SAT PREP
Vector Equation of Line

1 Write down a vector equation of the straight line

a parallel to the vector $(3\mathbf{i} - 2\mathbf{j})$ which passes through the point with position vector $(-\mathbf{i} + \mathbf{j})$,

b parallel to the x-axis which passes through the point with coordinates $(0, 4)$,

c parallel to the line $\mathbf{r} = 2\mathbf{i} + t(\mathbf{i} + 5\mathbf{j})$ which passes through the point with coordinates $(3, -1)$.

2 Find a vector equation of the straight line which passes through the points with position vectors

a $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$ **b** $\begin{pmatrix}-3\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ **c** $\begin{pmatrix}2\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\3\end{pmatrix}$

3 Find the value of the constant c such that line with vector equation $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \lambda(c\mathbf{i} + 2\mathbf{j})$

a passes through the point $(0, 5)$,

b is parallel to the line $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + \mu(6\mathbf{i} + 3\mathbf{j})$.

4 For each pair of lines, determine with reasons whether they are identical, parallel but not identical or not parallel.

a $\mathbf{r} = \begin{pmatrix}1\\2\end{pmatrix} + s\begin{pmatrix}3\\-1\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-2\\3\end{pmatrix} + t\begin{pmatrix}-6\\2\end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix}-1\\2\end{pmatrix} + s\begin{pmatrix}1\\4\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-2\\4\end{pmatrix} + t\begin{pmatrix}4\\1\end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix}2\\-5\end{pmatrix} + s\begin{pmatrix}2\\4\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-1\\1\end{pmatrix} + t\begin{pmatrix}3\\6\end{pmatrix}$

5 Find the position vector of the point of intersection of each pair of lines.

a $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{i}$
$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mu(3\mathbf{i} + \mathbf{j})$

b $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \lambda(-\mathbf{i} + \mathbf{j})$
$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + \mu(2\mathbf{i} - 3\mathbf{j})$

c $\mathbf{r} = \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j})$
$\mathbf{r} = 2\mathbf{i} + 10\mathbf{j} + \mu(-\mathbf{i} + 3\mathbf{j})$

d $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \lambda(-4\mathbf{i} + 6\mathbf{j})$
$\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + \mu(-\mathbf{i} + 2\mathbf{j})$

e $\mathbf{r} = -2\mathbf{i} + 11\mathbf{j} + \lambda(-3\mathbf{i} + 4\mathbf{j})$
$\mathbf{r} = -3\mathbf{i} - 7\mathbf{j} + \mu(5\mathbf{i} + 3\mathbf{j})$

f $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(3\mathbf{i} + 2\mathbf{j})$
$\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} + \mu(\mathbf{i} + 4\mathbf{j})$

6 Show that the lines with vector equations $\mathbf{r} = 4\mathbf{i} + 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} + t(-3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ intersect, and find the coordinates of their point of intersection.

7 Show that the lines with vector equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ are skew.

8 For each pair of lines, find the position vector of their point of intersection or, if they do not intersect, state whether they are parallel or skew.

a $\mathbf{r} = \begin{pmatrix}3\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\1\\-1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}3\\2\\-4\end{pmatrix} + \mu\begin{pmatrix}1\\0\\2\end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix}0\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-3\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}6\\-2\\-1\end{pmatrix} + \mu\begin{pmatrix}-4\\2\\6\end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix}8\\2\\-4\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}-2\\2\\8\end{pmatrix} + \mu\begin{pmatrix}4\\-3\\-4\end{pmatrix}$

d $\mathbf{r} = \begin{pmatrix}1\\5\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\-2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}7\\-6\\-5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-3\end{pmatrix}$

e $\mathbf{r} = \begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\-3\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}3\\-2\\1\end{pmatrix} + \mu\begin{pmatrix}5\\-3\\-4\end{pmatrix}$

f $\mathbf{r} = \begin{pmatrix}0\\7\\-2\end{pmatrix} + \lambda\begin{pmatrix}6\\-4\\8\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}-12\\-1\\11\end{pmatrix} + \mu\begin{pmatrix}5\\2\\-3\end{pmatrix}$

9 Write down a vector equation of the straight line

a parallel to the vector $(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ which passes through the point with position vector $(4\mathbf{i} + \mathbf{k})$,

b perpendicular to the xy-plane which passes through the point with coordinates (2, 1, 0),

c parallel to the line $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + t(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$ which passes through the point with coordinates (−1, 4, 2).

10 The points A and B have position vectors $(5\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and $(6\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ respectively.

a Find $\overrightarrow{AB}$ in terms of **i**, **j** and **k**.

b Write down a vector equation of the straight line l which passes through A and B.

c Show that l passes through the point with coordinates (3, 9, −8).

11 Find a vector equation of the straight line which passes through the points with position vectors

a $(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and $(5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})$

b $(3\mathbf{i} - 2\mathbf{k})$ and $(\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

c $\mathbf{0}$ and $(6\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

d $(-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and $(4\mathbf{i} - 7\mathbf{j} + \mathbf{k})$

12 Find the value of the constants a and b such that line $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + a\mathbf{j} + b\mathbf{k})$

a passes through the point (9, −2, −8),

b is parallel to the line $\mathbf{r} = 4\mathbf{j} - 2\mathbf{k} + \mu(8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$.

Answer

1 **a** $\mathbf{r} = -\mathbf{i} + \mathbf{j} + s(3\mathbf{i} - 2\mathbf{j})$

b $\mathbf{r} = 4\mathbf{j} + s\mathbf{i}$

c $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + s(\mathbf{i} + 5\mathbf{j})$

2 **a** $\text{dir}^{\text{n}} = \begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix}1\\0\end{pmatrix} + s\begin{pmatrix}2\\1\end{pmatrix}$

b $\text{dir}^{\text{n}} = \begin{pmatrix}-1\\1\end{pmatrix} - \begin{pmatrix}-3\\4\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix}-3\\4\end{pmatrix} + s\begin{pmatrix}2\\-3\end{pmatrix}$

c $\text{dir}^{\text{n}} = \begin{pmatrix}-2\\3\end{pmatrix} - \begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}-4\\5\end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix}2\\-2\end{pmatrix} + s\begin{pmatrix}-4\\5\end{pmatrix}$

3 **a** $-1 + 2\lambda = 5 \ \therefore\ \lambda = 3$

$3 + c\lambda = 3 + 3c = 0 \ \therefore\ c = -1$

b $c\mathbf{i} + 2\mathbf{j} = k(6\mathbf{i} + 3\mathbf{j})$

$\therefore\ k = \frac{2}{3}$

$\therefore\ c = 4$

4 **a** $\begin{pmatrix}3\\-1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}-6\\2\end{pmatrix}$

$\therefore$ parallel

(1, 2) lies on first line

when $x = 1$ on second line

$-2 - 6t = 1 \ \Rightarrow\ t = -\frac{1}{2}$

$\Rightarrow\ y = 3 + 2(-\frac{1}{2}) = 2$

parallel and common point

$\therefore$ identical

b $\begin{pmatrix}1\\4\end{pmatrix} \neq k\begin{pmatrix}4\\1\end{pmatrix}$

$\therefore$ not parallel

c $\begin{pmatrix}2\\4\end{pmatrix} = \frac{2}{3}\begin{pmatrix}3\\6\end{pmatrix}$

$\therefore$ parallel

(2, –5) lies on first line

when $x = 2$ on second line

$-1 + 3t = 2 \ \Rightarrow\ t = 1$

$\Rightarrow\ y = 1 + 6(1) = 7$

$\therefore$ (2, –5) not on second line

$\therefore$ parallel but not identical

5 **a** $1 + \lambda = 2 + 3\mu$ (1)

$2 = 1 + \mu$ (2)

(2) $\Rightarrow$ $\mu = 1$

$\therefore\ 5\mathbf{i} + 2\mathbf{j}$

b $4 - \lambda = 5 + 2\mu$ (1)

$1 + \lambda = -2 - 3\mu$ (2)

(1) + (2) $\Rightarrow$ $5 = 3 - \mu$

$\mu = -2$

$\therefore\ \mathbf{i} + 4\mathbf{j}$

c $2\lambda = 2 - \mu$ (1)

$1 - \lambda = 10 + 3\mu$ (2)

(1) + 2×(2) $\Rightarrow$ $2 = 22 + 5\mu$

$\mu = -4$

$\therefore\ 6\mathbf{i} - 2\mathbf{j}$

d $-1 - 4\lambda = 2 - \mu$ (1)

$5 + 6\lambda = -2 + 2\mu$ (2)

2×(1) + (2) $\Rightarrow$ $3 - 2\lambda = 2$

$\lambda = \frac{1}{2}$

$\therefore\ -3\mathbf{i} + 8\mathbf{j}$

e $-2 - 3\lambda = -3 + 5\mu$ (1)

$11 + 4\lambda = -7 + 3\mu$ (2)

4×(1) + 3×(2)

$\Rightarrow$ $25 = -33 + 29\mu$

$\mu = 2$

$\therefore\ 7\mathbf{i} - \mathbf{j}$

f $1 + 3\lambda = 3 + \mu$ (1)

$2 + 2\lambda = 5 + 4\mu$ (2)

2×(1) – 3×(2)

$\Rightarrow$ $-4 = -9 - 10\mu$

$\mu = -\frac{1}{2}$

$\therefore\ \frac{5}{2}\mathbf{i} + 3\mathbf{j}$

16 $4+s=7-3t$ (1)

$-2s=2+2t$ (2)

$3+2s=-5+t$ (3)

(2) + (3) $\Rightarrow$ $3=-3+3t$

$t=2,\ s=-3$

check (1) $4+(-3)=7-3(2)$

true $\therefore$ intersect

point of intersection: $(1, 6, -3)$

17 $2+\lambda=1+\mu$ (1)

$-1+\lambda=4-2\mu$ (2)

$4+3\lambda=3+\mu$ (3)

(1) − (2) $\Rightarrow$ $3=-3+3\mu$

$\mu=2,\ \lambda=1$

check (3) $4+3(1)=3+(2)$

false $\therefore$ do not intersect

$(\mathbf{i}+\mathbf{j}+3\mathbf{k}) \neq k(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ $\therefore$ not parallel

$\therefore$ skew

18 **a** $3+4\lambda=3+\mu$ (1)

$1+\lambda=2$ (2)

$5-\lambda=-4+2\mu$ (3)

(2) $\Rightarrow$ $\lambda=1$

sub. (1) $\mu=4$

check (3) $5-(1)=-4+2(4)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix}7\\2\\4\end{pmatrix}$

b $\begin{pmatrix}2\\-1\\-3\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}-4\\2\\6\end{pmatrix}$

$\therefore$ parallel

c $8+\lambda=-2+4\mu$ (1)

$2+3\lambda=2-3\mu$ (2)

$-4-2\lambda=8-4\mu$ (3)

(1) + (3) $\Rightarrow$ $4-\lambda=6$

$\lambda=-2,\ \mu=2$

check (2) $2+3(-2)=2-3(2)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix}6\\-4\\0\end{pmatrix}$

d $1+\lambda=7+2\mu$ (1)

$5+4\lambda=-6+\mu$ (2)

$2-2\lambda=-5-3\mu$ (3)

2×(1) + (3) $\Rightarrow$ $4=9+\mu$

$\mu=-5,\ \lambda=-4$

check (2) $5+4(-4)=-6+(-5)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix}-3\\-11\\10\end{pmatrix}$

e $4+2\lambda=3+5\mu$ (1)

$-1+5\lambda=-2-3\mu$ (2)

$3-3\lambda=1-4\mu$ (3)

3×(1) + 2×(3) $\Rightarrow$ $18=11+7\mu$

$\mu=1,\ \lambda=2$

check (2) $-1+5(2)=-2-3(1)$

false $\therefore$ do not intersect

$\begin{pmatrix}2\\5\\-3\end{pmatrix} \neq k\begin{pmatrix}5\\-3\\-4\end{pmatrix}$

$\therefore$ skew

f $6\lambda=-12+5\mu$ (1)

$7-4\lambda=-1+2\mu$ (2)

$-2+8\lambda=11-3\mu$ (3)

2×(2) + (3) $\Rightarrow$ $12=9+\mu$

$\mu=3,\ \lambda=\frac{1}{2}$

check (1) $6(\frac{1}{2})=-12+5(3)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix}3\\5\\2\end{pmatrix}$

9 **a** $\mathbf{r} = 4\mathbf{i} + \mathbf{k} + s(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

b $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + s\mathbf{k}$

c $\mathbf{r} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + s(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$

10 **a** $\overrightarrow{AB} = (6\mathbf{i} - 3\mathbf{j} + \mathbf{k}) - (5\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

b $\mathbf{r} = (5\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + s(\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$

c $5 + s = 3 \Rightarrow s = -2$
when $s = -2$, $\mathbf{r} = (5\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = 3\mathbf{i} + 9\mathbf{j} - 8\mathbf{k}$
$\therefore$ l passes through (3, 9, −8)

11 **a** direction $= (5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$
$= 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\therefore\ \mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + s(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b direction $= (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 2\mathbf{k})$
$= -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$
$\therefore\ \mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + s(-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

c $\mathbf{r} = s(6\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

d direction $= (4\mathbf{i} - 7\mathbf{j} + \mathbf{k}) - (-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$
$= 5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$
$\therefore\ \mathbf{r} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + s(5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$

12 **a** $3 + 2\lambda = 9\ \therefore\ \lambda = 3$
$-5 + a\lambda = -5 + 3a = -2\ \therefore\ a = 1$
$1 + b\lambda = 1 + 3b = -8\ \therefore\ b = -3$

b $2\mathbf{i} + a\mathbf{j} + b\mathbf{k} = k(8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$
$\therefore\ k = \frac{1}{4}$
$\therefore\ a = -1,\ b = \frac{1}{2}$