8823 – 7111M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

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Mathematics: analysis and approaches

Higher level

Paper 1

28 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

2. (a) $(\mathrm{P}(A \cup B) =) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

$= 0.8$ ***A1***

[2 marks]

(b) recognition that $A' \cap B' = (A \cup B)'$ OR $A' \cap B' = 1 - A \cup B$
(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$(= 1 - 0.8)$

$= 0.2$ ***A1***

Note: For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}(A' \cap B')$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3, \ q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1, \ d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =)65 - 40$ OR $(u_5 =)1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

4. **METHOD 1**

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2-1^2}=\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha+\sin^2\alpha=1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C}=1-\left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C}=\frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $=\frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$=\frac{1}{2}\times 10\times\sqrt{6}\times\frac{\sqrt{24}}{5}$ ***(A1)***

$=12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \text{B}\hat{\text{A}}\text{C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use 'Area $= \frac{1}{2} \text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm^2) ***A1***

[6 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1 kx + {}^nC_2 k^2x^2 + \ldots$ OR ${}^nC_1 k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve ***(M1)***

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

> **Note:** The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=(5^2)5^{2k}-(2^3)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8(5^{2k}-2^{3k})$ OR $25(5^{2k}-2^{3k})+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17(5^{2k}+8s)$ OR $17(25s+2^{3k})$ which is divisible by 17 ***A1***

OR

$(5^{2k}-2^{3k})\times 5^2=5^{2k+2}-25\times 2^{3k}=17s\times 25$ ***M1***

$=5^{2k+2}-8\times 2^{3k}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times 2^{3k}=17s\times 25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times 25+17\times 2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17(25s+2^{3k})$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

> **Note:** Only award ***R1*** if 4 of the previous 6 marks have been awarded
>
> **Note:** 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

7. METHOD 1

attempt to substitute solution into given equation ***(M1)***

$(5+q\mathrm{i})^2+\mathrm{i}(5+q\mathrm{i})=-p+25\mathrm{i}$

$25-q^2+10q\mathrm{i}-q+5\mathrm{i}+p-25\mathrm{i}=0$ OR $25-q^2+10q\mathrm{i}-q+5\mathrm{i}=-p+25\mathrm{i}$ ***A1***

$25-q^2+p-q+(10q-20)\mathrm{i}=0$

attempt to equate real or imaginary parts: ***(M1)***

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2$, $p=-19$ ***A1A1***

METHOD 2

$z^2+\mathrm{i}z+p-25\mathrm{i}=0$

sum of roots $=-\mathrm{i}$, product of roots $=p-25\mathrm{i}$ ***M1***

one root is $(5+q\mathrm{i})$ so other root is $(-5-q\mathrm{i}-\mathrm{i})$ ***A1***

product $(5+q\mathrm{i})(-5-q\mathrm{i}-\mathrm{i})=-25-5q\mathrm{i}-5\mathrm{i}-5q\mathrm{i}+q^2+q=p-25\mathrm{i}$

equating real and imaginary parts for product of roots ***(M1)***

Im: $-25=-10q-5$ Re: $p=-25+q^2+q$

$q=2$, $p=-19$ ***A1A1***

[5 marks]

8. (a) **METHOD 1**

attempt to integrate by parts *(M1)*

$u = (\ln x)^2$, $dv = x dx$ *(M1)*

$$\int x(\ln x)^2 \ dx = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ dx$$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ *(M1)*

$$\int x \ln x \ dx = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ dx\right]$$ ***A1***

$$\int x(\ln x)^2 \ dx = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ dx\right]$$

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} (+c)$$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ dx = x \ln x - x$ **)**

attempt to integrate by parts *(M1)*

$u = x \ln x$, $dv = \ln x dx$ *(M1)*

$$\int x \ln x (\ln x) \ dx = x \ln x (x \ln x - x) - \int (\ln x + 1)(x \ln x - x) dx$$ ***A1***

$$= x \ln x (x \ln x - x) - \int x (\ln x)^2 dx + \int x dx$$ ***A1***

$$2I = x \ln x (x \ln x - x) + \frac{x^2}{2} + c$$ ***M1***

$$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} (+c)$$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x \, dx$**)**

$$\int x\ln x \, dx = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts ***(M1)***

$u = \ln x$, $dv = x\ln x dx$ ***(M1)***

$$\int (x\ln x)\ln x \, dx = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)dx$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)dx$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x)=\dfrac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$=\dfrac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$=\dfrac{\sin^2(kx)}{x^2}\left(=f(x)\right)$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ ***(M1)***

attempt to differentiate numerator and denominator: ***M1***

$$\lim_{x\to 0}\left(f\left(x\right)\right)=\lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$$=\lim_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+...\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+...\right)$ ***A1***

$$\lim_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{k^2x^2\left(+...\right)}{x^2}\right)$$ ***M1***

$=\lim_{x\to 0}\left(k^2+\text{terms in } x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits ***(M1)***

$$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{k\cos kx}{1}\right)=k$$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx)=kx(+\ldots)$

$$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{kx+\ldots}{x}\right)=\lim_{x\to 0}(k+\text{terms in } x)=k$$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=k\times k=k^2$ ***A1***

$k^2=16\Rightarrow k=4\,(k>0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

10. (a) $x = 0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9) = \ln\frac{x^2}{d}$ OR $\ln\frac{x^2}{2x-9} = \ln d$ OR $\ln(2x-9)d = \ln x^2$

$\frac{x^2}{d} = 2x-9$ OR $\frac{x^2}{2x-9} = d$ OR $(2x-9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

(ii) discriminant = $(-2d)^2 - 4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4\times1\times9d > 0$ OR $(2d)^2 - 4\times9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graph
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9,\infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x =) \dfrac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10}\,(= p)$ or $x = 10 + \sqrt{10}\,(= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

$(a = 2, b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)e^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, e), \left(\frac{\pi}{2}, \frac{1}{e}\right), (\pi, e)$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***   For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)e^{\cos 2x} - (4\cos 2x)e^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4e < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4e < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{e} > 0$ so minimum ***R1***

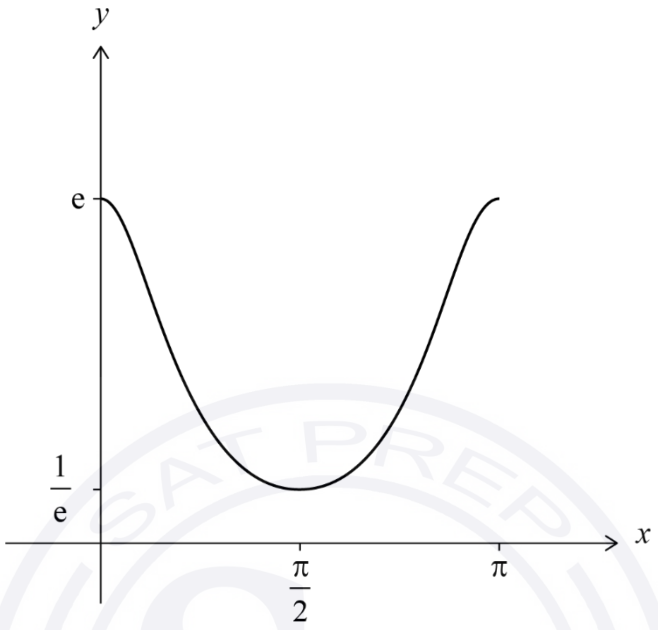
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0, \mathrm{e}), (\pi, \mathrm{e})$ and minimum point $\left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note:	Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots$; $e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + \ldots$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f\left(x\right)\approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x\approx\mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note:	If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta+{}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta+{}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$$ ***A1***

$$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)+\mathrm{i}\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right)$$ ***A1A1***

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos 5\theta+\mathrm{i}\sin 5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$ ***A1***

substitute $\cos^2\theta=1-\sin^2\theta$ ***(M1)***

$\sin 5\theta=5\left(1-\sin^2\theta\right)^2\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$\sin 5\theta=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ ***AG***

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \dfrac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \dfrac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\dfrac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\dfrac{\pi}{5}\sin\dfrac{3\pi}{5} = \sqrt{\dfrac{5+\sqrt{5}}{8}}\sqrt{\dfrac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\dfrac{20}{64}}$ ***A1***

$= \dfrac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\dfrac{\pi}{5}, \sin\dfrac{2\pi}{5}, \sin\dfrac{3\pi}{5}, \sin\dfrac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\dfrac{5}{16}$ ***M1***

$\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$ ***A1***

recognition that $\sin\dfrac{\pi}{5} = \sin\dfrac{4\pi}{5}$ and $\sin\dfrac{2\pi}{5} = \sin\dfrac{3\pi}{5}$

$\sin^2\dfrac{\pi}{5}\sin^2\dfrac{3\pi}{5} = \dfrac{5}{16}$ ***A1***

$\sin\dfrac{\pi}{5}\sin\dfrac{3\pi}{5} = \dfrac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]

8823 – 7106M

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 1

28 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

2. (a) $(\mathrm{P}(A \cup B) =) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

$= 0.8$ ***A1***

[2 marks]

(b) recognition that $A' \cap B' = (A \cup B)'$ OR $A' \cap B' = 1 - A \cup B$
(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$(= 1 - 0.8)$

$= 0.2$ ***A1***

Note: For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}(A' \cap B')$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3$, $q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1$, $d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =)65 - 40$ OR $(u_5 =)1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

4. **METHOD 1**

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2 - 1^2} =\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C} = 1 - \left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C} = \frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $= \frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$= \frac{1}{2} \times 10 \times \sqrt{6} \times \frac{\sqrt{24}}{5}$ ***(A1)***

$= 12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \text{B}\hat{\text{A}}\text{C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use 'Area $= \frac{1}{2}\text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm^2) ***A1***

[6 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1 kx + {}^nC_2 k^2 x^2 + \ldots$ OR ${}^nC_1 k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve (***M1***)

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

> **Note:** The assumption of truth must be clear. Do not award the ***M1*** for statements such
> as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=(5^2)5^{2k}-(2^3)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8(5^{2k}-2^{3k})$ OR $25(5^{2k}-2^{3k})+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17(5^{2k}+8s)$ OR $17(25s+2^{3k})$ which is divisible by 17 ***A1***

OR

$(5^{2k}-2^{3k})\times 5^2=5^{2k+2}-25\times 2^{3k}=17s\times 25$ ***M1***

$=5^{2k+2}-8\times 2^{3k}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times 2^{3k}=17s\times 25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times 25+17\times 2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17(25s+2^{3k})$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

> **Note:** Only award ***R1*** if 4 of the previous 6 marks have been awarded
>
> **Note:** 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

7. **METHOD 1**

attempt to substitute solution into given equation ***(M1)***

$(5+qi)^2 + i(5+qi) = -p + 25i$

$25 - q^2 + 10qi - q + 5i + p - 25i = 0$ OR $25 - q^2 + 10qi - q + 5i = -p + 25i$ ***A1***

$25 - q^2 + p - q + (10q - 20)i = 0$

attempt to equate real or imaginary parts: ***(M1)***

$10q - 20 = 0$ OR $25 - q^2 + p - q = 0$

$q = 2$, $p = -19$ ***A1A1***

METHOD 2

$z^2 + iz + p - 25i = 0$

sum of roots $= -i$, product of roots $= p - 25i$ ***M1***

one root is $(5+qi)$ so other root is $(-5-qi-i)$ ***A1***

product $(5+qi)(-5-qi-i) = -25 - 5qi - 5i - 5qi + q^2 + q = p - 25i$

equating real and imaginary parts for product of roots ***(M1)***

Im: $-25 = -10q - 5$ Re: $p = -25 + q^2 + q$

$q = 2$, $p = -19$ ***A1A1***

[5 marks]

8. (a) **METHOD 1**

attempt to integrate by parts **(M1)**

$u = (\ln x)^2$, $dv = x dx$ **(M1)**

$$\int x(\ln x)^2 \ dx = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ dx$$ **A1**

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ **(M1)**

$$\int x \ln x \ dx = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ dx\right]$$ **A1**

$$\int x(\ln x)^2 \ dx = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ dx\right]$$

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$$ **A1**

[6 marks]

METHOD 2 (knowing $\int \ln x \ dx = x \ln x - x$ **)**

attempt to integrate by parts **(M1)**

$u = x \ln x$, $dv = \ln x dx$ **(M1)**

$$\int x \ln x(\ln x) \ dx = x \ln x(x \ln x - x) - \int(\ln x + 1)(x \ln x - x) dx$$ **A1**

$$= x \ln x(x \ln x - x) - \int x(\ln x)^2 dx + \int x dx$$ **A1**

$$2I = x \ln x(x \ln x - x) + \frac{x^2}{2} + c$$ **M1**

$$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$$ **A1**

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x\,\mathrm{d}x$**)**

$\int x\ln x\,\mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$

attempt to integrate by parts ***(M1)***

$u = \ln x$, $\mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$\int (x\ln x)\ln x\,\mathrm{d}x = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)\mathrm{d}x$ ***A1***

$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int\left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)\mathrm{d}x$ ***A1***

$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$ ***A1***

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$ ***A1***

$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f\left(-x\right)=\dfrac{\sin^2\left(-kx\right)}{\left(-x\right)^2}$ ***M1***

$=\dfrac{\left(-\sin\left(kx\right)\right)^2}{\left(-x\right)^2}$ ***A1***

$=\dfrac{\sin^2\left(kx\right)}{x^2}\left(=f\left(x\right)\right)$

hence $f\left(x\right)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x \to 0}\left(f(x)\right) = \frac{0}{0}$ ***(M1)***

attempt to differentiate numerator and denominator: ***M1***

$$\lim_{x \to 0}\left(f(x)\right) = \lim_{x \to 0}\left(\frac{2k \sin kx \cos kx}{2x}\right)\left(= \lim_{x \to 0}\left(\frac{k \sin 2kx}{2x}\right)\right)$$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$$= \lim_{x \to 0}\left(\frac{2k^2\left(\cos^2 kx - \sin^2 kx\right)}{2}\right)\left(= \lim_{x \to 0}\left(\frac{2k^2 \cos 2kx}{2}\right)\right) = k^2$$ ***A1***

$\left(k^2 = 16 \Rightarrow\right) k = 4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin(kx)$ as a Maclaurin series ***M1***

$\sin(kx) = kx(+\ldots)$

$\sin^2(kx) = k^2x^2(+\ldots)$ ***A1***

$$\lim_{x \to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = \lim_{x \to 0}\left(\frac{k^2x^2(+\ldots)}{x^2}\right)$$ ***M1***

$$= \lim_{x \to 0}\left(k^2 + \text{terms in } x\right)$$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$= k^2$ ***A1***

$\left(k^2 = 16 \Rightarrow\right) k = 4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits **(*M1*)**

$$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{k\cos kx}{1}\right)=k$$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$$\sin(kx)=kx(+\ldots)$$

$$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)=\lim_{x\to 0}\left(\frac{kx+\ldots}{x}\right)=\lim_{x\to 0}(k+\text{terms in } x)=k$$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right)=k\times k=k^2$ ***A1***

$$k^2=16\Rightarrow k=4(k>0)$$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

10. (a) $x = 0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9) = \ln\frac{x^2}{d}$ OR $\ln\frac{x^2}{2x-9} = \ln d$ OR $\ln(2x-9)d = \ln x^2$

$\frac{x^2}{d} = 2x-9$ OR $\frac{x^2}{2x-9} = d$ OR $(2x-9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

(ii) discriminant = $(-2d)^2 - 4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4\times1\times9d > 0$ OR $(2d)^2 - 4\times9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graph
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9,\infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x =) \dfrac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10} \left(= p\right)$ or $x = 10 + \sqrt{10} \left(= q\right)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

$\left(a = 2, b = 10\right)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)e^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, e), \left(\frac{\pi}{2}, \frac{1}{e}\right), (\pi, e)$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.*** For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)e^{\cos 2x} - (4\cos 2x)e^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4e < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4e < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{e} > 0$ so minimum ***R1***

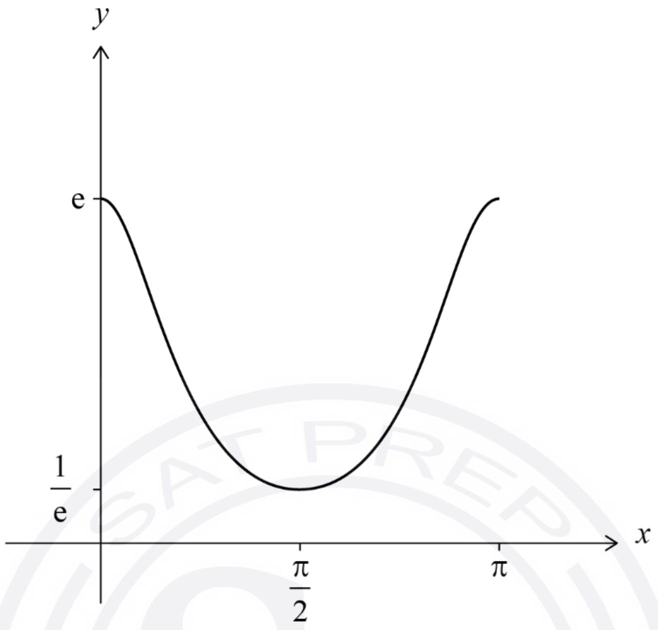
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maxima $(0, \mathrm{e}), (\pi, \mathrm{e})$ and minimum point $\left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots$; $e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + \ldots$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x) \approx\right) \mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}} \mathrm{e}^{\cos 2x}\mathrm{d}x \approx \mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$= \mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$= \mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$= \frac{298\mathrm{e}}{3000}$

$= \frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta+{}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta+{}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$$ ***A1***

$$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)+\mathrm{i}\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right)$$ ***A1A1***

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos5\theta+\mathrm{i}\sin5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$ ***A1***

substitute $\cos^2\theta=1-\sin^2\theta$ ***(M1)***

$\sin5\theta=5\left(1-\sin^2\theta\right)^2\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$\sin5\theta=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ ***AG***

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta(16\sin^4\theta - 20\sin^2\theta + 5)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \frac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \frac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\frac{20}{64}}$ ***A1***

$= \frac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5}, \sin\frac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\frac{5}{16}$ ***M1***

$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$ ***A1***

recognition that $\sin\frac{\pi}{5} = \sin\frac{4\pi}{5}$ and $\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$

$\sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempts to find perimeter *(M1)*

$\text{arc} + 2 \times \text{radius}$ OR $10 + 4 + 4$

$= 18 \text{ (cm)}$ ***A1***

[2 marks]

(b) $10 = 4\theta$ ***(A1)***

$\theta = \frac{10}{4} \left(= \frac{5}{2}, 2.5\right)$ ***A1***

[2 marks]

(c) area $= \frac{1}{2}\left(\frac{10}{4}\right)\left(4^2\right) \ (= 1.25 \times 16)$ ***(A1)***

$= 20 \text{ (cm}^2\text{)}$ ***A1***

[2 marks]

Total [6 marks]

2. (a) (i) $x = 2$ ***A1***

(ii) $y = 1$ ***A1***

[2 marks]

(b) (i) $\left(0, \frac{3}{2}\right)$ ***A1***

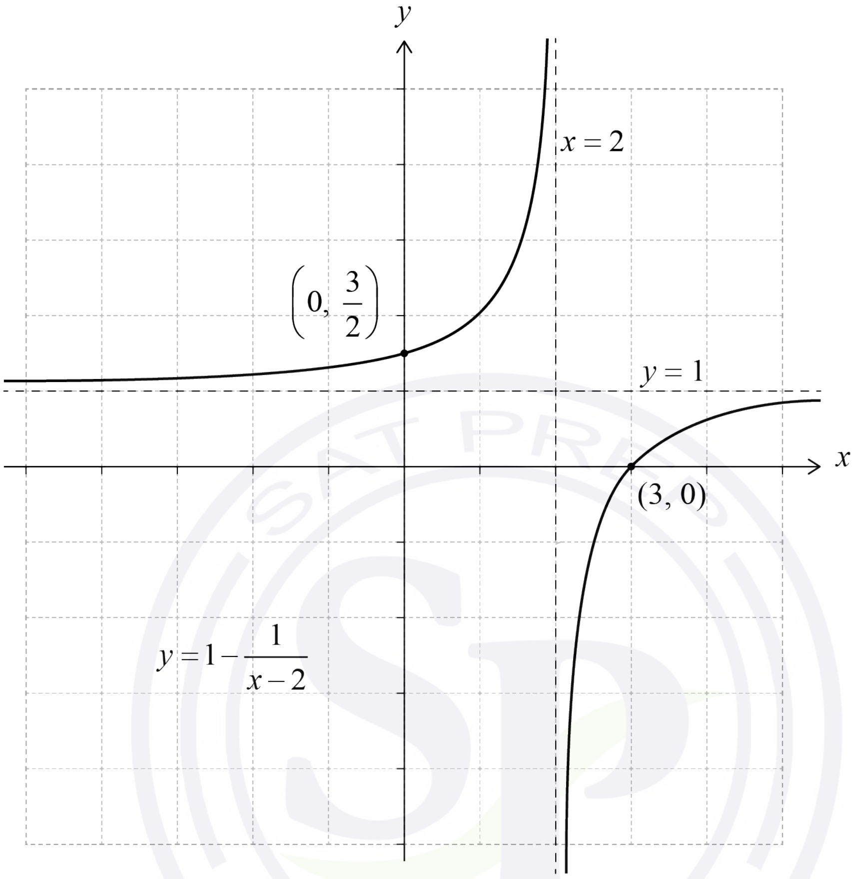
(ii) $(3, 0)$ ***A1***

[2 marks]

continued…

Question 2 continued

(c)



two correct branches with correct asymptotic behaviour and intercepts clearly shown ***A1***

[1 mark]

Total [5 marks]

3. substitutes into $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ to form

$0.55=0.4+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ (or equivalent) ***(A1)***

substitutes into $\mathrm{P}(A\,|\,B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ to form $0.25=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ (or equivalent) ***(A1)***

attempts to combine their two probability equations to form an equation in $\mathrm{P}(B)$ ***(M1)***

Note: The above two ***A*** marks are awarded independently.

correct equation in $\mathrm{P}(B)$ ***A1***

$0.55=0.4+\mathrm{P}(B)-0.25\mathrm{P}(B)$ OR $\dfrac{\mathrm{P}(B)-0.15}{\mathrm{P}(B)}=0.25$ OR $\mathrm{P}(B)-0.15=0.25\mathrm{P}(B)$

(or equivalent)

$\mathrm{P}(B)=\dfrac{15}{75}\ \left(=\dfrac{1}{5}=0.2\right)$ ***A1***

Total [5 marks]

4. $A = \int_0^c \frac{x}{x^2+2}\,\mathrm{d}x$

EITHER

attempts to integrate by inspection or substitution using $u = x^2 + 2$ or $u = x^2$ ***(M1)***

> **Note:** If candidate simply states $u = x^2 + 2$ or $u = x^2$, but does not attempt to integrate, do not award the ***(M1)*** .

> **Note:** If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k \ln u$ or $k \ln(u+2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln(u+2)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln(x^2+2)\right]_0^{c}$ ***A1***

> **Note:** Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

> **Note:** Award the ***(M1)*** only for expressions of the form $k\ln(x^2+2)$.

$\left[\frac{1}{2}\ln(x^2+2)\right]_0^{c}$ ***A1***

> **Note:** Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln(c^2+2) - \ln 2\right)\ (= \ln 3)$ OR $\frac{1}{2}\ln(c^2+2) - \frac{1}{2}\ln 2\,(= \ln 3)$

continued…

Question 4 continued

correctly applies at least one log law to their expression ***(M1)***

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)(=\ln 3)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}(=\ln 3)$ OR $\ln\left(\frac{c^2+2}{2}\right)=\ln 9$

OR $\ln\left(c^2+2\right)-\ln 2=\ln 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}(=\ln 3)$ OR $\ln\frac{\sqrt{c^2+2}}{\sqrt{2}}(=\ln 3)$

Note: Condone the absence of $\ln 3$ up to this stage.

$\frac{c^2+2}{2}=9$ OR $\sqrt{\frac{c^2+2}{2}}=3$ ***A1***

$c^2=16$

$c=4$ ***A1***

Note: Award ***A0*** for $c=\pm 4$ as a final answer.

Total [6 marks]

5. attempts to form $(g \circ f)(x)$ ***(M1)***

$[f(x)]^2 + f(x) + 3$ OR $(ax+b)^2 + ax + b + 3$

$a^2x^2 + 2abx + b^2 + ax + b + 3 \left(= 4x^2 - 14x + 15\right)$ ***(A1)***

equates their corresponding terms to form at least one equation ***(M1)***

$a^2x^2 = 4x^2$ OR $a^2 = 4$ OR $2abx + ax = -14x$ OR $2ab + a = -14$ OR $b^2 + b + 3 = 15$

$a = \pm 2$ (seen anywhere) ***A1***

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) ***(M1)***

$f(x) = 2x - 4$ (accept $a = 2$ with $b = -4$), $f(x) = -2x + 3$ (accept $a = -2$ with $b = 3$) ***A1A1***

[7 marks]

6. (a) $\mathrm{E}(X) = 2a$ (by symmetry) ***A1***

[1 mark]

(b) **METHOD 1**

uses $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ ***(M1)***

$$\mathrm{Var}(X) = \int_a^{3a} \frac{x^2}{2a} \mathrm{d}x - (2a)^2$$

$$= \left[\frac{x^3}{6a}\right]_a^{3a} - (2a)^2$$ ***A1***

$$= \frac{13a^2}{3} - (2a)^2$$ ***(A1)***

$$= \frac{a^2}{3}$$ ***A1***

Note: Award as above if $\mathrm{E}(X^2)$ and $[\mathrm{E}(X)]^2$ are calculated separately leading to $\mathrm{Var}(X) = \frac{a^2}{3}$. Award ***(M1)A0A0A0*** for $\mathrm{Var}(X) = \frac{13a^2}{3}$.

continued…

Question 6 continued

METHOD 2

uses $\mathrm{Var}(X) = \mathrm{E}\left(X - \mathrm{E}(X)\right)^2$ ***(M1)***

$$\mathrm{Var}(X) = \int_a^{3a} \frac{(x-2a)^2}{2a}\,\mathrm{d}x$$

$$= \left[\frac{(x-2a)^3}{6a}\right]_a^{3a}$$ ***A1***

$$= \frac{a^3 - (-a^3)}{6a}$$ or equivalent ***(A1)***

$$= \frac{a^2}{3}$$ ***A1***

[4 marks]

Total [5 marks]

7. let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers, $n \geq 1$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = \frac{1}{2}$ and $\mathrm{RHS} = \frac{1}{2}$ and so $\mathrm{P}(1)$ is true **_R1_**

assume $\mathrm{P}(k)$ is true *ie*, $\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$ **_M1_**

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = \sum_{r=1}^{k} \frac{r}{(r+1)!} + \frac{k+1}{((k+1)+1)!}$$ ***(M1)***

$$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$ ***A1***

$$= 1 - \frac{(k+2)-(k+1)}{(k+2)!}$$ ***A1***

$$= 1 - \frac{1}{(k+2)!} \left(= 1 - \frac{1}{((k+1)+1)!} \right)$$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true (for all integers, $n \geq 1$) **_R1_**

Note: To obtain the final ***R1***, any four of the previous marks must have been awarded.

[7 marks]

8. (a) $\cos k = \frac{\sin k}{\cos k}$ ***A1***

$\cos^2 k = \sin k$ ***AG***

[1 mark]

(b) $f'(k) = -\sin k$ and $g'(k) = \sec^2 k$ ***A1***

Note: Award ***A1*** for $f'(x) = -\sin x$ and $g'(x) = \sec^2 x$.

EITHER

$f'(k)g'(k) = -\frac{\sin k}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow f'(k)g'(k)\left(= -\frac{\sin k}{\sin k}\right) = -1$ ***R1***

OR

$g'(k) = \frac{1}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow g'(k) = \frac{1}{\sin k} = -\frac{1}{f'(k)}$ ***R1***

Note: Accept showing that $f'(k) = -\frac{1}{g'(k)}$.

Note: Allow'backwards methods' such as starting with $f'(k) = -\frac{1}{g'(k)}$ leading to $\cos^2 k = \sin k$

THEN

$\Rightarrow$ the two tangents intersect at right angles at P ***AG***

Note: To obtain the final ***R1***, all of the previous marks must have been awarded.

[3 marks]

continued…

Question 8 continued

(c) $1-\sin^2 k = \sin k$ (from part (a)) ***A1***

$\sin^2 k + \sin k - 1 = 0$

attempts to solve for $\sin k$ ***(M1)***

$$\sin k = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

(for $0 < k < \frac{\pi}{2}$, $\sin k > 0$) $\Rightarrow \sin k = \frac{-1+\sqrt{5}}{2}$ ***A1***

$(a = -1, b = 5, c = 2)$

Note: Award ***A0*** if more than one solution is given

[3 marks]

Total [7 marks]

9. (a) $\vec{OM} = \boldsymbol{a} + k\boldsymbol{c}$ ***A1***

$\vec{MC} = (1-k)\boldsymbol{c} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) attempts to expand their dot product $\vec{OM} \bullet \vec{MC} = (\boldsymbol{a} + k\boldsymbol{c}) \bullet ((1-k)\boldsymbol{c} - \boldsymbol{a})$ ***M1***

$= (1-2k)(\boldsymbol{a} \bullet \boldsymbol{c}) - |\boldsymbol{a}|^2 + k(1-k)|\boldsymbol{c}|^2$ (or equivalent)

uses $|\boldsymbol{c}| = 2|\boldsymbol{a}|$ and $\boldsymbol{a} \bullet \boldsymbol{c} = 2|\boldsymbol{a}|^2 \cos\theta$ ***M1***

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - |\boldsymbol{a}|^2 + 4k(1-k)|\boldsymbol{a}|^2$

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - (1-2k)^2 |\boldsymbol{a}|^2$ ***A1***

$|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ ***AG***

[3 marks]

(c) attempts to solve $|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ for k ***(M1)***

$k = \frac{1}{2}$ or $k = \frac{1}{2} - \cos\theta$ $\left(|\boldsymbol{a}|^2 > 0\right)$

> **Note:** Award ***(M1)*** for their '$k =$' or their '$\cos\theta =$'. For example, $\cos\theta = \frac{1-2k}{2}$ or equivalent.

as $0 \le k \le 1$, $0 \le \frac{1}{2} - \cos\theta \le 1$

$-\frac{1}{2} \le \cos\theta \le \frac{1}{2}$ ***A1***

$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\theta \ne \frac{\pi}{2}$ ***A1A1***

($\theta = \frac{\pi}{2}$ corresponds to only one possible position for M when $k = \frac{1}{2}$)

[4 marks]

Total [9 marks]

Section B

10. (a) $y^2 = 9 - x^2$ OR $y = \pm\sqrt{9-x^2}$ ***A1***

(since $y > 0$) $\Rightarrow y = \sqrt{9-x^2}$ ***AG***

[1 mark]

(b) $b = 2y\ \left(= 2\sqrt{9-x^2}\right)$ or $h = x+3$ ***(A1)***

attempts to substitute their base expression and height expression into $A = \frac{1}{2}bh$ ***(M1)***

$A = \sqrt{9-x^2}(x+3)$ (or equivalent) $\left(= \frac{2(x+3)\sqrt{9-x^2}}{2} = x\sqrt{9-x^2} + 3\sqrt{9-x^2}\right)$ ***A1***

[3 marks]

(c) **METHOD 1**

attempts to use the product rule to find $\frac{\mathrm{d}A}{\mathrm{d}x}$ ***(M1)***

attempts to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^2}$ ***(M1)***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) \sqrt{9-x^2} + (3+x)\left(\frac{1}{2}\right)\left(9-x^2\right)^{-\frac{1}{2}}(-2x)\ \left(= \sqrt{9-x^2} - \frac{x^2+3x}{\sqrt{9-x^2}}\right)$ ***A1***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) \frac{9-x^2}{\sqrt{9-x^2}} - \frac{x^2+3x}{\sqrt{9-x^2}}\ \left(= \frac{9-x^2-\left(x^2+3x\right)}{\sqrt{9-x^2}}\right)$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

continued…

Question 10 continued

METHOD 2

$\frac{dA}{dx} = \frac{dA}{dy} \times \frac{dy}{dx}$

attempts to find $\frac{dA}{dy}$ where $A = y(x+3)$ and $\frac{dy}{dx}$ where $y^2 = 9 - x^2$ ***(M1)***

$\frac{dA}{dy} = y\frac{dx}{dy} + x + 3$ and $\frac{dy}{dx} = -\frac{x}{y}$ (or equivalent) ***A1***

substitutes their $\frac{dA}{dy}$ and their $\frac{dy}{dx}$ into $\frac{dA}{dx} = \frac{dA}{dy} \times \frac{dy}{dx}$ ***(M1)***

$\frac{dA}{dx} = \left(y\left(-\frac{y}{x}\right) + x + 3\right)\left(-\frac{x}{y}\right)$ (or equivalent)

$= \frac{9 - x^2 - x^2 - 3x}{\sqrt{9 - x^2}}$ (or equivalent) ***A1***

$\frac{dA}{dx} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$ ***AG***

[4 marks]

continued…

Question 10 continued

(d) $\frac{\mathrm{d}A}{\mathrm{d}x}=0 \left(\frac{9-3x-2x^2}{\sqrt{9-x^2}}=0\right)$ ***(M1)***

attempts to solve $9-3x-2x^2=0$ (or equivalent) ***(M1)***

$-(2x-3)(x+3)(=0)$ OR $x=\frac{3\pm\sqrt{(-3)^2-4(-2)(9)}}{2(-2)}$ (or equivalent) ***(A1)***

$x=\frac{3}{2}$ ***A1***

Note: Award the above ***A1*** if $x=-3$ is also given.

substitutes their value of x into either $y=\sqrt{9-x^2}$ or $y=-\sqrt{9-x^2}$ ***(M1)***

Note: Do not award the above ***(M1)*** if $x\le 0$.

$y=-\sqrt{9-\left(\frac{3}{2}\right)^2}$

$=-\frac{\sqrt{27}}{2}\left(=-\frac{3\sqrt{3}}{2},=-\sqrt{\frac{27}{4}},=-\sqrt{6.75}\right)$ ***A1***

[6 marks]

Total [14 marks]

11. (a) **METHOD 1**

$|u| = \sqrt{(-1)^2 + (\sqrt{3})^2} \left(= \sqrt{1+3}\right)$ ***A1***

$= 2$ ***AG***

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}(\sqrt{3})$ OR $\arg u = \pi + \tan^{-1}(-\sqrt{3})$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

Note: Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ and so $u = 2e^{i\frac{2\pi}{3}}$ ***AG***

[3 marks]

METHOD 2

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}(\sqrt{3})$ OR $\arg u = \pi + \tan^{-1}(-\sqrt{3})$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

Note: Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ ***AG***

$r\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

$r = \frac{-1}{\cos\frac{2\pi}{3}} = \frac{-1}{-\frac{1}{2}}$ OR $r = \frac{\sqrt{3}}{\sin\frac{2\pi}{3}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}}$ ***A1***

$= 2$ and so $u = 2e^{i\frac{2\pi}{3}}$ ***AG***

[3 marks]

continued…

Question 11 continued

(b) (i) $u^n \in \mathbb{R} \Rightarrow \frac{2n\pi}{3} = k\pi \ \ (k \in \mathbb{Z})$ ***(M1)(A1)***

Note: Award ***M1*** for noting that $\sin\frac{2n\pi}{3} = 0$ from $u^n = 2^n\left(\cos\frac{2n\pi}{3} + \mathrm{i}\sin\frac{2n\pi}{3}\right)$.
Award ***(A1)*** for a multiple of 3 considered.

$n = 3$ ***A1***

(ii) substitutes their value (must be a multiple of 3) for n into u^n ***(M1)***

$u^3 = 2^3\cos 2\pi$

$= 8$ ***A1***

[5 marks]

continued…

Question 11 continued

(c) (i) $-1-\sqrt{3}\mathrm{i}$ is a root (by the conjugate root theorem) ***A1***

Note: Accept $2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$.

let $z=c$ be the real root

EITHER

uses sum of roots (equated to ± 5) ***(M1)***

$\left(\left(-1+\sqrt{3}\mathrm{i}\right)+\left(-1-\sqrt{3}\mathrm{i}\right)+c\right)=-5$ ***(A1)***

$-2+c=-5$ ***(A1)***

OR

uses product of roots (equated to ± 12) ***(M1)***

$\left(-1+\sqrt{3}\mathrm{i}\right)\left(-1-\sqrt{3}\mathrm{i}\right)c=-12$ ***(A1)***

$4c=-12$ ***(A1)***

OR

$\left(z-\left(-1+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(-1-\sqrt{3}\mathrm{i}\right)\right)=z^2+2z+4$ ***(A1)***

compares coefficients *eg* ***(M1)***

$(z-c)\left(z^2+2z+4\right)=z^3+5z^2+10z+12$

$-4c=12$ ***(A1)***

THEN

$c=-3$ (and so $z=-3$ is a root) ***A1***

continued…

Question 11 continued

(ii) **METHOD 1**

compares $z^3+5z^2+10z+12=0$ and $1+5w+10w^2+12w^3=0$

$z=\frac{1}{w}\Rightarrow w=\frac{1}{z}$ ***A2***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\text{i}}\left(=\frac{-1\pm\sqrt{3}\text{i}}{4}\right)$ ***A1A1***

METHOD 2

attempts to factorize into a product of a linear factor and a quadratic factor ***(M1)***

$1+5w+10w^2+12w^3=(3w+1)(4w^2+2w+1)$ ***A1***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\text{i}}\left(=\frac{-1\pm\sqrt{3}\text{i}}{4}\right)$ ***A1A1***

[9 marks]

(d) $(a+b\text{i})^2=2(a-b\text{i})$ ***A1***

attempts to expand and equate real and imaginary parts: ***M1***

$a^2-b^2+2ab\text{i}=2a-2b\text{i}$

$a^2-b^2=2a$ and $2ab=-2b$

attempts to find the value of a or b ***M1***

$2b(a+1)=0$

$b=0\Rightarrow a^2=2a\Rightarrow a=2$ (real root) ***A1***

$a=-1\Rightarrow 1-b^2=-2\Rightarrow b=\pm\sqrt{3}$ (complex roots $-1\pm\sqrt{3}\text{i}$) ***A1***

[5 marks]

Total [22 marks]

12. (a) let $t=\sqrt{x}$ ***M1***

$t^2=x\Rightarrow 2t\ \mathrm{d}t=\mathrm{d}x$ (or equivalent) ***A1***

so $\int\cos\sqrt{x}\ \mathrm{d}x=2\int t\cos t\ \mathrm{d}t$ ***A1***

attempts integration by parts ***(M1)***

$u=2t$, $\mathrm{d}v=\cos t\ \mathrm{d}t$, $\mathrm{d}u=2\ \mathrm{d}t$, $v=\sin t$

$2\int t\cos t\ \mathrm{d}t=2t\sin t-2\int\sin t\ \mathrm{d}t$ ***(A1)***

$=2t\sin t+2\cos t+C$ ***A1***

substitution of $t=\sqrt{x}\ \Rightarrow\int\cos\sqrt{x}\ \mathrm{d}x=2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}+C$ ***AG***

[6 marks]

(b) $x_{n+1}=\dfrac{\left(2\left(n+1\right)-1\right)^2\pi^2}{4}\ \left(=\dfrac{\left(2n+1\right)^2\pi^2}{4}\right)$ ***A1***

[1 mark]

continued…

Question 12 continued

(c) area of R_n is $\left|\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x\right|$ ***(M1)***

Note: Modulus may be seen at a later stage.

$$=\left|\left[2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}\right]_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}}\right|$$ ***A1***

Note: Condone $+C$ at this stage.

attempts to substitute their limits into their integrated expression ***(M1)***

$$=2\left|\frac{(2n+1)\pi}{2}\times\sin\frac{(2n+1)\pi}{2}+\cos\frac{(2n+1)\pi}{2}-\left(\frac{(2n-1)\pi}{2}\times\sin\frac{(2n-1)\pi}{2}+\cos\frac{(2n-1)\pi}{2}\right)\right|$$

A1

$$=2\left|(-1)^n\frac{(2n+1)\pi}{2}-\left((-1)^{n+1}\frac{(2n-1)\pi}{2}\right)\right|$$ (or equivalent) ***A1***

$$=2\left|(-1)^n\frac{(2n+1)\pi}{2}+(-1)^n\frac{(2n-1)\pi}{2}\right|$$ ***A1***

$$=2\left|(-1)^n\frac{4n\pi}{2}\right|$$

$$=4n\pi$$ ***A1***

Note: Award a maximum of ***(M1)A1M1A1A1A0A0*** for only attempting to calculate $\int_{x_n}^{x_{n+1}}\cos\sqrt{x}\ \mathrm{d}x$, and not applying the modulus.

[7 marks]

continued…

Question 12 continued

(d) **EITHER**

attempts to find $(d =)R_{n+1} - R_n$ ***M1***

$(d =)4(n+1)\pi - 4n\pi$

$= 4\pi$ ***A1***

Note: Award ***M0*** for consideration of special cases for example R_3 and R_2. Accept $d = k\pi$.

which is a constant (common difference is 4π) ***R1***

OR

an arithmetic sequence is of the form $u_n = dn + c$ $(u_n = dn + u_1 - d)$ ***M1***

attempts to compare $u_n = dn + c$ $(u_n = dn + u_1 - d)$ and $R_n = 4n\pi$ ***M1***

$d = 4\pi$ and $c = 0$ $(u_1 - d = 0)$ ***A1***

Note: Accept $d = k\pi$.

THEN

so the areas of the regions form an arithmetic sequence ***AG***

[3 marks]

Total [17 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) recognizing $f\left(x\right)=0$ ***(M1)***

$x=-1$ ***A1***

[2 marks]

(b) (i) $x=2$ (must be an equation with x) ***A1***

(ii) $y=\frac{7}{2}$ (must be an equation with y) ***A1***

[2 marks]

(c) **EITHER**

interchanging x and y ***(M1)***

$2xy-4x=7y+7$

correct working with y terms on the same side: $2xy-7y=4x+7$ ***(A1)***

OR

$2yx-4y=7x+7$

correct working with x terms on the same side: $2yx-7x=4y+7$ ***(A1)***

interchanging x and y OR making x the subject $x=\frac{4y+7}{2y-7}$ ***(M1)***

THEN

$f^{-1}(x)=\frac{4x+7}{2x-7}$ (or equivalent) $\left(x\neq\frac{7}{2}\right)$ ***A1***

[3 marks]

Total [7 marks]

2. (a) (i) summing frequencies of riders or finding complement ***(M1)***

probability $= \frac{34}{40}$ ***A1***

(ii) attempt to find expected value ***(M1)***

$$\frac{16}{40}+\left(2\times\frac{13}{40}\right)+\left(3\times\frac{2}{40}\right)+\left(4\times\frac{3}{40}\right)$$

$\frac{60}{40}(=1.5)$ ***A1***

[4 marks]

(b) evidence of **their** rides/visitor $\times 1000$ or $\div 10$ ***(M1)***

1500 OR 0.15

150 (times) ***A1***

[2 marks]

Total [6 marks]

3. $1-2\sin^2 x=\sin x$ ***A1***

$2\sin^2 x+\sin x-1=0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x-1)(\sin x+1)$ OR $\frac{-1\pm\sqrt{1-4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x=\frac{1}{2}$ OR $\sin x=-1$

any correct solution from $\sin x=-1$ ***A1***

any correct solution from $\sin x=\frac{1}{2}$ ***A1***

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x=-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ ***A1***

Note: If no working shown, award no marks for a final value(s).
Award ***A0*** for $-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ if additional values also given.

Total [6 marks]

4. recognition of quadratic in e^x ***(M1)***

$(e^x)^2 - 3e^x + \ln k (= 0)$ OR $A^2 - 3A + \ln k (= 0)$

recognizing discriminant ≥ 0 (seen anywhere) (***M1)***

$(-3)^2 - 4(1)(\ln k)$ OR $9 - 4\ln k$ ***(A1)***

$\ln k \le \frac{9}{4}$ ***(A1)***

$e^{9/4}$ (seen anywhere) ***A1***

$0 < k \le e^{9/4}$ ***A1***

[6 marks]

5. (a) recognition that period is $4m$ OR substitution of a point on f (except the origin) ***(M1)***

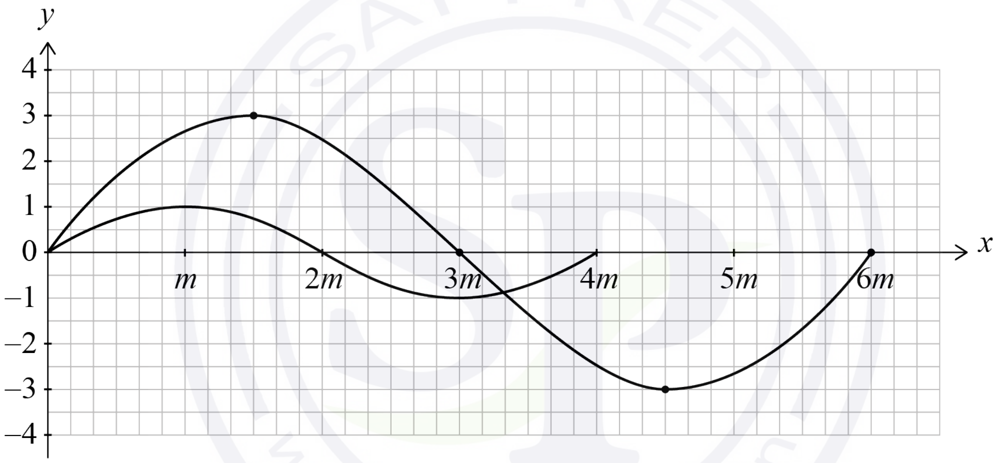
$4m = \frac{2\pi}{q}$ OR $1 = \sin qm$

$m = \frac{\pi}{2q}$ ***A1***

[2 marks]

(b) horizontal scale factor is $\frac{3}{2}$ (seen anywhere) ***(A1)***

Note: This ***(A1)*** may be earned by seeing a period of $6m$, half period of $3m$ or the correct x-coordinate of the maximum/minimum point.



A1A1A1

Note: Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
A1 for correct amplitude.
A1 for correct domain.
A1 for correct max and min points **and** correct x-intercepts.

[4 marks]

Total [6 marks]

6. $A=\frac{1}{2}x^2\sin\frac{\pi}{3}$ OR $A=\frac{1}{2}x^2\sin 60^\circ$ OR triangle height $h=\sqrt{x^2-\left(\frac{x}{2}\right)^2}$ $\left(=\frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$=\frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A=\frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(=\frac{\sqrt{3}}{4}x^2\right)$ ***A1***

> **Note:** Award ***A1*** for $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$

$\frac{dA}{dt}=\frac{\sqrt{3}}{4}\times 2x\frac{dx}{dt}$ OR $\frac{dA}{dt}=\frac{1}{2}\times\sin\frac{\pi}{3}\times 2x\frac{dx}{dt}$ ***(A1)***

$=\frac{2\sqrt{3}}{4}\times 5\sqrt{3}\times 4$

$\frac{dA}{dt}=30\,(\text{cm}^2\text{s}^{-1})$ ***A1***

> **Note:** Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

7. **METHOD 1**

3i (is a root) ***A1***

(other complex root is) -3i ***A1***

> **Note:** Award ***A1A1*** for $P(3\text{i})=0$ and $P(-3\text{i})=0$ seen in their working.
> Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\text{i})=0$ or $P(-3\text{i})=0$ ***(M1)***

$$4m-3m\text{i}+\frac{36}{m}(3\text{i})^2-(3\text{i})^3=0$$

$$4m-3m\text{i}-\frac{36}{m}(-9)+27\text{i}=0$$

attempt to equate the real or imaginary parts ***(M1)***

$27-3m=0$ OR $9\times\frac{36}{m}=4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

> **Note:** Accept sum of three roots set to $-\frac{36}{m}$.
>
> Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$$3\text{i}-3\text{i}+r=\frac{36}{m}\left(\Rightarrow r=\frac{36}{m}\right)$$

substitute their r into product of roots ***(M1)***

$(3\text{i})(-3\text{i})\left(\frac{36}{m}\right)=4m$ OR $(z^2+9)\left(\frac{36}{m}-z\right)$

$9\times\frac{36}{m}=4m$ OR $\frac{4m}{9}=\frac{36}{m}$

continued…

Question 7 continued

OR

attempt to equate product of three roots to $4m$ ***(M1)***

Note: Accept product of three roots set to $-4m$.   Award ***M0*** for stating product of roots is $\pm 4m$.

$(3\text{i})(-3\text{i}) \times r = 4m \left(\Rightarrow r = \frac{4m}{9} \right)$

substitute their r into sum of roots ***(M1)***

$3\text{i} - 3\text{i} + \frac{4m}{9} = \frac{36}{m}$ OR $(z^2 + 9)\left(\frac{4m}{9} - z\right)$

$\frac{4m}{9} = \frac{36}{m}$

THEN

$m = 9$ ***(A1)***

third root is 4 ***A1***

[6 marks]

Question 7 continued

METHOD 2

3i (is a root) ***A1***

(other complex root is) -3i ***A1***

recognition that the other factor is $(z+3\text{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\text{i})(z+3\text{i})(r-z)$ OR $(z-3\text{i})(z+3\text{i})=z^2+9 \Rightarrow P(z)=(z^2+9)\left(\frac{4m}{9}-z\right)$

Note:	Accept any attempt at long division of $P(z)$ by z^2+9. Award ***M0*** for stating other factor is $(z+3\text{i})$ or obtaining z^2+9 with no further working.

attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

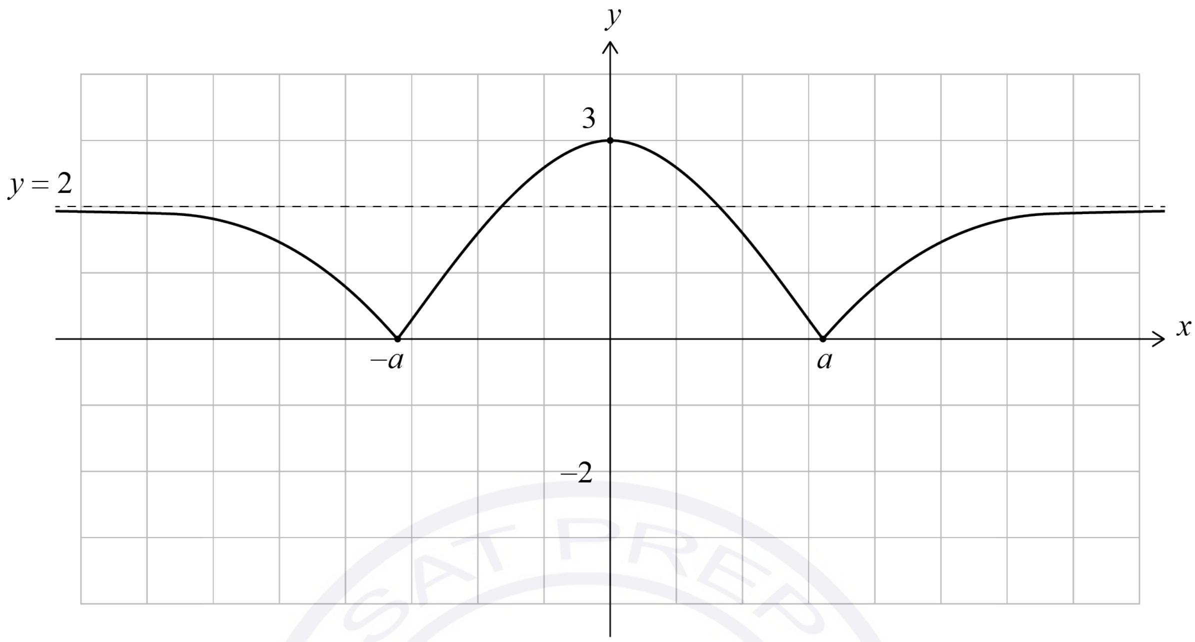
$m=9$ ***(A1)***

third root is 4 ***A1***

Note:	Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\text{i})(z+3\text{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

8. (a) attempt to reflect f in the x OR y axis ***(M1)***



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:
A1 for correct asymptotic behaviour at $y = 2$ (either side)
A1 for correctly reflected RHS of the graph in the *y*-axis with smooth maximum at (0, 3).
A1 for labelled *x*-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

[4 marks]

(b) $k = 0$ ***A1***

$4 \le k < 9$ ***A2***

Note: If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.
Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the *y*-axis.

[3 marks]

Total [7 marks]

9. **METHOD 1 (subtracting volumes)**

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $= \pi\int\left(r^2 - y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2 h}{2} - \frac{h^3}{24}$

recognition that the volume of the ring is $\pi\int\left(r^2 - y^2\right)\mathrm{d}y - \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi\int\left(r^2 - y^2\right)\mathrm{d}y - \pi\left(r^2 - \frac{h^2}{4}\right)h$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{r^2 h}{2} - \frac{h^3}{24}\right) - \pi r^2 h + \frac{\pi h^3}{4} = \pi$ OR $\frac{h^3}{4} - \frac{h^3}{12} = 1$ (or equivalent)

OR

recognition that the volume of the ring is $\pi\int\left(\left(r^2 - y^2\right) - \left(r^2 - \frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$\frac{h^2}{4}y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{h^3}{8} - \frac{h^3}{24}$

correct equation ***(A1)***

$2\pi\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = \pi$ OR $2\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = 1$ (or equivalent)

THEN

$h = \sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $= \pi\int\left(r^2 - y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi\int\left(r^2 - y^2\right)\mathrm{d}y + \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi\int\left(r^2 - y^2\right)\mathrm{d}y + \pi\left(r^2 - \frac{h^2}{4}\right)h \quad \left(= \frac{4}{3}\pi r^3 - \pi\right)$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}\right) + \pi r^2 h - \frac{\pi h^3}{4} = \frac{4}{3}\pi r^3 - \pi$ OR $\frac{h^3}{12} - \frac{h^3}{4} = -1$ (or equivalent)

$h = \sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$$2\pi\int x\sqrt{r^2-x^2}\,\mathrm{d}x$$

correct limits *r* and $\sqrt{r^2-\frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$$-\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}$$

attempt to substitute their limits into their integrated expression ***(M1)***

$$-\frac{1}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$$

correct equation ***(A1)***

$$2\times\frac{-2\pi}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)=\pi \quad \text{OR} \quad 2\left(\frac{2\pi}{3}\times\frac{h^3}{8}\right)=\pi$$

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

Section B

10. (a) (i) recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

(ii) **METHOD 1**

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

[4 marks]

(b) recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 ***(M1)***

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}(u_1 + 15)$

$u_1 = 5$ ***A1***

[2 marks]

continued…

Question 10 continued

(c) **EITHER**

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}(5 + u_n)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n-1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

(d) recognition that $v_2 r^2 = v_4$ OR $(v_3)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3} \quad \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

Total [14 marks]

11. (a) $L = \text{AC} + \text{CB}$

$\dfrac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\alpha \left(\Rightarrow \text{AC} = \dfrac{\frac{3}{4}}{\cos\alpha} \Rightarrow \text{AC} = \dfrac{3}{4}\sec\alpha \right)$ ***A1***

$\dfrac{6}{\text{CB}} = \sin\alpha \left(\Rightarrow \text{CB} = \dfrac{6}{\sin\alpha} \Rightarrow \text{CB} = 6\operatorname{cosec}\alpha \right)$ ***A1***

so $L = \dfrac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$ ***AG***

[2 marks]

(b) (i) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{3}{4}\sec\alpha\tan\alpha - 6\operatorname{cosec}\alpha\cot\alpha$ ***A1***

(ii) attempt to write $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha}$ in terms of $\sin\alpha, \cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \dfrac{6\cos\alpha}{\sin^2\alpha}$ OR $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \dfrac{6}{\sin\alpha\tan\alpha}\left(= \dfrac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha} \right)$

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \dfrac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\dfrac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

[5 marks]

continued…

Question 11 continued

(c) (i) attempt to use product rule (at least once) ***(M1)***

$$\frac{d^2L}{d\alpha^2} = \frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$. Allow unsimplified correct answer.

$$\left(\frac{d^2L}{d\alpha^2} = \frac{3}{4}\sec\alpha\tan^2\alpha + \frac{3}{4}\sec^3\alpha + 6\operatorname{cosec}\alpha\cot^2\alpha + 6\operatorname{cosec}^3\alpha\right)$$

(ii) attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha = \arctan 2$ substituted into their $\frac{d^2L}{d\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\alpha = \sqrt{5}$ OR $\operatorname{cosec}\alpha = \frac{\sqrt{5}}{2}$ OR $\cot\alpha = \frac{1}{2}$ OR $\cos\alpha = \frac{1}{\sqrt{5}}$ OR $\sin\alpha = \frac{2}{\sqrt{5}}$

Note: ***M1A1*** may be seen in part (d).

$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3$ (or equivalent) ***A2***

$$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$$

Note: Award ***A1*** for only two or three correct terms.
Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$\frac{d^2L}{d\alpha^2} = \frac{45}{4}\sqrt{5}$ ***AG***

[7 marks]

continued…

Question 11 continued

(d) (i) $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

(ii) $\left(L_{\min} =\right)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

(e) $\left(11.25 =\right)\frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [19 marks]

12. (a) $2t+1\times 0+0\times(3+t)$ $(=2t)$ (seen anywhere) ***(A1)***

one correct magnitude $\sqrt{1^2+1^2+0^2}, \sqrt{(2t)^2+(3+t)^2}$ ***(A1)***

correct substitution of their magnitudes and scalar product ***M1***

$2t=\sqrt{2}\times\sqrt{(2t)^2+(3+t)^2}\times\cos\frac{\pi}{3}$ OR $\cos\frac{\pi}{3}=\frac{2t}{\sqrt{2}\times\sqrt{5t^2+6t+9}}$

$4t=\sqrt{2\left(4t^2+9+6t+t^2\right)}$ OR $\frac{1}{2}=\frac{2t}{\sqrt{2\left(5t^2+6t+9\right)}}$ (or equivalent) ***A1***

$4t=\sqrt{10t^2+12t+18}$ ***AG***

[4 marks]

(b) correct quadratic equation ***A1***

$16t^2=10t^2+12t+18$, $6t^2-12t-18=0$, $t^2-2t-3=0$

valid attempt to solve their quadratic set =0 ***(M1)***

$(t+1)(t-3)$ OR $\frac{12\pm\sqrt{(-12)^2-4\times 6\times(-18)}}{12}$ OR $(t-1)^2-4$ ***(A1)***

$t=3$ ***A1***

Note: Award ***A0*** if additional answer(s) given.

[4 marks]

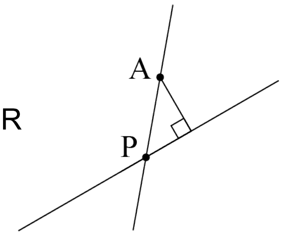
continued…

Question 12 continued

(c) **METHOD 1**

recognizing shortest distance from A is perpendicular to L_1 ***(M1)***

OR



$$\sin\frac{\pi}{3} = \frac{x}{\left|\overrightarrow{\mathrm{PA}}\right|}$$

$\left|\overrightarrow{\mathrm{PA}}\right| = \sqrt{6^2+6^2}\quad\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

$\dfrac{\sqrt{3}}{2} = \dfrac{x}{\sqrt{72}}$ ***(A1)***

$x = \dfrac{\sqrt{216}}{2}\quad\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$

shortest distance is $\dfrac{\sqrt{216}}{2}\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

METHOD 2

recognition that the distance required is $\dfrac{\left|\boldsymbol{v}\times\overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$=\dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$ ***(A1)***

$=\dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}6\\-6\\-6\end{pmatrix}\right|$ ***(A1)***

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

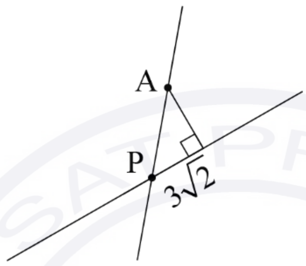
Question 12 continued

METHOD 3

recognition that the base of the triangle is $\dfrac{\left|\boldsymbol{v}\cdot\overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$\dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$

$=\dfrac{6}{\sqrt{2}}\left(=3\sqrt{2}\right)$ OR



(A1)

$\left|\overrightarrow{\mathrm{PA}}\right|=\sqrt{6^2+6^2}\quad\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

Note: The value of $\left\|\overrightarrow{\mathrm{PA}}\right\|=\sqrt{6^2+6^2}$ may be seen as part of the working of their shortest distance, $d=\sqrt{\left\|\overrightarrow{\mathrm{PA}}\right\|^2-b^2}=\sqrt{\left(\sqrt{72}\right)^2-\left(3\sqrt{2}\right)^2}$

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

METHOD 4

Let B be a general point on L_1 $(\lambda, 8+\lambda, -3)$ such that AB is perpendicular to L_1

attempt to find vector $\overrightarrow{\mathrm{AB}}$ OR $\left|\overrightarrow{\mathrm{AB}}\right|$ (the shortest distance from A to L_1) ***(M1)***

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}} = \overrightarrow{\mathrm{OP}} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \overrightarrow{\mathrm{OA}} \left(= \begin{pmatrix}0\\8\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}6\\8\\3\end{pmatrix} = \overrightarrow{\mathrm{AP}} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right) \quad (\lambda \in \mathbb{R})$$

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix}-6\\0\\-6\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ OR $\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(\lambda-6)^2 + (8+\lambda-8)^2 + (-3-3)^2}$ ***A1***

$$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(\lambda-6)^2 + \lambda^2 + (-6)^2} \left(= \sqrt{2\lambda^2 - 12\lambda + 72}\right)$$

EITHER

$\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\overrightarrow{\mathrm{AB}}\right|^2\right) = 0 \Rightarrow 4\lambda - 12 = 0 \Rightarrow \lambda = 3$ ***A1***

OR

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{2(\lambda-3)^2 + 54}$ to obtain $\lambda = 3$ ***A1***

OR

$\begin{pmatrix}-6+\lambda\\\lambda\\-6\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = 0 \Rightarrow -6 + \lambda + \lambda = 0 \Rightarrow \lambda = 3$ ***A1***

THEN

shortest distance is $\sqrt{54}$ $\left(= 3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

(d) attempt to find the vector product of two direction vectors ***(M1)***

$$\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}$$

$\boldsymbol{n}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ (or any scalar multiple of this) (accept $\boldsymbol{n}=<1,-1,-1>$ or equivalent) ***A1***

Note: Award ***A0*** for a final answer given in coordinate form.

[2 marks]

continued…

Question 12 continued

(e) substituting their x into volume formula and equating **(M1)**

$$\frac{1}{3}\pi\left(3\sqrt{6}\right)^2 h = 90\sqrt{3}\pi$$

$h = 5\sqrt{3}$ (seen anywhere) **A1**

recognition that the position vector of vertex is given by $\overrightarrow{\mathrm{OA}} + \mu \boldsymbol{n}$ OR $\overrightarrow{\mathrm{OA}} + h \times \hat{\boldsymbol{n}}$ **(M1)**

$$\begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \text{ OR } \left(6+\mu, 8-\mu, 3-\mu\right)$$

EITHER

recognition that $\mu|\boldsymbol{n}| = h$ (where μ is a parameter) **(M1)**

$\mu|\boldsymbol{n}| = 5\sqrt{3}$ OR $\sqrt{\mu^2 + (-\mu)^2 + (-\mu)^2} = 5\sqrt{3}$ OR $3\mu^2 = 75$ $\left(\Rightarrow \sqrt{3}\mu = 5\sqrt{3}\right)$

$\mu = \pm 5$ (accept $\mu = 5$) **(A1)**

OR

attempt to find cone's height vector $h \times \hat{\boldsymbol{n}}$ **(M1)**

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$ **(A1)**

$$5\sqrt{3} \times \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

THEN

$$= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm 5 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \left(= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm \begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} \right)$$

vertex = $(11, 3, -2)$ and $(1, 13, 8)$ (accept position vectors) **A1A1**

Note: Award a maximum of ***(M0)A0(M1)(M1)(A1)A1A1FT*** for $\left(\frac{39}{4}, \frac{17}{4}, -\frac{3}{4}\right)$ and $\left(\frac{9}{4}, \frac{47}{4}, \frac{27}{4}\right)$ obtained using $x = \left|\overrightarrow{\mathrm{PA}}\right|$ from part (c).

[7 marks]

Total [21 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Monday 31 October 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8822 – 7101
12 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The function g is defined by $g(x) = e^{x^2+1}$, where $x \in \mathbb{R}$.

Find $g'(-1)$.

2. [Maximum mark: 7]

Consider a circle with a diameter AB, where A has coordinates $(1, 4, 0)$ and B has coordinates $(-3, 2, -4)$.

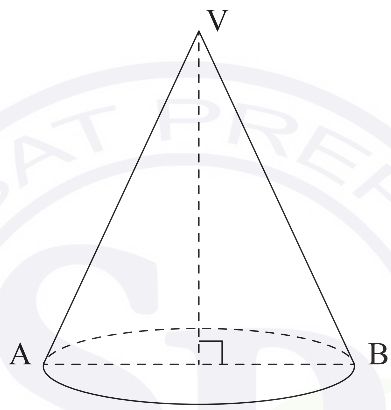
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex V has coordinates $(-1, -1, 0)$.

diagram not to scale



(b) Find the exact volume of the cone. [3]

3. [Maximum mark: 5]

Let a be a constant, where $a > 1$.

(a) Show that $a^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2+1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length a, $\left(\frac{a^2-1}{2}\right)$ and $\left(\frac{a^2+1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of a. [2]

4. [Maximum mark: 5]

The derivative of the function f is given by $f'(x) = \frac{6x}{x^2 + 1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

5. [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

(b) Find the value of p. [3]

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6. [Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.8$.

(a) Determine the value of $\mathrm{P}(A \cap B)$ in the case where the events A and B are independent. [1]

(b) Determine the minimum possible value of $\mathrm{P}(A \cap B)$. [3]

(c) Determine the maximum possible value of $\mathrm{P}(A \cap B)$, justifying your answer. [2]

7. [Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$.

8. [Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \le x \le a$ and $a, k \in \mathbb{R}^+$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [2]

(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k. [2]

9. [Maximum mark: 10]

Consider the homogeneous differential equation $\frac{dy}{dx} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of m. [2]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 20]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x$, $0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

11. [Maximum mark: 16]

Consider a three-digit code abc, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

Do **not** write solutions on this page.

12. [Maximum mark: 18]

Let z_n be the complex number defined as $z_n = (n^2 + n + 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_0)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of n. [3]

Let $w_n = z_0 z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

14 pages

2222 – 7111

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The n^{th} term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, u_1. [1]

(b) Given that the n^{th} term of this sequence is -33, find the value of n. [2]

(c) Find the common difference, d. [2]

2. [Maximum mark: 6]

Consider any three consecutive integers, $n-1$, n and $n+1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

3. [Maximum mark: 8]

A function f is defined by $f\left(x\right)=\frac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

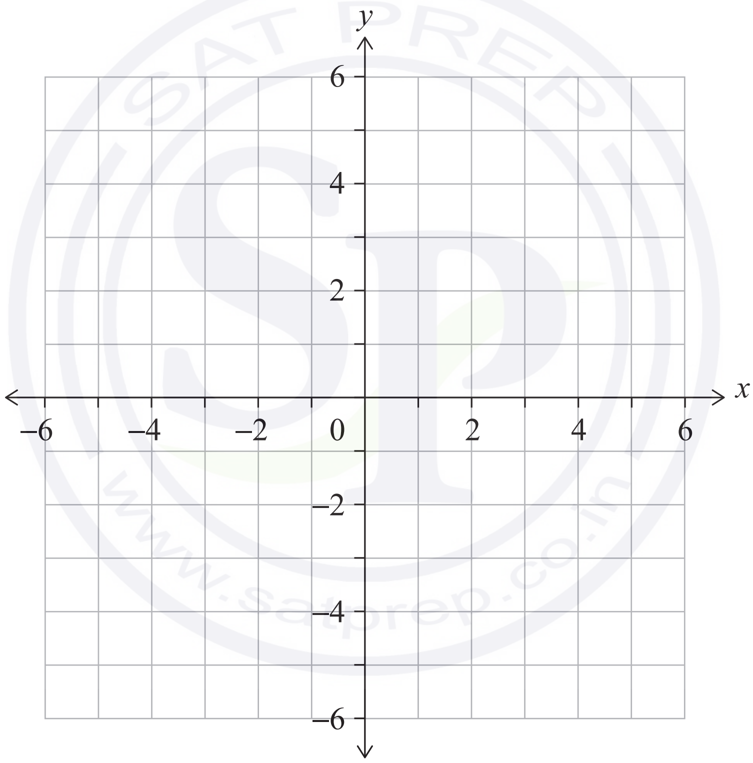
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]



(c) Hence, solve the inequality $0<\frac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\frac{2|x|-1}{|x|+1}<2$. [2]

(This question continues on the following page)

(Question 3 continued)

4. [Maximum mark: 5]

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

5. [Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of x. [5]

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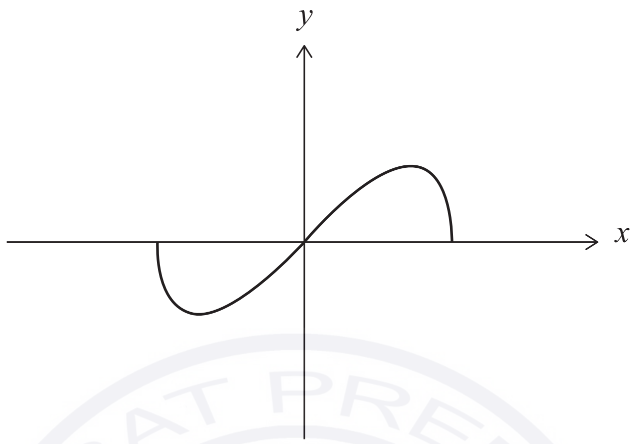
6. [Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

The range of f is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of a and the value of b. [6]

7. [Maximum mark: 6]

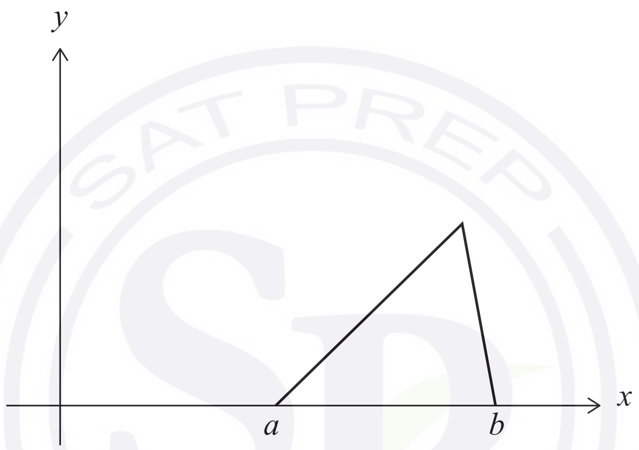
By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, \mathrm{d}x$ in terms of n, where n is a non-zero real number.

8. [Maximum mark: 6]

A continuous random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\ \dfrac{2}{(b-a)(b-c)}(b-x), & c< x\le b\\ 0, & \text{otherwise}\end{cases}.$$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.



Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of X in terms of a, b and c.

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9. [Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die with faces labelled 1, 2, 3 and 4 is rolled and the result recorded. Let X be the result obtained when the die is rolled. The probability distribution for X is given in the following table where p and q are constants.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	0.3	q	0.1

For this probability distribution, it is known that $\mathrm{E}(X)=2$.

(a) Show that $p=0.4$ and $q=0.2$. [5]

(b) Find $\mathrm{P}(X>2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four.

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair.
Both yellow dice have faces labelled 1, 2, 3 and 4. Let S represent the sum obtained by rolling the two yellow dice. The probability distribution for S is shown below.

s	2	3	4	5	6	7	8
$\mathrm{P}(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

The first red die has faces labelled 1, 2, 2 and 3. The second red die has faces labelled 1, a, a and b, where $a<b$ and $a, b \in \mathbb{Z}^+$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of b. [2]

(e) Find the value of a, providing evidence for your answer. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

A function f is defined by $f(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x > 3$.

(b) The inverse of g is g^{-1}.

(i) Show that $g^{-1}(x)=1+\dfrac{\sqrt{4x^2+x}}{x}$.

(ii) State the domain of g^{-1}. [7]

A function h is defined by $h(x)=\arctan\dfrac{x}{2}$, where $x \in \mathbb{R}$.

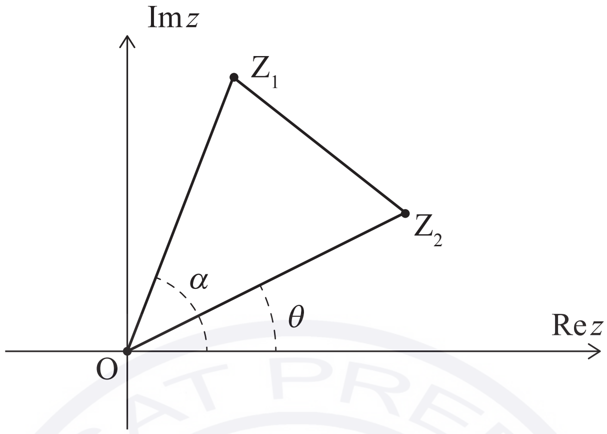
(c) Given that $(h \circ g)(a)=\dfrac{\pi}{4}$, find the value of a.

Give your answer in the form $p+\dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

Do **not** write solutions on this page.

12. [Maximum mark: 18]

In the following Argand diagram, the points Z_1, O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2e^{i\theta}$, where $r_2 > 0$.

Angles α, θ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha$, $\theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1z_2^* = r_1r_2e^{i(\alpha-\theta)}$ where z_2^* is the complex conjugate of z_2. [2]

(b) Given that $\text{Re}(z_1 z_2^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2.

(ii) Hence show that $z_1^2 + z_2^2 = z_1z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
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- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2222 – 7106

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

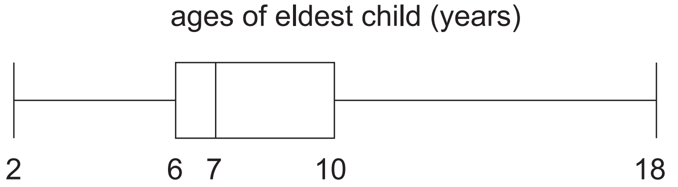
Find the value of $\int_1^9 \left(\frac{3\sqrt{x}-5}{\sqrt{x}}\right) \mathrm{d}x$.

2. [Maximum mark: 7]

A survey at a swimming pool is given to one adult in each family. The age of the adult, a years old, and of their eldest child, c years old, are recorded.

The ages of the eldest child are summarized in the following box and whisker diagram.

diagram not to scale



(a) Find the largest value of c that would not be considered an outlier. [3]

The regression line of a on c is $a = \frac{7}{4}c + 20$. The regression line of c on a is $c = \frac{1}{2}a - 9$.

(b) (i) One of the adults surveyed is 42 years old. Estimate the age of their eldest child.

(ii) Find the mean age of all the adults surveyed. [4]

3. [Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

4. [Maximum mark: 5]

Consider the curve with equation $y = (2x - 1)e^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5e^{k}x$.

Find the value of k.

5. [Maximum mark: 7]

Consider $f(x) = 4\sin x + 2.5$ and $g(x) = 4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q$, where $x \in \mathbb{R}$ and $q > 0$.

The graph of g is obtained by two transformations of the graph of f.

(a) Describe these two transformations. [2]

The y-intercept of the graph of g is at $(0, r)$.

(b) Given that $g(x) \geq 7$, find the smallest value of r. [5]

6. [Maximum mark: 5]

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

7. [Maximum mark: 8]

The continuous random variable X has probability density function

$$f(x)=\begin{cases}\dfrac{k}{\sqrt{4-3x^2}}, & 0\le x\le 1\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the value of k. [4]

(b) Find $\mathrm{E}(X)$. [4]

8. [Maximum mark: 6]

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

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9. [Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = (1 - b^2) - 2b\mathrm{i}$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the series $\ln x + p\ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R}, x > 1$ and $p \in \mathbb{R}, p \neq 0$.

(a) Consider the case where the series is geometric.

(i) Show that $p = \pm\frac{1}{\sqrt{3}}$.

(ii) Hence or otherwise, show that the series is convergent.

(iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [6]

(b) Now consider the case where the series is arithmetic with common difference d.

(i) Show that $p = \frac{2}{3}$.

(ii) Write down d in the form $k\ln x$, where $k \in \mathbb{Q}$.

(iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n. [12]

11. [Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : -9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $\mathrm{P}(1, -2, 0)$ lies on both Π_1 and Π_2.

(ii) Find a vector equation of L, the line of intersection of Π_1 and Π_2. [5]

(c) Find the distance between L and Π_3. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

The function f is defined by $f(x) = e^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 e^{x^2} \sin(x^2)\,dx$. [4]

The function g is defined by $g(x) = e^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$.

(ii) Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim_{x\to 0} \frac{e^x \cos x - 1 - x}{x^3}$. [3]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Monday 1 November 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
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- Section A: answer all questions. Answers must be written within the answer boxes provided.
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- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8821 – 7101

14 pages

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Given that $\frac{dy}{dx} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find y in terms of x.

2. [Maximum mark: 9]

The function f is defined by $f(x) = \frac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of f;

(ii) the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates where the graph of f crosses

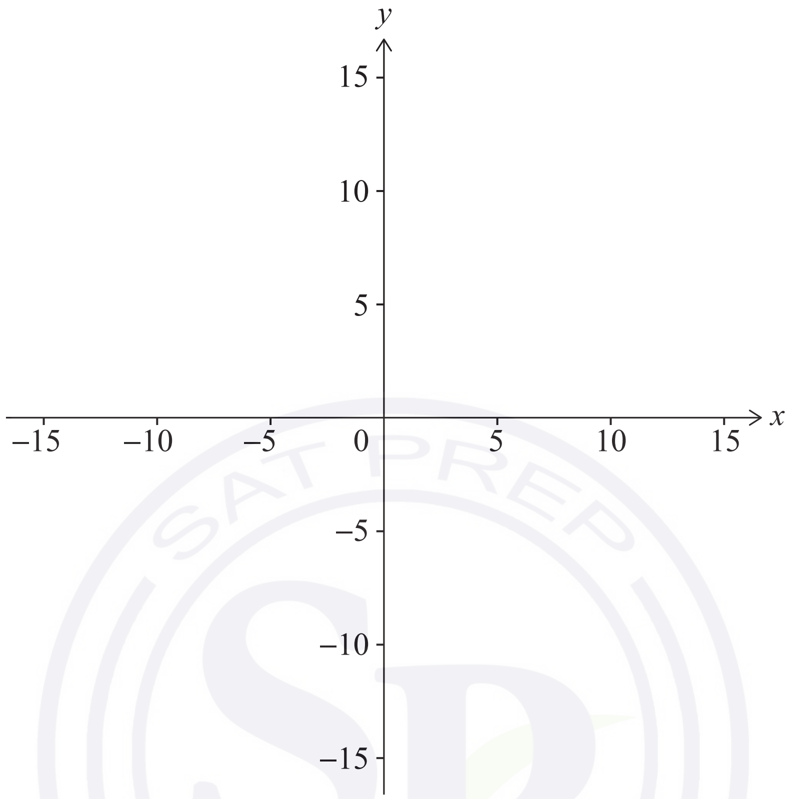
(i) the x-axis;

(ii) the y-axis. [2]

(This question continues on the following page)

(Question 2 continued)

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x) = \dfrac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of a. [4]

3. [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

4. [Maximum mark: 5]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that "box 1 is chosen" and let R be the event that "a red ball is drawn".

(b) Determine whether events A and R are independent. [2]

5. [Maximum mark: 7]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of h at $x = 4$. [3]

6. [Maximum mark: 7]

(a) Show that $2x-3-\frac{6}{x-1}=\frac{2x^2-5x-3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{4}$. [5]

7. [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

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8. [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = f(x)$.

9. [Maximum mark: 7]

Consider the expression $\dfrac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q},\ a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of a and the value of b. [6]

(b) State the restriction which must be placed on x for this expansion to be valid. [1]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A particle P moves along the x-axis. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ at time t seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0$, P is at the origin O.

(a) (i) Find the value of t when P reaches its maximum velocity.

(ii) Show that the distance of P from O at this time is $\frac{88}{27}$ metres. [7]

(b) Sketch a graph of v against t, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by P. [5]

11. [Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right) = \left[x^2 + 2nx + n(n-1)\right]\mathrm{e}^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2\mathrm{e}^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim_{x\to 0}\left[\frac{\left(x^2\mathrm{e}^x - x^2\right)^3}{x^9}\right]$. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
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- Section A: answer all questions. Answers must be written within the answer boxes provided.
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- The maximum mark for this examination paper is **[110 marks]**.

15 pages

2221 – 7111

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Consider two consecutive positive integers, n and $n + 1$.

Show that the difference of their squares is equal to the sum of the two integers.

2. [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

3. [Maximum mark: 5]

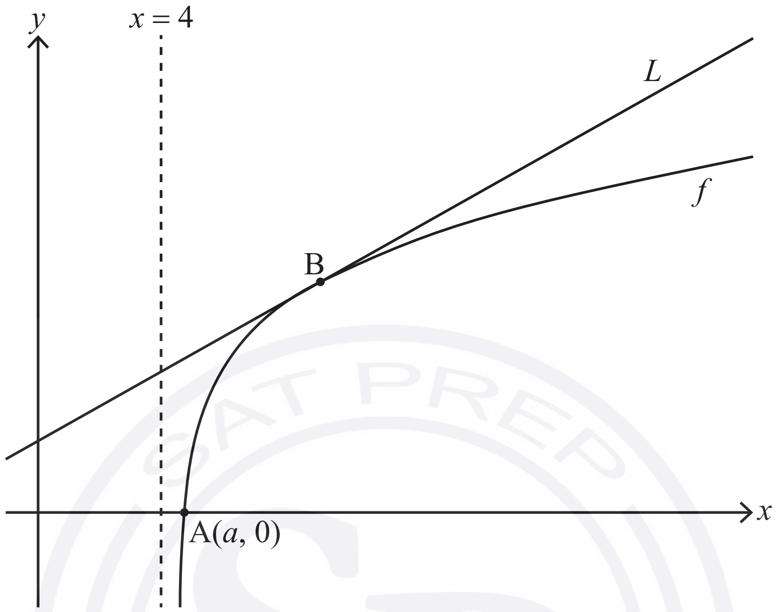
In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k.

4. [Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



(a) Find the exact value of a. [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

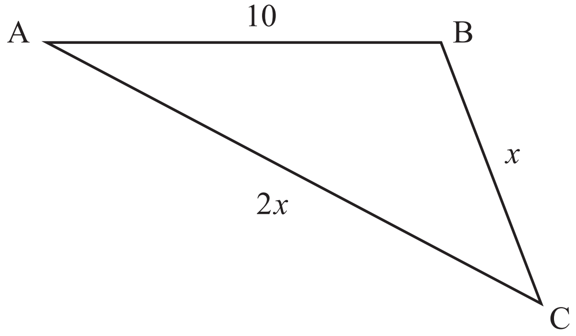
5. [Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

6. [Maximum mark: 7]

The following diagram shows triangle ABC, with $AB = 10$, $BC = x$ and $AC = 2x$.

diagram not to scale



Given that $\cos \hat{C} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

7. [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

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8. [Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not intersect. [3]

(b) Find the minimum distance between l_1 and l_2. [5]

9. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} \mathrm{d}x$.

Do **not** write solutions on this page.

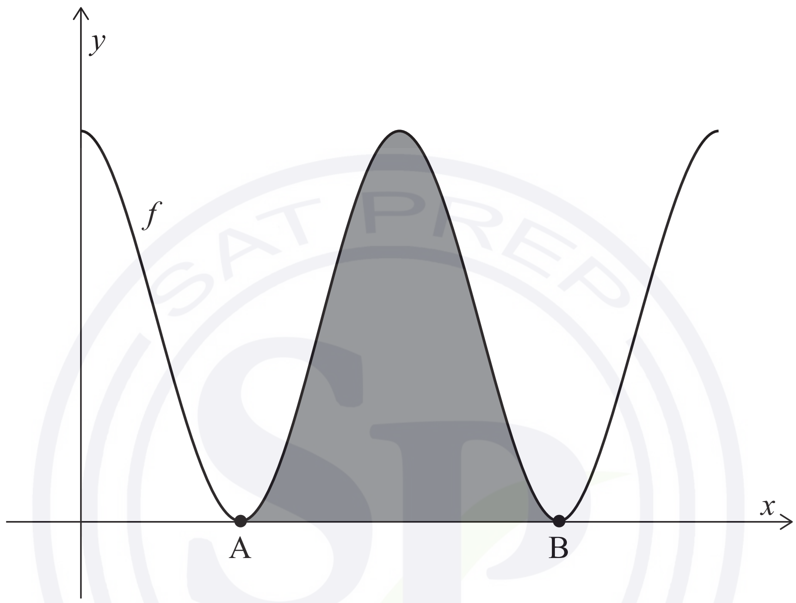
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \le x \le 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π. [5]

(This question continues on the following page)

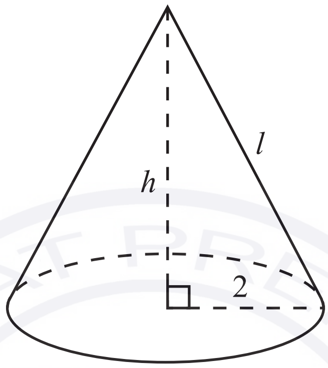
Do **not** write solutions on this page.

(Question 10 continued)

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

diagram not to scale



(c) Find the value of l. [3]

(d) Hence, find the volume of the cone. [4]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

The acceleration, $a\,\text{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\text{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\text{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\,e^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $e^T = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\,e^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T-k) = e^k - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

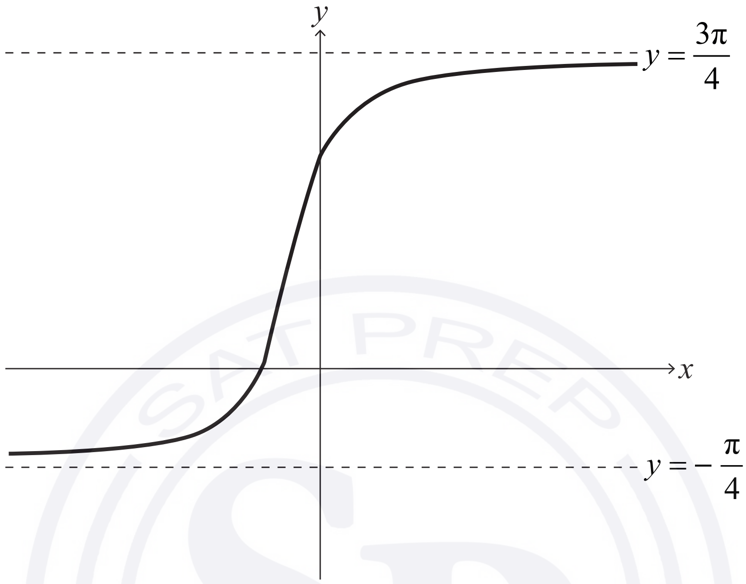
(d) Deduce a similar expression for $v(T+k)$ in terms of k. [2]

(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

Do **not** write solutions on this page.

12. [Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

References:

Please **do not** write on this page.

Answers written on this page
will not be marked.

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2221 – 7106

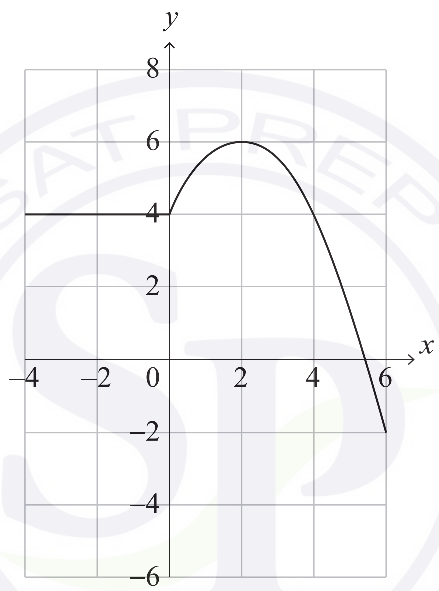
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.



(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g. [3]

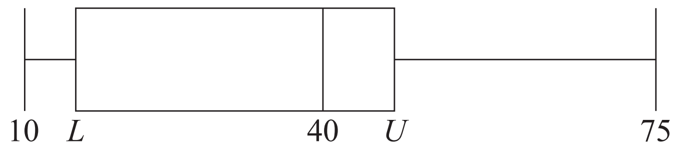
2. [Maximum mark: 5]

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, u_1, and the value of the common difference, d.

3. [Maximum mark: 5]

A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where L and U are the lower and upper quartiles respectively.

diagram not to scale



The interquartile range is 20 grams and there are no outliers in the results.

(a) Find the minimum possible value of U. [3]

(b) Hence, find the minimum possible value of L. [2]

4. [Maximum mark: 7]

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h,\ k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x = 3$.

(b) Show that $h = \frac{e+6}{2}$. [3]

(c) Hence, show that $k = e + \frac{e^2}{4}$. [3]

5. [Maximum mark: 8]

(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

6. [Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\ \theta$.

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7. [Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

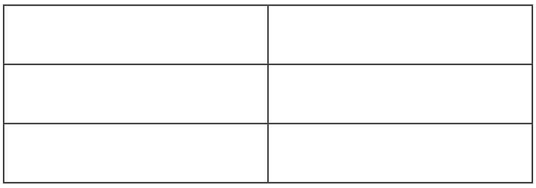
Find the possible values of a.

8. [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$.

9. [Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die, A, is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	p	p	$\frac{1}{2}p$

(a) Find the value of p. [2]

(b) Hence, find the value of $\mathrm{E}(X)$. [2]

A second biased four-sided die, B, is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$\mathrm{P}(Y=y)$	q	q	q	r

(c) (i) State the range of possible values of r.

(ii) Hence, find the range of possible values of q. [3]

(d) Hence, find the range of possible values for $\mathrm{E}(Y)$. [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is $\frac{1}{2}$.

(e) Find the value of $\mathrm{E}(Y)$. [6]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on L_1.

(ii) Find a vector equation of L_1. [4]

Consider a second line L_2 defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of a when the acute angle between L_1 and L_2 is 45°. [8]

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of k, and find the coordinates of the point A in terms of a. [7]

12. [Maximum mark: 20]

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$

for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = e^{mx}$, $m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

(c) Find the possible values of m. [8]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Specimen paper

Candidate session number

2 hours

Instructions to candidates

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

2. [Maximum mark: 5]

(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n \in \mathbb{Z}$. [2]

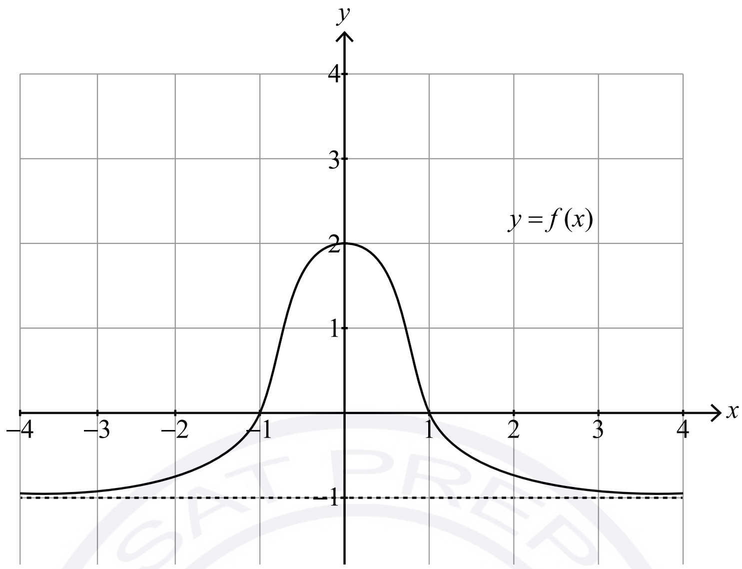
(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

3. [Maximum mark: 5]

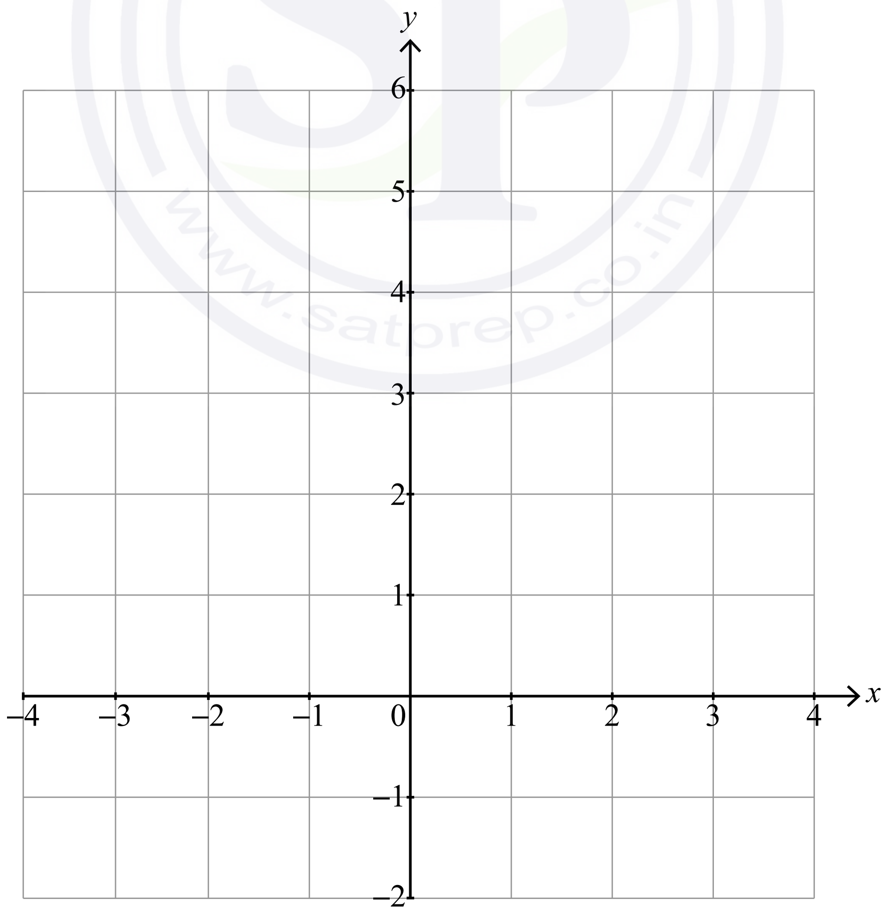
Let $f'(x) = \dfrac{8x}{\sqrt{2x^2 + 1}}$. Given that $f(0) = 5$, find $f(x)$.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



5. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = \dfrac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

6. [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3 \sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{\pi x}{36}\sin\left(\frac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$.

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8. [Maximum mark: 7]

The plane Π has the Cartesian equation $2x + y + 2z = 3$.

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

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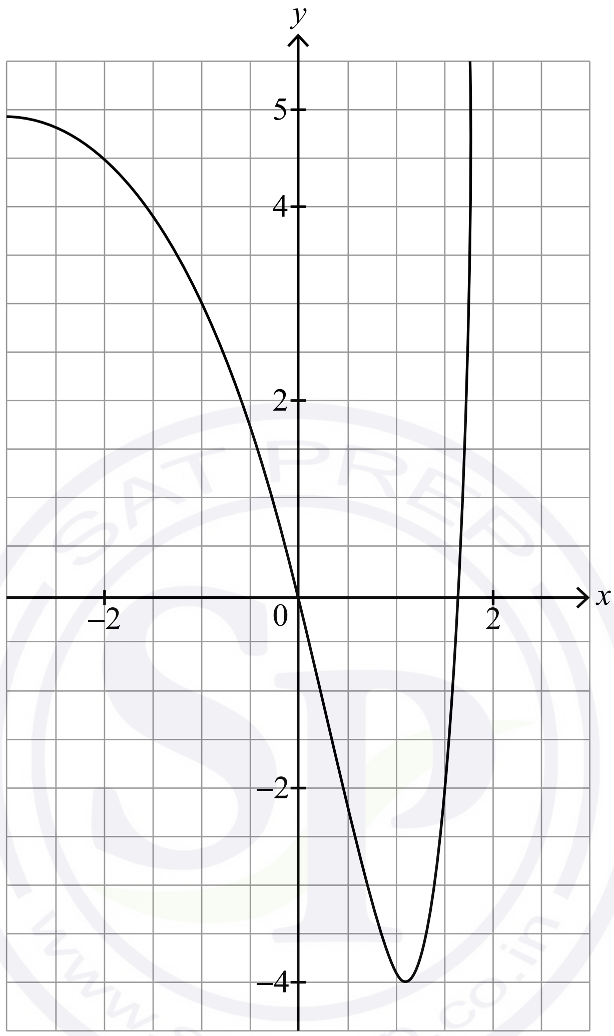
9. [Maximum mark: 8]

The function f is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

(This question continues on the following page)

(Question 9 continued)

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^{+}$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

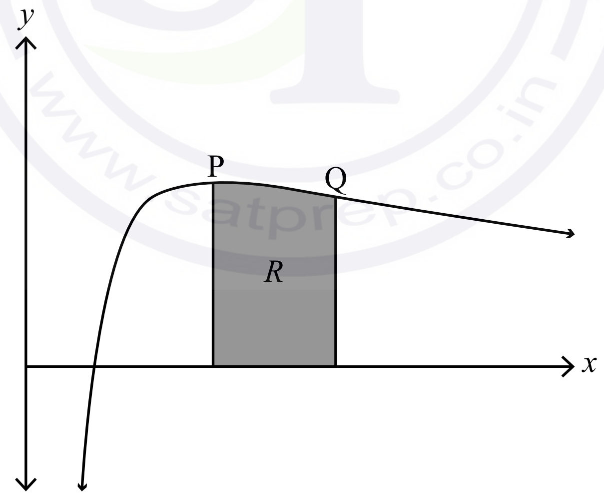
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\dfrac{1}{5}\mathrm{e}^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k. [7]

Do **not** write solutions on this page.

11. [Maximum mark: 18]

(a) Express $-3 + \sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [5]

Let the roots of the equation $z^3 = -3 + \sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that $\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$. [4]

12. [Maximum mark: 21]

The function f is defined by $f(x) = \mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x} - 1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x} - 1\right)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim_{x \to 0} \frac{f(x) - 1}{\arctan\left(\mathrm{e}^{3x} - 1\right)}$. [3]