Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 1

30 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

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- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8823 – 7111

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Answers written on this page
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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

2. [Maximum mark: 4]

Events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.75$ and $\mathrm{P}(A \cap B) = 0.6$.

(a) Find $\mathrm{P}(A \cup B)$. [2]

(b) Hence, or otherwise, find $\mathrm{P}(A' \cap B')$. [2]

3. [Maximum mark: 7]

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

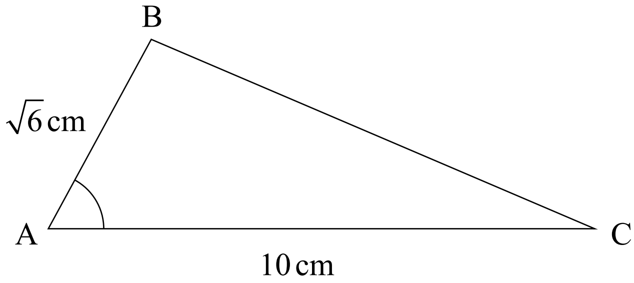
(a) Find the value of p and the value of q. [5]

(b) Find the value of u_5. [2]

4. [Maximum mark: 6]

In the following triangle ABC, $\mathrm{AB} = \sqrt{6}\ \mathrm{cm}$, $\mathrm{AC} = 10\,\mathrm{cm}$ and $\cos \mathrm{B\hat{A}C} = \frac{1}{5}$.

diagram not to scale



Find the area of triangle ABC.

5. [Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + ... + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k.

6. [Maximum mark: 7]

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by 17 for all $n \in \mathbb{Z}^+$.

7. [Maximum mark: 5]

It is given that $z = 5 + q\mathrm{i}$ satisfies the equation $z^2 + \mathrm{i}z = -p + 25\mathrm{i}$, where $p,\ q \in \mathbb{R}$.

Find the value of p and the value of q.

8. [Maximum mark: 9]

(a) Find $\int x(\ln x)^2 \mathrm{d}x$. [6]

(b) Hence, show that $\int_1^4 x(\ln x)^2 \mathrm{d}x = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$. [3]

9. [Maximum mark: 8]

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x \to 0} f(x) = 16$, find the value of k. [6]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 9)$, where $x > \frac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of d. [9]

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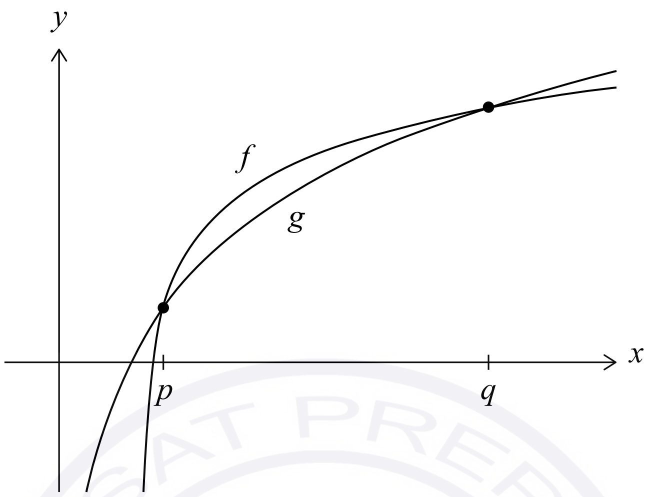
Do **not** write solutions on this page.

(Question 10 continued)

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{Z}^+$. [5]

Do **not** write solutions on this page.

11. [Maximum mark: 21]

Consider the function $f(x) = e^{\cos 2x}$, where $-\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \le x \le \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4.

(ii) Hence, find the Maclaurin series for $e^{\cos 2x - 1}$, up to and including the term in x^4.

(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [6]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} e^{\cos 2x}\,dx \approx \frac{149e}{1500}$. [3]

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12. [Maximum mark: 17]

(a) Find the binomial expansion of $(\cos\theta + i\sin\theta)^5$. Give your answer in the form $a + bi$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. [7]

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15 pages

8823 – 7106

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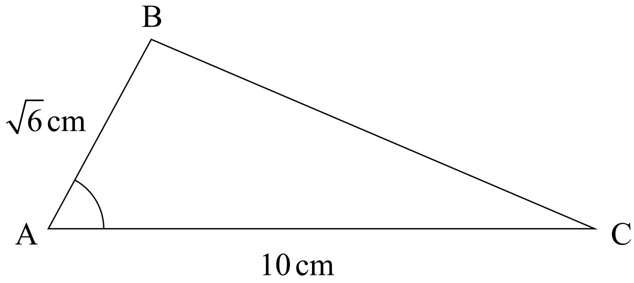

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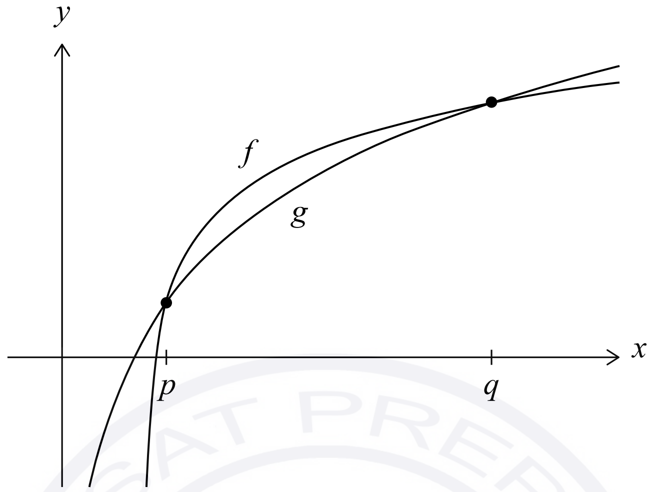

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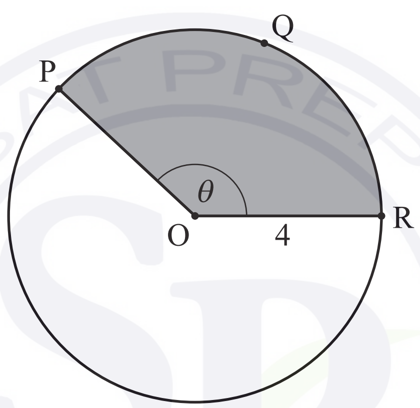
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius $4\,\mathrm{cm}$.

diagram not to scale



The points P, Q and R lie on the circumference of the circle and $\mathrm{P\hat{O}R} = \theta$, where θ is measured in radians.

The length of arc PQR is $10\,\mathrm{cm}$.

(a) Find the perimeter of the shaded sector. [2]

(b) Find θ. [2]

(c) Find the area of the shaded sector. [2]

(This question continues on the following page)

(Question 1 continued)

2. [Maximum mark: 5]

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}$, $x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(i) the y-axis;

(ii) the x-axis. [2]

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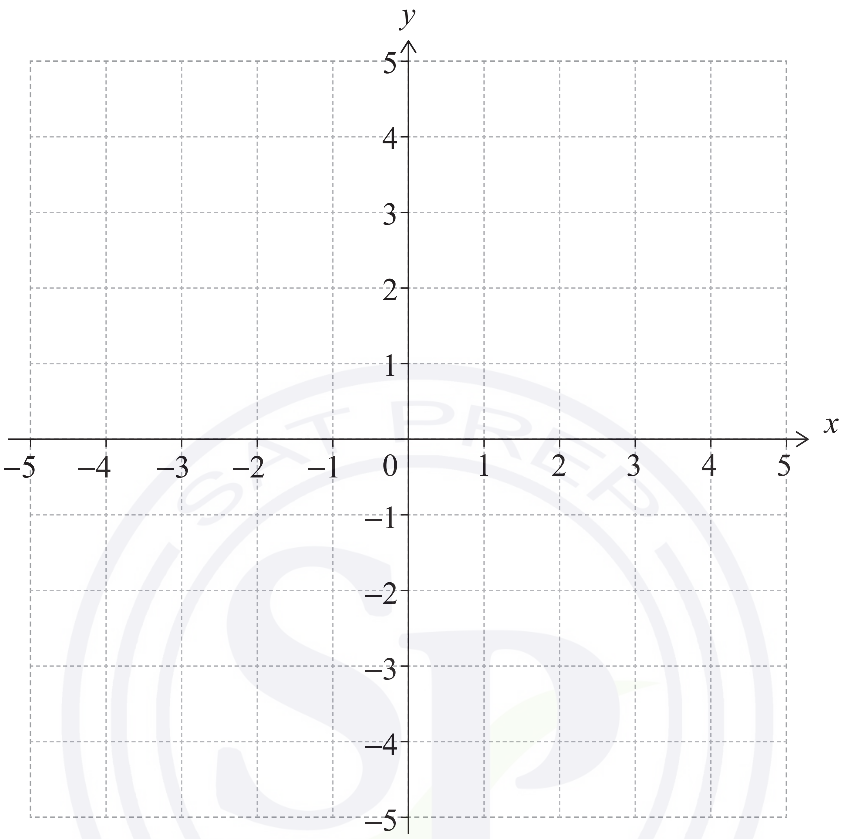
(Question 2 continued)

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b). [1]



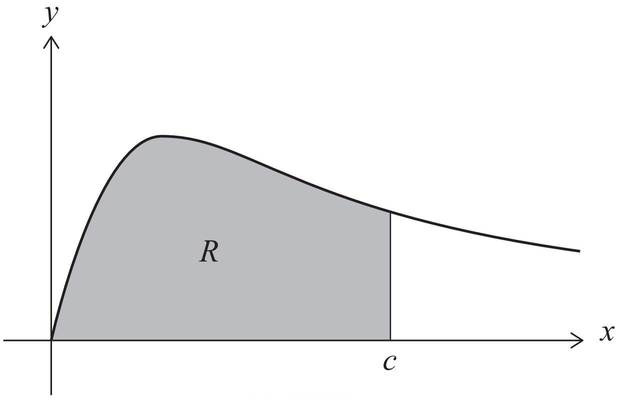
3. [Maximum mark: 5]

Events A and B are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(A|B) = 0.25$ and $\mathrm{P}(A \cup B) = 0.55$.

Find $\mathrm{P}(B)$.

4. [Maximum mark: 6]

The following diagram shows part of the graph of $y = \frac{x}{x^2 + 2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

Find the value of c.

5. [Maximum mark: 7]

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b\text{, where } a,\ b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3\,.$$

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

6. [Maximum mark: 5]

A continuous random variable X has probability density function f defined by

$$f(x)=\begin{cases}\dfrac{1}{2a}, & a \le x \le 3a\\ 0, & \text{otherwise}\end{cases}$$

where a is a positive real number.

(a) State $\mathrm{E}(X)$ in terms of a. [1]

(b) Use integration to find $\mathrm{Var}(X)$ in terms of a. [4]

7. [Maximum mark: 7]

Use mathematical induction to prove that $\sum_{r=1}^{n}\frac{r}{(r+1)!}=1-\frac{1}{(n+1)!}$ for all integers $n \geq 1$.

8. [Maximum mark: 7]

The functions f and g are defined by

$$f(x) = \cos x,\ 0 \le x \le \frac{\pi}{2}$$

$$g(x) = \tan x,\ 0 \le x < \frac{\pi}{2}.$$

The curves $y = f(x)$ and $y = g(x)$ intersect at a point P whose x-coordinate is k, where $0 < k < \frac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

(b) Hence, show that the tangent to the curve $y = f(x)$ at P and the tangent to the curve $y = g(x)$ at P intersect at right angles. [3]

(c) Find the value of $\sin k$. Give your answer in the form $\frac{a+\sqrt{b}}{c}$, where $a,\ c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

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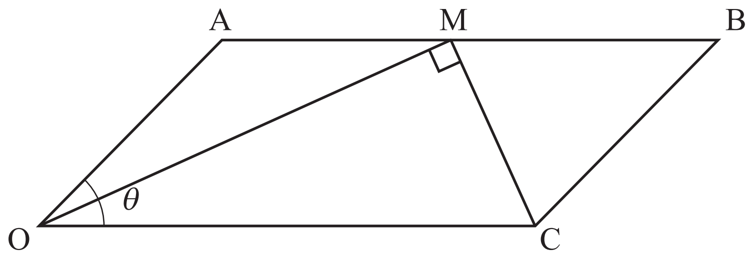
9. [Maximum mark: 9]

The following diagram shows parallelogram OABC with $\vec{OA} = \boldsymbol{a}$, $\vec{OC} = \boldsymbol{c}$ and $|\boldsymbol{c}| = 2|\boldsymbol{a}|$, where $|\boldsymbol{a}| \neq 0$.



The angle between $\vec{OA}$ and $\vec{OC}$ is θ, where $0 < \theta < \pi$.

Point M is on [AB] such that $\vec{AM} = k\vec{AB}$, where $0 \leq k \leq 1$ and $\vec{OM} \bullet \vec{MC} = 0$.

(a) Express $\vec{OM}$ and $\vec{MC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$. [2]

(b) Hence, use a vector method to show that $|\boldsymbol{a}|^2\,(1-2k)\big(2\cos\theta - (1-2k)\big) = 0$. [3]

(c) Find the range of values for θ such that there are two possible positions for M. [4]

Do **not** write solutions on this page.

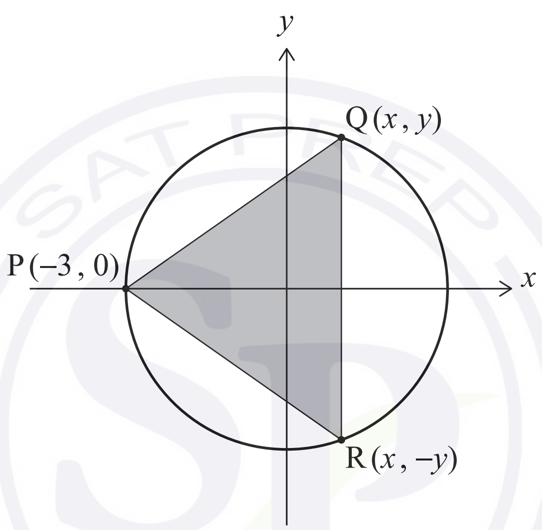
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0, 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $\text{P}(-3, 0)$, $\text{Q}(x, y)$ and $\text{R}(x, -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



(a) For point Q, show that $y = \sqrt{9 - x^2}$. [1]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

(c) Show that $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$. [4]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

Do **not** write solutions on this page.

11. [Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}\,\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

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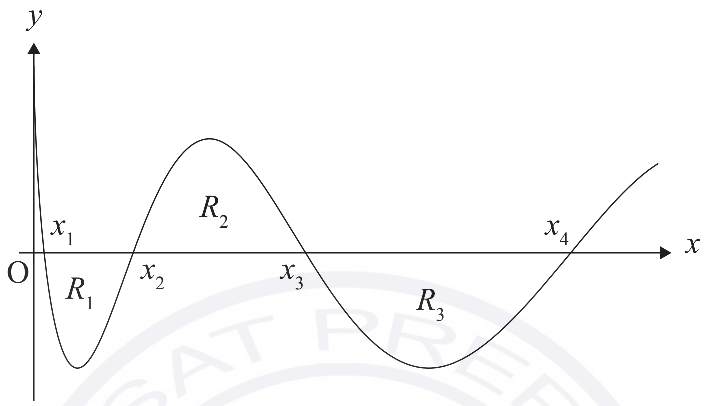
12. [Maximum mark: 17]

(a) By using an appropriate substitution, show that $\int \cos\sqrt{x}\ \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos\sqrt{x}$ for $x \geq 0$.



The curve intersects the x-axis at $x_1, x_2, x_3, x_4, \ldots$.

The nth x-intercept of the curve, x_n, is given by $x_n = \frac{(2n-1)^2\pi^2}{4}$, where $n \in \mathbb{Z}^+$.

(b) Write down a similar expression for x_{n+1}. [1]

The regions bounded by the curve and the x-axis are denoted by $R_1, R_2, R_3, \ldots$, as shown on the above diagram.

(c) Calculate the area of region R_n.

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

(d) Hence, show that the areas of the regions bounded by the curve and the x-axis, $R_1, R_2, R_3, \ldots$, form an arithmetic sequence. [3]

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- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2223 – 7106

16 pages

Please **do not** write on this page.

Answers written on this page
will not be marked.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

The function f is defined by $f(x) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}$, $x \neq 2$.

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y = f(x)$, write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

2. [Maximum mark: 6]

On a Monday at an amusement park, a sample of 40 visitors was randomly selected as they were leaving the park. They were asked how many times that day they had been on a ride called *The Dragon*. This information is summarized in the following frequency table.

Number of times on *The Dragon*	**Frequency**
0	6
1	16
2	13
3	2
4	3

It can be assumed that this sample is representative of all visitors to the park for the following day.

(a) For the following day, Tuesday, estimate

(i) the probability that a randomly selected visitor will ride *The Dragon*;

(ii) the expected number of times a visitor will ride *The Dragon*. [4]

It is known that 1000 visitors will attend the amusement park on Tuesday. *The Dragon* can carry a maximum of 10 people each time it runs.

(b) Estimate the minimum number of times *The Dragon* must run to satisfy demand. [2]

3. [Maximum mark: 6]

Solve $\cos 2x = \sin x$, where $-\pi \leq x \leq \pi$.

4. [Maximum mark: 6]

Find the range of possible values of k such that $e^{2x} + \ln k = 3e^{x}$ has at least one real solution.

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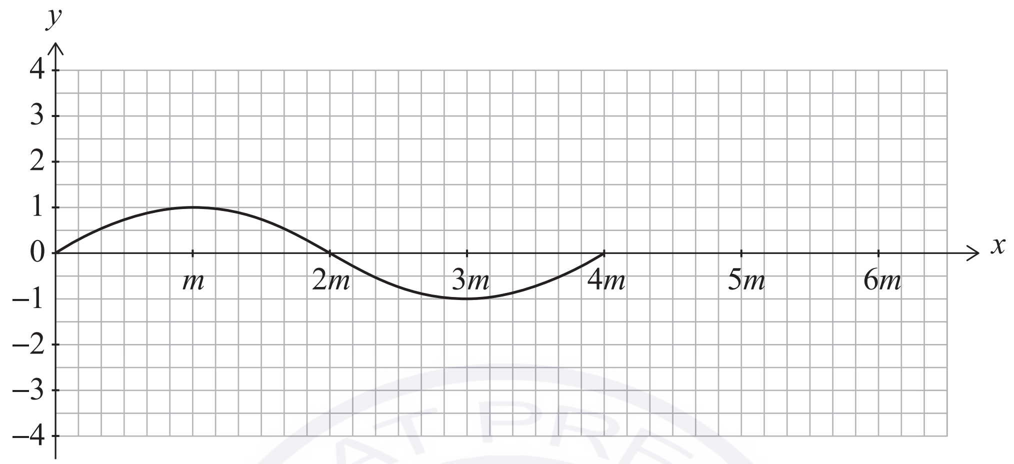
5. [Maximum mark: 6]

The function f is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of f for $0 \le x \le 4m$, where x is in radians. There are x-intercepts at $x = 0$, $2m$ and $4m$.



(a) Find an expression for m in terms of q. [2]

The function g is defined by $g(x) = 3\sin\frac{2qx}{3}$, for $0 \le x \le 6m$.

(b) On the axes above, sketch the graph of g. [4]

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6. [Maximum mark: 5]

The side lengths, $x\,\text{cm}$, of an equilateral triangle are increasing at a rate of $4\,\text{cm}\,\text{s}^{-1}$.

Find the rate at which the area of the triangle, $A\,\text{cm}^2$, is increasing when the side lengths are $5\sqrt{3}$ cm.

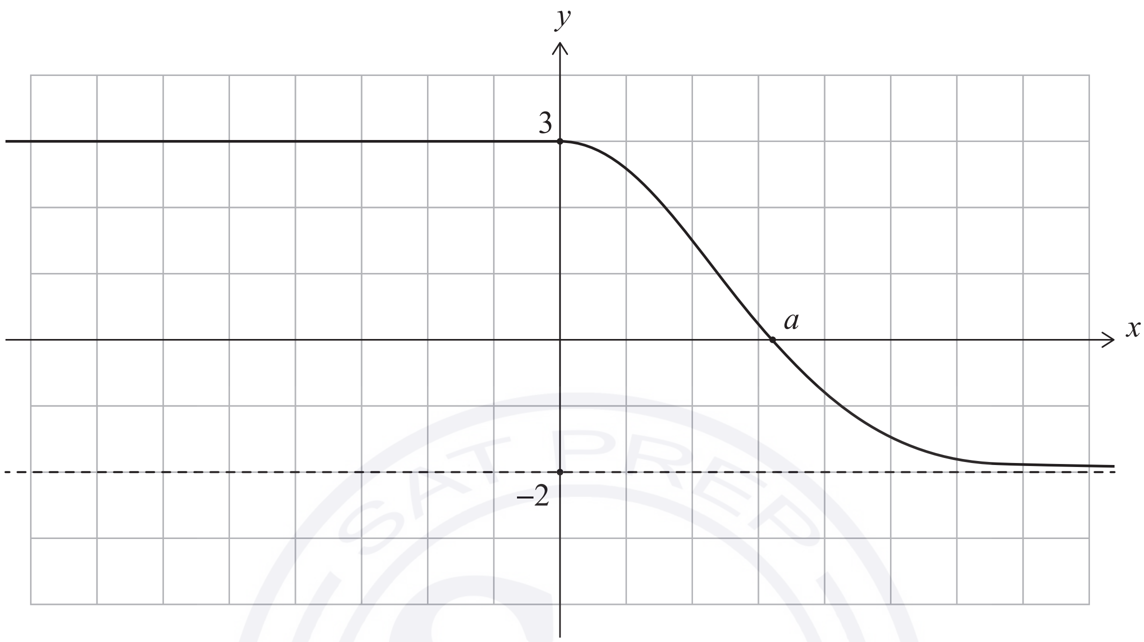
7. [Maximum mark: 6]

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\text{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$.

8. [Maximum mark: 7]

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0, 3)$, an x-intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.

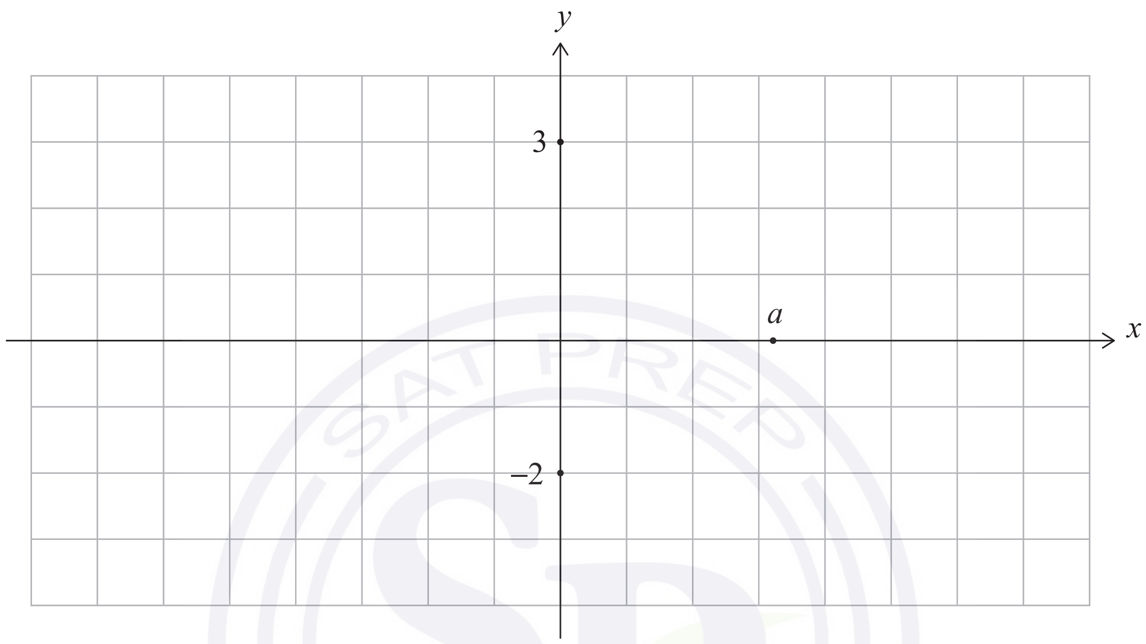


(This question continues on the following page)

(Question 8 continued)

Consider the function $g(x) = \left| f(|x|) \right|$.

(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote. [4]



(b) Find the possible values of k such that $(g(x))^2 = k$ has exactly two solutions. [3]

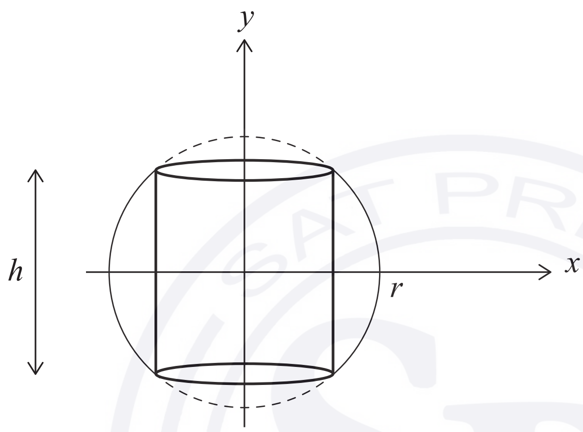
9. [Maximum mark: 7]

The function f is defined by $f(y) = \sqrt{r^2 - y^2}$ for $-r \le y \le r$.

The region enclosed by the graph of $x = f(y)$ and the y-axis is rotated by $360°$ about the y-axis to form a solid sphere. The sphere is drilled through along the y-axis, creating a cylindrical hole. The resulting spherical ring has height, h.

This information is shown in the following diagrams.

diagram not to scale



The spherical ring has a volume of π cubic units. Find the value of h.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a) (i) Find the sum of the first five terms.

(ii) Given that $S_6 = 60$, find u_6. [4]

(b) Find u_1. [2]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

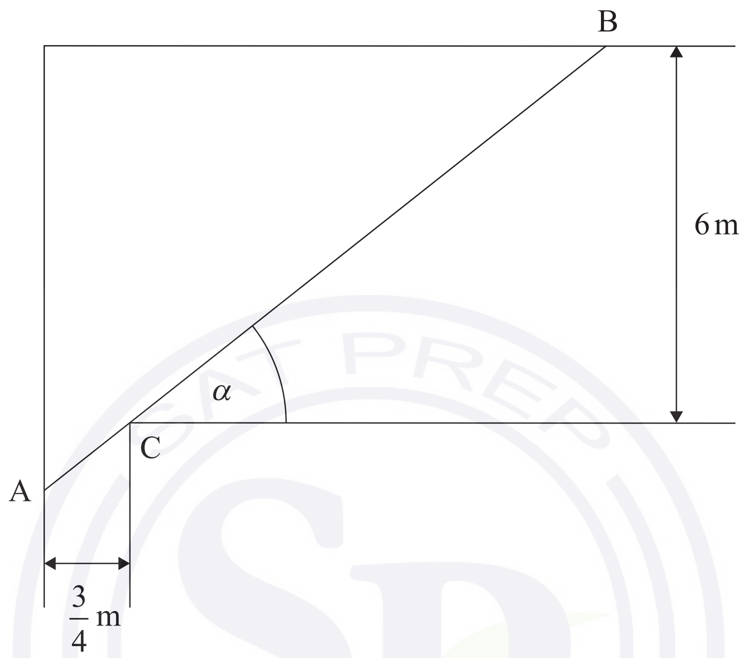
(e) Given that $v_{99} < 0$, find v_5. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with width $\frac{3}{4}$ m is perpendicular to a room of width $6\,\text{m}$. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that [AB] makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$. [2]

(b) (i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$.

(ii) When $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, show that $\alpha = \arctan 2$. [5]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 11 continued)

(c) (i) Find $\frac{d^2L}{d\alpha^2}$.

(ii) When $\alpha = \arctan 2$, show that $\frac{d^2L}{d\alpha^2} = \frac{45}{4}\sqrt{5}$. [7]

(d) (i) Hence, justify that L is a minimum when $\alpha = \arctan 2$.

(ii) Determine this minimum value of L. [3]

Two people need to carry a pole of length $11.25\,\text{m}$ from the passageway into the room. It must be carried horizontally.

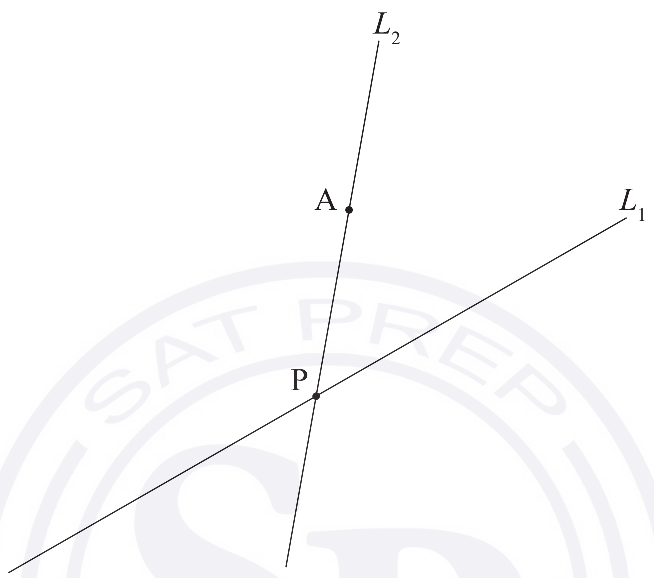
(e) Determine whether this is possible, giving a reason for your answer. [2]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Two lines, L_1 and L_2, intersect at point P. Point $\mathrm{A}(2t, 8, 3)$, where $t > 0$, lies on L_2. This is shown in the following diagram.

diagram not to scale



The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of L_1 is $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\overrightarrow{\mathrm{PA}} = \begin{pmatrix}2t\\0\\3+t\end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$. [4]

(b) Find the value of t. [4]

(c) Hence or otherwise, find the shortest distance from A to L_1. [4]

A plane, Π, contains L_1 and L_2.

(d) Find a normal vector to Π. [2]

The base of a right cone lies in Π, centred at A such that L_1 is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Monday 31 October 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8822 – 7101

12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The function g is defined by $g(x) = e^{x^2+1}$, where $x \in \mathbb{R}$.

Find $g'(-1)$.

2. [Maximum mark: 7]

Consider a circle with a diameter AB, where A has coordinates $(1, 4, 0)$ and B has coordinates $(-3, 2, -4)$.

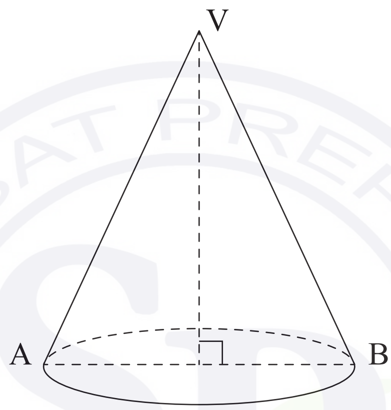
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex V has coordinates $(-1, -1, 0)$.

diagram not to scale



(b) Find the exact volume of the cone. [3]

3. [Maximum mark: 5]

Let a be a constant, where $a > 1$.

(a) Show that $a^2 + \left(\frac{a^2 - 1}{2}\right)^2 = \left(\frac{a^2 + 1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length a, $\left(\frac{a^2 - 1}{2}\right)$ and $\left(\frac{a^2 + 1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of a. [2]

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4. [Maximum mark: 5]

The derivative of the function f is given by $f'(x) = \frac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

5. [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

(b) Find the value of p. [3]

6. [Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.8$.

(a) Determine the value of $\mathrm{P}(A \cap B)$ in the case where the events A and B are independent. [1]

(b) Determine the minimum possible value of $\mathrm{P}(A \cap B)$. [3]

(c) Determine the maximum possible value of $\mathrm{P}(A \cap B)$, justifying your answer. [2]

7. [Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$.

8. [Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \le x \le a$ and $a, k \in \mathbb{R}^+$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [2]

(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k. [2]

9. [Maximum mark: 10]

Consider the homogeneous differential equation $\frac{dy}{dx} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of m. [2]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 20]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x$, $0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

11. [Maximum mark: 16]

Consider a three-digit code abc, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

Do **not** write solutions on this page.

12. [Maximum mark: 18]

Let z_n be the complex number defined as $z_n = (n^2 + n + 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_0)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of n. [3]

Let $w_n = z_0 z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\dfrac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2222 – 7111

14 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The n^{th} term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, u_1. [1]

(b) Given that the n^{th} term of this sequence is -33, find the value of n. [2]

(c) Find the common difference, d. [2]

2. [Maximum mark: 6]

Consider any three consecutive integers, $n-1$, n and $n+1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

3. [Maximum mark: 8]

A function f is defined by $f(x)=\dfrac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

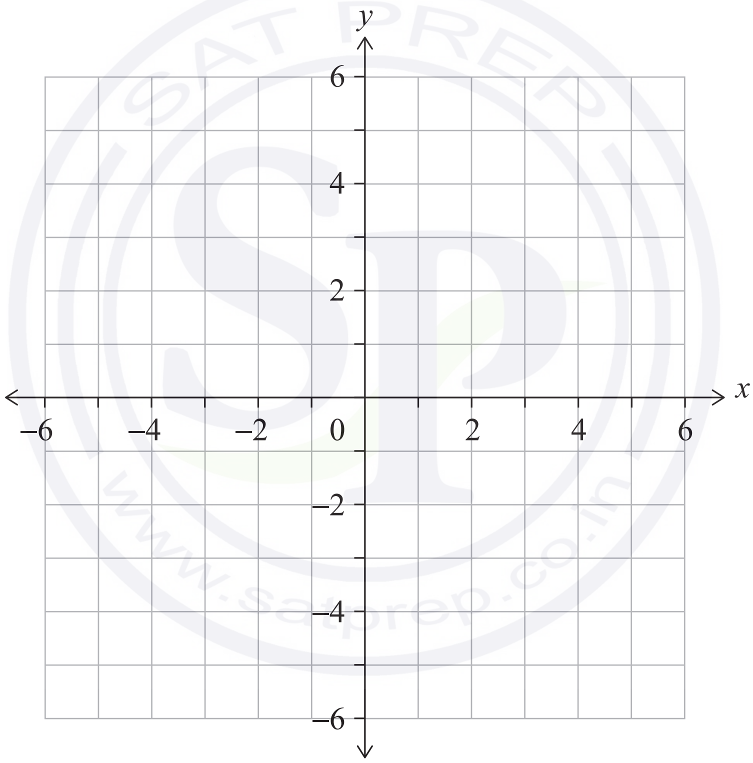
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]



(c) Hence, solve the inequality $0<\dfrac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\dfrac{2|x|-1}{|x|+1}<2$. [2]

(This question continues on the following page)

(Question 3 continued)

4. [Maximum mark: 5]

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

5. [Maximum mark: 7]

Consider the binomial expansion $(x + 1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + ... + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of x. [5]

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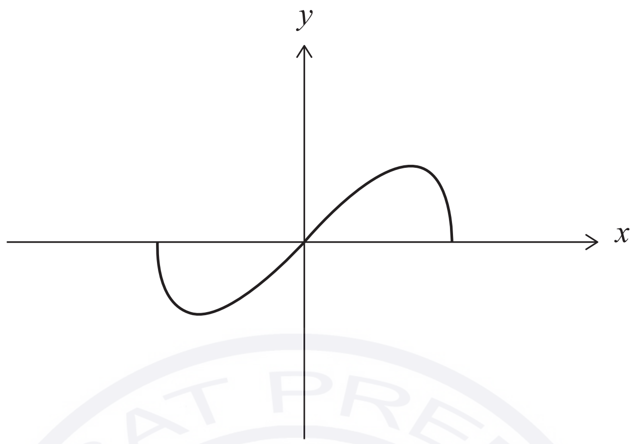
6. [Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

The range of f is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of a and the value of b. [6]

7. [Maximum mark: 6]

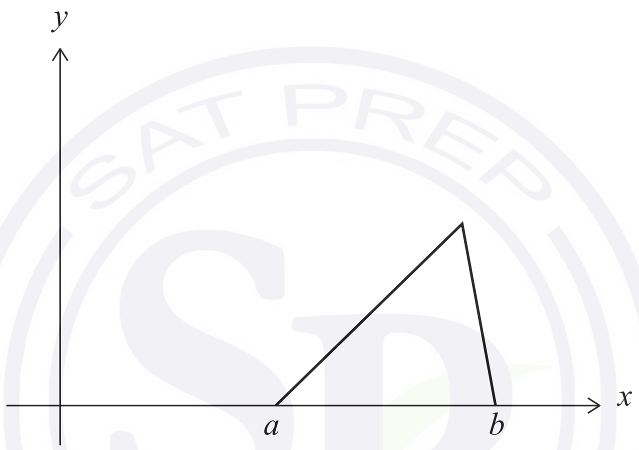
By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, dx$ in terms of n, where n is a non-zero real number.

8. [Maximum mark: 6]

A continuous random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\[2mm] \dfrac{2}{(b-a)(b-c)}(b-x), & c<x\le b\\[2mm] 0, & \text{otherwise}\end{cases}.$$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.



Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of X in terms of a, b and c.

9. [Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die with faces labelled 1, 2, 3 and 4 is rolled and the result recorded. Let X be the result obtained when the die is rolled. The probability distribution for X is given in the following table where p and q are constants.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	0.3	q	0.1

For this probability distribution, it is known that $\mathrm{E}(X)=2$.

(a) Show that $p=0.4$ and $q=0.2$. [5]

(b) Find $\mathrm{P}(X>2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four.

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair.
Both yellow dice have faces labelled 1, 2, 3 and 4. Let S represent the sum obtained by rolling the two yellow dice. The probability distribution for S is shown below.

s	2	3	4	5	6	7	8
$\mathrm{P}(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

The first red die has faces labelled 1, 2, 2 and 3. The second red die has faces labelled 1, a, a and b, where $a<b$ and $a, b \in \mathbb{Z}^{+}$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of b. [2]

(e) Find the value of a, providing evidence for your answer. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

A function f is defined by $f(x) = \dfrac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x) = \dfrac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x > 3$.

(b) The inverse of g is g^{-1}.

(i) Show that $g^{-1}(x) = 1 + \dfrac{\sqrt{4x^2 + x}}{x}$.

(ii) State the domain of g^{-1}. [7]

A function h is defined by $h(x) = \arctan \dfrac{x}{2}$, where $x \in \mathbb{R}$.

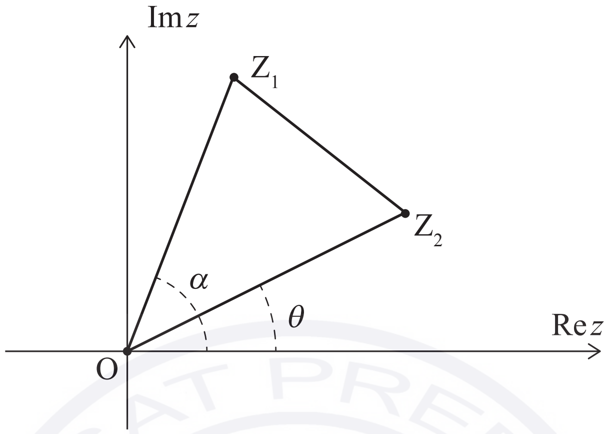
(c) Given that $(h \circ g)(a) = \dfrac{\pi}{4}$, find the value of a.

Give your answer in the form $p + \dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

Do **not** write solutions on this page.

12. [Maximum mark: 18]

In the following Argand diagram, the points Z_1, O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles α, θ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha,\ \theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2{}^* = r_1 r_2 e^{i(\alpha - \theta)}$ where $z_2{}^*$ is the complex conjugate of z_2. [2]

(b) Given that $\mathrm{Re}(z_1 z_2{}^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2.

(ii) Hence show that $z_1{}^2 + z_2{}^2 = z_1 z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
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2222 – 7106

12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

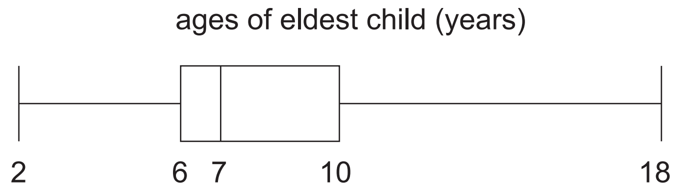
Find the value of $\int_1^9 \left(\frac{3\sqrt{x} - 5}{\sqrt{x}} \right) \mathrm{d}x$.

2. [Maximum mark: 7]

A survey at a swimming pool is given to one adult in each family. The age of the adult, a years old, and of their eldest child, c years old, are recorded.

The ages of the eldest child are summarized in the following box and whisker diagram.

diagram not to scale



(a) Find the largest value of c that would not be considered an outlier. [3]

The regression line of a on c is $a = \frac{7}{4}c + 20$. The regression line of c on a is $c = \frac{1}{2}a - 9$.

(b) (i) One of the adults surveyed is 42 years old. Estimate the age of their eldest child.

(ii) Find the mean age of all the adults surveyed. [4]

3. [Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

4. [Maximum mark: 5]

Consider the curve with equation $y = (2x - 1)e^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5e^{k}x$.

Find the value of k.

5. [Maximum mark: 7]

Consider $f(x) = 4\sin x + 2.5$ and $g(x) = 4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q$, where $x \in \mathbb{R}$ and $q > 0$.

The graph of g is obtained by two transformations of the graph of f.

(a) Describe these two transformations. [2]

The y-intercept of the graph of g is at $(0, r)$.

(b) Given that $g(x) \geq 7$, find the smallest value of r. [5]

6. [Maximum mark: 5]

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

7. [Maximum mark: 8]

The continuous random variable X has probability density function

$$f(x)=\begin{cases}\dfrac{k}{\sqrt{4-3x^2}}, & 0\le x\le 1\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the value of k. [4]

(b) Find $\mathrm{E}(X)$. [4]

8. [Maximum mark: 6]

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

9. [Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = (1 - b^2) - 2b\mathrm{i}$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the series $\ln x + p\ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R}, x > 1$ and $p \in \mathbb{R}, p \neq 0$.

(a) Consider the case where the series is geometric.

(i) Show that $p = \pm\frac{1}{\sqrt{3}}$.

(ii) Hence or otherwise, show that the series is convergent.

(iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [6]

(b) Now consider the case where the series is arithmetic with common difference d.

(i) Show that $p = \frac{2}{3}$.

(ii) Write down d in the form $k\ln x$, where $k \in \mathbb{Q}$.

(iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n. [12]

11. [Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : -9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $\mathrm{P}(1, -2, 0)$ lies on both Π_1 and Π_2.

(ii) Find a vector equation of L, the line of intersection of Π_1 and Π_2. [5]

(c) Find the distance between L and Π_3. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

The function f is defined by $f(x) = \mathrm{e}^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 \mathrm{e}^{x^2} \sin(x^2)\,\mathrm{d}x$. [4]

The function g is defined by $g(x) = \mathrm{e}^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2\left(g'(x) - g(x)\right)$.

(ii) Hence, deduce that $g^{(4)}(x) = 2\left(g'''(x) - g''(x)\right)$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim\limits_{x \to 0} \dfrac{\mathrm{e}^x \cos x - 1 - x}{x^3}$. [3]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Monday 1 November 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

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8821 – 7101

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find y in terms of x.

2. [Maximum mark: 9]

The function f is defined by $f\left(x\right)=\dfrac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of f;

(ii) the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates where the graph of f crosses

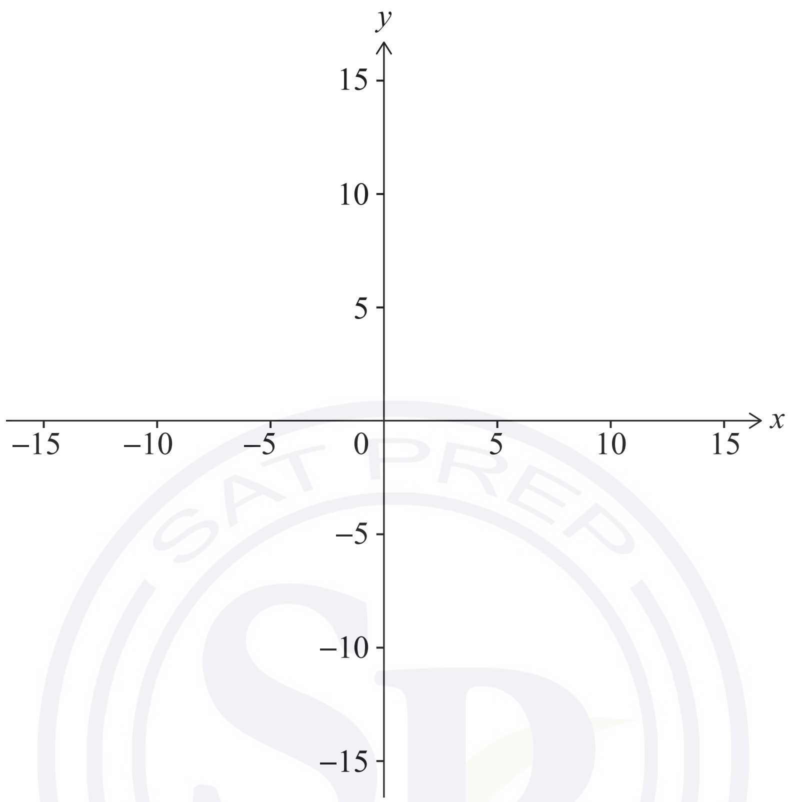
(i) the x-axis;

(ii) the y-axis. [2]

(This question continues on the following page)

(Question 2 continued)

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x)=\dfrac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x)=g^{-1}(x)$, determine the value of a. [4]

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3. [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

4. [Maximum mark: 5]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that "box 1 is chosen" and let R be the event that "a red ball is drawn".

(b) Determine whether events A and R are independent. [2]

5. [Maximum mark: 7]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of h at $x = 4$. [3]

6. [Maximum mark: 7]

(a) Show that $2x-3-\frac{6}{x-1}=\frac{2x^2-5x-3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{4}$. [5]

7. [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

8. [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = f(x)$.

9. [Maximum mark: 7]

Consider the expression $\dfrac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q},\ a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of a and the value of b. [6]

(b) State the restriction which must be placed on x for this expansion to be valid. [1]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A particle P moves along the x-axis. The velocity of P is $v\,\text{m}\,\text{s}^{-1}$ at time t seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0$, P is at the origin O.

(a) (i) Find the value of t when P reaches its maximum velocity.

(ii) Show that the distance of P from O at this time is $\dfrac{88}{27}$ metres. [7]

(b) Sketch a graph of v against t, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by P. [5]

11. [Maximum mark: 14]

(a) Prove by mathematical induction that $\dfrac{\text{d}^n}{\text{d}x^n}\left(x^2\text{e}^x\right) = \left[x^2 + 2nx + n(n-1)\right]\text{e}^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2\text{e}^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim\limits_{x \to 0}\left[\dfrac{\left(x^2\text{e}^x - x^2\right)^3}{x^9}\right]$. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

References:

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Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

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2221 – 7111

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Consider two consecutive positive integers, n and $n+1$.

Show that the difference of their squares is equal to the sum of the two integers.

2. [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

3. [Maximum mark: 5]

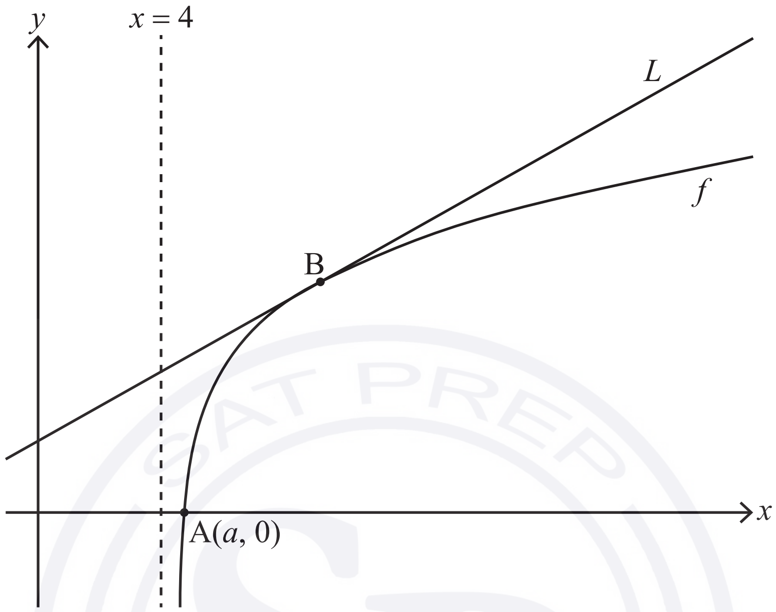
In the expansion of $(x + k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k.

4. [Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



(a) Find the exact value of a. [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

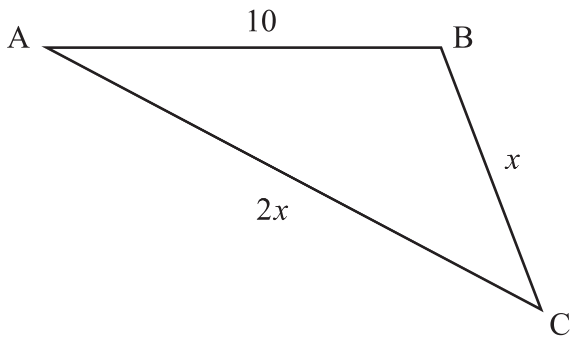
5. [Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

6. [Maximum mark: 7]

The following diagram shows triangle ABC, with $\mathrm{AB} = 10$, $\mathrm{BC} = x$ and $\mathrm{AC} = 2x$.

diagram not to scale



Given that $\cos \hat{\mathrm{C}} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

7. [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

8. [Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not intersect. [3]

(b) Find the minimum distance between l_1 and l_2. [5]

9. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} \mathrm{d}x$.

Do **not** write solutions on this page.

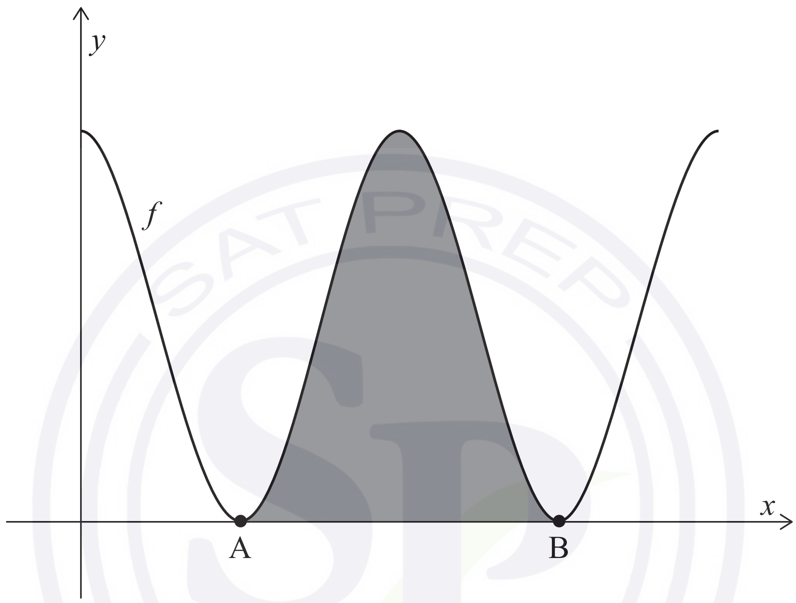
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \le x \le 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π. [5]

(This question continues on the following page)

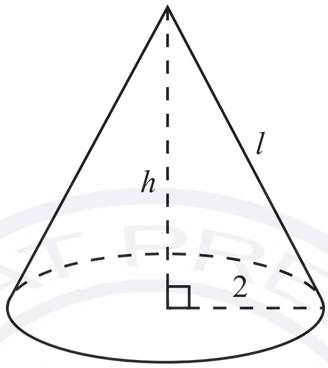
Do **not** write solutions on this page.

(Question 10 continued)

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

diagram not to scale



(c) Find the value of l. [3]

(d) Hence, find the volume of the cone. [4]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

The acceleration, $a\,\text{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\text{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\text{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\text{e}^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\text{e}^{T} = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T - k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T - k) = (1 + v_0)\text{e}^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T - k) = \text{e}^{k} - 1$. [2]

Similarly, let $v(T + k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

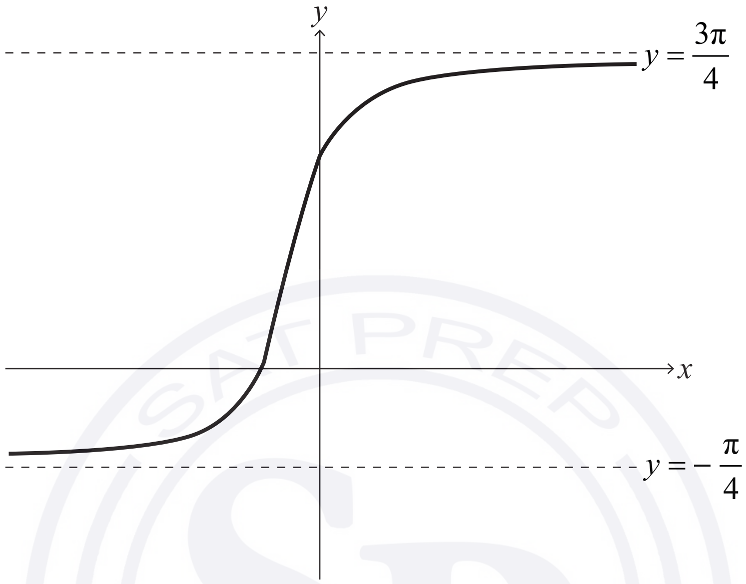
(d) Deduce a similar expression for $v(T + k)$ in terms of k. [2]

(e) Hence, show that $v(T - k) + v(T + k) \geq 0$. [3]

Do **not** write solutions on this page.

12. [Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

References:

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Diploma Programme
Programme du diplôme
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Mathematics: analysis and approaches
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2221 – 7106

12 pages

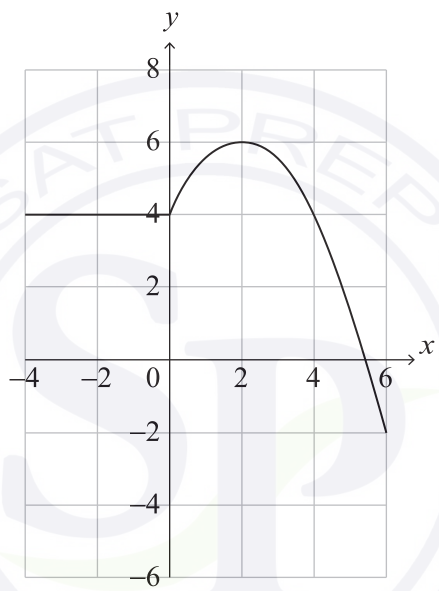
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.



(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2}f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g. [3]

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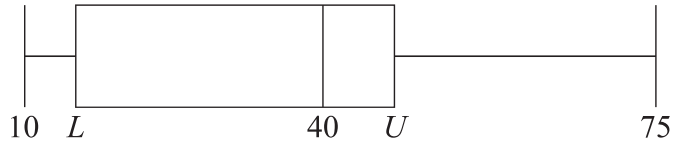
2. [Maximum mark: 5]

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, u_1, and the value of the common difference, d.

3. [Maximum mark: 5]

A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where L and U are the lower and upper quartiles respectively.

diagram not to scale



The interquartile range is 20 grams and there are no outliers in the results.

(a) Find the minimum possible value of U. [3]

(b) Hence, find the minimum possible value of L. [2]

4. [Maximum mark: 7]

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h, k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x = 3$.

(b) Show that $h = \frac{e+6}{2}$. [3]

(c) Hence, show that $k = e + \frac{e^2}{4}$. [3]

5. [Maximum mark: 8]

(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

6. [Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\ \theta$.

7. [Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

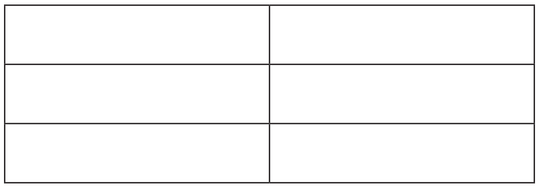
Find the possible values of a.

8. [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$.

9. [Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die, A, is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$P(X = x)$	p	p	p	$\frac{1}{2}p$

(a) Find the value of p. [2]

(b) Hence, find the value of $E(X)$. [2]

A second biased four-sided die, B, is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$P(Y = y)$	q	q	q	r

(c) (i) State the range of possible values of r.

(ii) Hence, find the range of possible values of q. [3]

(d) Hence, find the range of possible values for $E(Y)$. [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is $\frac{1}{2}$.

(e) Find the value of $E(Y)$. [6]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on L_1.

(ii) Find a vector equation of L_1. [4]

Consider a second line L_2 defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of a when the acute angle between L_1 and L_2 is 45°. [8]

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of k, and find the coordinates of the point A in terms of a. [7]

12. [Maximum mark: 20]

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$

for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = \mathrm{e}^{mx}$, $m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

(c) Find the possible values of m. [8]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Specimen paper

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2 hours

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13 pages

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

2. [Maximum mark: 5]

(a) Show that $(2n-1)^2 + (2n+1)^2 = 8n^2 + 2$, where $n \in \mathbb{Z}$. [2]

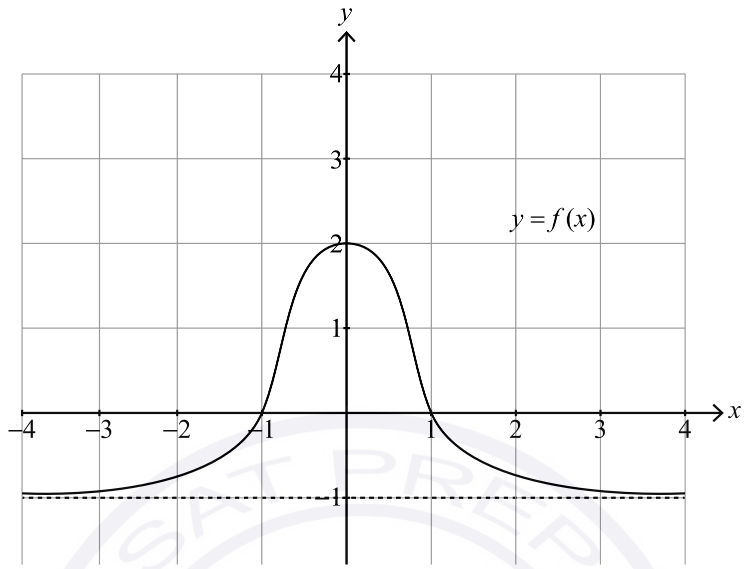
(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

3. [Maximum mark: 5]

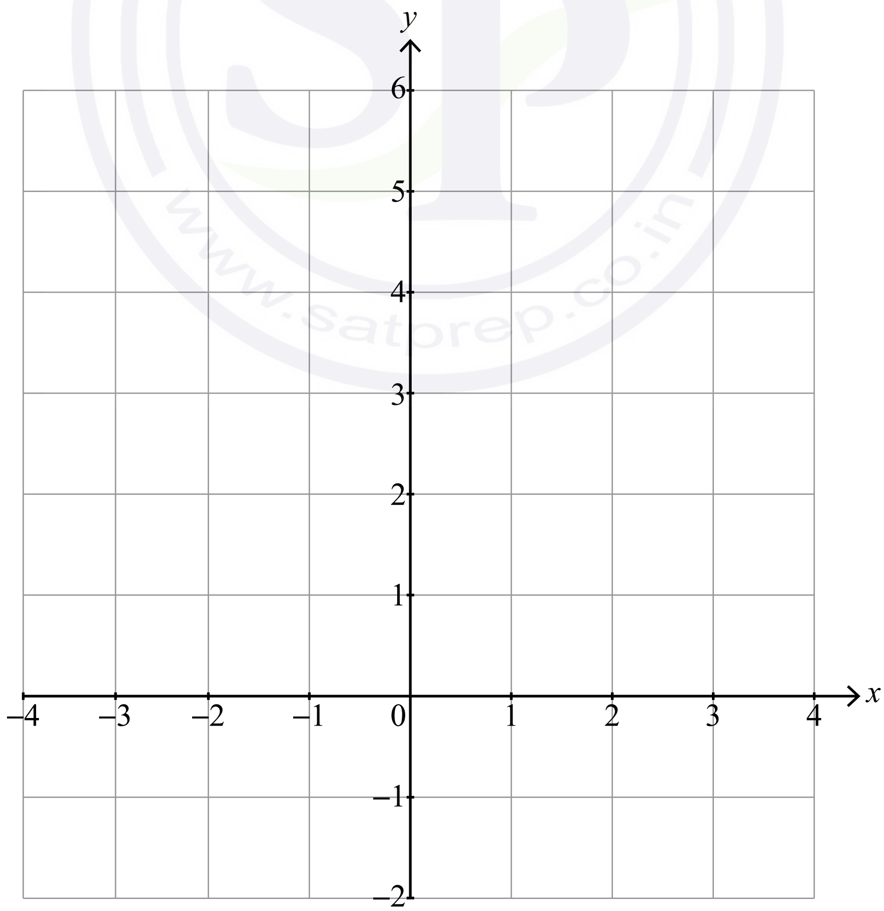
Let $f'(x) = \dfrac{8x}{\sqrt{2x^2+1}}$. Given that $f(0) = 5$, find $f(x)$.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



5. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

6. [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3\sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{\pi x}{36}\sin\left(\dfrac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$.

8. [Maximum mark: 7]

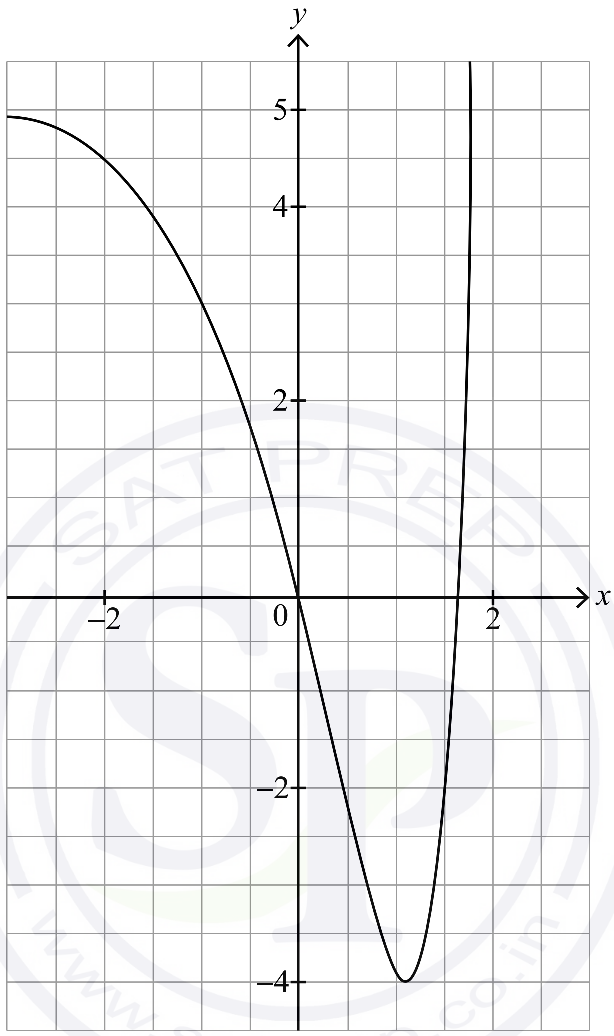
The plane Π has the Cartesian equation $2x + y + 2z = 3$.

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

9. [Maximum mark: 8]

The function f is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

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(Question 9 continued)

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

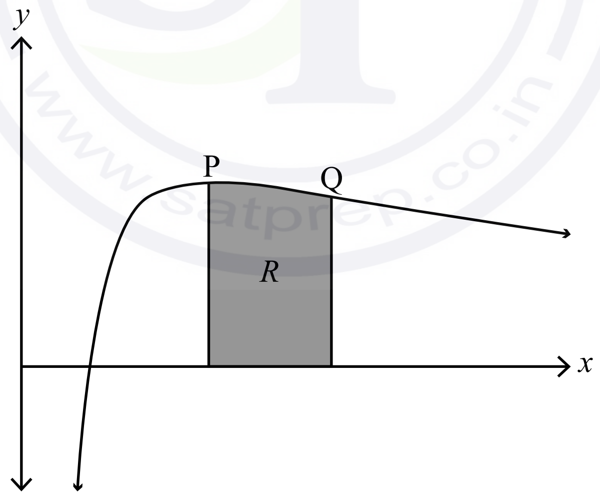
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\dfrac{1}{5}\mathrm{e}^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k. [7]

Do **not** write solutions on this page.

11. [Maximum mark: 18]

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that
$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0$. [4]

12. [Maximum mark: 21]

The function f is defined by $f(x)=\mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x}-1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x}-1\right)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x\to 0}\frac{f(x)-1}{\arctan\left(\mathrm{e}^{3x}-1\right)}$. [3]