equation $x + 2y + 2z = 6$. Find the value or values of a in each of the following cases.

(i) The line l is parallel to the plane p. [2]

(ii) The line l intersects the line passing through the points with position vectors $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \mathbf{k}$. [4]

(iii) The acute angle between the line l and the plane p is $\tan^{-1} 2$. [5]

Question 4



The diagram shows three points A, B and C whose position vectors with respect to the origin O are given by $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$. The point D lies on BC, between B and C, and is such that $CD = 2DB$.

(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Find the position vector of D. [1]

(iii) Show that the length of the perpendicular from A to OD is $\frac{1}{3}\sqrt{(65)}$. [4]

Question 5

Two planes have equations $3x - y + 2z = 9$ and $x + y - 4z = -1$.

(i) Find the acute angle between the planes. [3]

(ii) Find a vector equation of the line of intersection of the planes. [6]

Question 6

The straight line l has equation $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$. The plane p passes through the point $(4, -1, 2)$ and is perpendicular to l.

(i) Find the equation of p, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Find the perpendicular distance from the origin to p. [3]

(iii) A second plane q is parallel to p and the perpendicular distance between p and q is 14 units. Find the possible equations of q. [3]

Question 7

Referred to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

(i) Find the exact value of the cosine of angle BAC. [4]

(ii) Hence find the exact value of the area of triangle ABC. [3]

(iii) Find the equation of the plane which is parallel to the y-axis and contains the line through B and C. Give your answer in the form $ax + by + cz = d$. [5]

Question 8

The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and the plane p has equation $2x + 3y - 5z = 18$.

(i) Find the position vector of the point of intersection of l and p. [3]

(ii) Find the acute angle between l and p. [4]

(iii) A second plane q is perpendicular to the plane p and contains the line l. Find the equation of q, giving your answer in the form $ax + by + cz = d$. [5]

Question 9

The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

(ii) The line l lies in the plane with equation $ax + by - 3z + 1 = 0$, where a and b are constants. Find the values of a and b. [5]

Question 10

The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = a\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3a\mathbf{k}),$$

where a is a constant.

(i) Show that the lines intersect for all values of a. [4]

(ii) Given that the point of intersection is at a distance of 9 units from the origin, find the possible values of a. [4]

Question 11

The straight line l_1 passes through the points $(0,\ 1,\ 5)$ and $(2,\ -2,\ 1)$. The straight line l_2 has equation $\mathbf{r} = 7\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$.

(i) Show that the lines l_1 and l_2 are skew. [6]

(ii) Find the acute angle between the direction of the line l_2 and the direction of the x-axis. [3]

Question 12

The points A and B have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

(ii) Find the equation of the plane containing the line l and the point A. Give your answer in the form $ax + by + cz = d$. [6]

Question 13

Two planes have equations $x + 3y - 2z = 4$ and $2x + y + 3z = 5$. The planes intersect in the straight line l.

(i) Calculate the acute angle between the two planes. [4]

(ii) Find a vector equation for the line l. [6]

Question 14

The points A, B and C have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix}1\\2\\0\end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix}3\\0\\1\end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix}1\\1\\4\end{pmatrix}.$$

The plane m is perpendicular to AB and contains the point C.

(i) Find a vector equation for the line passing through A and B. [2]

(ii) Obtain the equation of the plane m, giving your answer in the form $ax + by + cz = d$. [2]

(iii) The line through A and B intersects the plane m at the point N. Find the position vector of N and show that $CN = \sqrt{(13)}$. [5]

Question 15

A plane has equation $4x - y + 5z = 39$. A straight line is parallel to the vector $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and passes through the point $A\,(0, 2, -8)$. The line meets the plane at the point B.

(i) Find the coordinates of B. [3]

(ii) Find the acute angle between the line and the plane. [4]

(iii) The point C lies on the line and is such that the distance between C and B is twice the distance between A and B. Find the coordinates of each of the possible positions of the point C. [3]

Question 16

The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. The plane p has equation $\mathbf{r}.\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 6$.

(i) Show that l is parallel to p. [3]

(ii) A line m lies in the plane p and is perpendicular to l. The line m passes through the point with coordinates $(5, 3, 1)$. Find a vector equation for m. [6]

Question 17

With respect to the origin O, the points A, B, C, D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \overrightarrow{OC} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OD} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

(i) Find the equation of the plane containing A, B and C, giving your answer in the form $ax + by + cz = d$. [6]

(ii) The line through D parallel to OA meets the plane with equation $x + 2y - z = 7$ at the point P. Find the position vector of P and show that the length of DP is $2\sqrt{(14)}$. [5]

Question 18

The points A, B and C have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$. A fourth point D is such that the quadrilateral $ABCD$ is a parallelogram.

(i) Find the position vector of D and verify that the parallelogram is a rhombus. [5]

(ii) The plane p is parallel to OA and the line BC lies in p. Find the equation of p, giving your answer in the form $ax + by + cz = d$. [5]

Question 19

The points A and B have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{k}$. The line l has vector equation $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Show that the line passing through A and B does not intersect l. [4]

(ii) Show that the length of the perpendicular from A to l is $\frac{1}{\sqrt{2}}$. [5]

Question 20

Two planes have equations $3x + y - z = 2$ and $x - y + 2z = 3$.

(i) Show that the planes are perpendicular. [3]

(ii) Find a vector equation for the line of intersection of the two planes. [6]

Question 21

The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

(i) Find the position vectors of the two points on the line whose distance from the origin is $\sqrt{(10)}$. [5]

(ii) The plane p has equation $ax + y + z = 5$, where a is a constant. The acute angle between the line l and the plane p is equal to $\sin^{-1}\left(\frac{2}{3}\right)$. Find the possible values of a. [5]

Question 22

(ii) A second plane is parallel to l, perpendicular to p and contains the point with position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find the equation of this plane, giving your answer in the form $ax + by + cz = d$. [5]

The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. The plane p has equation $3x + y - 5z = 20$.

(i) Show that the line l lies in the plane p. [3]

Question 23

The plane with equation $2x + 2y - z = 5$ is denoted by m. Relative to the origin O, the points A and B have coordinates $(3, 4, 0)$ and $(-1, 0, 2)$ respectively.

(i) Show that the plane m bisects AB at right angles. [5]

A second plane p is parallel to m and nearer to O. The perpendicular distance between the planes is 1.

(ii) Find the equation of p, giving your answer in the form $ax + by + cz = d$. [3]

Question 24

The plane with equation $2x + 2y - z = 5$ is denoted by m. Relative to the origin O, the points A and B have coordinates $(3, 4, 0)$ and $(-1, 0, 2)$ respectively.

(i) Show that the plane m bisects AB at right angles. [5]

A second plane p is parallel to m and nearer to O. The perpendicular distance between the planes is 1.

(ii) Find the equation of p, giving your answer in the form $ax + by + cz = d$. [3]

(iii) Find the exact value of the perpendicular distance of A from this plane. [3]

Question 25

The points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$, where m is a constant.

(i) Given that the line l intersects the line passing through A and B, find the value of m. [5]

(ii) Find the equation of the plane which is parallel to $\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and contains the points A and B. Give your answer in the form $ax + by + cz = d$. [5]

Question 26

The equations of two lines l and m are $\mathbf{r} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ respectively.

(i) Show that the lines do not intersect. [3]

(ii) Calculate the acute angle between the directions of the lines. [3]

(iii) Find the equation of the plane which passes through the point $(3, -2, -1)$ and which is parallel to both l and m. Give your answer in the form $ax + by + cz = d$. [5]

Question 27

Two planes p and q have equations $x + y + 3z = 8$ and $2x - 2y + z = 3$ respectively.

(i) Calculate the acute angle between the planes p and q. [4]

(ii) The point A on the line of intersection of p and q has y-coordinate equal to 2. Find the equation of the plane which contains the point A and is perpendicular to both the planes p and q. Give your answer in the form $ax + by + cz = d$. [7]

Question 28

The line l has equation $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$. The plane p has equation $2x - 3y - z = 4$.

(i) Find the position vector of the point of intersection of l and p. [3]

(ii) Find the acute angle between l and p. [3]

(iii) A second plane q is parallel to l, perpendicular to p and contains the point with position vector $4\mathbf{j} - \mathbf{k}$. Find the equation of q, giving your answer in the form $ax + by + cz = d$. [5]

Question 29

The point P has position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(i) Find the length of the perpendicular from P to l, giving your answer correct to 3 significant figures. [5]

(ii) Find the equation of the plane containing l and P, giving your answer in the form $ax + by + cz = d$. [5]

Question 30

Two lines l and m have equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + s(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ respectively.

(i) Show that the lines are skew. [4]

A plane p is parallel to the lines l and m.

(ii) Find a vector that is normal to p. [3]

(iii) Given that p is equidistant from the lines l and m, find the equation of p. Give your answer in the form $ax + by + cz = d$. [3]

Question 31

The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

The point P, with parameter t, lies on l and is such that angle PAB is equal to $120°$.

(ii) Show that $3t^2 + 8t + 4 = 0$. Hence find the position vector of P. [6]

Question 32

The planes m and n have equations $3x + y - 2z = 10$ and $x - 2y + 2z = 5$ respectively. The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

(i) Show that l is parallel to m. [3]

(ii) Calculate the acute angle between the planes m and n. [3]

(iii) A point P lies on the line l. The perpendicular distance of P from the plane n is equal to 2. Find the position vectors of the two possible positions of P. [4]

Question 33

The line l has equation $\mathbf{r} = 5\mathbf{i} - 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. The plane p has equation

$$(\mathbf{r} - \mathbf{i} - 2\mathbf{j}).(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0.$$

The line l intersects the plane p at the point A.

(i) Find the position vector of A. [3]

(ii) Calculate the acute angle between l and p. [4]

(iii) Find the equation of the line which lies in p and intersects l at right angles. [4]

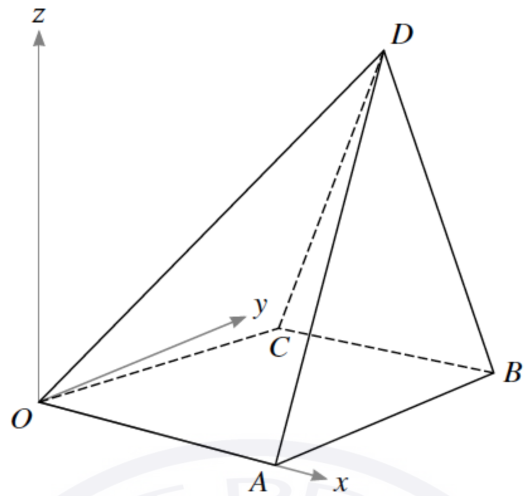
Question 34

Two planes have equations $2x + 3y - z = 1$ and $x - 2y + z = 3$.

(i) Find the acute angle between the planes. [4]

(ii) Find a vector equation for the line of intersection of the planes. [6]

Question 35



The diagram shows a set of rectangular axes Ox, Oy and Oz, and four points A, B, C and D with position vectors $\overrightarrow{OA} = 3\mathbf{i}$, $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{OC} = \mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OD} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$.

(i) Find the equation of the plane BCD, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Calculate the acute angle between the planes BCD and $OABC$. [4]

Question 36

The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

(ii) The plane m is perpendicular to AB and passes through the mid-point of AB. The plane m intersects the line l at the point P. Find the equation of m and the position vector of P. [5]

Question 37

The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$.

(i) The point P has position vector $4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Find the length of the perpendicular from P to l. [5]

(ii) It is given that l lies in the plane with equation $ax + by + 2z = 13$, where a and b are constants. Find the values of a and b. [6]

Question 38

The line l has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$. The plane p has equation $2x + y - 3z = 5$.

(i) Find the position vector of the point of intersection of l and p. [3]

(ii) When a has this value, find the equation of the plane containing l and m. [5]

Question 39

Two lines l and m have equations $\mathbf{r} = a\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ respectively, where a is a constant. It is given that the lines intersect.

(i) Find the value of a. [4]

(ii) Calculate the acute angle between l and p. [3]

(iii) A second plane q is perpendicular to the plane p and contains the line l. Find the equation of q, giving your answer in the form $ax + by + cz = d$. [5]

Question 40

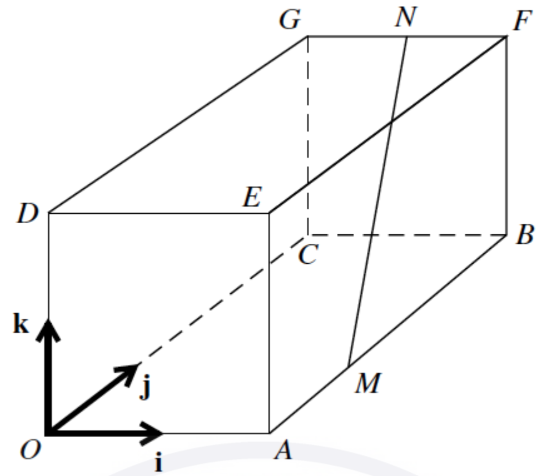
The plane m has equation $x + 4y - 8z = 2$. The plane n is parallel to m and passes through the point P with coordinates $(5, 2, -2)$.

(i) Find the equation of n, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Calculate the perpendicular distance between m and n. [3]

(iii) The line l lies in the plane n, passes through the point P and is perpendicular to OP, where O is the origin. Find a vector equation for l. [4]

Question 41



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 3$ units and $OD = 2$ units. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OD respectively. The point M on AB is such that $MB = 2AM$. The midpoint of FG is N.

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(b) Find a vector equation for the line through M and N. [2]

(c) Find the position vector of P, the foot of the perpendicular from D to the line through M and N. [4]

Question 42

With respect to the origin O, the vertices of a triangle ABC have position vectors

$$\overrightarrow{OA} = 2\mathbf{i} + 5\mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

(a) Using a scalar product, show that angle ABC is a right angle. [3]

(b) Show that triangle ABC is isosceles. [2]

(c) Find the exact length of the perpendicular from O to the line through B and C. [4]

Question 43

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. The midpoint of OA is M. The point N lying on AB, between A and B, is such that $AN = 2NB$.

(a) Find a vector equation for the line through M and N. [5]

The line through M and N intersects the line through O and B at the point P.

(b) Find the position vector of P. [3]

(c) Calculate angle OPM, giving your answer in degrees. [3]

Question 44

Relative to the origin O, the points A, B and D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{k}.$$

A fourth point C is such that $ABCD$ is a parallelogram.

(a) Find the position vector of C and verify that the parallelogram is not a rhombus. [5]

(b) Find angle BAD, giving your answer in degrees. [3]

(c) Find the area of the parallelogram correct to 3 significant figures. [2]

Question 45

Two lines have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$, where a is a constant.

(a) Given that the two lines intersect, find the value of a and the position vector of the point of intersection. [5]

(b) Given instead that the acute angle between the directions of the two lines is $\cos^{-1}\left(\frac{1}{6}\right)$, find the two possible values of a. [6]

Question 46

With respect to the origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Show that $AB = 2CD$. [3]

(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3]

(c) Show that the line through A and B does not intersect the line through C and D. [4]

Question 47

Two lines have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

(b) Find the acute angle between the directions of the two lines. [3]

Question 48

The quadrilateral $ABCD$ is a trapezium in which AB and DC are parallel. With respect to the origin O, the position vectors of A, B and C are given by $\overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(a) Given that $\overrightarrow{DC} = 3\overrightarrow{AB}$, find the position vector of D. [3]

(b) State a vector equation for the line through A and B. [1]

(c) Find the distance between the parallel sides and hence find the area of the trapezium. [5]

Question 49

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OB} = \mathbf{j} - 2\mathbf{k}$.

(a) Show that $OA = OB$ and use a scalar product to calculate angle AOB in degrees. [4]

The midpoint of AB is M. The point P on the line through O and M is such that $PA : OA = \sqrt{7} : 1$.

(b) Find the possible position vectors of P. [6]

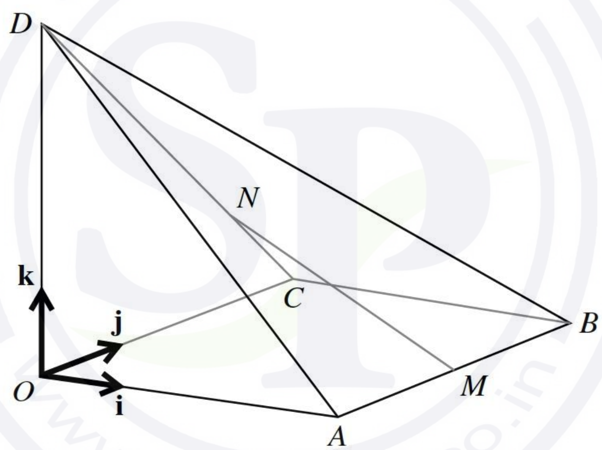
Question 50

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(a) Find the acute angle between the directions of AB and l. [4]

(b) Find the position vector of the point P on l such that $AP = BP$. [5]

Question 51



In the diagram, $OABCD$ is a pyramid with vertex D. The horizontal base $OABC$ is a square of side 4 units. The edge OD is vertical and $OD = 4$ units. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OD respectively.

The midpoint of AB is M and the point N on CD is such that $DN = 3NC$.

(a) Find a vector equation for the line through M and N. [5]

(b) Show that the length of the perpendicular from O to MN is $\frac{1}{3}\sqrt{82}$. [4]

Question 52

With respect to the origin O, the position vectors of the points A and B are given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

(a) Find a vector equation for the line l through A and B. [3]

(b) The point C lies on l and is such that $\overrightarrow{AC} = 3\overrightarrow{AB}$.

Find the position vector of C. [2]

(c) Find the possible position vectors of the point P on l such that $OP = \sqrt{14}$. [5]

Question 53

Two lines l and m have equations $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + s(4\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + t(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ respectively.

(a) Show that l and m are perpendicular. [2]

(b) Show that l and m intersect and state the position vector of the point of intersection. [5]

(c) Show that the length of the perpendicular from the origin to the line m is $\frac{1}{3}\sqrt{5}$. [4]

Question 54

The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ respectively. The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$.

(a) Find a vector equation for the line through A and B. [3]

(b) Find the acute angle between the directions of AB and l, giving your answer in degrees. [3]

(c) Show that the line through A and B does not intersect the line l. [4]

Question 55

With respect to the origin O, the point A has position vector given by $\overrightarrow{OA} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. The line l has vector equation $\mathbf{r} = 4\mathbf{i} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(a) Find in degrees the acute angle between the directions of OA and l. [3]

(b) Find the position vector of the foot of the perpendicular from A to l. [4]

(c) Hence find the position vector of the reflection of A in l. [2]

Question 56

The lines l and m have vector equations

$$\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

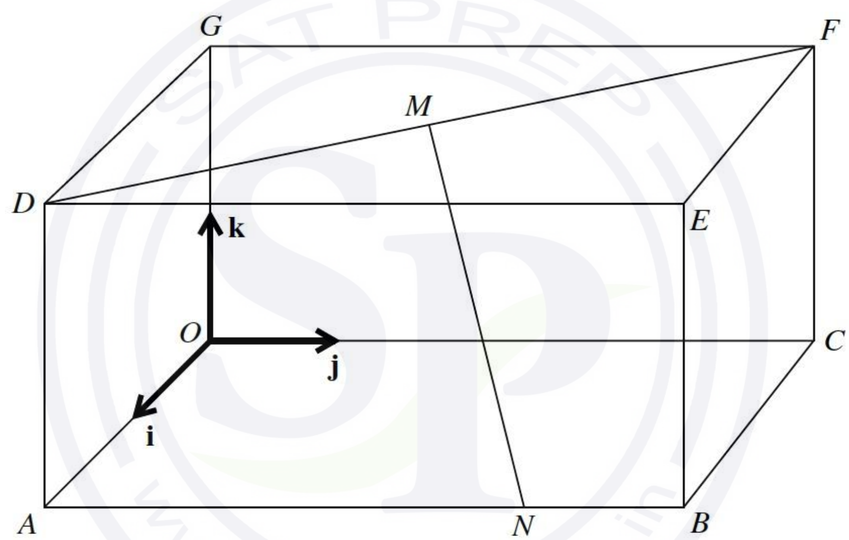
respectively, where a and b are constants.

(a) Given that l and m intersect, show that $2b - a = 4$. [4]

(b) Given also that l and m are perpendicular, find the values of a and b. [4]

(c) When a and b have these values, find the position vector of the point of intersection of l and m. [2]

Question 57



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 4$ units and $OG = 2$ units. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OG respectively. The point M is the midpoint of DF. The point N on AB is such that $AN = 3NB$.

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(b) Find a vector equation for the line through M and N. [2]

(c) Show that the length of the perpendicular from O to the line through M and N is $\sqrt{\frac{53}{6}}$. [4]

Question 58

With respect to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}.$$

The midpoint of AC is M and the point N lies on BC, between B and C, and is such that $BN = 2NC$.

(a) Find the position vectors of M and N. [3]

(b) Find a vector equation for the line through M and N. [2]

(c) Find the position vector of the point Q where the line through M and N intersects the line through A and B. [4]

Question 59

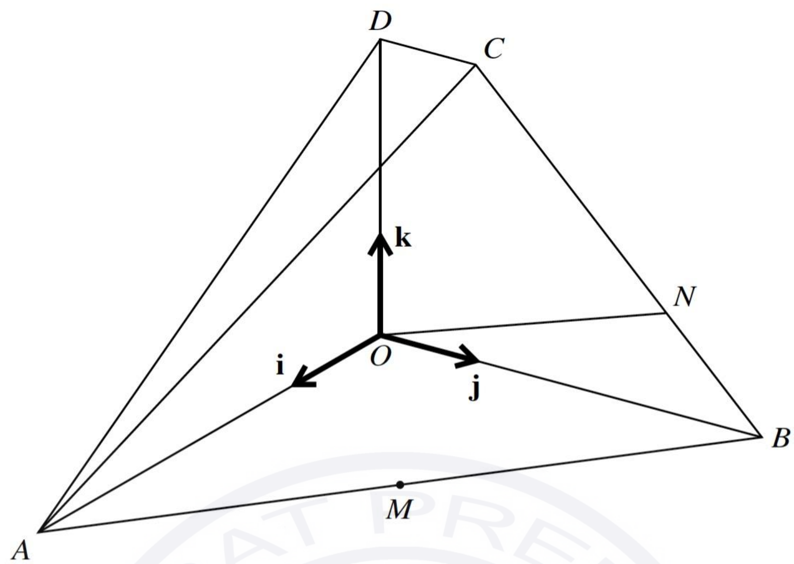
Relative to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}.$$

(a) Using a scalar product, find the cosine of angle BAC. [4]

(b) Hence find the area of triangle ABC. Give your answer in a simplified exact form. [4]

Question 60



In the diagram, $OABCD$ is a solid figure in which $OA = OB = 4$ units and $OD = 3$ units. The edge OD is vertical, DC is parallel to OB and $DC = 1$ unit. The base, OAB, is horizontal and angle $AOB = 90°$. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OB and OD respectively. The midpoint of AB is M and the point N on BC is such that $CN = 2NB$.

(a) Express vectors $\overrightarrow{MD}$ and $\overrightarrow{ON}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [4]

(b) Calculate the angle in degrees between the directions of $\overrightarrow{MD}$ and $\overrightarrow{ON}$. [3]

(c) Show that the length of the perpendicular from M to ON is $\sqrt{\frac{22}{5}}$. [4]

Question 61

With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix}.$$

(a) Find the obtuse angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [3]

The line l passes through the points A and B.

(b) Find a vector equation for the line l. [2]

(c) Find the position vector of the point of intersection of the line l and the line passing through C and D. [4]

Question 62

The lines l and m have equations

$$l: \quad \mathbf{r} = a\mathbf{i} + 3\mathbf{j} + b\mathbf{k} + \lambda(c\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}),$$
$$m: \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Relative to the origin O, the position vector of the point P is $4\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$.

(a) Given that l is perpendicular to m and that P lies on l, find the values of the constants a, b and c. [4]

(b) The perpendicular from P meets line m at Q. The point R lies on PQ extended, with $PQ : QR = 2 : 3$.

Find the position vector of R. [6]

Question 63

The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$.

(a) Show that l does not intersect the line passing through A and B. [5]

(b) Find the position vector of the foot of the perpendicular from A to l. [4]

Question 64

Relative to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

The quadrilateral $ABCD$ is a parallelogram.

(a) Find the position vector of D. [3]

(b) The angle between BA and BC is θ.

Find the exact value of $\cos\theta$. [3]

(c) Hence find the area of $ABCD$, giving your answer in the form $p\sqrt{q}$, where p and q are integers. [4]