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Prove each of the following identities.

1. $\tan x \sin x + \cos x = \sec x$

2. $\dfrac{1}{\tan x} + \tan x = \dfrac{1}{\sin x \cos x}$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

4. $\dfrac{\cos\alpha}{1+\sin\alpha} + \dfrac{1+\sin\alpha}{\cos\alpha} = 2\sec\alpha$

5. $\dfrac{\cos x}{1-\sin x} - \dfrac{\cos x}{1+\sin x} = 2\tan x$

6. $\cos^2 x = \dfrac{\csc x \cos x}{\tan x + \cot x}$

7. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$

8. $\dfrac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$

9. $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$

10. $1 - 2\cos^2 x = \dfrac{\tan^2 x - 1}{\tan^2 x + 1}$

11. $\tan^2\theta = \csc^2\theta \tan^2\theta - 1$

12. $\sec x + \tan x = \dfrac{\cos x}{1-\sin x}$

13. $\dfrac{\csc\beta}{\sin\beta} - \dfrac{\cot\beta}{\tan\beta} = 1$

14. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

15. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$

16. $\dfrac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \dfrac{3+\sin x}{1-\sin x}$

17. $\dfrac{\cos x}{1-\sin x} - \tan x = \sec x$

18. $\tan^2 x + 1 + \tan x \sec x = \dfrac{1+\sin x}{\cos^2 x}$

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1. $\tan x \sin x + \cos x = \sec x$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\begin{aligned} \text{LHS} &= \tan x \sin x + \cos x = \frac{\sin x}{\cos x} \cdot \sin x + \cos x = \frac{\sin^2 x}{\cos x} + \cos x = \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} = \text{RHS} \end{aligned}$$

2. $\dfrac{1}{\tan x} + \tan x = \dfrac{1}{\sin x \cos x}$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\text{LHS} = \frac{1}{\tan x} + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \text{RHS}$$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\text{LHS} = \sin x - \sin x \cos^2 x = \sin x \left(1 - \cos^2 x\right) = \sin x \cdot \sin^2 x = \text{RHS}$$

4. $\dfrac{\cos\alpha}{1+\sin\alpha} + \dfrac{1+\sin\alpha}{\cos\alpha} = 2\sec\alpha$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\begin{aligned} \text{LHS} &= \frac{\cos\alpha}{1+\sin\alpha} + \frac{1+\sin\alpha}{\cos\alpha} = \frac{\cos^2\alpha}{(1+\sin\alpha)\cos\alpha} + \frac{(1+\sin\alpha)^2}{(1+\sin\alpha)\cos\alpha} = \frac{\cos^2\alpha + (1+\sin\alpha)^2}{(1+\sin\alpha)\cos\alpha} \\ &= \frac{\cos^2\alpha + 1 + 2\sin\alpha + \sin^2\alpha}{(1+\sin\alpha)\cos\alpha} = \frac{\cos^2\alpha + \sin^2\alpha + 1 + 2\sin\alpha}{(1+\sin\alpha)\cos\alpha} = \frac{2+2\sin\alpha}{(1+\sin\alpha)\cos\alpha} \\ &= \frac{2(1+\sin\alpha)}{(1+\sin\alpha)\cos\alpha} = \frac{2}{\cos\alpha} = 2 \cdot \frac{1}{\cos\alpha} = 2\sec\alpha = \text{RHS} \end{aligned}$$

5. $\dfrac{\cos x}{1-\sin x} - \dfrac{\cos x}{1+\sin x} = 2\tan x$

Solution: We will start with the left-hand side. First we bring the fractions to the common denominator. Recall that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\begin{aligned} \text{LHS} &= \frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x} = \frac{\cos x\,(1+\sin x)}{(1-\sin x)(1+\sin x)} - \frac{\cos x\,(1-\sin x)}{(1-\sin x)(1+\sin x)} \\ &= \frac{\cos x\,(1+\sin x) - \cos x\,(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{\cos x + \cos x \sin x - \cos x + \cos x \sin x}{1-\sin^2 x} = \frac{2\sin x \cos x}{\cos^2 x} \\ &= \frac{2\sin x}{\cos x} = 2\tan x = \text{RHS} \end{aligned}$$

6. $\cos^2 x = \dfrac{\csc x \cos x}{\tan x + \cot x}$

Solution: We will start with the right-hand side. We will re-write everything in terms of $\sin x$ and $\cos x$ and simplify. We will again run into the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}\text{RHS} &= \frac{\csc x \cos x}{\tan x + \cot x} = \frac{\frac{1}{\sin x}\cdot \cos x}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{\frac{1}{\sin x}\cdot\frac{\cos x}{1}}{\frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}} = \frac{\frac{\cos x}{\sin x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x \cos x}} \\ &= \frac{\cos x}{\sin x}\cdot\frac{\cos x \sin x}{1} = \frac{\cos^2 x}{1} = \cos^2 x = \text{LHS}\end{aligned}$$

7. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$

Solution: We can factor the numerator via the difference of squares theorem.

$$\begin{aligned}\text{LHS} &= \frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = \frac{\left(\sin^2 x\right)^2 - \left(\cos^2 x\right)^2}{\sin^2 x - \cos^2 x} = \frac{\left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)}{\sin^2 x - \cos^2 x} \\ &= \sin^2 x + \cos^2 x = 1 = \text{RHS}\end{aligned}$$

8. $\dfrac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$

Solution:

$$\begin{aligned}\text{LHS} &= \frac{\tan^2 x}{\tan^2 x + 1} = \frac{\left(\frac{\sin x}{\cos x}\right)^2}{\left(\frac{\sin x}{\cos x}\right)^2 + 1} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + 1} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} \\ &= \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{\sin^2 x}{\cos^2 x}\cdot\frac{\cos^2 x}{1} = \sin^2 x = \text{RHS}\end{aligned}$$

9. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

Solution:

$$\begin{aligned}\text{LHS} &= \frac{1 - \sin x}{\cos x} = \frac{1 - \sin x}{\cos x}\cdot 1 = \frac{1 - \sin x}{\cos x}\cdot\frac{1 + \sin x}{1 + \sin x} = \frac{(1 - \sin x)(1 + \sin x)}{\cos x\,(1 + \sin x)} = \frac{1 - \sin^2 x}{\cos x\,(1 + \sin x)} \\ &= \frac{\cos^2 x}{\cos x\,(1 + \sin x)} = \frac{\cos x}{1 + \sin x} = \text{RHS}\end{aligned}$$

10. $1-2\cos^2 x = \dfrac{\tan^2 x - 1}{\tan^2 x + 1}$

Solution:

$$\begin{aligned}
\text{RHS} &= \frac{\tan^2 x - 1}{\tan^2 x + 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x} - 1}{\dfrac{\sin^2 x}{\cos^2 x} + 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x} - \dfrac{\cos^2 x}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x}} = \frac{\dfrac{\sin^2 x - \cos^2 x}{\cos^2 x}}{\dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}} \\
&= \frac{\sin^2 x - \cos^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{1} = \sin^2 x - \cos^2 x \\
&= \left(1 - \cos^2 x\right) - \cos^2 x = 1 - 2\cos^2 x = \text{LHS}
\end{aligned}$$

11. $\tan^2\theta = \csc^2\theta\tan^2\theta - 1$

$$\begin{aligned}
\text{RHS} &= \csc^2\theta\tan^2\theta - 1 = \frac{1}{\sin^2\theta} \cdot \left(\frac{\sin\theta}{\cos\theta}\right)^2 - 1 = \frac{1}{\sin^2\theta} \cdot \frac{\sin^2\theta}{\cos^2\theta} - 1 = \frac{1}{\cos^2\theta} - 1 \\
&= \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta = \text{LHS}
\end{aligned}$$

12. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Solution:

$$\begin{aligned}
\text{RHS} &= \frac{\cos x}{1-\sin x} = \frac{\cos x}{1-\sin x} \cdot 1 = \frac{\cos x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} = \frac{\cos x\,(1+\sin x)}{(1-\sin x)\,(1+\sin x)} \\
&= \frac{\cos x\,(1+\sin x)}{1-\sin^2 x} = \frac{\cos x\,(1+\sin x)}{\cos^2 x} = \frac{1+\sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \text{LHS}
\end{aligned}$$

13. $\dfrac{\csc\beta}{\sin\beta} - \dfrac{\cot\beta}{\tan\beta} = 1$

Solution: We will start with the left-hand side. We will re-write everything in terms of $\sin\beta$ and $\cos\beta$ and simplify. We will again run into the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$ for all angles x.

$$\begin{aligned}
\text{LHS} &= \frac{\csc\beta}{\sin\beta} - \frac{\cot\beta}{\tan\beta} = \frac{\dfrac{1}{\sin\beta}}{\dfrac{\sin\beta}{1}} - \frac{\dfrac{\cos\beta}{\sin\beta}}{\dfrac{\sin\beta}{\cos\beta}} = \frac{1}{\sin\beta} \cdot \frac{1}{\sin\beta} - \frac{\cos\beta}{\sin\beta} \cdot \frac{\cos\beta}{\sin\beta} = \frac{1}{\sin^2\beta} - \frac{\cos^2\beta}{\sin^2\beta} \\
&= \frac{1-\cos^2\beta}{\sin^2\beta} = \frac{\left(\sin^2\beta + \cos^2\beta\right) - \cos^2\beta}{\sin^2\beta} = \frac{\sin^2\beta}{\sin^2\beta} = 1 = \text{RHS}
\end{aligned}$$

14. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

Solution:

$$\begin{aligned}
\text{LHS} &= \sin^4 x - \cos^4 x = \left(\sin^2 x\right)^2 - \left(\cos^2 x\right)^2 = \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right) \\
&= 1 \cdot \left(\sin^2 x - \cos^2 x\right) = \left(1 - \cos^2 x\right) - \cos^2 x = 1 - 2\cos^2 x = \text{RHS}
\end{aligned}$$

15. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$

Solution:

$$\begin{aligned} \text{LHS} &= (\sin x - \cos x)^2 + (\sin x + \cos x)^2 \\ &= \left(\sin^2 x + \cos^2 x - 2\sin x \cos x\right) + \left(\sin^2 x + \cos^2 x + 2\sin x \cos x\right) = 2\sin^2 x + 2\cos^2 x \\ &= 2\left(\sin^2 x + \cos^2 x\right) = 2 \cdot 1 = 2 = \text{RHS} \end{aligned}$$

16. $\dfrac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \dfrac{3 + \sin x}{1 - \sin x}$

Solution:

$$\text{LHS} = \frac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \frac{(\sin x + 1)(\sin x + 3)}{1 - \sin^2 x} = \frac{(\sin x + 1)(\sin x + 3)}{(1 + \sin x)(1 - \sin x)} = \frac{\sin x + 3}{1 - \sin x} = \text{RHS}$$

17. $\dfrac{\cos x}{1 - \sin x} - \tan x = \sec x$

Solution:

$$\begin{aligned} \text{LHS} &= \frac{\cos x}{1 - \sin x} - \tan x = \frac{\cos x}{1 - \sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin x\,(1 - \sin x)}{\cos x\,(1 - \sin x)} = \frac{\cos^2 x - \sin x + \sin^2 x}{\cos x\,(1 - \sin x)} \\ &= \frac{\left(\cos^2 x + \sin^2 x\right) - \sin x}{\cos x\,(1 - \sin x)} = \frac{1 - \sin x}{\cos x\,(1 - \sin x)} = \frac{1}{\cos x} = \text{RHS} \end{aligned}$$

18. $\tan^2 x + 1 + \tan x \sec x = \dfrac{1 + \sin x}{\cos^2 x}$

Solution:

$$\begin{aligned} \text{LHS} &= \tan^2 x + 1 + \tan x \sec x = \frac{\sin^2 x}{\cos^2 x} + 1 + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \\ &= \frac{\sin^2 x + \cos^2 x + \sin x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x} = \text{RHS} \end{aligned}$$

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