Permutations & Combinations

Multiplication Rule

If one event can occur in **m** ways, a second event in **n** ways and a third event in **r**, then the three events can occur in **m × n × r** ways.

Example Erin has 5 tops, 6 skirts and 4 caps from which to choose an outfit.
In how many ways can she select one top, one skirt and one cap?

Solution: **Ways = 5 × 6 × 4**

Repetition of an Event

If one event with **n** outcomes occurs **r** times with repetition allowed, then the number of ordered arrangements is n^r

Example 1 What is the number of arrangements if a die is rolled

(a) 2 times ? $6 \times 6 = 6^2$

(b) 3 times ? $6 \times 6 \times 6 = 6^3$

(b) r times ? $6 \times 6 \times 6 \times \ldots\ldots = 6^r$

Repetition of an Event

Example 2

(a) How many different car number plates are possible with 3 letters followed by 3 digits?

Solution: $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3$

(b) How many of these number plates begin with ABC ?

Solution: $1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3$

(c) If a plate is chosen at random, what is the probability that it begins with ABC?

Solution: $$\frac{10^3}{26^3 \times 10^3} = \frac{1}{26^3}$$

Factorial Representation

$n! = n(n - 1)(n - 2)\ldots\ldots\ldots\ldots 3 \times 2 \times 1$

For example $5! = 5.4.3.2.1$ **Note** $0! = 1$

Example

a) In how many ways can 6 people be arranged in a row?

Solution : $6.5.4.3.2.1 = 6!$

b) How many arrangements are possible if only 3 of them are chosen?

Solution: $6.5.4 = 120$

Arrangements or Permutations

Distinctly ordered sets are called **arrangements** or **permutations**.

The number of permutations of **n** objects taken **r** at a time is given by:

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}}$$

where n = number of objects
r = number of positions

Arrangements or Permutations

Eg 1. A maths debating team consists of 4 speakers.

a) In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : $4.3.2.1 = 4!$ or ${}^4P_4$

b) How many ways can the captain and vice-captain be chosen?

Solution : $4.3 = 12$ or ${}^4P_2$

Arrangements or Permutations

Eg 2. A flutter on the horses

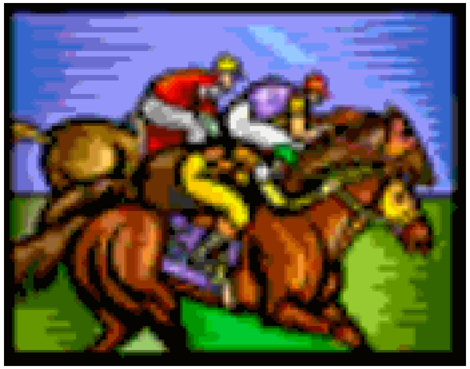
There are 7 horses in a race.

a) In how many different orders can the horses finish?

Solution : $7.6.5.4.3.2.1 = 7!$ or ${}^{7}P_{7}$

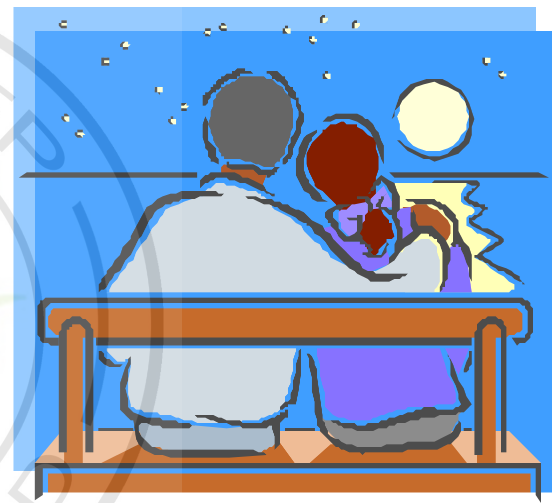
b) How many trifectas (1^{st}, 2^{nd} and 3^{rd}) are possible?

Solution : $7.6.5 = 210$ or ${}^{7}P_{3}$



Permutations with Restrictions

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if



a) there are no restrictions?

Solution : $9!$ or ${}^9P_9$

c) boys and girls alternate?

Solution : A boy will be on each end

$$BGBGBGBGB = 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$$

$$= 5! \times 4! \text{ or } {}^5P_5 \times {}^4P_4$$

Permutations with Restrictions

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if

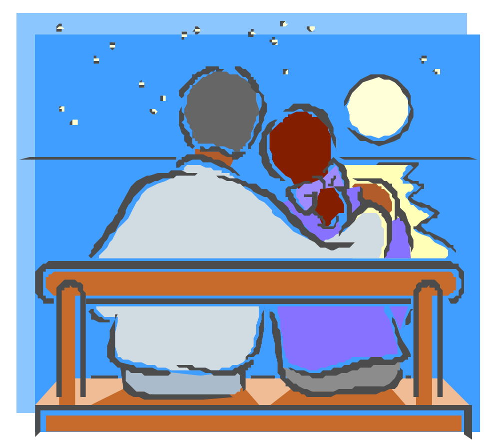
c) boys and girls are in separate groups?

Solution : **Boys & Girls or Girls & Boys**

$$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$$

$$\text{or } {}^{5}P_{5} \times {}^{4}P_{4} \times 2$$

d) Anne and Jim wish to stay together?



Solution : **(AJ) _ _ _ _ _ _ _**

$$= 2 \times 8! \text{ or } 2 \times {}^{8}P_{8}$$

Arrangements with Repetitions

If we have **n** elements of which x are alike of one kind, y are alike of another kind, z are alike of another kind,
………… then the number of ordered selections or permutations is given by:

$$\boxed{\frac{n!}{x!\,y!\,z!}}$$

Arrangements with Repetitions

Eg.1 How many different arrangements of the word **PARRAMATTA** are possible?

Solution :

10 letters but note repetition (4 A's, 2 R's, 2 T's)

P

A A A A

R R

M

T T

No. of arrangements = $\frac{10!}{4!\,2!\,2!}$

$= 37\ 800$

Arrangements with Restrictions

Eg 1. How many arrangements of the letters of the word REMAND are possible if:

a) there are no restrictions?

Solution : $^6P_6 = 720$ or $6!$

b) they begin with RE?

Solution : $\underline{R}\ \underline{E}\ \_\ \_\ \_\ \_ = {}^4P_4 = 24$ or $4!$

c) they do not begin with RE?

Solution : Total – (b) = $6! - 4! = 696$

Arrangements with Restrictions

Eg 1. How many arrangements of the letters of the word REMAND are possible if:

d) they have RE together in order?

Solution : (RE) _ _ _ _ $= {}^{5}P_{5} = 120$ or $5!$

e) they have REM together in any order?

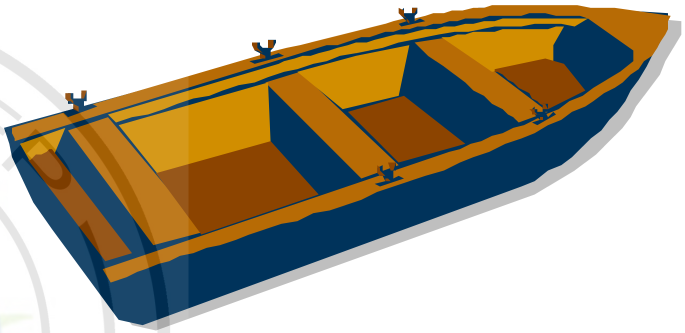
Solution : (REM) _ _ _ $= {}^{3}P_{3} \times {}^{4}P_{4} = 144$

f) R, E and M are not to be together?

Solution : Total – (e) $= 6! - 144 = 576$

Arrangements with Restrictions

Eg 2. There are 6 boys who enter a boat with 8 seats, 4 on each side. In how many ways can



a) they sit anywhere?

Solution : $^8P_6$

b) two boys A and B sit on the port side and another boy W sit on the starboard side?

Solution : $A \,\&\, B = {}^4P_2$

$W = {}^4P_1$

$\text{Others} = {}^5P_3$

$\text{Total} = {}^4P_2 \times {}^4P_1 \times {}^5P_3$

Arrangements with Restrictions

Eg 3. From the digits 2, 3, 4, 5, 6

a) how many numbers greater than 4 000 can be formed?

Solution : **5 digits (any) = ${}^5P_5$**

4 digits (must start with digit $\geq$ 4) = ${}^3P_1 \times {}^4P_3$

Total = ${}^5P_5 + {}^3P_1 \times {}^4P_3$

b) how many 4 digit numbers would be even?

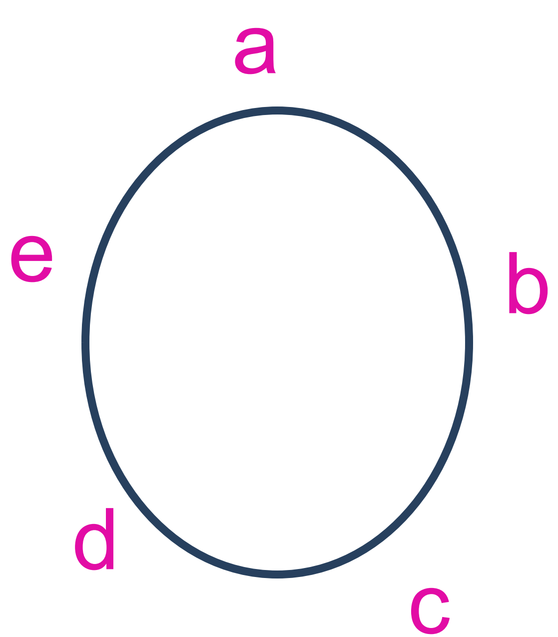
Even (ends with 2, 4 or 6) = $\_\ \_\ \_\ {}^3P_1$

$= {}^4P_3 \times {}^3P_1$

Circular Arrangements

Circular arrangements are permutations in which objects are arranged in a circle.

Consider arranging 5 objects (a, b, c, d, e) around a circular table. The arrangements



abcde

bcdea

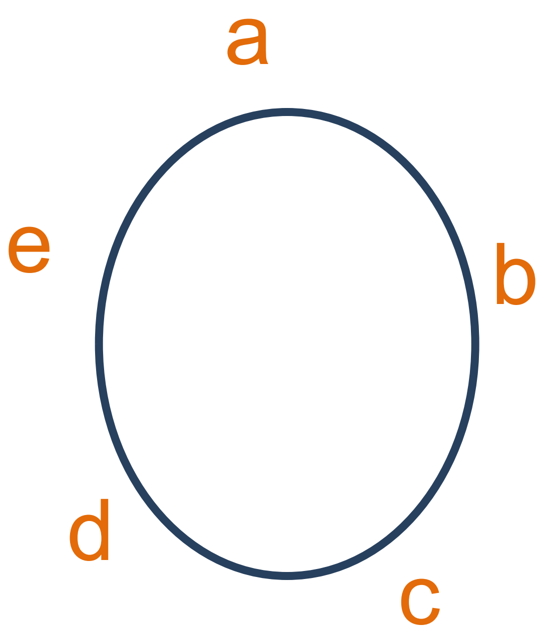
cdeab

deabc

eabcd

are different in a line, but are **identical** around a circle.

Circular Arrangements



To calculate the number of ways in which n objects can be arranged in a circle, we arbitrarily fix the position of one object, so the remaining (n-1) objects can be arranged as if they were on a straight line in (n-1)! ways.

i.e. the number of arrangements = (n – 1) !
in a circle

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

a) there are no restrictions

Solution :

$(12 - 1)! = 11!$

b) men and women alternate

Solution : $(6 - 1)! \times 6! = 5! \times 6!$

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

c) Ted and Carol must sit together

Solution : **(TC) & other 10 = $2! \times 10!$**

d) Bob, Ted and Carol must sit together

Solution : **(BTC) & other 9 = $3! \times 9!$**

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

d) Neither Bob nor Carol can sit next to Ted.

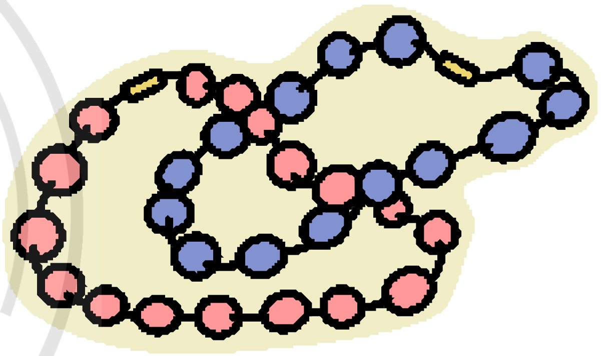
Solution : Seat 2 of the other 9 people next to Ted in (9×8) ways or $^{9}P_{2}$

Then sit the remaining 9 people (including Bob and Carol) in $9!$ ways

Ways $= (9 \times 8) \times 9!$ or $^{9}P_{2} \times 9!$

Circular Arrangements

Eg 2. In how many ways can 8 differently coloured beads be threaded on a string?



Solution :

As necklace can be turned over, clockwise and anti-clockwise arrangements are the same

$= (8-1)! \div 2 \;\; = 7! \div 2$

Unordered Selections

The number of different **combinations** (i.e. unordered sets) of **r** objects from **n** distinct objects is represented by :

$$\text{No. of Combinations} = \frac{\text{number of permutations}}{\text{arrangements of r objects}}$$

and is denoted by

$$\boxed{{}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}}$$

Combinations

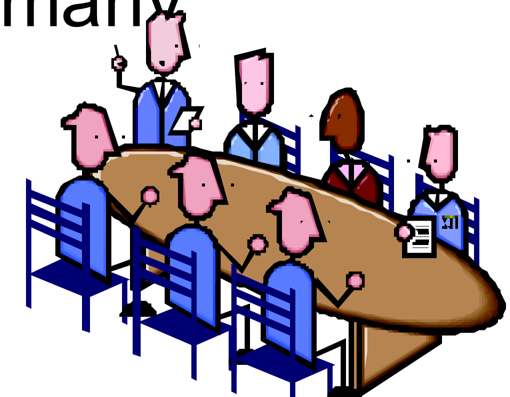
Eg 1. How many ways can a basketball team of 5 players be chosen from 8 players?

Solution :

$^{8}C_{5}$

Combinations

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if



a) there are no restrictions?

Solution : $^{10}C_5$

b) one particular person must be chosen on the committee?

Solution : $\underline{1} \times {}^{9}C_4$

c) one particular woman must be excluded from the committee?

Solution : $^{9}C_5$

Combinations

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if:

d) there are to be 3 men and 2 women?

Solution : **Men & Women = $^6C_3 \times {}^4C_2$**

e) there are to be men only?

Solution : $^6C_5$

f) there is to be a majority of women?

Solution :

3 Women & 2 men Or 4 Women & 1 man

$$= {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$$

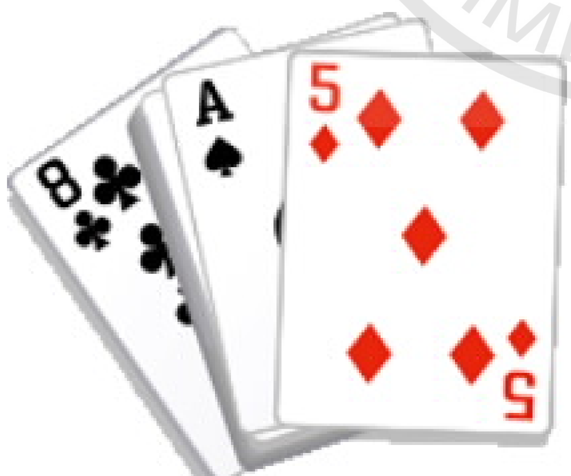
Combinations

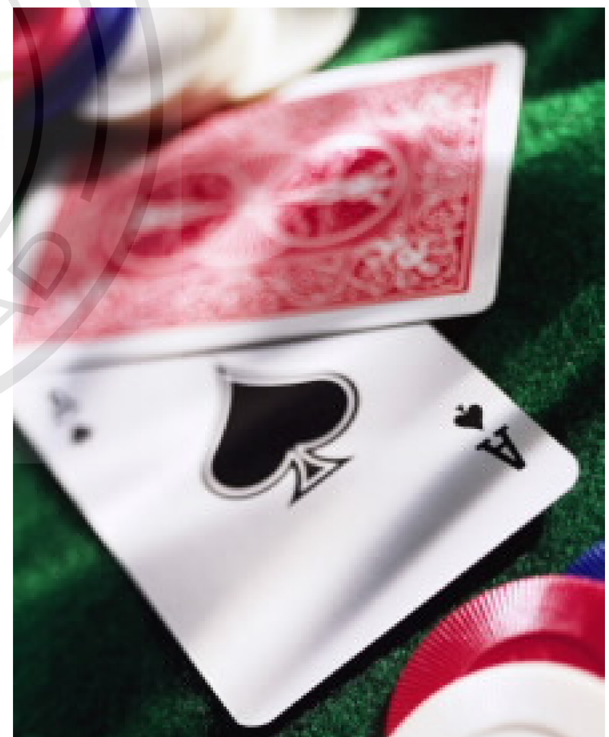
Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

(i) What is the total possible number of hands if there are no restrictions?

Solution :

$^{52}C_5$





Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

a) 4 Kings?

Solution : $^{4}C_{4} \times {}^{48}C_{1}$ or 1×48

b) 2 Clubs and 3 Hearts?

Solution : $^{13}C_{2} \times {}^{13}C_{3}$

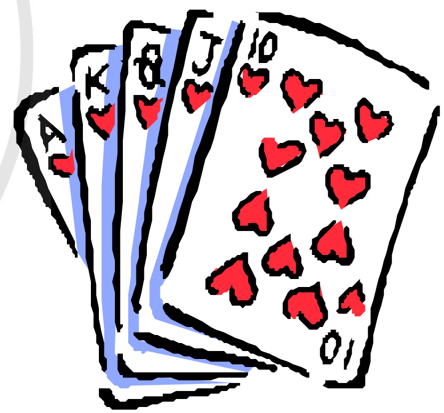
Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

c) all Hearts?

Solution : $^{13}C_5$



d) all the same colour?

Solution : **Red or Black =** $^{26}C_5 + {}^{26}C_5 = 2 \times {}^{26}C_5$

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

e) four of the same kind?

Solution :

$^{4}C_{4} \times {}^{48}C_{1} \times 13 = 1 \times 48 \times 13$

f) 3 Aces and two Kings?

Solution : $^{4}C_{3} \times {}^{4}C_{2}$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf:

a) If there are no restrictions?

Solution : $^{6}C_{4} \times {}^{5}C_{3} \times 7!$

c) If the 4 Maths books remain together?

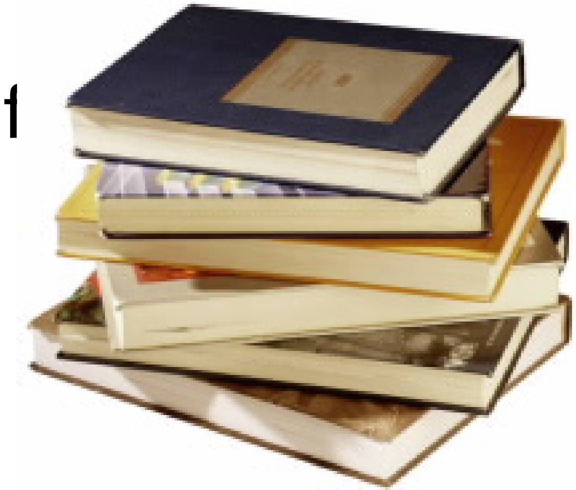
Solution : = (MMMM) _ _ _

$= {}^{6}P_{4} \times {}^{5}C_{3} \times 4!$ or $({}^{6}C_{4} \times 4!) \times {}^{5}C_{3} \times 4!$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

c) a Maths book is at the beginning of the shelf?



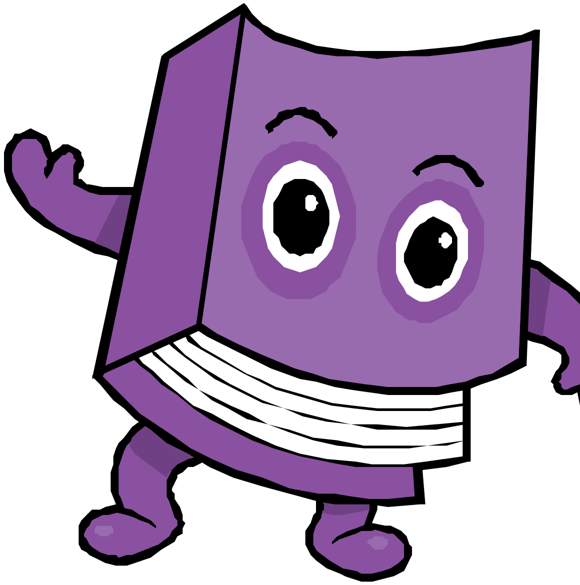
Solution : $= \text{M}\_\ \_\ \_\ \_\ \_\ \_$

$$= 6 \times {}^5C_3 \times {}^5C_3 \times 6!$$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

d) Maths and English books alternate



Solution : = M E M E M E M

$= {}^{6}P_{4} \times {}^{5}P_{3}$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

e) A Maths is at the beginning and an English book is in the middle of the shelf.

Solution : M _ _ E _ _ _

$$= 6 \times 5 \times {}^5C_3 \times {}^4C_2 \times 5!$$

Further Permutations and Combinations

Eg 2. (i) How many different 8 letter words are possible using the letters of the word SYLLABUS ?

Solution : **2 S's & 2 L's**

$$\text{Words} = \frac{8!}{2! \times 2!}$$

$$= 10\,080$$

Further Permutations and Combinations

SYLLABUS **= 10 080 permutations**

(ii) If a word is chosen at random, find the probability that the word:

a) contains the two S's together

Solution : (SS) _ _ _ _ _ _ (Two L's)

Words = $\frac{7!}{2!} = 2520$ **Prob =** $\frac{2520}{10080} = \frac{1}{4}$

b) begins and ends with L

Solution : L _ _ _ _ _ _ L (Two S's)

Words = $\frac{6!}{2!} = 360$ **Prob =** $\frac{360}{10080} = \frac{1}{28}$