AS-Level

Pure Mathematics P1

Topic : Functions and Transformation

May 2013- May 2023

Question 1

The function f is defined by $f : x \mapsto 2x + k,\ x \in \mathbb{R}$, where k is a constant.

(i) In the case where $k = 3$, solve the equation $ff(x) = 25$. [2]

The function g is defined by $g : x \mapsto x^2 - 6x + 8,\ x \in \mathbb{R}$.

(ii) Find the set of values of k for which the equation $f(x) = g(x)$ has no real solutions. [3]

The function h is defined by $h : x \mapsto x^2 - 6x + 8,\ x > 3$.

(iii) Find an expression for $h^{-1}(x)$. [4]

Question 2

(i) Express $2x^2 - 12x + 13$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

(ii) The function f is defined by $f(x) = 2x^2 - 12x + 13$ for $x \geqslant k$, where k is a constant. It is given that f is a one-one function. State the smallest possible value of k. [1]

The value of k is now given to be 7.

(iii) Find the range of f. [1]

(iv) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [5]

Question 3

The function f is defined by $f : x \mapsto x^2 + 4x$ for $x \geqslant c$, where c is a constant. It is given that f is a one-one function.

(i) State the range of f in terms of c and find the smallest possible value of c. [3]

The function g is defined by $g : x \mapsto ax + b$ for $x \geqslant 0$, where a and b are positive constants. It is given that, when $c = 0$, $gf(1) = 11$ and $fg(1) = 21$.

(ii) Write down two equations in a and b and solve them to find the values of a and b. [6]

Question 4

Solve the inequality $x^2 - x - 2 > 0$. [3]

Question 5

A function f is defined by $f : x \mapsto 3\cos x - 2$ for $0 \leqslant x \leqslant 2\pi$.

(i) Solve the equation $f(x) = 0$. [3]

(ii) Find the range of f. [2]

(iii) Sketch the graph of $y = f(x)$. [2]

A function g is defined by $g : x \mapsto 3\cos x - 2$ for $0 \leqslant x \leqslant k$.

(iv) State the maximum value of k for which g has an inverse. [1]

(v) Obtain an expression for $g^{-1}(x)$. [2]

Question 6

The function f is defined by

$$f : x \mapsto x^2 + 1 \quad \text{for} \quad x \geqslant 0.$$

(i) Define in a similar way the inverse function f^{-1}. [3]

(ii) Solve the equation $ff(x) = \frac{185}{16}$. [3]

Question 7

Functions f and g are defined by

$$f : x \mapsto 2x - 3, \quad x \in \mathbb{R},$$
$$g : x \mapsto x^2 + 4x, \quad x \in \mathbb{R}.$$

(i) Solve the equation $ff(x) = 11$. [2]

(ii) Find the range of g. [2]

(iii) Find the set of values of x for which $g(x) > 12$. [3]

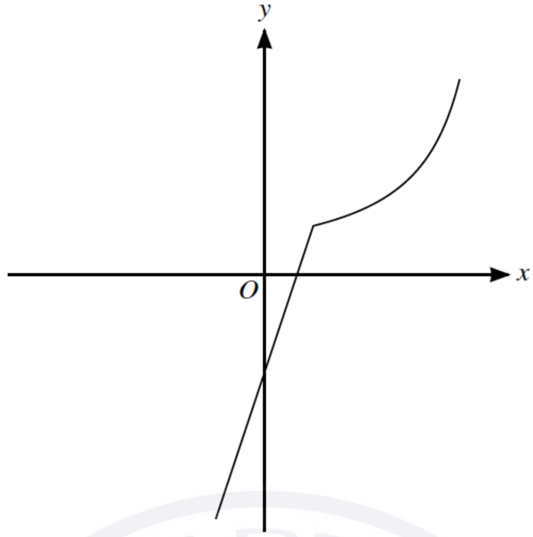
(iv) Find the value of the constant p for which the equation $gf(x) = p$ has two equal roots. [3]

Function h is defined by $h : x \mapsto x^2 + 4x$ for $x \geqslant k$, and it is given that h has an inverse.

(v) State the smallest possible value of k. [1]

(vi) Find an expression for $h^{-1}(x)$. [4]

Question 8



The diagram shows the function f defined for $-1 \leqslant x \leqslant 4$, where

$$f(x) = \begin{cases} 3x - 2 & \text{for } -1 \leqslant x \leqslant 1, \\ \dfrac{4}{5 - x} & \text{for } 1 < x \leqslant 4. \end{cases}$$

(i) State the range of f. [1]

(ii) Copy the diagram and on your copy sketch the graph of $y = f^{-1}(x)$. [2]

(iii) Obtain expressions to define the function f^{-1}, giving also the set of values for which each expression is valid. [6]

Question 9

(a) The functions f and g are defined for $x \geqslant 0$ by

$$f : x \mapsto (ax + b)^{\frac{1}{3}}, \text{ where } a \text{ and } b \text{ are positive constants,}$$
$$g : x \mapsto x^2.$$

Given that $fg(1) = 2$ and $gf(9) = 16$,

(i) calculate the values of a and b, [4]

(ii) obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

(b) A point P travels along the curve $y = (7x^2 + 1)^{\frac{1}{3}}$ in such a way that the x-coordinate of P at time t minutes is increasing at a constant rate of 8 units per minute. Find the rate of increase of the y-coordinate of P at the instant when P is at the point $(3, 4)$. [5]

Question 10

The function $f : x \mapsto 6 - 4\cos(\frac{1}{2}x)$ is defined for $0 \leqslant x \leqslant 2\pi$.

(i) Find the exact value of x for which $f(x) = 4$. [3]

(ii) State the range of f. [2]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Find an expression for $f^{-1}(x)$. [3]

Question 11

(i) Express $x^2 - 2x - 15$ in the form $(x + a)^2 + b$. [2]

The function f is defined for $p \leqslant x \leqslant q$, where p and q are positive constants, by

$$f : x \mapsto x^2 - 2x - 15.$$

The range of f is given by $c \leqslant f(x) \leqslant d$, where c and d are constants.

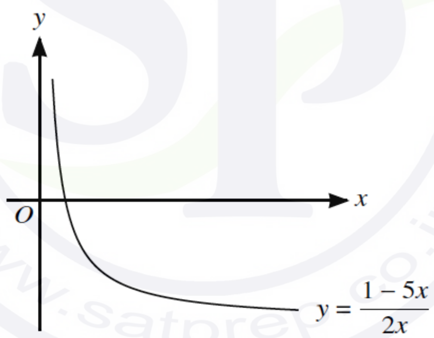
(ii) State the smallest possible value of c. [1]

For the case where $c = 9$ and $d = 65$,

(iii) find p and q, [4]

(iv) find an expression for $f^{-1}(x)$. [3]

Question 12



The diagram shows the graph of $y = f^{-1}(x)$, where f^{-1} is defined by $f^{-1}(x) = \frac{1 - 5x}{2x}$ for $0 < x \leqslant 2$.

(i) Find an expression for $f(x)$ and state the domain of f. [5]

(ii) The function g is defined by $g(x) = \frac{1}{x}$ for $x \geqslant 1$. Find an expression for $f^{-1}g(x)$, giving your answer in the form $ax + b$, where a and b are constants to be found. [2]

Question 13

The function f is defined by $f : x \mapsto 2x^2 - 6x + 5$ for $x \in \mathbb{R}$.

(i) Find the set of values of p for which the equation $f(x) = p$ has no real roots. [3]

The function g is defined by $g : x \mapsto 2x^2 - 6x + 5$ for $0 \leqslant x \leqslant 4$.

(ii) Express $g(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

(iii) Find the range of g. [2]

The function h is defined by $h : x \mapsto 2x^2 - 6x + 5$ for $k \leqslant x \leqslant 4$, where k is a constant.

(iv) State the smallest value of k for which h has an inverse. [1]

(v) For this value of k, find an expression for $h^{-1}(x)$. [3]

Question 14

A tourist attraction in a city centre is a big vertical wheel on which passengers can ride. The wheel turns in such a way that the height, h m, of a passenger above the ground is given by the formula $h = 60(1 - \cos kt)$. In this formula, k is a constant, t is the time in minutes that has elapsed since the passenger started the ride at ground level and kt is measured in radians.

(i) Find the greatest height of the passenger above the ground. [1]

One complete revolution of the wheel takes 30 minutes.

(ii) Show that $k = \frac{1}{15}\pi$. [2]

(iii) Find the time for which the passenger is above a height of 90 m. [3]

Question 15

The function $f : x \mapsto 5 + 3\cos(\frac{1}{2}x)$ is defined for $0 \leqslant x \leqslant 2\pi$.

(i) Solve the equation $f(x) = 7$, giving your answer correct to 2 decimal places. [3]

(ii) Sketch the graph of $y = f(x)$. [2]

(iii) Explain why f has an inverse. [1]

(iv) Obtain an expression for $f^{-1}(x)$. [3]

Question 16

The function f is defined by $f(x) = 3x + 1$ for $x \leqslant a$, where a is a constant. The function g is defined by $g(x) = -1 - x^2$ for $x \leqslant -1$.

(i) Find the largest value of a for which the composite function gf can be formed. [2]

For the case where $a = -1$,

(ii) solve the equation $fg(x) + 14 = 0$, [3]

(iii) find the set of values of x which satisfy the inequality $gf(x) \leqslant -50$. [4]

Question 17

The function f is defined, for $x \in \mathbb{R}$, by $\mathrm{f} : x \mapsto x^2 + ax + b$, where a and b are constants.

(i) In the case where $a = 6$ and $b = -8$, find the range of f. [3]

(ii) In the case where $a = 5$, the roots of the equation $\mathrm{f}(x) = 0$ are k and $-2k$, where k is a constant. Find the values of b and k. [3]

(iii) Show that if the equation $\mathrm{f}(x + a) = a$ has no real roots, then $a^2 < 4(b - a)$. [3]

Question 18

Functions f and g are defined by

$$\mathrm{f} : x \mapsto 3x + 2, \quad x \in \mathbb{R},$$
$$\mathrm{g} : x \mapsto 4x - 12, \quad x \in \mathbb{R}.$$

Solve the equation $\mathrm{f}^{-1}(x) = \mathrm{gf}(x)$. [4]

Question 19

(i) Express $-x^2 + 6x - 5$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

The function $\mathrm{f} : x \mapsto -x^2 + 6x - 5$ is defined for $x \geqslant m$, where m is a constant.

(ii) State the smallest value of m for which f is one-one. [1]

(iii) For the case where $m = 5$, find an expression for $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

Question 20

The function f is such that $\mathrm{f}(x) = a^2x^2 - ax + 3b$ for $x \leqslant \frac{1}{2a}$, where a and b are constants.

(i) For the case where $\mathrm{f}(-2) = 4a^2 - b + 8$ and $\mathrm{f}(-3) = 7a^2 - b + 14$, find the possible values of a and b. [5]

(ii) For the case where $a = 1$ and $b = -1$, find an expression for $\mathrm{f}^{-1}(x)$ and give the domain of f^{-1}. [5]

Question 21

The function f is such that $\mathrm{f}(x) = 2x + 3$ for $x \geqslant 0$. The function g is such that $\mathrm{g}(x) = ax^2 + b$ for $x \leqslant q$, where a, b and q are constants. The function fg is such that $\mathrm{fg}(x) = 6x^2 - 21$ for $x \leqslant q$.

(i) Find the values of a and b. [3]

(ii) Find the greatest possible value of q. [2]

It is now given that $q = -3$.

(iii) Find the range of fg. [1]

(iv) Find an expression for $(\mathrm{fg})^{-1}(x)$ and state the domain of $(\mathrm{fg})^{-1}$. [3]

Question 22

The function f is defined by $f : x \mapsto 6x - x^2 - 5$ for $x \in \mathbb{R}$.

(i) Find the set of values of x for which $f(x) \leqslant 3$. [3]

(ii) Given that the line $y = mx + c$ is a tangent to the curve $y = f(x)$, show that $4c = m^2 - 12m + 16$. [3]

The function g is defined by $g : x \mapsto 6x - x^2 - 5$ for $x \geqslant k$, where k is a constant.

(iii) Express $6x - x^2 - 5$ in the form $a - (x - b)^2$, where a and b are constants. [2]

(iv) State the smallest value of k for which g has an inverse. [1]

(v) For this value of k, find an expression for $g^{-1}(x)$. [2]

Question 23

Functions f and g are defined by

$$f : x \mapsto 10 - 3x, \quad x \in \mathbb{R},$$

$$g : x \mapsto \frac{10}{3 - 2x}, \quad x \in \mathbb{R},\ x \neq \tfrac{3}{2}.$$

Solve the equation $ff(x) = gf(2)$. [3]

Question 24

The function f is defined by $f : x \mapsto 4\sin x - 1$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) State the range of f. [2]

(ii) Find the coordinates of the points at which the curve $y = f(x)$ intersects the coordinate axes. [3]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Obtain an expression for $f^{-1}(x)$, stating both the domain and range of f^{-1}. [4]

Question 25

(i) Express $4x^2 + 12x + 10$ in the form $(ax + b)^2 + c$, where a, b and c are constants. [3]

(ii) Functions f and g are both defined for $x > 0$. It is given that $f(x) = x^2 + 1$ and $fg(x) = 4x^2 + 12x + 10$. Find $g(x)$. [1]

(iii) Find $(fg)^{-1}(x)$ and give the domain of $(fg)^{-1}$. [4]

Question 26

A function f is defined by $\mathrm{f} : x \mapsto 5 - 2\sin 2x$ for $0 \leqslant x \leqslant \pi$.

(i) Find the range of f. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$. [2]

(iii) Solve the equation $\mathrm{f}(x) = 6$, giving answers in terms of π. [3]

The function g is defined by $\mathrm{g} : x \mapsto 5 - 2\sin 2x$ for $0 \leqslant x \leqslant k$, where k is a constant.

(iv) State the largest value of k for which g has an inverse. [1]

(v) For this value of k, find an expression for $\mathrm{g}^{-1}(x)$. [3]

Question 27

The functions f and g are defined by

$$\mathrm{f}(x) = \frac{4}{x} - 2 \quad \text{for } x > 0,$$

$$\mathrm{g}(x) = \frac{4}{5x + 2} \quad \text{for } x \geqslant 0.$$

(i) Find and simplify an expression for $\mathrm{fg}(x)$ and state the range of fg. [3]

(ii) Find an expression for $\mathrm{g}^{-1}(x)$ and find the domain of g^{-1}. [5]

Question 28

The functions f and g are defined for $x \geqslant 0$ by

$$\mathrm{f} : x \mapsto 2x^2 + 3,$$

$$\mathrm{g} : x \mapsto 3x + 2.$$

(i) Show that $\mathrm{gf}(x) = 6x^2 + 11$ and obtain an unsimplified expression for $\mathrm{fg}(x)$. [2]

(ii) Find an expression for $(\mathrm{fg})^{-1}(x)$ and determine the domain of $(\mathrm{fg})^{-1}$. [5]

(iii) Solve the equation $\mathrm{gf}(2x) = \mathrm{fg}(x)$. [3]

Question 29

(i) Express $9x^2 - 6x + 6$ in the form $(ax + b)^2 + c$, where a, b and c are constants. [3]

The function f is defined by $\mathrm{f}(x) = 9x^2 - 6x + 6$ for $x \geqslant p$, where p is a constant.

(ii) State the smallest value of p for which f is a one-one function. [1]

(iii) For this value of p, obtain an expression for $\mathrm{f}^{-1}(x)$, and state the domain of f^{-1}. [4]

(iv) State the set of values of q for which the equation $\mathrm{f}(x) = q$ has no solution. [1]

Question 30

The function f is defined by $f(x) = 3\tan(\frac{1}{2}x) - 2$, for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Solve the equation $f(x) + 4 = 0$, giving your answer correct to 1 decimal place. [3]

(ii) Find an expression for $f^{-1}(x)$ and find the domain of f^{-1}. [5]

(iii) Sketch, on the same diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$. [3]

Question 31

The function f is defined by $f : x \mapsto \dfrac{2}{3 - 2x}$ for $x \in \mathbb{R},\ x \neq \frac{3}{2}$.

(i) Find an expression for $f^{-1}(x)$. [3]

The function g is defined by $g : x \mapsto 4x + a$ for $x \in \mathbb{R}$, where a is a constant.

(ii) Find the value of a for which $gf(-1) = 3$. [3]

(iii) Find the possible values of a given that the equation $f^{-1}(x) = g^{-1}(x)$ has two equal roots. [4]

Question 32

The functions f and g are defined by

$$f(x) = \frac{2}{x^2 - 1} \text{ for } x < -1,$$
$$g(x) = x^2 + 1 \text{ for } x > 0.$$

(i) Find an expression for $f^{-1}(x)$. [3]

(ii) Solve the equation $gf(x) = 5$. [4]

Question 33

(a) The function f, defined by $f : x \mapsto a + b\sin x$ for $x \in \mathbb{R}$, is such that $f(\frac{1}{6}\pi) = 4$ and $f(\frac{1}{2}\pi) = 3$.

(i) Find the values of the constants a and b. [3]

(ii) Evaluate $ff(0)$. [2]

(b) The function g is defined by $g : x \mapsto c + d\sin x$ for $x \in \mathbb{R}$. The range of g is given by $-4 \leqslant g(x) \leqslant 10$. Find the values of the constants c and d. [3]

Question 35

A function f is defined by $f : x \mapsto 4 - 5x$ for $x \in \mathbb{R}$.

(i) Find an expression for $f^{-1}(x)$ and find the point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$. [3]

(ii) Sketch, on the same diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the graphs. [3]

Question 36

Functions f and g are defined for $x > 3$ by

$$f : x \mapsto \frac{1}{x^2 - 9},$$
$$g : x \mapsto 2x - 3.$$

(i) Find and simplify an expression for $gg(x)$. [2]

(ii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Solve the equation $fg(x) = \frac{1}{7}$. [4]

Question 37

Functions f and g are defined by

$$f(x) = \frac{8}{x-2} + 2 \quad \text{for } x > 2,$$
$$g(x) = \frac{8}{x-2} + 2 \quad \text{for } 2 < x < 4.$$

(i) **(a)** State the range of the function f. [1]

(b) State the range of the function g. [1]

(c) State the range of the function fg. [1]

(ii) Explain why the function gf cannot be formed. [1]

Question 38

The one-one function f is defined by $f(x) = (x-2)^2 + 2$ for $x \geqslant c$, where c is a constant.

(i) State the smallest possible value of c. [1]

In parts **(ii)** and **(iii)** the value of c is 4.

(ii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) Solve the equation $ff(x) = 51$, giving your answer in the form $a + \sqrt{b}$. [5]

Question 39

The function f is defined by $f : x \mapsto 7 - 2x^2 - 12x$ for $x \in \mathbb{R}$.

(i) Express $7 - 2x^2 - 12x$ in the form $a - 2(x+b)^2$, where a and b are constants. [2]

(ii) State the coordinates of the stationary point on the curve $y = f(x)$. [1]

The function g is defined by $g : x \mapsto 7 - 2x^2 - 12x$ for $x \geqslant k$.

(iii) State the smallest value of k for which g has an inverse. [1]

(iv) For this value of k, find $g^{-1}(x)$. [3]

Question 40

Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto \tfrac{1}{2}x - 2,$$
$$g : x \mapsto 4 + x - \tfrac{1}{2}x^2.$$

(i) Find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$. [3]

(ii) Find the set of values of x for which $f(x) > g(x)$. [2]

(iii) Find an expression for $fg(x)$ and deduce the range of fg. [4]

The function h is defined by $h : x \mapsto 4 + x - \tfrac{1}{2}x^2$ for $x \geqslant k$.

(iv) Find the smallest value of k for which h has an inverse. [2]

Question 41

(i) Express $2x^2 - 12x + 11$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

The function f is defined by $f(x) = 2x^2 - 12x + 11$ for $x \leqslant k$.

(ii) State the largest value of the constant k for which f is a one-one function. [1]

(iii) For this value of k find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

The function g is defined by $g(x) = x + 3$ for $x \leqslant p$.

(iv) With k now taking the value 1, find the largest value of the constant p which allows the composite function fg to be formed, and find an expression for $fg(x)$ whenever this composite function exists. [3]

Question 42

The function f is defined by $f : x \mapsto 2x^2 - 12x + 7$ for $x \in \mathbb{R}$.

(i) Express $2x^2 - 12x + 7$ in the form $2(x + a)^2 + b$, where a and b are constants. [2]

(ii) State the range of f. [1]

The function g is defined by $g : x \mapsto 2x^2 - 12x + 7$ for $x \leqslant k$.

(iii) State the largest value of k for which g has an inverse. [1]

(iv) Given that g has an inverse, find an expression for $g^{-1}(x)$. [3]

Question 43

(a) The one-one function f is defined by $f(x) = (x-3)^2 - 1$ for $x < a$, where a is a constant.

(i) State the greatest possible value of a. [1]

(ii) It is given that a takes this greatest possible value. State the range of f and find an expression for $f^{-1}(x)$. [3]

(b) The function g is defined by $g(x) = (x-3)^2$ for $x \geqslant 0$.

(i) Show that $gg(2x)$ can be expressed in the form $(2x-3)^4 + b(2x-3)^2 + c$, where b and c are constants to be found. [2]

(ii) Hence expand $gg(2x)$ completely, simplifying your answer. [4]

Question 44

(i) Express $x^2 - 4x + 7$ in the form $(x+a)^2 + b$. [2]

The function f is defined by $f(x) = x^2 - 4x + 7$ for $x < k$, where k is a constant.

(ii) State the largest value of k for which f is a decreasing function. [1]

The value of k is now given to be 1.

(iii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(iv) The function g is defined by $g(x) = \dfrac{2}{x-1}$ for $x > 1$. Find an expression for $gf(x)$ and state the range of gf. [4]

Question 45

The function f is defined by $f(x) = x^2 - 4x + 8$ for $x \in \mathbb{R}$.

(i) Express $x^2 - 4x + 8$ in the form $(x-a)^2 + b$. [2]

(ii) Hence find the set of values of x for which $f(x) < 9$, giving your answer in exact form. [3]

Question 46

The function f is defined by $f(x) = \dfrac{48}{x-1}$ for $3 \leqslant x \leqslant 7$. The function g is defined by $g(x) = 2x - 4$ for $a \leqslant x \leqslant b$, where a and b are constants.

(i) Find the greatest value of a and the least value of b which will permit the formation of the composite function gf. [2]

It is now given that the conditions for the formation of gf are satisfied.

(ii) Find an expression for $gf(x)$. [1]

(iii) Find an expression for $(gf)^{-1}(x)$. [2]

Question 47

Functions f and g are defined by

$$\mathrm{f} : x \mapsto 3x - 2, \quad x \in \mathbb{R},$$
$$\mathrm{g} : x \mapsto \frac{2x+3}{x-1}, \quad x \in \mathbb{R},\ x \neq 1.$$

(i) Obtain expressions for $\mathrm{f}^{-1}(x)$ and $\mathrm{g}^{-1}(x)$, stating the value of x for which $\mathrm{g}^{-1}(x)$ is not defined. [4]

(ii) Solve the equation $\mathrm{fg}(x) = \frac{7}{3}$. [3]

Question 48

The function f is defined by $\mathrm{f}(x) = -2x^2 + 12x - 3$ for $x \in \mathbb{R}$.

(i) Express $-2x^2 + 12x - 3$ in the form $-2(x+a)^2 + b$, where a and b are constants. [2]

(ii) State the greatest value of $\mathrm{f}(x)$. [1]

The function g is defined by $\mathrm{g}(x) = 2x + 5$ for $x \in \mathbb{R}$.

(iii) Find the values of x for which $\mathrm{gf}(x) + 1 = 0$. [3]

Question 49

The function g is defined by $\mathrm{g}(x) = x^2 - 6x + 7$ for $x > 4$. By first completing the square, find an expression for $\mathrm{g}^{-1}(x)$ and state the domain of g^{-1}. [5]

Question 50

Functions f and g are defined by

$$\mathrm{f}(x) = 2x^2 + 8x + 1 \quad \text{for } x \in \mathbb{R},$$
$$\mathrm{g}(x) = 2x - k \quad \text{for } x \in \mathbb{R},$$

where k is a constant.

(i) Find the value of k for which the line $y = \mathrm{g}(x)$ is a tangent to the curve $y = \mathrm{f}(x)$. [3]

(ii) In the case where $k = -9$, find the set of values of x for which $\mathrm{f}(x) < \mathrm{g}(x)$. [3]

(iii) In the case where $k = -1$, find $\mathrm{g}^{-1}\mathrm{f}(x)$ and solve the equation $\mathrm{g}^{-1}\mathrm{f}(x) = 0$. [3]

(iv) Express $\mathrm{f}(x)$ in the form $2(x+a)^2 + b$, where a and b are constants, and hence state the least value of $\mathrm{f}(x)$. [3]

Question 51

Functions f and g are defined by

$$f : x \mapsto \frac{3}{2x+1} \quad \text{for } x > 0,$$

$$g : x \mapsto \frac{1}{x} + 2 \quad \text{for } x > 0.$$

(i) Find the range of f and the range of g. [3]

(ii) Find an expression for $fg(x)$, giving your answer in the form $\frac{ax}{bx+c}$, where a, b and c are integers. [2]

(iii) Find an expression for $(fg)^{-1}(x)$, giving your answer in the same form as for part **(ii)**. [3]

Question 52

The graph of $y = f(x)$ is transformed to the graph of $y = 1 + f(\frac{1}{2}x)$.

Describe fully the two single transformations which have been combined to give the resulting transformation. [4]

Question 53

(a) Express $2x^2 + 12x + 11$ in the form $2(x+a)^2 + b$, where a and b are constants. [2]

The function f is defined by $f(x) = 2x^2 + 12x + 11$ for $x \leqslant -4$.

(b) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

The function g is defined by $g(x) = 2x - 3$ for $x \leqslant k$.

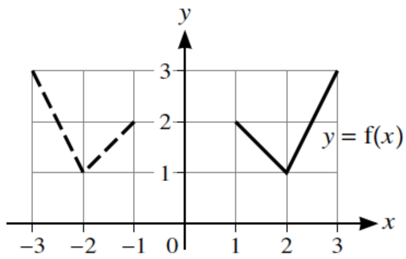
(c) For the case where $k = -1$, solve the equation $fg(x) = 193$. [2]

(d) State the largest value of k possible for the composition fg to be defined. [1]

Question 54

In each of parts **(a)**, **(b)** and **(c)**, the graph shown with solid lines has equation $y = \mathrm{f}(x)$. The graph shown with broken lines is a transformation of $y = \mathrm{f}(x)$.

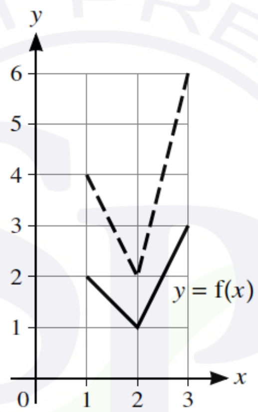
(a)



State, in terms of f, the equation of the graph shown with broken lines. [1]

....

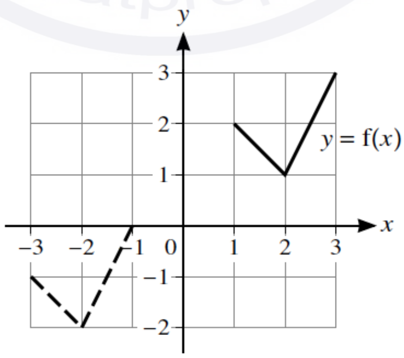
(b)



State, in terms of f, the equation of the graph shown with broken lines. [1]

....

(c)



State, in terms of f, the equation of the graph shown with broken lines. [2]

Question 55

The functions f and g are defined by

$$f(x) = x^2 - 4x + 3 \quad \text{for } x > c, \text{ where } c \text{ is a constant,}$$

$$g(x) = \frac{1}{x+1} \quad \text{for } x > -1.$$

(a) Express $f(x)$ in the form $(x-a)^2 + b$. [2]

It is given that f is a one-one function.

(b) State the smallest possible value of c. [1]

It is now given that $c = 5$.

(c) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(d) Find an expression for $gf(x)$ and state the range of gf. [3]

Question 56

The function f is defined for $x \in \mathbb{R}$ by

$$f : x \mapsto a - 2x,$$

where a is a constant.

(a) Express $ff(x)$ and $f^{-1}(x)$ in terms of a and x. [4]

(b) Given that $ff(x) = f^{-1}(x)$, find x in terms of a. [2]

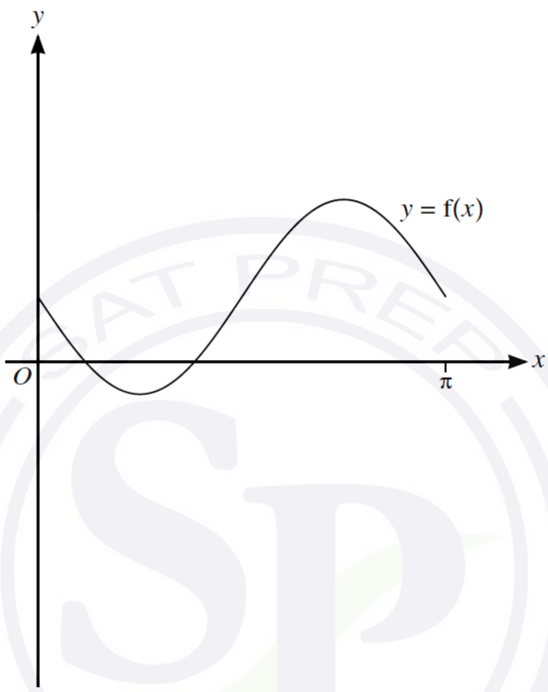
Question 57

Functions f and g are such that

$$f(x) = 2 - 3\sin 2x \quad \text{for } 0 \leqslant x \leqslant \pi,$$
$$g(x) = -2f(x) \quad \text{for } 0 \leqslant x \leqslant \pi.$$

(a) State the ranges of f and g. [3]

The diagram below shows the graph of $y = f(x)$.



(b) Sketch, on this diagram, the graph of $y = g(x)$. [2]

The function h is such that

$$h(x) = g(x + \pi) \quad \text{for } -\pi \leqslant x \leqslant 0.$$

(c) Describe fully a sequence of transformations that maps the curve $y = f(x)$ on to $y = h(x)$. [3]

Question 58

Functions f and g are defined for $x \in \mathbb{R}$ by

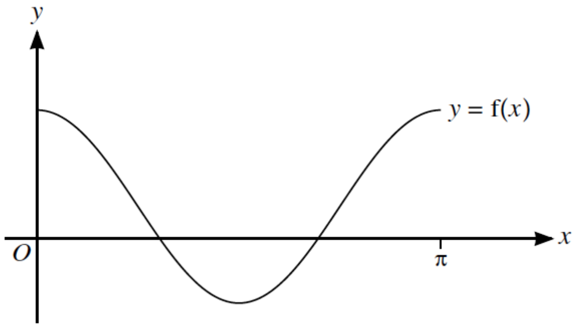
$$f : x \mapsto \tfrac{1}{2}x - a,$$
$$g : x \mapsto 3x + b,$$

where a and b are constants.

(a) Given that $gg(2) = 10$ and $f^{-1}(2) = 14$, find the values of a and b. [4]

(b) Using these values of a and b, find an expression for $gf(x)$ in the form $cx + d$, where c and d are constants. [2]

Question 59



The diagram shows the graph of $y = f(x)$, where $f(x) = \frac{3}{2}\cos 2x + \frac{1}{2}$ for $0 \leqslant x \leqslant \pi$.

(a) State the range of f. [2]

A function g is such that $g(x) = f(x) + k$, where k is a positive constant. The x-axis is a tangent to the curve $y = g(x)$.

(b) State the value of k and hence describe fully the transformation that maps the curve $y = f(x)$ on to $y = g(x)$. [2]

(c) State the equation of the curve which is the reflection of $y = f(x)$ in the x-axis. Give your answer in the form $y = a\cos 2x + b$, where a and b are constants. [1]

Question 60

The function f is defined by $f(x) = \dfrac{2x}{3x-1}$ for $x > \frac{1}{3}$.

(a) Find an expression for $f^{-1}(x)$. [3]

(b) Show that $\dfrac{2}{3} + \dfrac{2}{3(3x-1)}$ can be expressed as $\dfrac{2x}{3x-1}$. [2]

(c) State the range of f. [1]

Question 61

(a) Express $x^2 + 6x + 5$ in the form $(x + a)^2 + b$, where a and b are constants. [2]

(b) The curve with equation $y = x^2$ is transformed to the curve with equation $y = x^2 + 6x + 5$.

Describe fully the transformation(s) involved. [2]

Question 62

A curve has equation $y = 3\cos 2x + 2$ for $0 \leqslant x \leqslant \pi$.

(a) State the greatest and least values of y. [2]

....

....

....

....

(b) Sketch the graph of $y = 3\cos 2x + 2$ for $0 \leqslant x \leqslant \pi$. [2]

(c) By considering the straight line $y = kx$, where k is a constant, state the number of solutions of the equation $3\cos 2x + 2 = kx$ for $0 \leqslant x \leqslant \pi$ in each of the following cases.

(i) $k = -3$ [1]

....

....

(ii) $k = 1$ [1]

....

....

(iii) $k = 3$ [1]

....

....

Functions f, g and h are defined for $x \in \mathbb{R}$ by

$$f(x) = 3\cos 2x + 2,$$
$$g(x) = f(2x) + 4,$$
$$h(x) = 2f\left(x + \tfrac{1}{2}\pi\right).$$

(d) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ on to $y = g(x)$. [2]

(e) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ on to $y = h(x)$. [2]

Question 63

Functions f and g are defined by

$$f(x) = 4x - 2, \quad \text{for } x \in \mathbb{R},$$
$$g(x) = \frac{4}{x+1}, \quad \text{for } x \in \mathbb{R},\ x \neq -1.$$

(a) Find the value of fg(7). [1]

(b) Find the values of x for which $f^{-1}(x) = g^{-1}(x)$. [5]

Question 64

The functions f and g are defined by

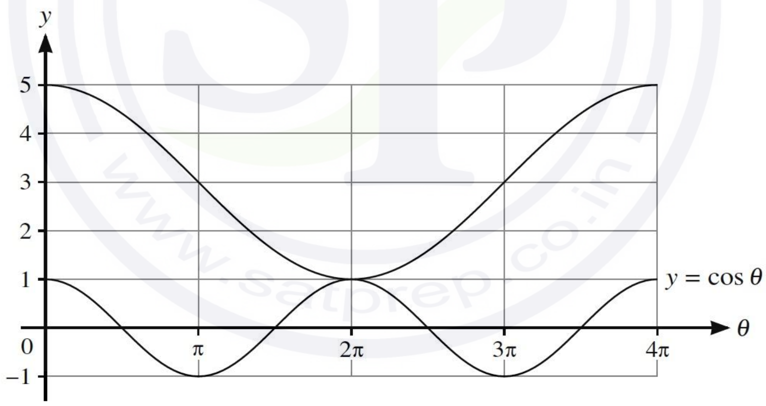
$$f(x) = x^2 + 3 \quad \text{for } x > 0,$$
$$g(x) = 2x + 1 \quad \text{for } x > -\tfrac{1}{2}.$$

(a) Find an expression for $fg(x)$. [1]

(b) Find an expression for $(fg)^{-1}(x)$ and state the domain of $(fg)^{-1}$. [4]

(c) Solve the equation $fg(x) - 3 = gf(x)$. [4]

Question 65



In the diagram, the lower curve has equation $y = \cos\theta$. The upper curve shows the result of applying a combination of transformations to $y = \cos\theta$.

Find, in terms of a cosine function, the equation of the upper curve. [3]

Question 66

Functions f and g are defined as follows:

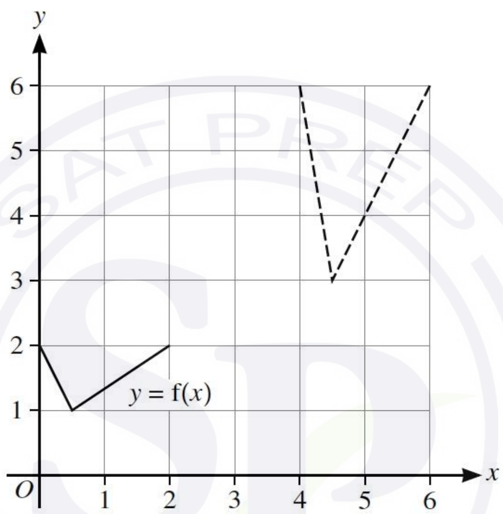
$$\mathrm{f} : x \mapsto x^2 + 2x + 3 \text{ for } x \leqslant -1,$$
$$\mathrm{g} : x \mapsto 2x + 1 \text{ for } x \geqslant -1.$$

(a) Express $\mathrm{f}(x)$ in the form $(x + a)^2 + b$ and state the range of f. [3]

(b) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

(c) Solve the equation $\mathrm{gf}(x) = 13$. [3]

Question 67



In the diagram, the graph of $y = \mathrm{f}(x)$ is shown with solid lines. The graph shown with broken lines is a transformation of $y = \mathrm{f}(x)$.

(a) Describe fully the two single transformations of $y = \mathrm{f}(x)$ that have been combined to give the resulting transformation. [4]

(b) State in terms of y, f and x, the equation of the graph shown with broken lines. [2]

Question 68

Functions f and g are defined as follows:

$$\mathrm{f} : x \mapsto x^2 - 1 \text{ for } x < 0,$$
$$\mathrm{g} : x \mapsto \frac{1}{2x+1} \text{ for } x < -\tfrac{1}{2}.$$

(a) Solve the equation $\mathrm{fg}(x) = 3$. [4]

(b) Find an expression for $(\mathrm{fg})^{-1}(x)$. [3]

Question 69

Functions f and g are both defined for $x \in \mathbb{R}$ and are given by

$$f(x) = x^2 - 2x + 5,$$
$$g(x) = x^2 + 4x + 13.$$

(a) By first expressing each of $f(x)$ and $g(x)$ in completed square form, express $g(x)$ in the form $f(x + p) + q$, where p and q are constants. [4]

(b) Describe fully the transformation which transforms the graph of $y = f(x)$ to the graph of $y = g(x)$. [2]

Question 70

The function f is defined by $f(x) = 2x^2 + 3$ for $x \geqslant 0$.

(a) Find and simplify an expression for $ff(x)$. [2]

(b) Solve the equation $ff(x) = 34x^2 + 19$. [4]

Question 71

(a) The graph of $y = f(x)$ is transformed to the graph of $y = 2f(x - 1)$.

Describe fully the two single transformations which have been combined to give the resulting transformation. [3]

(b) The curve $y = \sin 2x - 5x$ is reflected in the y-axis and then stretched by scale factor $\frac{1}{3}$ in the x-direction.

Write down the equation of the transformed curve. [2]

Question 72

Functions f and g are defined as follows:

$$f(x) = (x - 2)^2 - 4 \text{ for } x \geqslant 2,$$
$$g(x) = ax + 2 \text{ for } x \in \mathbb{R},$$

where a is a constant.

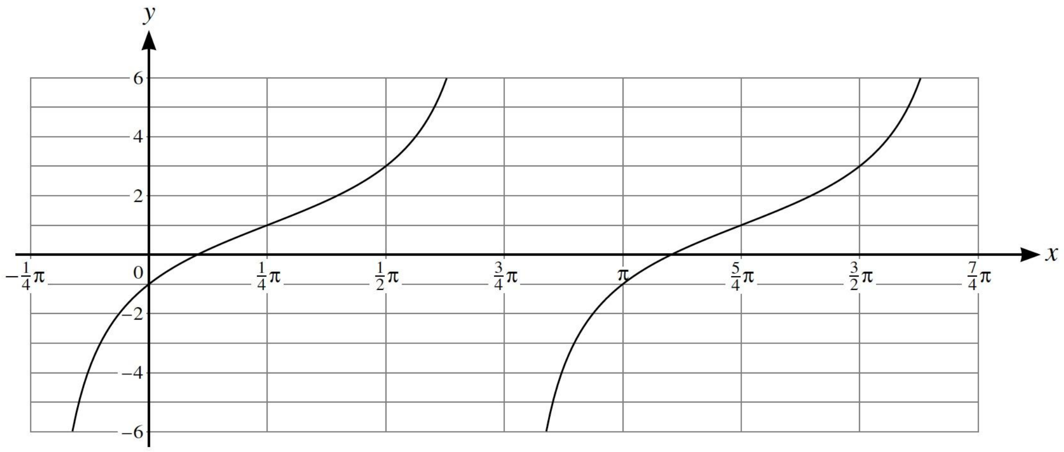
(a) State the range of f. [1]

(b) Find $f^{-1}(x)$. [2]

(c) Given that $a = -\frac{5}{3}$, solve the equation $f(x) = g(x)$. [3]

(d) Given instead that $ggf^{-1}(12) = 62$, find the possible values of a. [5]

Question 73



The diagram shows part of the graph of $y = a\tan(x - b) + c$.

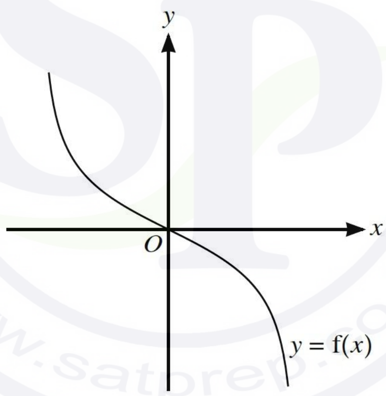
Given that $0 < b < \pi$, state the values of the constants a, b and c. [3]

Question 74

The graph of $y = \mathrm{f}(x)$ is transformed to the graph of $y = 3 - \mathrm{f}(x)$.

Describe fully, in the correct order, the two transformations that have been combined. [4]

Question 75



The diagram shows the graph of $y = \mathrm{f}(x)$.

(a) On this diagram sketch the graph of $y = \mathrm{f}^{-1}(x)$. [1]

It is now given that $\mathrm{f}(x) = -\dfrac{x}{\sqrt{4 - x^2}}$ where $-2 < x < 2$.

(b) Find an expression for $\mathrm{f}^{-1}(x)$. [4]

The function g is defined by $\mathrm{g}(x) = 2x$ for $-a < x < a$, where a is a constant.

(c) State the maximum possible value of a for which fg can be formed. [1]

(d) Assuming that fg can be formed, find and simplify an expression for $\mathrm{fg}(x)$. [2]

Question 76

The graph of $y = f(x)$ is transformed to the graph of $y = 3 - f(x)$.

Describe fully, in the correct order, the two transformations that have been combined. [4]

Question 77

The function f is defined as follows:

$$f(x) = \frac{x+3}{x-1} \text{ for } x > 1.$$

(a) Find the value of ff(5). [2]

(b) Find an expression for $f^{-1}(x)$. [3]

Question 78

The graph of $y = f(x)$ is transformed to the graph of $y = f(2x) - 3$.

(a) Describe fully the two single transformations that have been combined to give the resulting transformation. [3]

The point $P(5, 6)$ lies on the transformed curve $y = f(2x) - 3$.

(b) State the coordinates of the corresponding point on the original curve $y = f(x)$. [2]

Question 79

(a) Express $-3x^2 + 12x + 2$ in the form $-3(x-a)^2 + b$, where a and b are constants. [2]

The one-one function f is defined by $f : x \mapsto -3x^2 + 12x + 2$ for $x \leqslant k$.

(b) State the largest possible value of the constant k. [1]

It is now given that $k = -1$.

(c) State the range of f. [1]

(d) Find an expression for $f^{-1}(x)$. [3]

The result of translating the graph of $y = f(x)$ by $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is the graph of $y = g(x)$.

(e) Express $g(x)$ in the form $px^2 + qx + r$, where p, q and r are constants. [3]

Question 80

Functions f, g and h are defined as follows:

$$f : x \mapsto x - 4x^{\frac{1}{2}} + 1 \quad \text{for } x \geqslant 0,$$
$$g : x \mapsto mx^2 + n \quad \text{for } x \geqslant -2, \text{ where } m \text{ and } n \text{ are constants},$$
$$h : x \mapsto x^{\frac{1}{2}} - 2 \quad \text{for } x \geqslant 0.$$

(a) Solve the equation $f(x) = 0$, giving your solutions in the form $x = a + b\sqrt{c}$, where a, b and c are integers. [4]

(b) Given that $f(x) \equiv gh(x)$, find the values of m and n. [4]

Question 81

(a) Express $2x^2 - 8x + 14$ in the form $2[(x-a)^2 + b]$. [2]

The functions f and g are defined by

$$f(x) = x^2 \quad \text{for } x \in \mathbb{R},$$
$$g(x) = 2x^2 - 8x + 14 \quad \text{for } x \in \mathbb{R}.$$

(b) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$, making clear the order in which the transformations are applied. [4]

Question 82

The function f is defined by $f(x) = 2x^2 - 16x + 23$ for $x < 3$.

(a) Express $f(x)$ in the form $2(x+a)^2 + b$. [2]

(b) Find the range of f. [1]

(c) Find an expression for $f^{-1}(x)$. [3]

The function g is defined by $g(x) = 2x + 4$ for $x < -1$.

(d) Find and simplify an expression for $fg(x)$. [2]

Question 83

(a) The curve with equation $y = x^2 + 2x - 5$ is translated by $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Find the equation of the translated curve, giving your answer in the form $y = ax^2 + bx + c$. [3]

(b) The curve with equation $y = x^2 + 2x - 5$ is transformed to a curve with equation $y = 4x^2 + 4x - 5$.

Describe fully the single transformation that has been applied. [2]

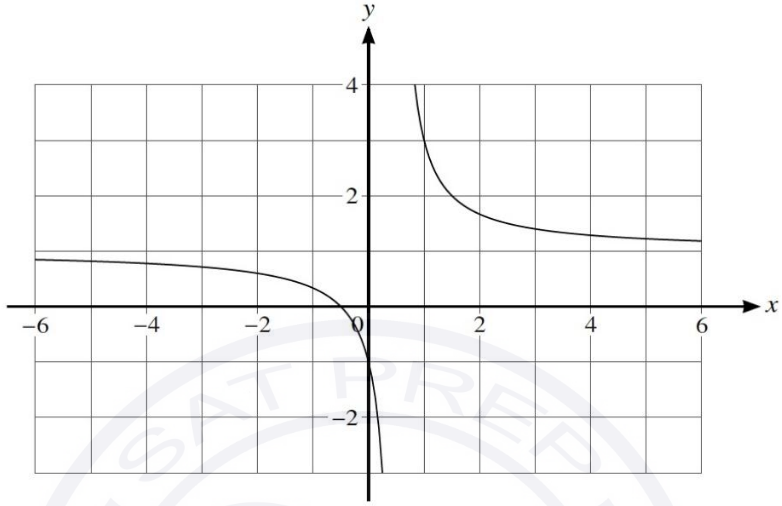
Question 84

Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \quad \text{for } x \neq \tfrac{1}{2},$$

$$g(x) = x^2 + 4 \quad \text{for } x \in \mathbb{R}.$$

(a)



The diagram shows part of the graph of $y = f(x)$.

State the domain of f^{-1}. [1]

(b) Find an expression for $f^{-1}(x)$. [3]

(c) Find $gf^{-1}(3)$. [2]

(d) Explain why $g^{-1}(x)$ cannot be found. [1]

(e) Show that $1 + \dfrac{2}{2x-1}$ can be expressed as $\dfrac{2x+1}{2x-1}$. Hence find the area of the triangle enclosed by the tangent to the curve $y = f(x)$ at the point where $x = 1$ and the x- and y-axes. [6]

Question 85

The function f is defined as follows:

$$f(x) = \frac{x^2-4}{x^2+4} \quad \text{for } x > 2.$$

(a) Find an expression for $f^{-1}(x)$. [3]

(b) Show that $1 - \dfrac{8}{x^2+4}$ can be expressed as $\dfrac{x^2-4}{x^2+4}$ and hence state the range of f. [4]

(c) Explain why the composite function ff cannot be formed. [1]

Question 86

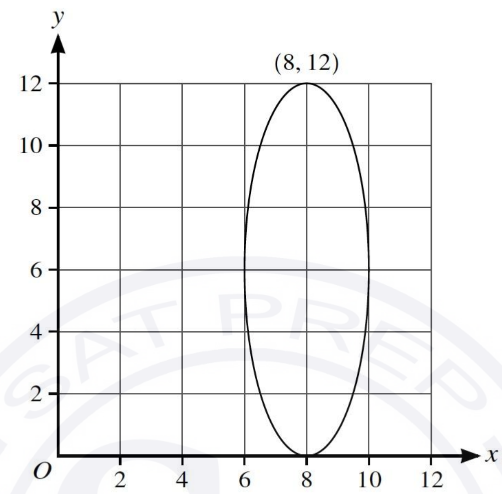
The function f is defined by $f(x) = -2x^2 - 8x - 13$ for $x < -3$.

(a) Express $f(x)$ in the form $-2(x + a)^2 + b$, where a and b are integers. [2]

(b) Find the range of f. [1]

(c) Find an expression for $f^{-1}(x)$. [3]

Question 87



The diagram shows a curve which has a maximum point at (8, 12) and a minimum point at (8, 0). The curve is the result of applying a combination of two transformations to a circle. The first transformation applied is a translation of $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$. The second transformation applied is a stretch in the y-direction.

(a) State the scale factor of the stretch. [1]

(b) State the radius of the original circle. [1]

(c) State the coordinates of the centre of the circle after the translation has been completed but before the stretch is applied. [2]

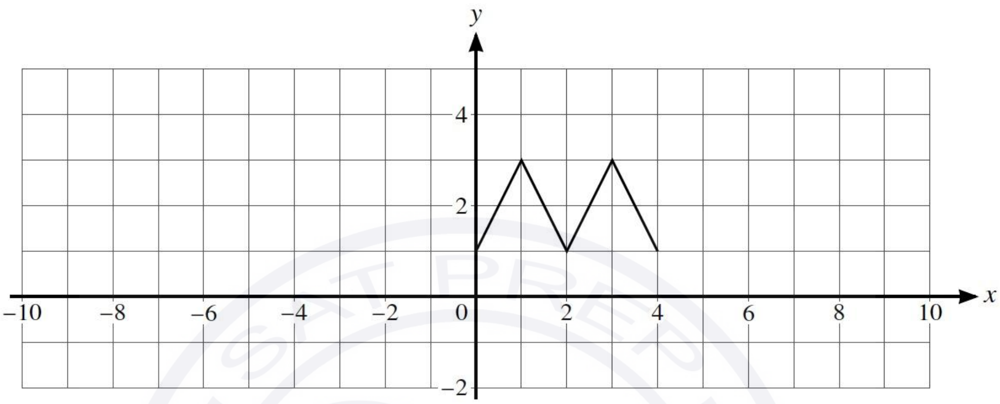
(d) State the coordinates of the centre of the original circle. [2]

Question 88

The graph with equation $y = \mathrm{f}(x)$ is transformed to the graph with equation $y = \mathrm{g}(x)$ by a stretch in the x-direction with factor 0.5, followed by a translation of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) The diagram below shows the graph of $y = \mathrm{f}(x)$.

On the diagram sketch the graph of $y = \mathrm{g}(x)$. [3]



(b) Find an expression for $\mathrm{g}(x)$ in terms of $\mathrm{f}(x)$. [2]

Question 89

Functions f and g are defined by

$$\mathrm{f}(x) = x + \frac{1}{x} \quad \text{for } x > 0,$$
$$\mathrm{g}(x) = ax + 1 \quad \text{for } x \in \mathbb{R},$$

where a is a constant.

(a) Find an expression for $\mathrm{gf}(x)$. [1]

(b) Given that $\mathrm{gf}(2) = 11$, find the value of a. [2]

(c) Given that the graph of $y = \mathrm{f}(x)$ has a minimum point when $x = 1$, explain whether or not f has an inverse. [1]

It is given instead that $a = 5$.

(d) Find and simplify an expression for $\mathrm{g}^{-1}\mathrm{f}(x)$. [3]

(e) Explain why the composite function fg cannot be formed. [1]

Question 90

Functions f and g are both defined for $x \in \mathbb{R}$ and are given by

$$f(x) = x^2 - 4x + 9,$$
$$g(x) = 2x^2 + 4x + 12.$$

(a) Express $f(x)$ in the form $(x - a)^2 + b$. [1]

(b) Express $g(x)$ in the form $2[(x + c)^2 + d]$. [2]

(c) Express $g(x)$ in the form $kf(x + h)$, where k and h are integers. [1]

(d) Describe fully the two transformations that have been combined to transform the graph of $y = f(x)$ to the graph of $y = g(x)$. [4]

Question 91

A function f is defined by $f(x) = x^2 - 2x + 5$ for $x \in \mathbb{R}$. A sequence of transformations is applied in the following order to the graph of $y = f(x)$ to give the graph of $y = g(x)$.

Stretch parallel to the x-axis with scale factor $\frac{1}{2}$

Reflection in the y-axis

Stretch parallel to the y-axis with scale factor 3

Find $g(x)$, giving your answer in the form $ax^2 + bx + c$, where a, b and c are constants. [4]

Question 92

The function f is defined by $f(x) = -3x^2 + 2$ for $x \leqslant -1$.

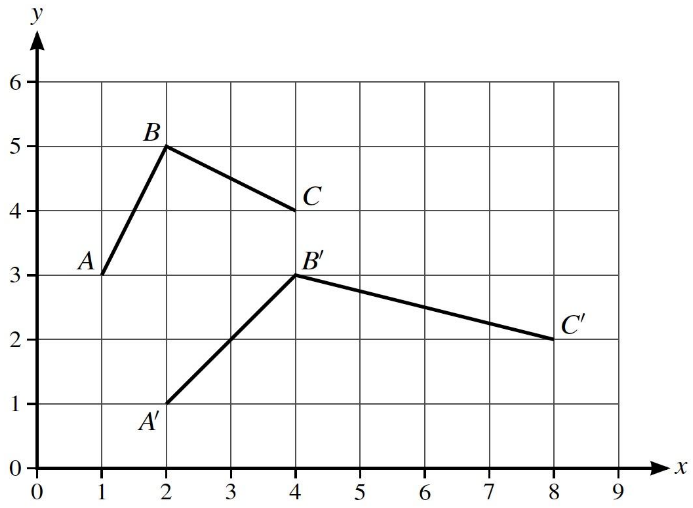
(a) State the range of f. [1]

(b) Find an expression for $f^{-1}(x)$. [3]

The function g is defined by $g(x) = -x^2 - 1$ for $x \leqslant -1$.

(c) Solve the equation $fg(x) - gf(x) + 8 = 0$. [5]

Question 93



The diagram shows the graph of $y = f(x)$, which consists of the two straight lines AB and BC. The lines $A'B'$ and $B'C'$ form the graph of $y = g(x)$, which is the result of applying a sequence of two transformations, in either order, to $y = f(x)$.

State fully the two transformations. [4]

Question 94

The function f is defined by $f(x) = 2 - \dfrac{5}{x+2}$ for $x > -2$.

(a) State the range of f. [1]

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

The function g is defined by $g(x) = x + 3$ for $x > 0$.

(c) Obtain an expression for $fg(x)$ giving your answer in the form $\dfrac{ax+b}{cx+d}$, where a, b, c and d are integers. [3]

Question 95

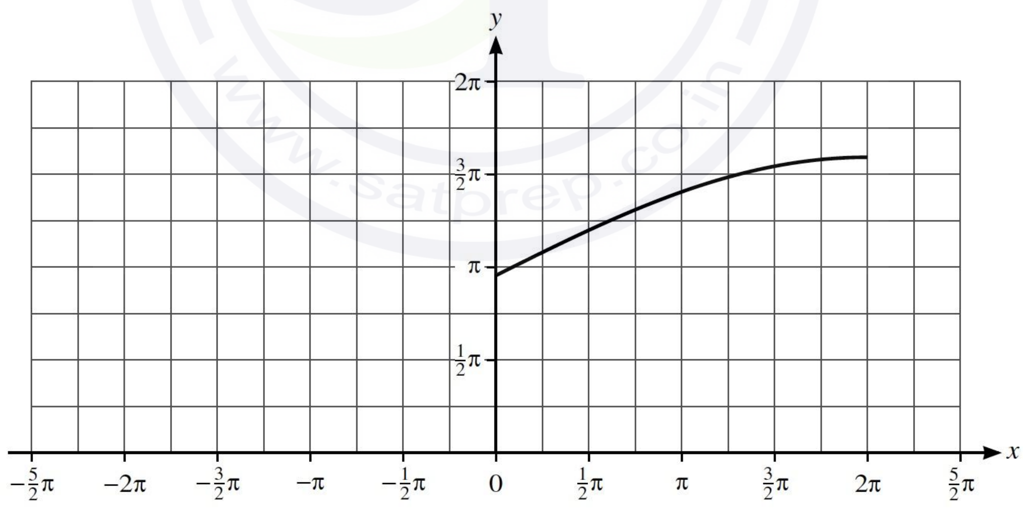
The diagram shows the graph of $y = f(x)$ where the function f is defined by

$$f(x) = 3 + 2\sin\tfrac{1}{4}x \text{ for } 0 \leqslant x \leqslant 2\pi.$$

(a) On the diagram above, sketch the graph of $y = f^{-1}(x)$. [2]

(b) Find an expression for $f^{-1}(x)$. [2]

(c)

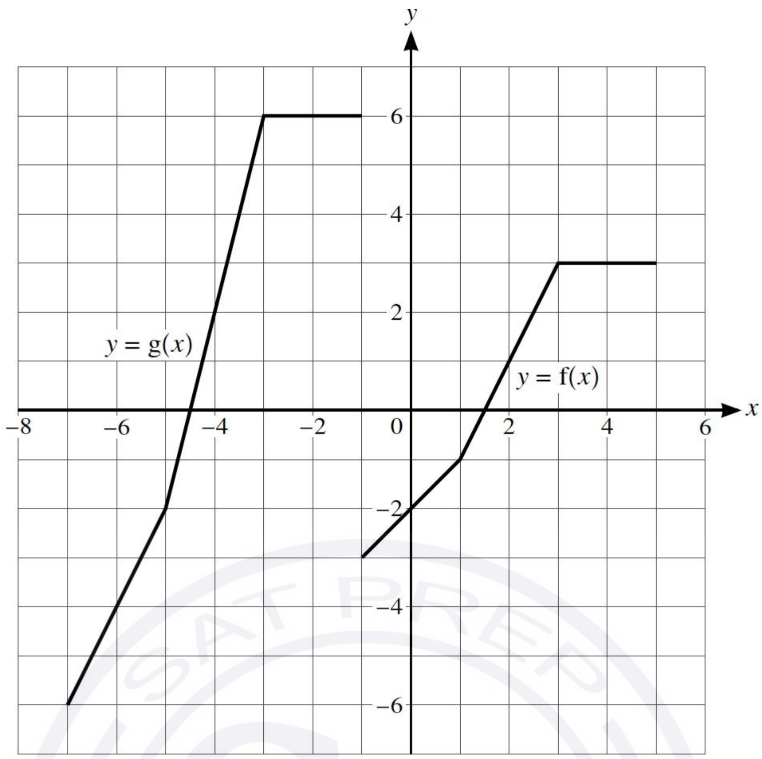


The diagram above shows part of the graph of the function $g(x) = 3 + 2\sin\frac{1}{4}x$ for $-2\pi \leqslant x \leqslant 2\pi$.

Complete the sketch of the graph of $g(x)$ on the diagram above and hence explain whether the function g has an inverse. [2]

(d) Describe fully a sequence of three transformations which can be combined to transform the graph of $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ to the graph of $y = f(x)$, making clear the order in which the transformations are applied. [6]

Question 96



The diagram shows graphs with equations $y = f(x)$ and $y = g(x)$.

Describe fully a sequence of two transformations which transforms the graph of $y = f(x)$ to $y = g(x)$. [4]

Question 97

A curve has equation $y = 2 + 3\sin\frac{1}{2}x$ for $0 \leqslant x \leqslant 4\pi$.

(a) State greatest and least values of y. [2]

(b) Sketch the curve. [2]

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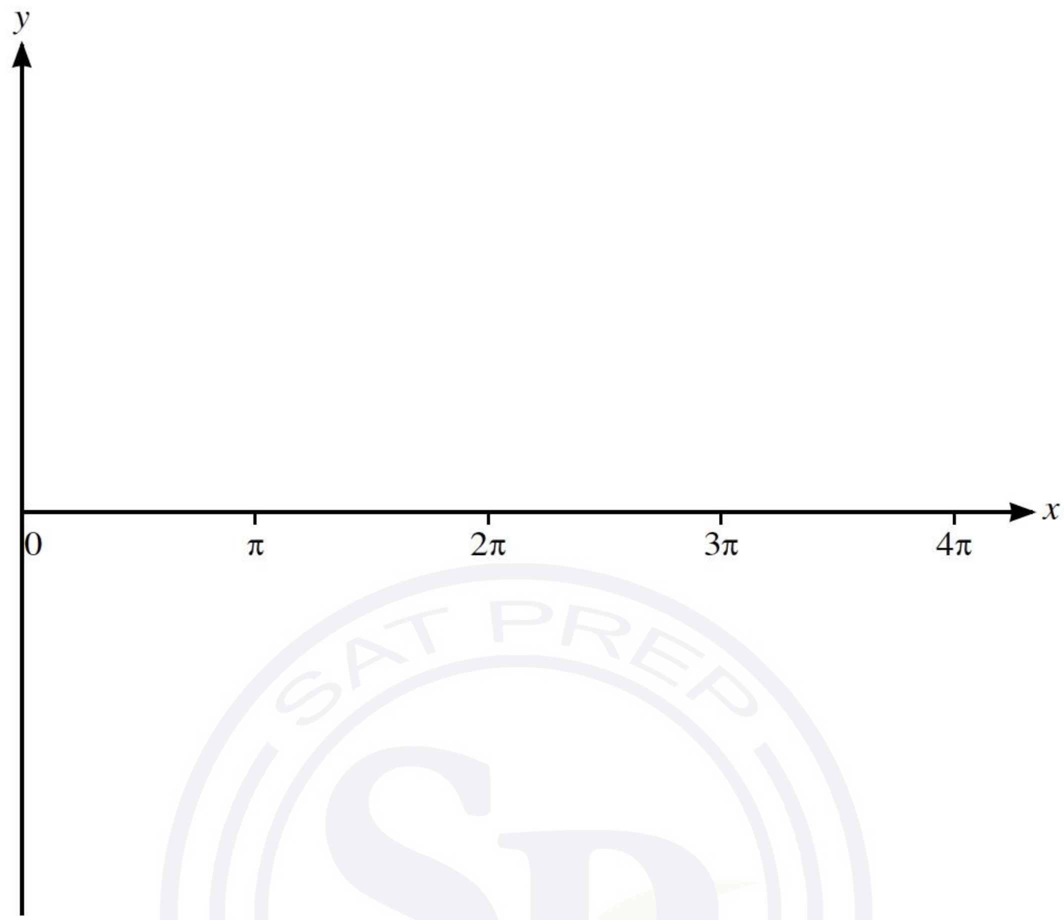

(c) State the number of solutions of the equation

$$2 + 3\sin\tfrac{1}{2}x = 5 - 2x$$

for $0 \leqslant x \leqslant 4\pi$. [1]

Question 98

The functions f and g are defined as follows, where a and b are constants.

$$f(x) = 1 + \frac{2a}{x-a} \text{ for } x > a$$

$$g(x) = bx - 2 \text{ for } x \in \mathbb{R}$$

(a) Given that $f(7) = \frac{5}{2}$ and $gf(5) = 4$, find the values of a and b. [4]

For the rest of this question, you should use the value of a which you found in **(a)**.

(b) Find the domain of f^{-1}. [1]

(c) Find an expression for $f^{-1}(x)$. [3]