Diploma Programme
Programme du diplôme
Programa del Diploma

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Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 2

16 May 2025

Zone A morning | **Zone B** morning | **Zone C** morning

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

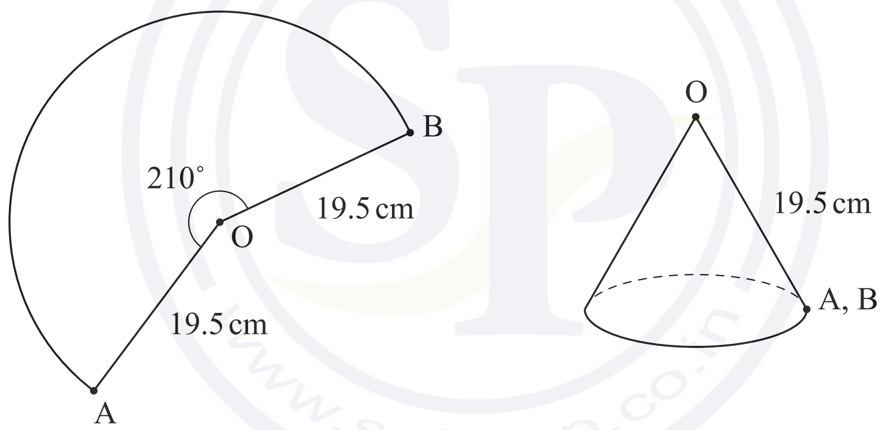
1. [Maximum mark: 6]

The points A and B lie on a circle, with centre O and radius $19.5\,\mathrm{cm}$, such that $\mathrm{B\hat{O}A} = 210^\circ$.

A piece of paper is cut into the shape of the sector BOA.

A hollow cone with no base is constructed from the sector by joining the points A and B. The sector forms the curved surface of the cone.

This is shown in the following diagrams.



Find

(a) the area of the sector BOA; [3]

(b) the radius of the cone. [3]

(This question continues on the following page)

(Question 1 continued)

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2. [Maximum mark: 4]

Consider the function $f(x) = a\tan(2x) + b$, where $x \neq \frac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$ and $a, b \in \mathbb{R}$.

(a) Write down the period of f. [1]

The graph of $y = f(x)$ passes through the points $\left(\frac{\pi}{12}, 5\right)$ and $\left(\frac{\pi}{3}, 7\right)$.

(b) Find the value of a and the value of b. [3]

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3. [Maximum mark: 6]

A population, P, has a rate of change modelled by $\frac{\mathrm{d}P}{\mathrm{d}t} = -104000\mathrm{e}^{-0.0145t}$, where t is the time measured in years since the **start** of 2022.

At the start of 2022, the population was 6.78×10^{6}.

Based on this model, find the predicted population at the start of 2026.

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4. [Maximum mark: 8]

In a study, measurements for arm span, A cm, and foot length, F cm, are taken from a large group of adults.

For this group, the regression line of F on A is found to be $F = 0.335A - 32.6$, and the regression line of A on F is found to be $A = 2.89F + 99.3$. Each regression line passes through the mean point.

(a) By using an appropriate regression line, find an estimate of the arm span for an adult with a foot length of 19.8 cm. [2]

(b) For this group of adults, find the mean arm span and the mean foot length. [3]

The heights, H cm, of adults in the group can be modelled by a normal distribution with mean 163 cm and standard deviation σ cm.

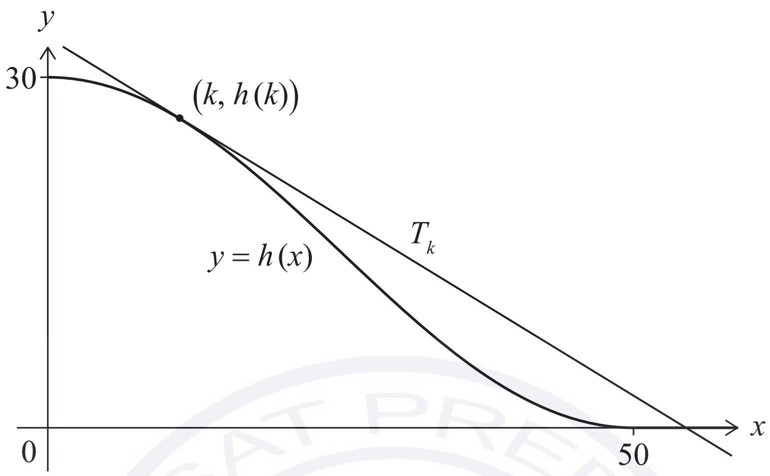
It is found that 88% of the group have a height between 153 cm and 173 cm.

(c) Find the value of σ. [3]

5. [Maximum mark: 6]

Consider the function $h(x) = 15\cos\left(\frac{\pi x}{50}\right) + 15$, where $0 \le x \le 50$.

The tangent, T_k, to the curve $y = h(x)$ at the point $(k, h(k))$ is shown on the following diagram.



(a) Find the gradient of T_k in terms of k. [3]

Consider the case where the angle between T_k and the x-axis is $\frac{\pi}{8}$ radians.

(b) Find the possible values of k. [3]

6. [Maximum mark: 7]

Consider the function $f(x) = 4\cot x + \sin x$, where $0 < x < \pi$.

(a) (i) Write $f(x)$ in terms of $\sin x$ and $\cos x$.

(ii) Hence or otherwise, sketch the graph of $y = f(x)$, showing the value of the x-intercept. [3]

(b) Find the value of $f^{-1}(2)$. [1]

It is given that $\sec\alpha = 1.5$, where $0 < \alpha < \pi$.

(c) Find the value of $f(\alpha)$. [3]

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7. [Maximum mark: 5]

At $09{:}00$ a helicopter is located at a point $(10, 3, 0.5)$ relative to a point O on horizontal ground. The x-direction is due east, the y-direction is due north and the z-direction is vertically upwards.

All distances are measured in kilometres.

The helicopter is flying at a constant height.

The helicopter's position relative to the point O is given by $\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 0.5 \end{pmatrix} + 4t\begin{pmatrix} 10 \\ -25 \\ 0 \end{pmatrix}$, where t represents the time in hours since $09{:}00$.

(a) Find the speed of the helicopter. [2]

At $10{:}00$ the helicopter begins to descend.

During descent the helicopter's vertical height decreases at a constant rate of $16\,\mathrm{km\,h^{-1}}$ and its horizontal velocity remains unchanged.

The angle of descent, β, is defined as the angle between the helicopter's direction of travel and the horizontal.

(b) Find β, giving your answer in degrees. [3]

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8. [Maximum mark: 7]

Consider the functions f, g and h defined as follows for $t \in \mathbb{R}$.

$$f(t) = \sin(2t+1)$$

$$g(t) = \sin(2t+3)$$

$$h(t) = f(t) + g(t)$$

(a) Show that $h(t) = \mathrm{Im}\left(\mathrm{e}^{2ti}(\mathrm{e}^{i} + \mathrm{e}^{3i})\right)$. [2]

(b) Write $\mathrm{e}^{i} + \mathrm{e}^{3i}$ in the form $r\mathrm{e}^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(c) Hence or otherwise, write $h(t)$ in the form $p\sin(2t+q)$, where $p > 0$ and $0 < q < 2\pi$. [3]

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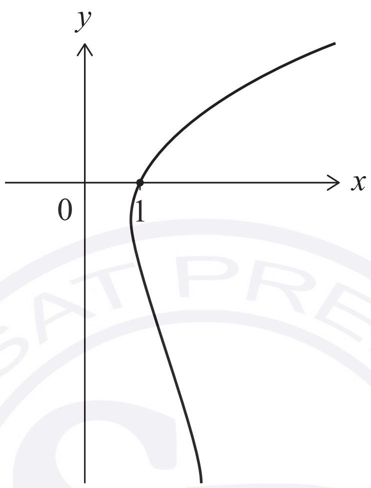
9. [Maximum mark: 6]

Consider the differential equation $\frac{dy}{dx} = \frac{2x}{x^2 + y}$.

The solution curve passes through the point $(1, 0)$.

(a) Use Euler's method with a step value of 0.25 to estimate the value of y when $x = 2$. [3]

Part of the solution curve is shown in the following diagram.



(b) (i) Determine whether your answer to part (a) is an overestimate or an underestimate, justifying your answer.

(ii) Justify why the use of Euler's method starting at $(1, 0)$ does not lead to an estimate of the negative value of y when $x = 2$. [3]

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Do **not** write solutions on this page.

Section B

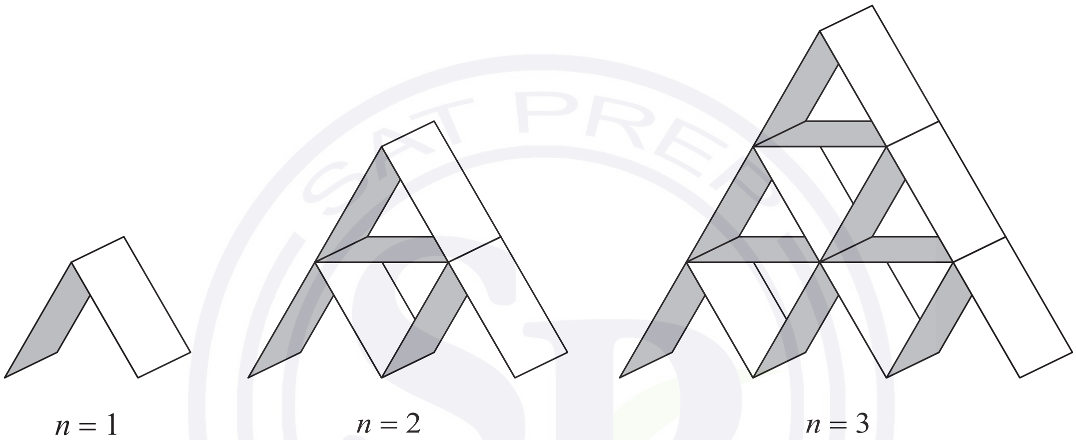
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Rectangular playing cards are stacked in the shape of a pyramid with n rows, where $n \geq 1$.

Some cards are placed horizontally and some cards are stacked at an angle of $60°$ to the horizontal.

The following diagrams represent pyramid stacks for $n = 1$, $n = 2$ and $n = 3$.



Let t_n represent the number of cards used to create a pyramid stack with n rows.

(a) Write down t_3. [1]

(b) Find t_4. [2]

(c) Show that $t_n = \dfrac{n(3n+1)}{2}$. [3]

There are 52 cards in a full pack of playing cards.

(d) A complete pyramid stack is created using playing cards taken from 14 full packs. Find the maximum number of rows in this stack. [3]

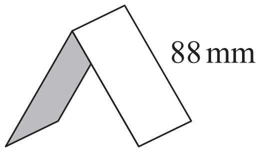
(e) A complete pyramid stack is created using playing cards taken from full packs with no cards left over. Find the minimum number of rows in this stack. [2]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The long edge of each playing card measures $88\,\text{mm}$ as illustrated in the following diagram.



(f) Find the minimum number of cards needed to create a complete pyramid stack with a vertical height of more than 2 metres. The thickness of the cards may be ignored. [5]

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Do **not** write solutions on this page.

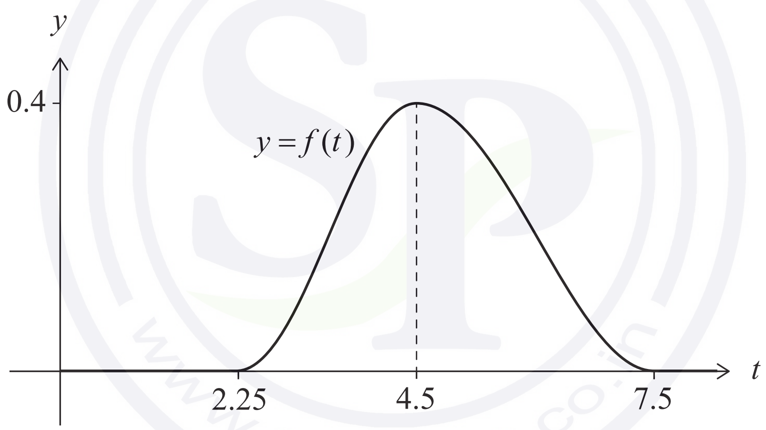
11. [Maximum mark: 19]

In a marathon race, the random variable T represents the time, in hours, taken for a runner to complete the race. No runner completes the race in less than 2.25 hours, and no runner completes it in more than 7.5 hours.

The probability density function for T is modelled by f, defined by

$$f(t)=\begin{cases}\frac{4}{21}\left(1-\cos\left(\frac{4\pi}{9}(t-2.25)\right)\right), & 2.25\le t<4.5\\ \frac{4}{21}\left(1+\cos\left(\frac{\pi}{3}(t-4.5)\right)\right), & 4.5\le t\le 7.5\\ 0, & \text{otherwise.}\end{cases}$$

The graph of f has a maximum point at $t=4.5$ as shown in the following diagram:



(a) (i) Find the value of $\int_{2.25}^{4.5} f(t)\,\mathrm{d}t$.

(ii) Write down the mode of T.

(iii) Determine which is greater, the mode of T or the median of T, justifying your answer. [4]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 11 continued)

The runners who finish the race in 3.5 hours or less are considered to be fast runners.

(b) Find the probability that a runner chosen at random is a fast runner. [2]

(c) Find the probability that a fast runner chosen at random finishes the race in 3 hours or less. [3]

(d) Find the lower quartile of T. [3]

Each runner's time is converted to a score which is calculated as $a - bt$, where t represents their time in hours, and a, $b > 0$.

Consider the random variable P which represents the score of a runner. It is given that $\mathrm{E}(P) = 100$ and the maximum possible score is 150.

(e) Use $\mathrm{E}(T) = 4.723$ to determine the value of a and the value of b, giving your answers to the nearest integer. [5]

(f) Given also that $\mathrm{Var}(T) = 0.906$, find $\mathrm{Var}(P)$. [2]

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Do **not** write solutions on this page.

12. [Maximum mark: 20]

Consider the family of functions f_n defined by $f_n(x) = \sum_{r=0}^{n} \left(-2x^2\right)^r$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Show that f_n is an even function for all values of n. [3]

(b) (i) Show that $f_3(x) = 1 - 2x^2 + 4x^4 - 8x^6$.

(ii) Write down a similar expression for $f_4(x)$ in ascending powers of x. [2]

Consider the function $f(x) = \lim_{n \to \infty} f_n(x)$ defined over the domain $-k < x < k$ where $k > 0$.

The largest possible value of k is K.

(c) (i) Find the value of K, giving your answer in exact form.

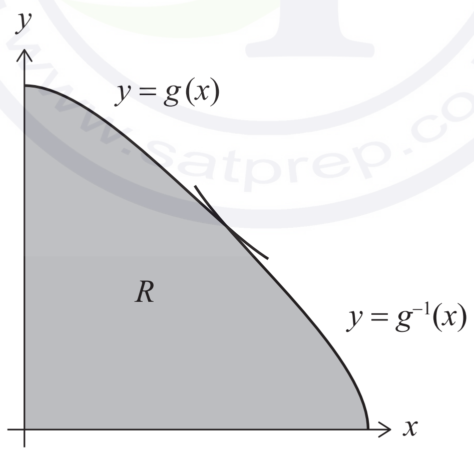
(ii) Express $f(x)$ as a rational function in the form $\frac{1}{a + bx^2}$, where a and b are constants to be determined. [5]

The function g is defined as $g(x) = f(x)$ for $0 \le x < K$.

(d) (i) Justify that g^{-1} exists.

(ii) Find $g^{-1}(x)$, giving its domain. [6]

The region R is completely enclosed by the curves $y = g(x)$, $y = g^{-1}(x)$ and the x- and y-axes, as shown on the following diagram.



(e) Find the area of R. [4]

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Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 2

16 May 2025

Zone A morning | **Zone B** morning | **Zone C** morning

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The following table shows the number of hours of play time, x, and sleep time, y, for a group of six children, over the period of one week.

Play time (x)	11	13	14	17	22	24
Sleep time (y)	62	65	68	75	84	87

The regression line of y on x for this data can be written in the form $y = ax + b$.

(a) Find the value of a and the value of b. [2]

(b) Use the equation of the regression line to estimate the sleep time of a child whose weekly play time is 20 hours. [2]

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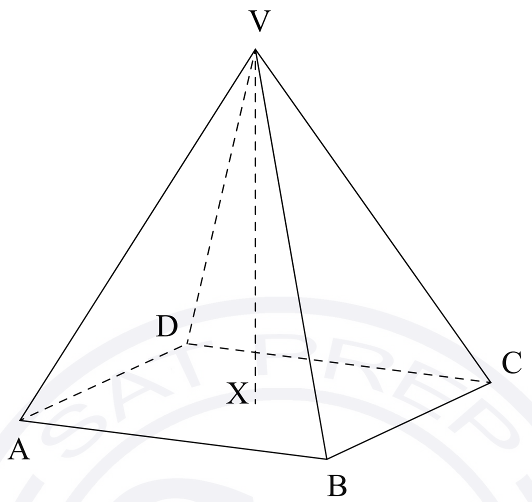
2. [Maximum mark: 6]

The following diagram shows a square-based right-pyramid with vertex $V(1, 7, 0)$.

Point $X(-3, 4, 2)$ is the centre of the base ABCD.

diagram not to scale



(a) Find VX. [2]

The square base has side length 5 cm.

(b) Find AC. [2]

(c) Find the size of the angle between the edge [VC] and the base of the pyramid. [2]

3. [Maximum mark: 6]

The derivative of a function f is given by $f'(x) = 4 + 2x - 3e^x$, where $x \in \mathbb{R}$.

(a) Find the values of x for which f is decreasing. [3]

(b) Find the values of x for which the graph of f is concave-up. [3]

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4. [Maximum mark: 6]

Alex purchases a car for €$30\,000$. The value of the car depreciates at 15% per annum.

(a) Find the value of the car after ten years. Give your answer to two decimal places. [2]

Alex invests €$50\,000$ in a bank account that pays a compound interest rate of 1.5% per month.

Inflation over the same time period was 0.8% per month.

(b) Find the number of months required for the real value of the investment to first exceed €$55\,000$. [4]

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5. [Maximum mark: 7]

A particle P moves in a straight line. The velocity $v\,\mathrm{m\,s^{-1}}$ of P, at time t seconds is given by $v(t) = \mathrm{e}^{-\sin t}\cos(2t)$, for $0 \le t \le 5$.

(a) Find the maximum speed of P. [2]

(b) Find the total distance travelled by P. [2]

(c) Find the acceleration when P changes direction for the **second** time. [3]

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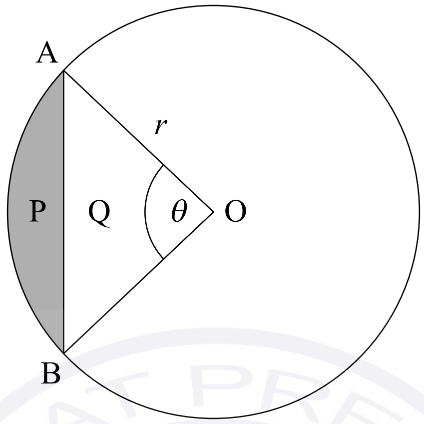
6. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius r cm. Points A and B lie on the circle and $A\hat{O}B = \theta$ radians.

Sector OAB is divided into two regions, a shaded segment P and a triangle Q.



The area of the shaded segment P is $12.8\,\text{cm}^2$.

The areas of P and Q are in the ratio $3:5$.

Find the value of r.

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7. [Maximum mark: 7]

A geometric sequence has first term 80 and fourth term 0.74088.

(a) Find the second term. [3]

The first two terms of this geometric sequence are also the first term and eleventh term respectively, of an arithmetic sequence.

Let S_n denote the sum of the first n terms of the arithmetic sequence.

(b) Find the greatest value of S_n, giving your answer to two decimal places. [4]

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8. [Maximum mark: 7]

The marks obtained by students in a class quiz are shown in the following table where $p, q \in \mathbb{Z}^+$.

Marks	Frequency
20	12
35	q
p	8

The mean and variance of the marks are 31 and 124 respectively.

Find the value of p and the value of q.

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9. [Maximum mark: 8]

A line L_1 has vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ where $t \in \mathbb{R}$.

The plane Π_1 contains the line L_1 and passes through the point $(2, 1, 5)$.

(a) Show that the Cartesian equation of the plane Π_1 is $x + y - z = -2$. [4]

Consider the three planes

$$\Pi_1 : x + y - z = -2$$
$$\Pi_2 : 2x + by - z = 3$$
$$\Pi_3 : x - y + 2z = d$$

where $b, d \in \mathbb{Q}^+$.

The three planes intersect in a line.

(b) Find the value of b and the value of d. [4]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

At Adam's Apple Orchard the weights of apples, W, in grams, are normally distributed with a mean 175 grams and standard deviation 8 grams.

(a) Find the probability that a randomly chosen apple weighs less than 170 grams. [2]

(b) It is found that 20% of the apples weigh more than w grams. Find w, correct to four significant figures. [2]

All orchards classify an apple as premium when its weight is between 170 and 185 grams.

(c) Find the percentage of apples that are classified as premium at Adam's Apple Orchard. [2]

After orders are completed, there are many apples left over. Boxes are filled with randomly chosen left-over apples. Each box contains 40 apples.

(d) Find the probability that a randomly chosen box contains at least 30 premium apples. [3]

(e) If 10 of these boxes are randomly selected, find the probability that exactly 4 boxes have at least 30 premium apples. [2]

At a neighbouring orchard the weights of apples, M, in grams, are normally distributed with mean μ and standard deviation σ. It is known that:

- 82% of their apples are classified as premium
- the percentage of apples that weigh less than 170 grams is twice the percentage of apples that weigh more than 185 grams.

(f) Find the value of μ. [6]

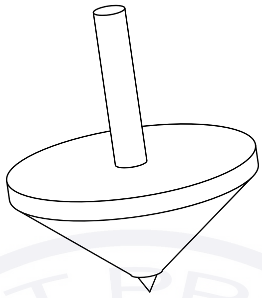
Do **not** write solutions on this page.

11. [Maximum mark: 14]

A mathematics class of 15 students plays a game which requires three equal size teams.

(a) Find the total number of ways that the three teams can be chosen. [3]

The game involves the spinning of a top.



The time, T, in minutes that the spinning top is in motion can be modelled by the probability density function f where

$$f(t)=\begin{cases} kt\mathrm{e}^{-3t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $k \in \mathbb{Z}^+$.

(b) Show that $\int_0^a f(t)\,\mathrm{d}t = \frac{k}{9}\left[1-(3a+1)\mathrm{e}^{-3a}\right]$, where $a \in \mathbb{R}^+$. [4]

(c) (i) Use l'Hôpital's rule to find $\lim_{x\to\infty}(3x+1)\mathrm{e}^{-3x}$.

(ii) Hence, by considering $\lim_{a\to\infty}\int_0^a f(t)\,\mathrm{d}t$, find the value of k. [5]

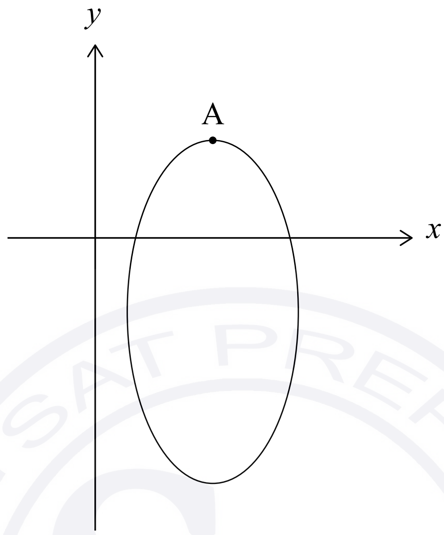
(d) Find the median length of time that a spinning top is in motion. [2]

Do **not** write solutions on this page.

12. [Maximum mark: 22]

The curve C has equation $4x^2 + y^2 - 24x + 4y + 20 = 0$.

The following diagram shows C with a maximum point at A.



(a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{4(3-x)}{y+2}$. [4]

(b) Hence, determine the domain of C. Give your answer in the form $3-\sqrt{a} \le x \le 3+\sqrt{a}$, where $a \in \mathbb{Z}^+$. [4]

(c) Find (x_A, y_A), the coordinates of A. [3]

A line $y = mx$ is a tangent to C, where $m \in \mathbb{Z}$.

(d) Find the possible values of m. [4]

The line $y = -4x$ touches C at point B.

(e) Find y_B, the y-coordinate of B. [3]

The region bounded by the curve C, the y-axis and the lines $y = y_A$ and $y = y_B$, is rotated $360°$ about the y-axis to form a solid of revolution.

(f) Find the volume of the solid formed. [4]

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

16 May 2025

Zone A morning | **Zone B** morning | **Zone C** morning

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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13 pages

2225 – 7117

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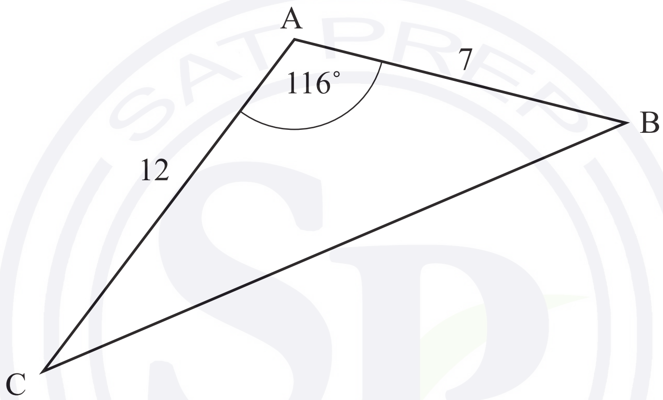
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a triangle ABC, with $\text{AB} = 7$, $\text{AC} = 12$ and $\text{B}\hat{\text{A}}\text{C} = 116°$.

diagram not to scale



(a) Find BC. [3]

(b) Find $\text{A}\hat{\text{C}}\text{B}$. [3]

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2. [Maximum mark: 5]

Consider the function $f(x) = 2x^4 - 6x^3 + px^2 + qx - 2$, where p, $q \in \mathbb{R}$.
A factor of $f(x)$ is $(x - 1)$, and when $f(x)$ is divided by $(x - 3)$ the remainder is -2.

Find the value of p and of q.

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3. [Maximum mark: 6]

A supermarket analyses the shopping habits of its customers.

The number of times, X, each customer visits the supermarket in a week is given by the following probability distribution.

x	1	2	3	4	5	≥ 6
$\mathrm{P}(X = x)$	$1.5a$	$2a$	0.281	a	0.026	0

(a) (i) Find the value of a.

(ii) Write down the mode of X. [3]

(b) Find the mean of X. [2]

The manager wants to know why customers come to their supermarket. They survey the first 50 customers to arrive at the supermarket on a particular day.

(c) Identify which one of the following best describes the manager's sampling method. Circle your answer. [1]

Simple random / Systematic / Convenience / Quota / Stratified

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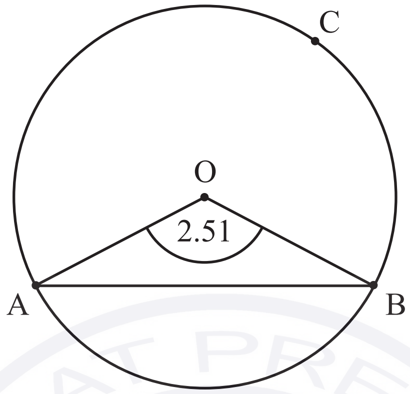
4. [Maximum mark: 5]

The following diagram shows a circle with centre O.

Points A, B and C lie on the circle.

diagram not to scale



The area of triangle AOB is $26\,\text{cm}^2$ and $\text{A}\hat{\text{O}}\text{B} = 2.51$ radians.

Find the length of arc ACB.

5. [Maximum mark: 7]

Consider the function $f(x) = \frac{(2x+a)^3}{(x+5)^2}$, where $x \neq -5$ and $a \in \mathbb{R}^+$.

(a) Find an expression for $f'(x)$, in terms of a. [3]

When $x = 1$, the tangent to the graph of f makes an angle of $70°$ to the horizontal.

(b) Find the two possible values of a. [4]

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6. [Maximum mark: 8]

Consider the vectors $\boldsymbol{a} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -8 \\ 9 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} p \\ -6 \end{pmatrix}$, where $p \in \mathbb{R}$.

(a) Find an expression, in terms of p, for

(i) $\boldsymbol{a} \bullet \boldsymbol{c}$;

(ii) $\boldsymbol{b} \bullet \boldsymbol{c}$. [3]

The angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ is equal to the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$.

(b) Find the value of p. [5]

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7. [Maximum mark: 6]

In the expansion of $\frac{1}{\sqrt{q-x^2}}$, where $q \in \mathbb{Q}^+$, the coefficient of x^6 is 5120.

Find the value of q.

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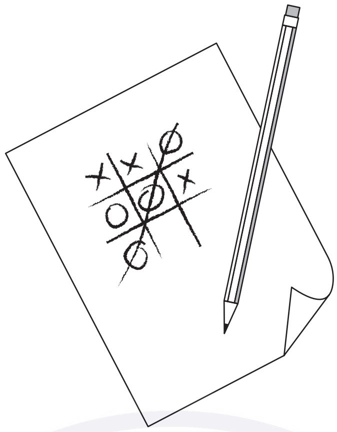
8. [Maximum mark: 5]

A class of students plays a tic-tac-toe competition among themselves. Each individual game in the competition involves only two students.



Every student in the class is to play every other student twice. However, Stephen left the class after he had played only seven games. All other games, not involving Stephen, were played.

By the end of the competition a total of 513 games had been played.

Determine the number of students that were originally in the class.

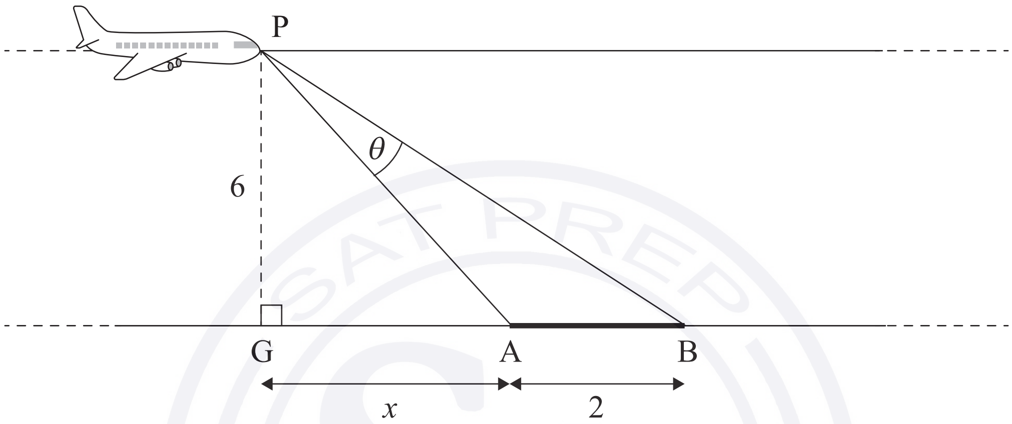
9. [Maximum mark: 9]

An airplane, P, is flying over horizontal ground at a constant height of $6\,\text{km}$ and travelling at a constant speed. It is approaching a runway, [AB], which is $2\,\text{km}$ in length.

Let G be the point on the ground directly below the airplane. When $\text{GA} = x\,\text{km}$, the pilot's viewing angle of the runway, $\text{A}\hat{\text{P}}\text{B}$, is θ.

This is shown in the following diagram.

diagram not to scale



(a) Show that $\theta = \arctan\left(\dfrac{x+2}{6}\right) - \arctan\left(\dfrac{x}{6}\right)$. [2]

When the viewing angle is 0.178 radians, the rate at which the viewing angle is changing is 12.5 radians per hour.

(b) Find the speed of the airplane. [7]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

Two athletes, Fiona and Lucy, compete in a 200 metres race along a straight track.

Fiona's velocity, in $\mathrm{m\,s^{-1}}$, during the race can be modelled by $v(t)=\dfrac{8.14\,t}{\sqrt{t^2+0.2}}$, where $t \ge 0$. Time, t, is measured in seconds from when the race starts.

(a) (i) Write down the value of $v(1)$.

(ii) Find the time when Fiona's velocity is $5\,\mathrm{m\,s^{-1}}$. [3]

(b) Find the time when Fiona's acceleration is $4\,\mathrm{m\,s^{-2}}$. [2]

(c) (i) Write down the limit of $v(t)$ as t approaches infinity.

(ii) State a reason why the value in part (c)(i) is not valid in the context of this question. [3]

Lucy's velocity, in $\mathrm{m\,s^{-1}}$, during the race can be modelled by $w(t)=\dfrac{8\,t}{\sqrt{t^2+0.3}}$, where $t \ge 0$.

Fiona completes the race and crosses the finishing line in front of Lucy.

(d) Find the distance Lucy is from the finishing line when Fiona completes the 200 metres. [6]

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11. [Maximum mark: 18]

Amanda enters data from surveys into a database. It can be assumed that the accuracy of any survey entered is independent of all other surveys entered.

From previous records, it is known that Amanda enters 8% of the surveys inaccurately.

(a) On a particular day Amanda enters data from 50 surveys.

(i) Find the probability that Amanda entered at most six surveys inaccurately.

(ii) Given that at most six surveys were entered inaccurately, find the probability that exactly four surveys were entered inaccurately. [5]

On a different day Amanda enters data from n surveys. On this day, the probability that at most six surveys were entered inaccurately is approximately 0.367.

(b) Find the value of n. [3]

Bryce and Carmen also enter data from surveys into the same database. It is known that surveys entered by Bryce and Carmen are inaccurate 6% and 11% of the time respectively. It can again be assumed that the accuracy of any survey entered is independent of all other surveys entered.

From the surveys assigned to the three of them, Amanda enters 55%, Bryce 25% and Carmen 20%.

(c) Find the probability that a randomly selected survey was

(i) entered inaccurately;

(ii) entered by Amanda, given that the survey was entered inaccurately. [6]

The following year, the accuracy of Amanda's and Bryce's work remained the same, as did the percentage of surveys entered by each of the three employees. However, Carmen's accuracy had improved and the probability that she entered a survey inaccurately was now $x\%$.

The probability that a randomly selected survey had been entered inaccurately was now the same as the probability that Carmen made an error when entering a survey.

(d) Find the value of x. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

(a) Find $\int (x^2 - 5)\,\mathrm{e}^x\,\mathrm{d}x$. [6]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y - 5$.

(b) By solving the differential equation, show that its solution can be expressed in the form $y = x^2 - 2x - 3 + C\mathrm{e}^{-x}$, where C is a constant. [4]

(c) Sketch the curve of the particular solution which passes through the point $(-3, 2)$, for $-4 \le x \le 4$, clearly labelling the coordinates of any local maximum and minimum points. [5]

Consider the family of curves that are solutions of the differential equation.

The tangent at $x = -3$ is drawn for each of these curves.

(d) By considering the curve which passes through the point $(-3, p)$ and the curve which passes through the point $(-3, q)$, where $p, q \in \mathbb{R}$, $p \neq q$, show that all these tangents intersect at a common point, and state its coordinates. [6]

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

25 October 2024

Zone A morning | **Zone B** morning | **Zone C** morning

Candidate session number

2 hours

Instructions to candidates

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- Section A: answer all questions. Answers must be written within the answer boxes provided.
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- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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17 pages

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

Consider the function $f(x) = 11\sqrt{x} - 2x - 11$, where $0 \le x \le 20$.

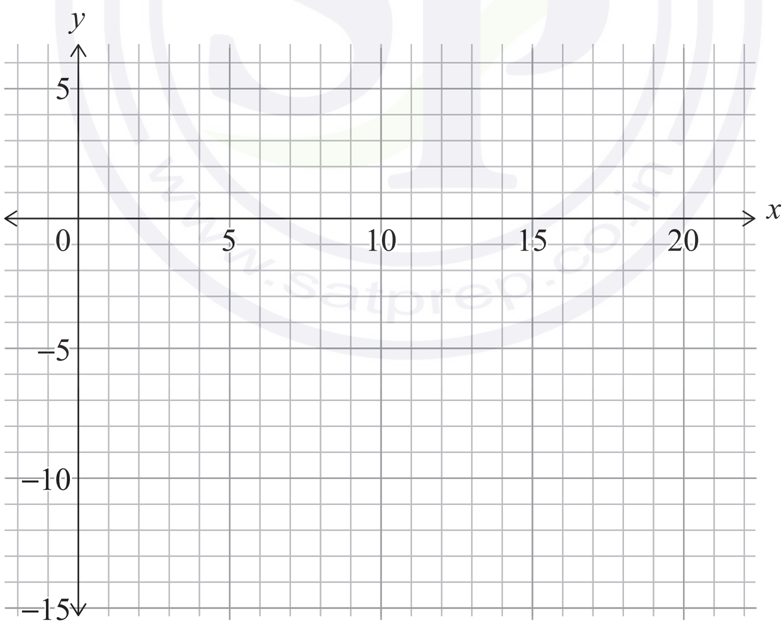
(a) Find the value of

(i) $f(0)$;

(ii) $f(20)$. [2]

(b) Find the two roots of $f(x) = 0$. [2]

(c) Sketch the graph of $y = f(x)$ on the following grid. [3]



(This question continues on the following page)

(Question 1 continued)

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2. [Maximum mark: 4]

Find the coefficient of x^6 in the expansion of $(2x-5)^9$.

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3. [Maximum mark: 5]

A discrete random variable, X, has the following probability distribution:

$$P(X = x) = \frac{kx}{20} \text{ for } x \in \{3, 5, 8, 11\}.$$

(a) Find the value of k. [2]

(b) Find $E(X)$. [3]

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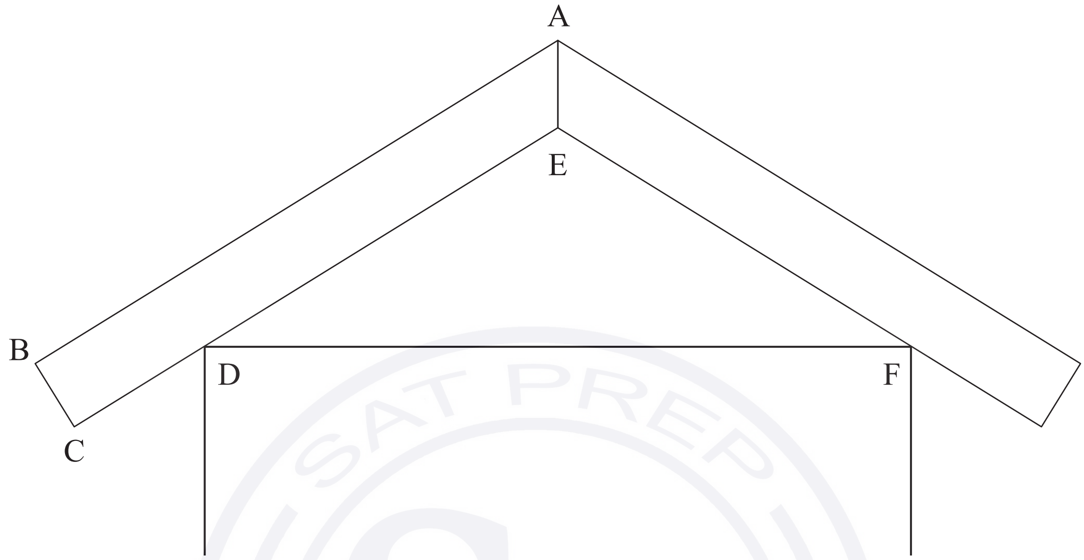
4. [Maximum mark: 8]

The following diagram shows the cross section of the roof of a house. The cross section is symmetrical about the vertical line through points A and E.

diagram not to scale



The gradient of $[\mathrm{BA}]$ is $\frac{7}{12}$.

(a) Find the size of $\mathrm{B\hat{A}E}$, expressing your answer in degrees. [3]

(This question continues on the following page)

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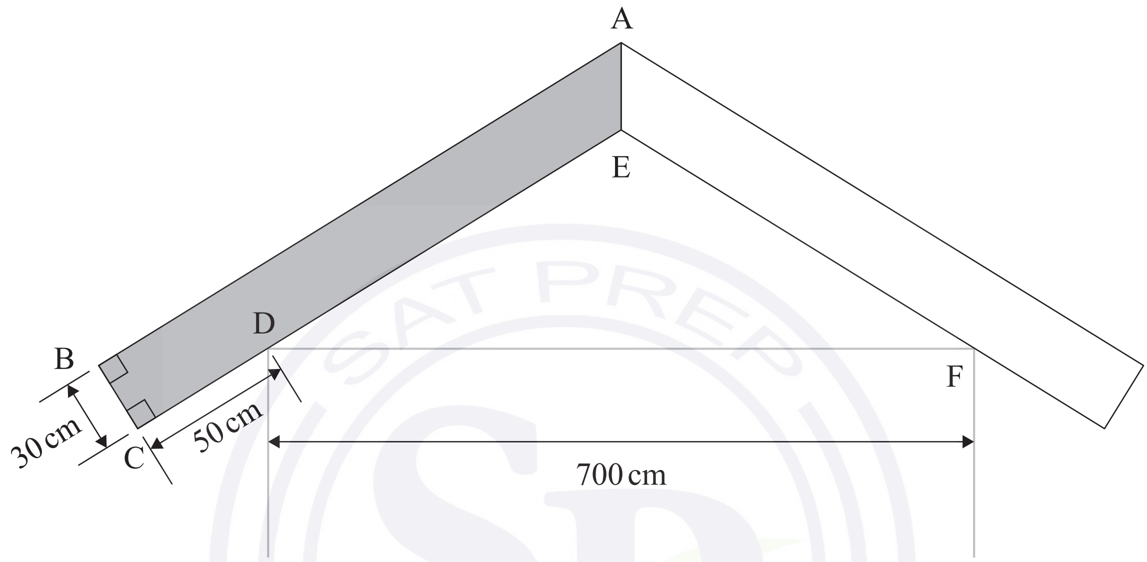
(Question 4 continued)

A builder requires the lengths of the sides [BA] and [CE].

The builder has the following measurements:

$\text{A}\hat{\text{B}}\text{C} = \text{B}\hat{\text{C}}\text{E} = 90°$, $\text{DC} = 50\text{ cm}$, $\text{BC} = 30\text{ cm}$, and $\text{DF} = 700\text{ cm}$.

diagram not to scale



(b) Find

(i) CE;

(ii) BA. [5]

5. [Maximum mark: 5]

Consider the function $h(x) = \log_{10}(4x^2 - rx + r - 1)$, where $x \in \mathbb{R}$.

Find the possible values of r. [5]

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6. [Maximum mark: 8]

A continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{5} & 0 \le x < 2 \\ -\frac{x}{30} + \frac{4}{15} & 2 \le x \le 8 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{E}(X)$. [3]

(b) Given that $\mathrm{E}(c - 2X) = 0$, where c is a constant, determine the value of c. [2]

(c) Find the median of X. [3]

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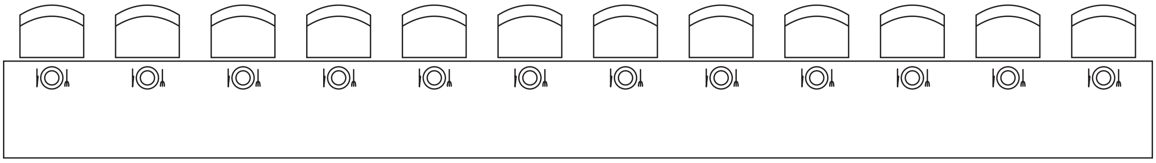
7. [Maximum mark: 6]

A self-service sushi restaurant has a row of 12 available seats, as shown in the following diagram.



Anvi, Vanya and Parita decide to go to the restaurant for lunch.

(a) Find the number of possible ways that they can be seated in this row, if they decide **not** to sit together as a group of 3. [3]

The next day, Anvi, Vanya and Parita are joined by 3 additional people in the same restaurant, and they sit in the same row of 12 available seats. Anvi, Vanya and Parita now decide to sit next to each other as a group of 3.

(b) Find the number of possible ways that these 6 people can be seated. [3]

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8. [Maximum mark: 6]

(a) Given that $w \in \mathbb{C}$, prove that $ww^* = |w|^2$. [2]

Two complex numbers z and w satisfy the following equations:

$$5w^* = (1 - 2\mathrm{i})z^2$$

$$zw = 10 + 10\mathrm{i}.$$

(b) Given that $|w| = 2\sqrt{5}$, find z expressing your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$. [4]

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9. [Maximum mark: 7]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=0}^{n-1} 5(\log_2 c)^r$.

(a) Given that S_n converges, find the range of possible values of c. [3]

(b) In the case where $c = 1.5$, find the least value of n such that $|S_\infty - S_n| < 0.1$. [4]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The following table shows the population of Canada t years after the year 2000.

t (years after 2000)	0	5	10	15	20
p (population in millions)	30.7	32.2	34.0	35.7	37.9

A student uses linear regression to model the population of Canada using these data. The student model is $p = at + b$.

(a) (i) Write down the value of a and the value of b.

(ii) Interpret, in context, the value of a. [3]

The student uses this model to predict the population of Canada in the year 2030, where $t = 30$, and calculates a population of approximately 41.3 million people.

(b) Comment on the reliability of the student's prediction. [1]

A data scientist, Benoit, uses additional information to develop an exponential model for Canada's future population.

In this model, $B(t) = 33.5(1.005)^t$ represents the millions of people in Canada t years after the year 2000, where $25 \le t \le 100$.

(c) (i) Use Benoit's model to predict the population of Canada in the year 2100.

(ii) Interpret, in context, the value 1.005 in Benoit's model. [3]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

Another data scientist, Cecilia, develops a third model for the Canadian population.

In this model, $C(t) = \frac{62}{1 + e^{-0.02t}}$ represents the millions of people in Canada t years after the year 2000, where $25 \le t \le 100$.

(d) Use Cecilia's model to predict the population of Canada in the year 2100. [1]

(e) Determine the year in which the difference between the predictions from Benoit's model and Cecilia's model is greatest. [3]

(f) Find the value of

(i) $B'(75)$;

(ii) $C'(75)$. [2]

(g) Compare and interpret, in context, the values of $B'(75)$ and $C'(75)$. [2]

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11. [Maximum mark: 18]

A line L is defined by $L: \ -\frac{x}{2}+1 = y+4 = \frac{z}{3}$.

(a) Find the equation of L, expressing your answer in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where $\lambda \in \mathbb{R}$. [3]

(b) Determine the minimum distance from the origin O to the line L. [5]

A plane Π is defined by $\Pi: \ 6x - 3y + 5z = 24$.

(c) Verify that Π contains L. [3]

A second line M is parallel to Π.

The line M passes through the point $(4, 1, 2)$ and intersects the z-axis.

(d) Find the equation of M, expressing your answer in the form $\boldsymbol{s} = \boldsymbol{c} + \mu\boldsymbol{d}$, where $\mu \in \mathbb{R}$. [7]

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12. [Maximum mark: 21]

A curve C has equation $y = \dfrac{2x^2 + 6x - 3}{x + k}$, $x \in \mathbb{R}$, $x \neq -k$, where k is a real positive constant.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2x^2 + 4kx + 6k + 3}{(x + k)^2}$. [4]

(b) Find the range of values of k for which a local minimum or maximum point exists. [4]

Consider the curve C, when $k = 2$.

(c) Write down the equation of the vertical asymptote. [1]

(d) Find the equation of the oblique asymptote. [4]

(e) Show that $\dfrac{dy}{dx} > 2$, for $x \in \mathbb{R}$, $x \neq -2$. [4]

(f) Sketch the curve C, showing clearly both asymptotes and the general behaviour of C as it approaches each asymptote. [You are not required to find any axes intercepts.] [4]

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Mathematics: analysis and approaches
Higher level
Paper 2

2 May 2024

Zone A morning | **Zone B** morning | **Zone C** morning

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

Darren buys a car for $\$35\,000$. The value of the car decreases by 15% in the first year.

(a) Find the value of the car at the end of the first year. [2]

After the first year, the value of the car decreases by 11% in each subsequent year.

(b) Find the value of Darren's car 10 years after he buys it, giving your answer to the nearest dollar. [2]

When Darren has owned the car for n complete years, the value of the car is less than 10% of its original value.

(c) Find the least value of n. [3]

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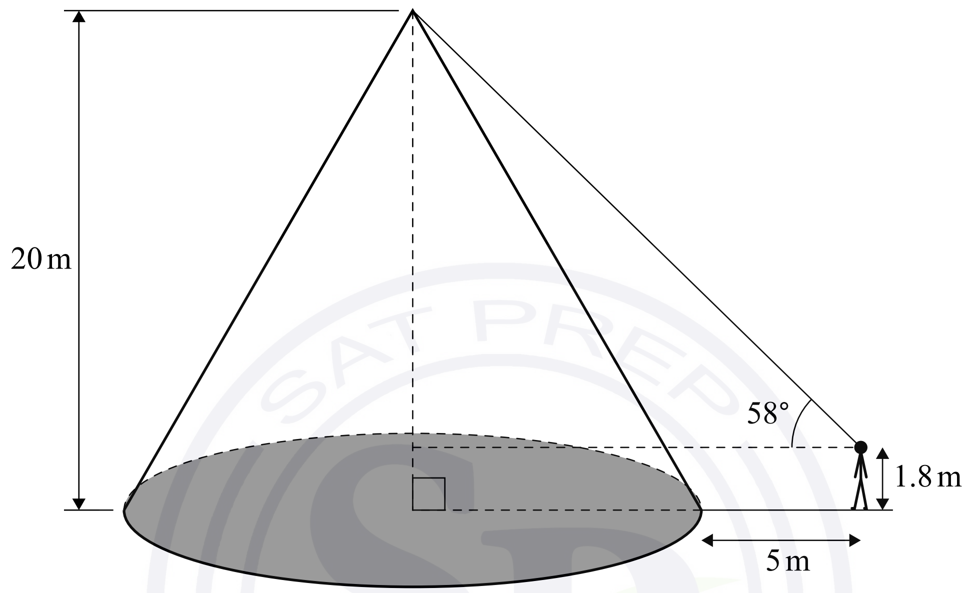
2. [Maximum mark: 5]

A monument is in the shape of a right cone with a vertical height of 20 metres. Oliver stands 5 metres from the base of the monument. His eye level is 1.8 metres above the ground and the angle of elevation from Oliver's eye level to the vertex of the cone is $58°$, as shown on the following diagram.

diagram not to scale



(a) Find the radius of the base of the cone. [3]

(b) Find the volume of the monument. [2]

3. [Maximum mark: 4]

The random variable X is normally distributed with mean 10 and standard deviation 2.

(a) Find the probability that X is more than 1.5 standard deviations above the mean. [2]

The probability that X is more than k standard deviations above the mean is 0.1, where $k \in \mathbb{R}$.

(b) Find the value of k. [2]

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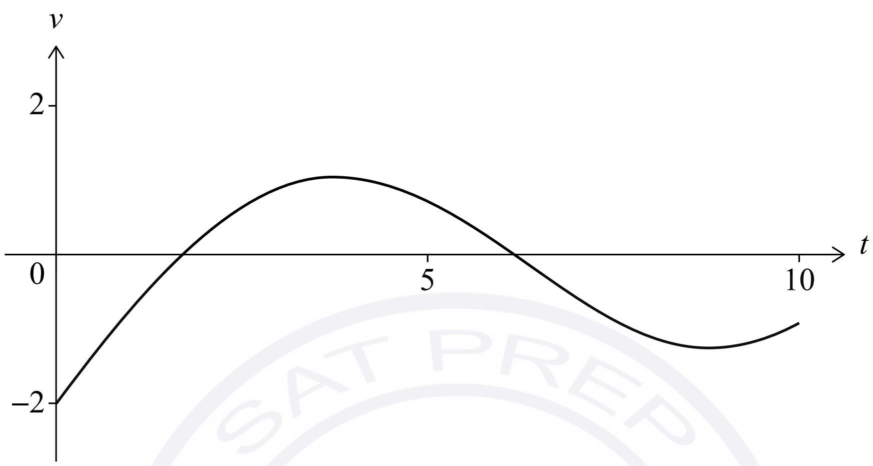
4. [Maximum mark: 6]

A particle moves in a straight line such that it passes through a fixed point O at time $t = 0$, where t represents time measured in seconds after passing O. For $0 \le t \le 10$ its velocity, v metres per second, is given by $v = 2\sin(0.5t) + 0.3t - 2$.

The graph of v is shown in the following diagram.



(a) Find the smallest value of t when the particle changes direction. [2]

The displacement of the particle is measured in metres from O.

(b) Find the range of values of t for which the displacement of the particle is increasing. [2]

(c) Find the displacement of the particle relative to O when $t = 10$. [2]

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5. [Maximum mark: 7]

A class is given two tests, Test A and Test B. Each test is scored out of a total of 100 marks. The scores of the students are shown in the following table.

Student	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Test A	52	71	100	93	81	80	88	100	70	61
Test B	58	80	92	98	90	82	100	100	65	74

Let x be the score on Test A and y be the score on Test B.

The teacher finds that the equation of the regression line of y on x for these scores is $y = 0.822x + 18.4$.

(a) Find the value of Pearson's product-moment correlation coefficient, r. [2]

Giovanni was absent for Test A and Paulo was absent for Test B.

The teacher uses the regression line of y on x to estimate the missing scores.

Paulo scored 10 on Test A.

The teacher estimated his score on Test B to be 27 to the nearest integer using the following calculation:

$$y = 0.822(10) + 18.4 \approx 27$$

(b) Give a reason why this method is not appropriate for Paulo. [1]

Giovanni scored 90 on Test B.

The teacher estimated his score on Test A to be 87 to the nearest integer using the following calculation:

$$90 = 0.822x + 18.4\text{, so } x = \frac{90 - 18.4}{0.822} \approx 87$$

(c) (i) Give a reason why this method is not appropriate for Giovanni.

(ii) Use an appropriate method to show that the estimated Test A score for Giovanni is 86 to the nearest integer. [4]

(This question continues on the following page)

(Question 5 continued)

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6. [Maximum mark: 6]

In Happyland, the weather on any given day is independent of the weather on any other day. On any day in May, the probability of rain is 0.2. May has 31 days.

Find the probability that

(a) it rains on exactly 10 days in May; [2]

(b) it rains on at least 10 days in May; [2]

(c) the first day that it rains in May is on the 10th day. [2]

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7. [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$, given that $y = 2$ when $x = 0$.

Give your answer in the form $y = f(x)$.

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8. [Maximum mark: 6]

A continuous random variable X has a probability density function f given by

$$f(x) = \begin{cases} kx & 0 \le x \le k \\ 2kx - x^2 & k < x \le 2k \\ 0, & \text{otherwise} \end{cases}$$

where $k > 0$.

(a) Show that k satisfies the equation $7k^3 = 6$. [2]

(b) Find the median of X. [4]

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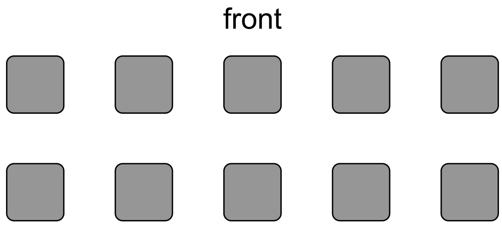
9. [Maximum mark: 7]

A group of 10 children includes one pair of brothers, Alvin and Bobby, and one pair of sisters, Catalina and Daniela.

The children are to be seated at 10 desks which are arranged in two rows of five as shown in the following diagram.



Alvin and Bobby must be seated next to each other in the same row.

(a) Find the total number of ways the children can be seated. [3]

After an argument, Catalina and Daniela must not be seated next to each other. Alvin and Bobby must still be seated next to each other.

(b) Find the total number of ways the children can be seated. [4]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Sule Skerry and Rockall are small islands in the Atlantic Ocean, in the same time zone.

On a given day, the height of water in metres at Sule Skerry is modelled by the function $H(t) = 1.63\sin(0.513(t - 8.20)) + 2.13$, where t is the number of hours after midnight.

The following graph shows the height of the water for 15 hours, starting at midnight.

At low tide the height of the water is $0.50\,\text{m}$. At high tide the height of the water is $3.76\,\text{m}$.

All heights are given correct to two decimal places.

(a) The length of time between the first low tide and the first high tide is 6 hours and m minutes. Find the value of m to the nearest integer. [3]

(b) Between two consecutive high tides, determine the length of time, in hours, for which the height of the water is less than 1 metre. [2]

(c) Find the rate of change of the height of the water when $t = 13$, giving your answer in metres per hour. [2]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

On the same day, the height of water at the second island, Rockall, is modelled by the function $h(t) = a\sin\left(b(t-c)\right) + d$, where t is the number of hours after midnight, and a, b, c, $d > 0$.

The first low tide occurs at 02:41 when the height of the water is $0.40\,\text{m}$.

The first high tide occurs at 09:02 when the height of the water is $2.74\,\text{m}$.

(d) Find the values of a, b, c and d. [7]

When $t = T$, the height of the water at Sule Skerry is the same as the height of the water at Rockall for the first time.

(e) Find the value of T. [2]

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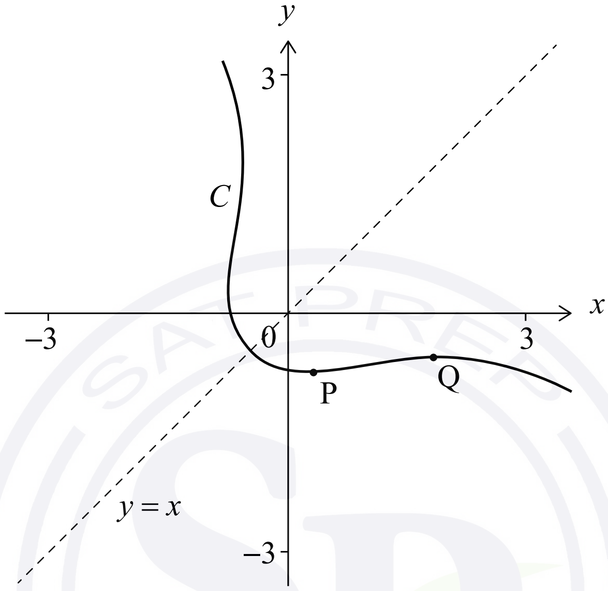
11. [Maximum mark: 19]

Consider the curve C defined by the equation $e^{x+y} = x^2 + y^2$, shown on the following diagram. The curve has a line of symmetry $y = x$.

There are two points on the curve C where the tangent is horizontal. These points are labelled P and Q.



(a) Show that $\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - 2y}$. [5]

(b) (i) Show that the x-coordinates of points P and Q satisfy the equation $2x^2 + (\ln(2x))^2 - 2x\ln(2x) - 2x = 0$.

(ii) Hence, find the coordinates of P and the coordinates of Q. [9]

(c) Using the line of symmetry, write down the coordinates of the points on the curve C where the tangent is vertical. [1]

(d) Find the coordinates of the point on the curve C where the tangent has a gradient of -1. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 20]

Consider the non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Using the definitions of $\boldsymbol{u} \bullet \boldsymbol{v}$ and $\boldsymbol{u} \times \boldsymbol{v}$ in terms of $|\boldsymbol{u}|$, $|\boldsymbol{v}|$ and θ, show that $(\boldsymbol{u} \bullet \boldsymbol{v})^2 + |\boldsymbol{u} \times \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 |\boldsymbol{v}|^2$. [2]

A triangle ABC has vertices $\mathrm{A}(0, 1, 2)$, $\mathrm{B}(p, q, 3)$ and $\mathrm{C}(3, 2, 1)$, $p, q \in \mathbb{Q}$.

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined as $\boldsymbol{u} = \overrightarrow{\mathrm{AB}}$ and $\boldsymbol{v} = \overrightarrow{\mathrm{AC}}$.

It is given that $\boldsymbol{u} \bullet \boldsymbol{v} = 3$ and the area of triangle ABC is $\sqrt{6}$.

(b) (i) Find the value of $|\boldsymbol{u} \times \boldsymbol{v}|$.

(ii) Hence, or otherwise, find the value of $|\boldsymbol{u}|$.

(iii) Hence, or otherwise, find the possible values of p and the corresponding values of q. [13]

Consider a new point D, the vector $\boldsymbol{w}$ is defined as $\boldsymbol{w} = \overrightarrow{\mathrm{CD}}$.

It is given that $\boldsymbol{u} \bullet \boldsymbol{w} = \boldsymbol{v} \bullet \boldsymbol{w} = 0$ and the area of triangle ACD is 5 square units.

(c) Assuming that $p = 1$, find the possible vectors for $\boldsymbol{w}$. [5]

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Mathematics: analysis and approaches
Higher level
Paper 2

2 May 2024

Zone A morning | **Zone B** morning | **Zone C** morning

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The functions f and g are both defined for $-1 \leq x \leq 0$ by

$$f(x) = 1 - x^2$$

$$g(x) = e^{2x}.$$

The graphs of f and g intersect at $x = a$ and $x = b$, where $a < b$.

(a) Find the value of a and the value of b. [3]

(b) Find the area of the region enclosed by the graphs of f and g. [3]

2. [Maximum mark: 5]

Consider the following bivariate data set where $p, q \in \mathbb{Z}^+$.

x	5	6	6	8	10
y	9	13	p	q	21

The regression line of y on x has equation $y = 2.1875x + 0.6875$.

The regression line passes through the mean point $(\bar{x}, \bar{y})$.

(a) Given that $\bar{x} = 7$, verify that $\bar{y} = 16$. [1]

(b) Given that $q - p = 3$, find the value of p and the value of q. [4]

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3. [Maximum mark: 6]

The loudness of a sound, L, measured in decibels, is related to its intensity, I units, by $L = 10\log_{10}(I \times 10^{12})$.

Consider two sounds, S_1 and S_2.

S_1 has an intensity of 10^{-6} units and a loudness of 60 decibels.

S_2 has an intensity that is twice that of S_1.

(a) State the intensity of S_2. [1]

(b) Determine the loudness of S_2. [2]

The maximum loudness of thunder in a thunderstorm was measured to be 115 decibels.

(c) Find the corresponding intensity, I, of the thunder. [3]

4. [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v(t) = 1 + \mathrm{e}^{-t} - \mathrm{e}^{-\sin 2t}$ for $0 \le t \le 2$.

(a) Find the velocity of the particle at $t = 2$. [1]

(b) Find the maximum velocity of the particle. [2]

(c) Find the acceleration of the particle at the instant it changes direction. [3]

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5. [Maximum mark: 5]

Consider a random variable X such that $X \sim \mathrm{B}(n, 0.25)$.

Determine the least value of n such that $\mathrm{P}(X \geq 1) > 0.99$.

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6. [Maximum mark: 6]

The volume of a spherical bubble increases at a constant rate of $5\,\mathrm{cm^3\,s^{-1}}$.

The initial volume of the bubble can be assumed to be zero.

Find the rate, in $\mathrm{cm\,s^{-1}}$, at which the radius of the bubble is increasing when the volume of the bubble is $20\,\mathrm{cm^3}$.

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7. [Maximum mark: 5]

The curve $y = 4\ln(x-2)$ for $0 \le y \le 4$ is rotated $360°$ about the y-axis to form a solid of revolution.

Find the volume of the solid formed.

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8. [Maximum mark: 10]

Let $z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Show that

(i) $\arg z = \theta$;

(ii) $|z| = 2\cos\theta$. [7]

(b) Hence or otherwise, find the value of θ such that $\arg(z^2) = |z^3|$. [3]

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9. [Maximum mark: 8]

Consider the curve $y = \dfrac{x-4}{ax^2+bx+c}$, where a, b and c are non-zero constants.

The curve has a local minimum point at $(2, 1)$ and a vertical asymptote with equation $x = 1$.

Find the values of a, b and c.

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

A shop sells chocolates. The weight, in kilograms, of chocolates bought by a random customer can be modelled by a continuous random variable X with probability density function f defined by

$$f(x) = \begin{cases} \frac{6}{85}\left(4 + 3x - x^2\right), & 0.5 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the mode of X. [2]

(b) Find $\mathrm{P}(1 \le X \le 2)$. [2]

(c) Find the median of X. [3]

The shop sells chocolates to customers at $\$25$ per kilogram.

However, if the weight of chocolate bought by a customer is at least 0.75 kilograms, the shop sells chocolate at a discounted rate of $\$24$ per kilogram.

(d) Find the probability that a randomly selected customer spends at most $\$48$. [3]

(e) Find the expected amount spent per customer. Give your answer correct to the nearest cent. [5]

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11. [Maximum mark: 17]

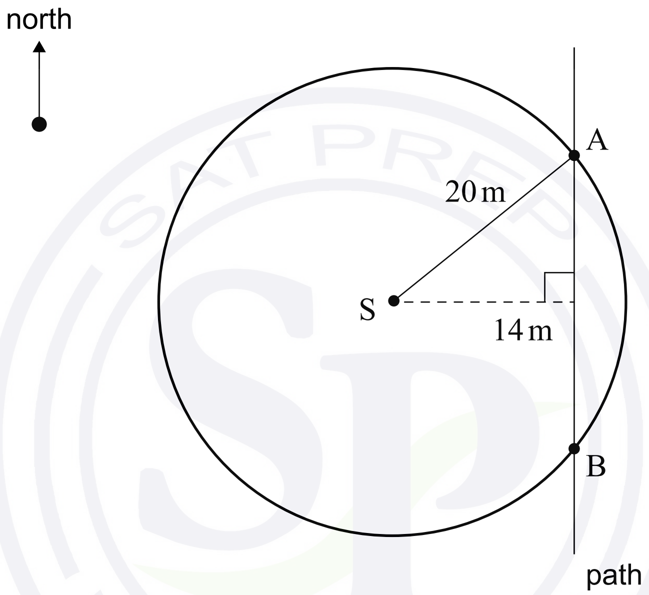
A rotating sprinkler is at a fixed point S.

It waters all points inside and on a circle of radius 20 metres.

Point S is 14 metres from the edge of a path which runs in a north-south direction.

The edge of the path intersects the circle at points A and B.

This information is shown in the following diagram.



(a) Show that $\mathrm{AB} = 28.57$, correct to four significant figures. [3]

The sprinkler rotates at a constant rate of one revolution every 16 seconds.

(b) Show that the sprinkler rotates through an angle of $\frac{\pi}{8}$ radians in one second. [1]

Let T seconds be the time that $[\mathrm{AB}]$ is watered in each revolution.

(c) Find the value of T. [4]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 11 continued)

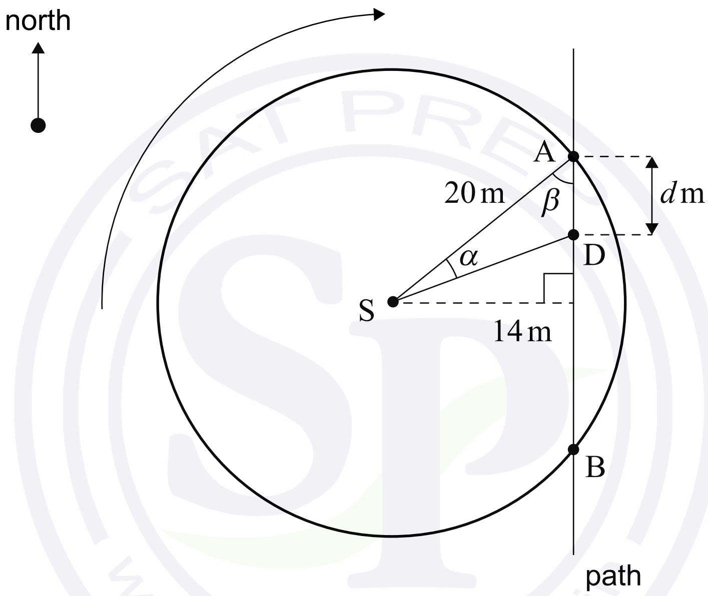
Consider one clockwise revolution of the sprinkler.

At $t = 0$, the water crosses the edge of the path at A.

At time t seconds, the water crosses the edge of the path at a movable point D which is a distance d metres south of point A.

Let $\alpha = \hat{\text{ASD}}$ and $\beta = \hat{\text{SAB}}$, where α, β are measured in radians.

This information is shown in the following diagram.



(d) Write down an expression for α in terms of t. [1]

It is known that $\beta = 0.7754$ radians, correct to four significant figures.

(e) By using the sine rule in ΔASD, show that the distance, d, at time t, can be modelled by

$$d(t) = \frac{20\sin\left(\frac{\pi t}{8}\right)}{\sin\left(2.37 - \frac{\pi t}{8}\right)}.$$ [3]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 11 continued)

A turtle walks south along the edge of the path.

At time t seconds, the turtle's distance, g metres south of A, can be modelled by

$$g(t) = 0.05t^2 + 1.1t + 18 \text{, where } t \geq 0.$$

(f) At $t = 0$, state how far south the turtle is from A. [1]

Let w represent the distance between the turtle and point D at time t seconds.

(g) (i) Use the expressions for $g(t)$ and $d(t)$ to write down an expression for w in terms of t.

(ii) Hence find when and where on the path the water first reaches the turtle. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec}2x = \sqrt{\tan x}$, where $0 < x < \frac{\pi}{2}$ and $y = \frac{\pi}{4}$ at $x = \frac{\pi}{4}$.

(a) Use Euler's method with step length $\frac{\pi}{12}$ to find an approximate value of y when $x = \frac{5\pi}{12}$.

Give your answer correct to three significant figures. [3]

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln(\cot x)\right) = -\operatorname{cosec}2x$. [4]

(c) Show that $\sqrt{\cot x}$ is an integrating factor for this differential equation. [4]

(d) Hence, by solving the differential equation, show that $y = x\sqrt{\tan x}$. [5]

(e) Consider the curve $y = x\sqrt{\tan x}$ for $0 < x < \frac{\pi}{2}$ and the Euler's method approximation calculated in part (a).

(i) Find the y-coordinate at $x = \frac{5\pi}{12}$. Give your answer correct to three significant figures.

(ii) By considering the gradient of the curve, suggest a reason why Euler's method does not give a good approximation for the y-coordinate at $x = \frac{5\pi}{12}$.

(iii) State why this approximation is less than the y-coordinate at $x = \frac{5\pi}{12}$. [3]

(f) By considering $\frac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec}2x = \sqrt{\tan x}$, deduce that the curve $y = x\sqrt{\tan x}$ has a positive gradient for $0 < x < \frac{\pi}{2}$. [2]

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Diploma Programme
Programme du diplôme
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Mathematics: analysis and approaches
Higher level
Paper 2

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

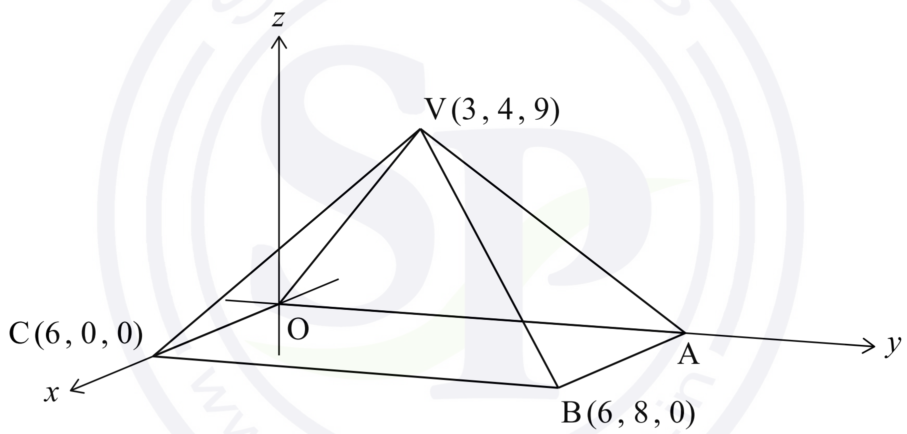
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a pyramid with vertex V and rectangular base OABC.

Point B has coordinates $(6, 8, 0)$, point C has coordinates $(6, 0, 0)$ and point V has coordinates $(3, 4, 9)$.

diagram not to scale



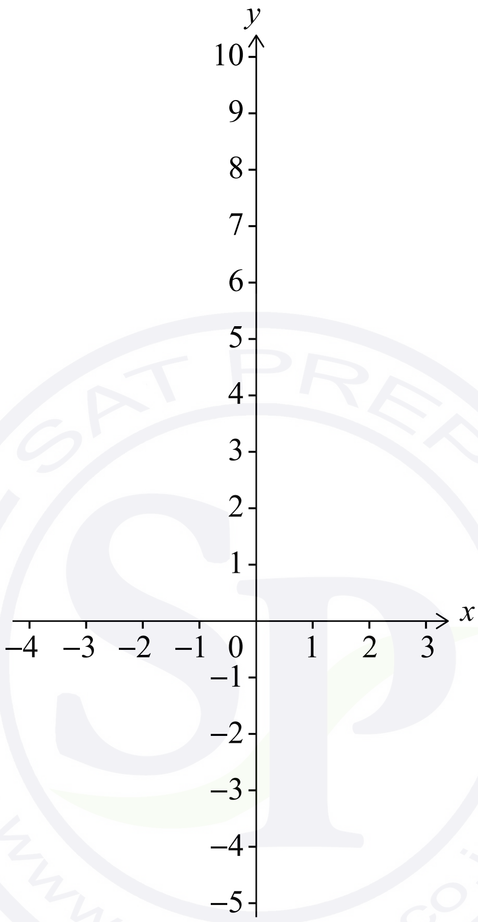
(a) Find BV. [2]

(b) Find the size of $\mathrm{B\hat{V}C}$. [4]

2. [Maximum mark: 5]

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of f for $-4 \le x \le 3$. [3]



The function g is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of g is obtained from the graph of f by a horizontal stretch with scale factor k, followed by a vertical translation of c units.

Find the value of k and the value of c. [2]

(This question continues on the following page)

(Question 2 continued)

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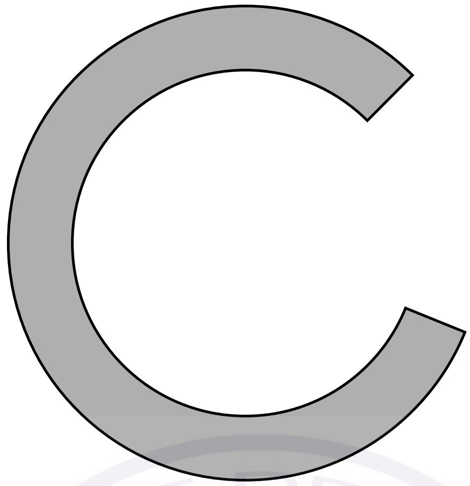
3. [Maximum mark: 7]

A company is designing a new logo in the shape of a letter "C".



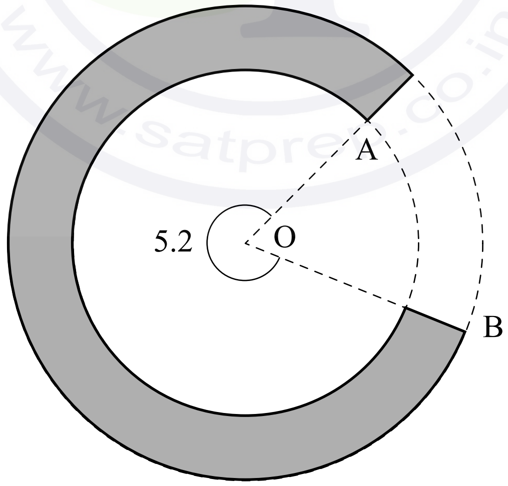
The letter "C" is formed between two circles with centre O.

The point A lies on the circumference of the inner circle with radius $r\,\text{cm}$, where $r < 10$.

The point B lies on the circumference of the outer circle with radius $10\,\text{cm}$.

The reflex angle $\text{A}\hat{\text{O}}\text{B}$ is 5.2 radians. The letter "C" is shown by the shaded area in the following diagram.

diagram not to scale



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(Question 3 continued)

(a) Show that the area of the "C" is given by $260 - 2.6r^2$. [2]

The area of the "C" is $64\,\text{cm}^2$.

(b) (i) Find the value of r.

(ii) Find the perimeter of the "C". [5]

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4. [Maximum mark: 5]

A particle moves along a straight line. Its displacement, s metres, from a fixed point O after time t seconds is given by $s(t) = 4.3\sin\left(\sqrt{3t+5}\right)$, where $0 \le t \le 10$.

The particle first comes to rest after q seconds.

(a) Find the value of q. [2]

(b) Find the total distance that the particle travels in the first q seconds. [3]

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5. [Maximum mark: 5]

The following table shows the probability distribution of a discrete random variable X, where $a, k \in \mathbb{R}^+$.

x	1	2	3	4
$P(X=x)$	k	k^2	a	k^3

Given that $E(X) = 2.3$, find the value of a.

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6. [Maximum mark: 5]

The random variable X is such that $X \sim B(25, p)$ and $\mathrm{Var}(X) = 5.75$.

(a) Find the possible values of p. [3]

The random variable Y is such that $Y = 1 - 2X$.

(b) Find $\mathrm{Var}(Y)$. [2]

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7. [Maximum mark: 6]

A junior baseball team consists of six boys and three girls.

The team members are to be placed in a line to have their photograph taken.

(a) In how many ways can the team members be placed if

(i) there are no restrictions;

(ii) the girls must be placed next to each other. [3]

(b) Five members of the team are selected to attend a baseball summer camp. Find the number of possible selections that contain at least two girls. [3]

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8. [Maximum mark: 9]

Three points are given by $A(0, p, 2)$, $B(1, 1, 1)$ and $C(p, 0, 4)$, where p is a positive constant.

(a) Show that $\vec{AB} \times \vec{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$. [4]

(b) Hence, find the smallest possible value of $\left|\vec{AB} \times \vec{AC}\right|^2$. [3]

(c) Hence, find the smallest possible area of triangle ABC. [2]

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9. [Maximum mark: 9]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a). [1]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A farmer is growing a field of wheat plants. The height, H cm, of each plant can be modelled by a normal distribution with mean μ and standard deviation σ.

It is known that $\mathrm{P}(H < 94.6) = 0.288$ and $\mathrm{P}(H > 98.1) = 0.434$.

(a) Find the probability that the height of a randomly selected plant is between 94.6 cm and 98.1 cm. [2]

(b) Find the value of μ and the value of σ. [5]

The farmer measures 100 randomly selected plants. Any plant with a height greater than 98.1 cm is considered ready to harvest. Heights of plants are independent of each other.

(c) (i) Find the probability that exactly 34 plants are ready to harvest.

(ii) Given that fewer than 49 plants are ready to harvest, find the probability that exactly 34 plants are ready to harvest. [6]

In another field, the farmer is growing the same variety of wheat, but is using a different fertilizer. The heights of these plants, F cm, are normally distributed with mean 98.6 and standard deviation d. The farmer finds the interquartile range to be 4.82 cm.

(d) Find the value of d. [3]

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11. [Maximum mark: 19]

Consider the function defined by $f(x)=\dfrac{x^2-14x+24}{2x+6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f. [1]

(b) Find the coordinates of the points where the graph of f crosses the x-axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where a, $b \in \mathbb{Q}$.

(c) Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f. [4]

(f) Solve the inequality $f(x) > x$. [4]

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12. [Maximum mark: 18]

Line L is given by the vector equation $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $s \in \mathbb{R}$.

Line M is given by the vector equation $\boldsymbol{r}_2 = \begin{pmatrix} 9 \\ 9 \\ 11 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ where $t \in \mathbb{R}$.

(a) Show that lines L and M intersect at a point A and find the position vector of A. [5]

(b) Verify that the lines L and M both lie in the plane Π given by $\boldsymbol{r} \bullet \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$. [3]

Point B has position vector $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$. A line through B perpendicular to Π intersects Π at point C.

(c) (i) Find the position vector of C.

(ii) Hence, find $\left|\overrightarrow{\mathrm{BC}}\right|$. [7]

(d) Find the reflection of the point B in the plane Π. [3]

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Mathematics: analysis and approaches
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

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- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
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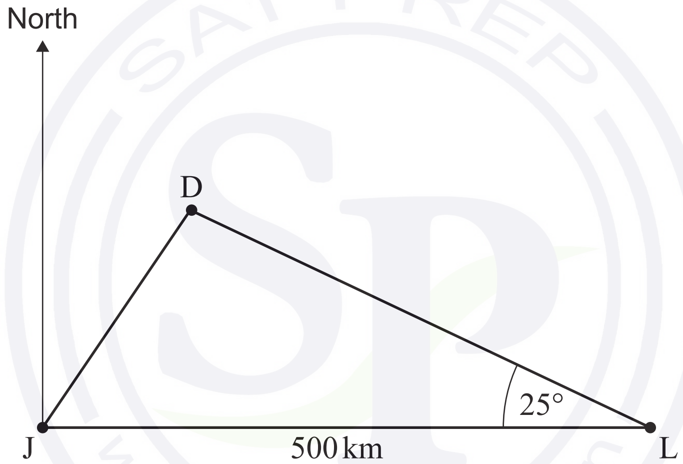
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The cities Lucknow (L), Jaipur (J) and Delhi (D) are represented in the following diagram. Lucknow lies $500\,\text{km}$ directly east of Jaipur, and $\text{JLD} = 25°$.

diagram not to scale



The bearing of D from J is $034°$.

(a) Find $\text{J}\hat{\text{D}}\text{L}$. [2]

(b) Find the distance between Lucknow and Delhi. [3]

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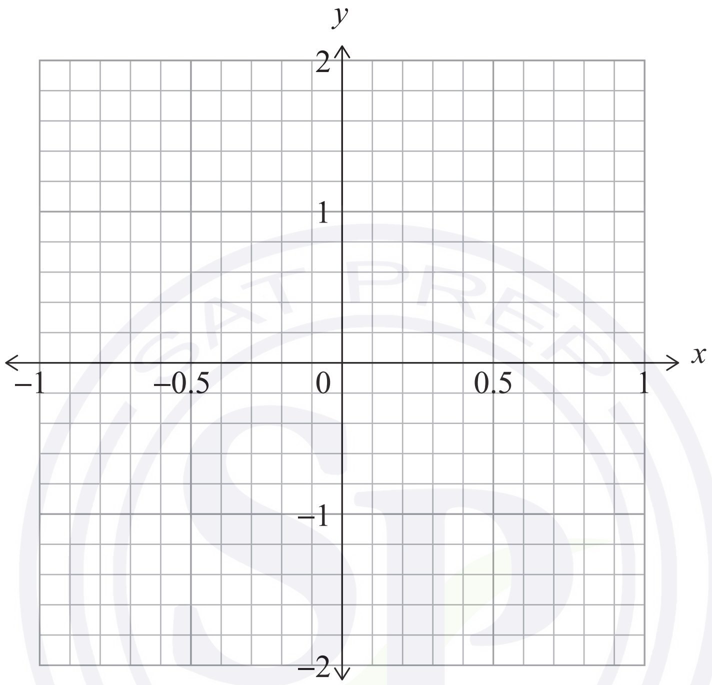
2. [Maximum mark: 5]

The functions f and g are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \le x \le 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]



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3. [Maximum mark: 7]

The total number of children, y, visiting a park depends on the highest temperature, T, in degrees Celsius (°C). A park official predicts the total number of children visiting his park on any given day using the model $y = -0.6T^2 + 23T + 110$, where $10 \le T \le 35$.

(a) Use this model to estimate the number of children in the park on a day when the highest temperature is $25\,°\text{C}$. [2]

An ice cream vendor investigates the relationship between the total number of children visiting the park and the number of ice creams sold, x. The following table shows the data collected on five different days.

Total number of children (y)	81	175	202	346	360
Ice creams sold (x)	15	27	23	35	46

(b) Find an appropriate regression equation that will allow the vendor to predict the number of ice creams sold on a day when there are y children in the park. [3]

(c) Hence, use your regression equation to predict the number of ice creams that the vendor sells on a day when the highest temperature is $25\,°\text{C}$. [2]

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4. [Maximum mark: 5]

A company manufactures metal tubes for bicycle frames. The diameters of the tubes, $D\,\text{mm}$, are normally distributed with mean 32 and standard deviation σ. The interquartile range of the diameters is 0.28.

Find the value of σ.

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5. [Maximum mark: 7]

The coefficient of x^6 in the expansion of $(ax^3 + b)^8$ is 448.

The coefficient of x^6 in the expansion of $(ax^3 + b)^{10}$ is 2880.

Find the value of a and the value of b, where a, $b > 0$.

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6. [Maximum mark: 7]

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of n that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

(b) Hence or otherwise, describe a single geometric transformation applied to z on the Argand diagram that results in z^{10}. [3]

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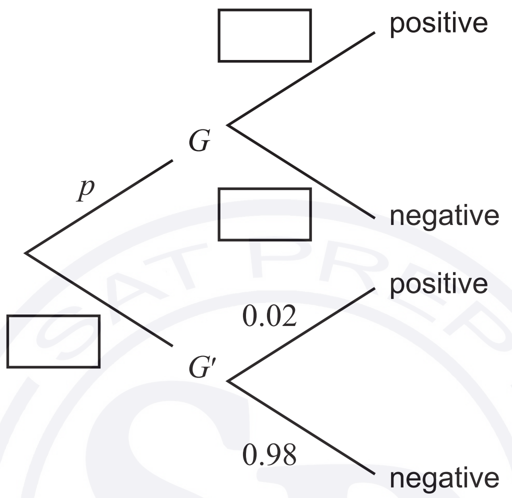
7. [Maximum mark: 6]

A new test has been developed to identify whether a particular gene, G, is present in a population of parrots. The test returns a correct positive result 95% of the time for parrots with the gene, and a false positive result 2% of the time for parrots without the gene.

The proportion of the population with the gene is p.

(a) Complete the tree diagram below. [2]



(b) A random sample of the population was tested. It was found that 150 tests returned a positive result. Out of the 150 parrots with a positive test result, 18 did not actually have the gene. Find an estimate for p. [4]

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8. [Maximum mark: 7]

The angle between a line and a plane is α, where $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

Find the value of α.

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9. [Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

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Do **not** write solutions on this page.

Section B

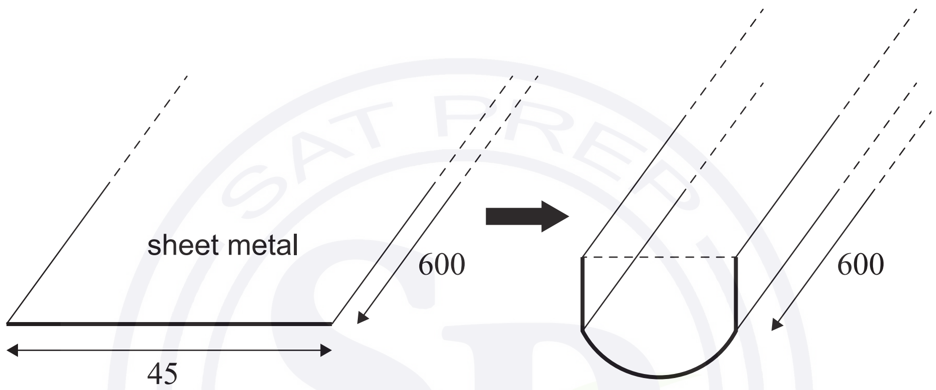
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

An engineer is designing a gutter to catch rainwater from the roof of a house.

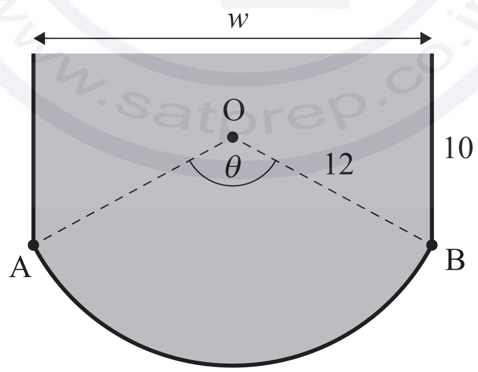
The gutter will be open at the top and is made by folding a piece of sheet metal $45\,\text{cm}$ wide and $600\,\text{cm}$ long.

diagram not to scale



The cross-section of the gutter is shaded in the following diagram.

diagram not to scale



The height of both vertical sides is $10\,\text{cm}$. The width of the gutter is $w\,\text{cm}$.

Arc AB lies on the circumference of a circle with centre O and radius $12\,\text{cm}$.

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

Let $\text{A}\hat{\text{O}}\text{B} = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in cm^3, of rainwater that enters the gutter can be modelled by a function $R(t)$, where t is the time, in seconds, since the start of the storm.

It was determined that the **rate** at which rainwater entered the gutter could be modelled by

$$R'(t) = 50\cos\left(\frac{2\pi t}{5}\right) + 3000, \ t \geq 0.$$

During any 60-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

A continuous random variable, X, has a probability density function defined by

$$f(x)=\begin{cases}\dfrac{6}{\pi\sqrt{16-x^2}}, & 0\le x\le 2\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the exact value of $\mathrm{E}(X)$. [5]

(b) Find $\mathrm{P}(X<0.5)$. [2]

A laboratory trial may require up to 2 millilitres of reagent. The amount of reagent used has been found to have a probability distribution that can be modelled by $f(x)$, where X is the amount of reagent in millilitres.

Each laboratory trial is independent. A trial is considered a success when $X<0.5$.

(c) Determine the least number of trials required to be 99% sure of at least one success. [3]

Ten trials were conducted.

(d) Find the probability that exactly three trials were successful. [2]

(e) Write down the number of ways these three successful trials could have occurred consecutively. [1]

Now consider n trials where it is given that exactly three successes have occurred.

(f) (i) Write down an expression for the number of ways these three successful trials could have occurred consecutively.

(ii) Find the greatest value of n such that the probability of three consecutive successful trials is more than 0.05. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2+1}$, where $y > 0$ and $y = 3$ when $x = 0$.

(a) Use Euler's method, with a step length of 0.03, to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [4]

(b) (i) Write down the value of $\frac{dy}{dx}$ when $x = 0$.

(ii) Show that $\frac{d^2y}{dx^2} = 3$ when $x = 0$. [5]

(c) (i) Given that $\frac{d^3y}{dx^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for y.

(ii) Use the Maclaurin series to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [3]

(d) (i) It is given that $\frac{x^2-1}{x^2+1} \equiv 1 - \frac{2}{x^2+1}$.

Solve the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2+1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(ii) Hence, find the value of y when $x = 0.15$. Give your answer correct to six significant figures. [7]

(e) For $0 \le x < 1$, explain why the approximate value for y obtained using Euler's method will always be less than the actual value for y. [2]

References:

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Mathematics: analysis and approaches
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

A botanist is conducting an experiment which studies the growth of plants.

The heights of the plants are measured on seven different days.

The following table shows the number of days, d, that the experiment has been running and the average height, h cm, of the plants on each of those days.

Number of days (d)	2	5	13	24	33	37	42
Average height (h)	10	16	30	59	76	79	82

The value of Pearson's product-moment correlation coefficient, r, for this data is 0.991, correct to three significant figures.

(a) The regression line of h on d for this data can be written in the form $h = ad + b$.

Find the value of a and the value of b. [2]

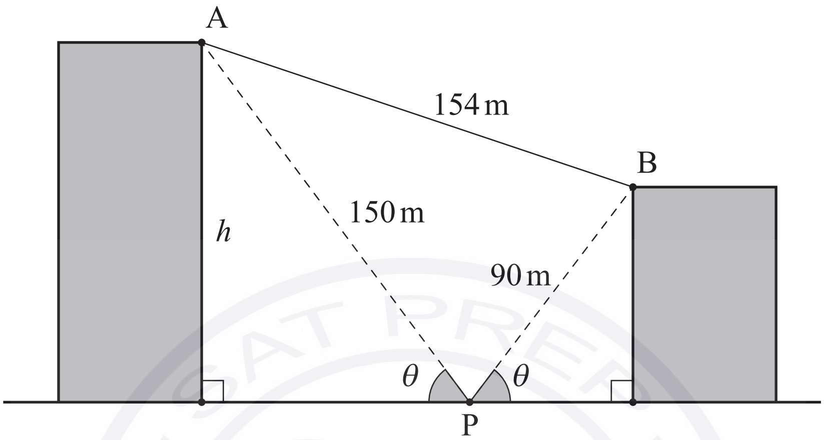
(b) Use your regression line to estimate the average height of the plants when the experiment has been running for 20 days. [2]

2. [Maximum mark: 6]

The following diagram shows two buildings situated on level ground.

From point P on the ground directly between the two buildings, the angle of elevation to the top of each building is θ.

diagram not to scale



The distance from point P to point A at the top of the taller building is 150 metres.

The distance from point P to point B at the top of the shorter building is 90 metres.

The distance between A and B is 154 metres.

Find the height, h, of the taller building.

3. [Maximum mark: 8]

The weights, W grams, of bags of rice packaged in a factory can be modelled by a normal distribution with mean 204 grams and standard deviation 5 grams.

(a) A bag of rice is selected at random.

Find the probability that it weighs more than 210 grams. [2]

According to this model, 80% of the bags of rice weigh between w grams and 210 grams.

(b) Find the probability that a randomly selected bag of rice weighs less than w grams. [2]

(c) Find the value of w. [2]

(d) Ten bags of rice are selected at random.

Find the probability that exactly one of the bags weighs less than w grams. [2]

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4. [Maximum mark: 7]

The expansion of $(x+h)^8$, where $h \in \mathbb{Q}^+$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

Given that the coefficients, a, b and d, are the first three terms of a geometric sequence, find the value of h.

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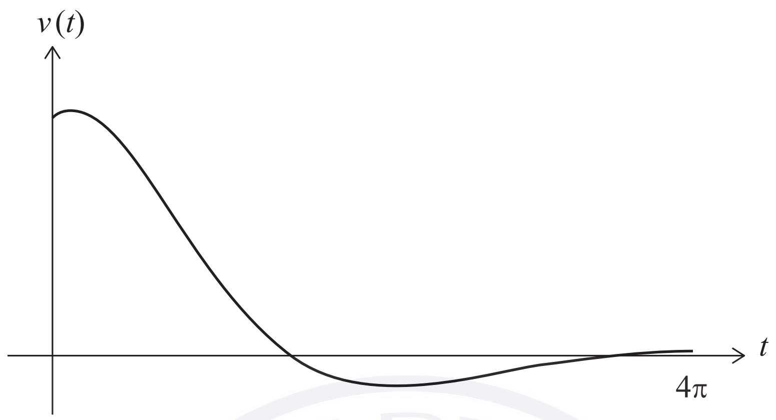
5. [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v(t) = 4e^{-\frac{t}{3}}\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \le t \le 4\pi$. The graph of v is shown in the following diagram.



Let t_1 be the first time when the particle's **acceleration** is zero.

(a) Find the value of t_1. [2]

Let t_2 be the **second** time when the particle is instantaneously at rest.

(b) Find the value of t_2. [2]

(c) Find the distance travelled by the particle between $t = t_1$ and $t = t_2$. [2]

6. [Maximum mark: 8]

Consider the two planes

$$\Pi_1 : 2x - y + 2z = 6$$

$$\Pi_2 : 4x + 3y - z = 2$$

Let L be the line of intersection of Π_1 and Π_2.

(a) Verify that a vector equation of L is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. [3]

(b) Find the coordinates of the point P on L that is nearest to the origin. [5]

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7. [Maximum mark: 5]

A function f is defined as $f(x) = \arctan(x - 2)$, where $2 \le x \le 2 + \sqrt{3}$.

The region bounded by the curve, the y-axis, the x-axis and the line $y = \frac{\pi}{3}$ is rotated 360° about the y-axis to form a solid of revolution.

Find the volume of the solid.

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8. [Maximum mark: 7]

A function g is defined by $g(x)=\dfrac{2x-5}{x^2-3}$, where $x\in\mathbb{R},\ x\neq\pm\sqrt{3}$.

(a) Determine the range of g. [4]

A function h is defined by $h(x)=g(|x|)\cos t$, where $x\in\mathbb{R},\ x\neq\pm\sqrt{3}$ and t is a constant where $\dfrac{\pi}{2}<t\leq\pi$.

(b) Find the set of values of x such that $h(x)\leq 0$. [3]

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9. [Maximum mark: 5]

Let S be the set of 30 positive integers $\{1, 2, 3, ..., 28, 29, 30\}$.

Raghu randomly selects three positive integers from S without replacement. He then adds them together and determines whether the sum is divisible by 3.

Determine the total number of selections Raghu can make to obtain a sum that is divisible by 3.

You may assume that order is not important, for example, $\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{2, 1, 3\}, \{3, 1, 2\}, \{3, 2, 1\}$ are all considered to be the same selection. [5]

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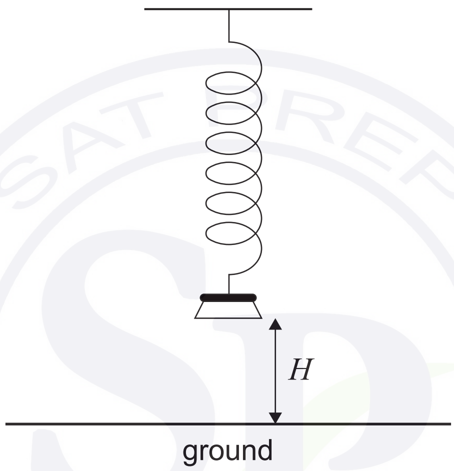
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 13]

A weight suspended on a spring is pulled down and released, so that it moves up and down vertically.

The height, H metres, of the base of the weight above the ground can be modelled by the function $H(t) = a\cos(7.8t) + b$, for a, $b \in \mathbb{R}$ and $0 \le t \le 10$, where t is the time in seconds after the weight is released.



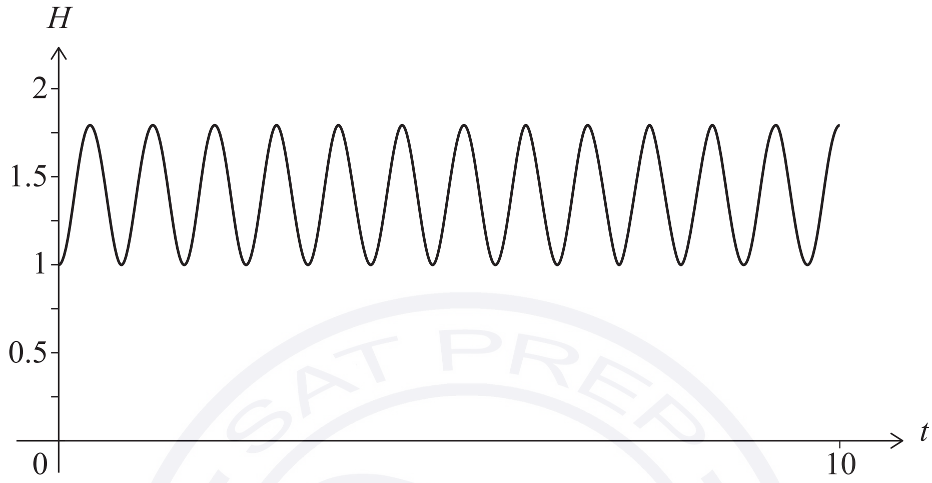
(a) Find the period of the function. [2]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The weight is released when its base is at a minimum height of 1 metre above the ground, and it reaches a maximum height of 1.8 metres above the ground. The graph of H is shown in the following diagram.



(b) Find the value of

(i) a;

(ii) b. [3]

(c) Find the number of times that the weight reaches its maximum height in the first five seconds of its motion. [2]

(d) Find the first time that the base of the weight reaches a height of 1.5 metres. [2]

A camera is set to take a picture of the weight at a random time during the first five seconds of its motion.

(e) Find the probability that the height of the base of the weight is greater than 1.5 metres at the time the picture is taken. [4]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

A game of chance involves drawing **two** balls at random out of a box without replacement. The box initially contains r red balls and y yellow balls.

Let $\mathrm{P}(YY)$ represent the probability of drawing two yellow balls from the box without replacement.

Consider a version of this game where it is known that $\mathrm{P}(YY) = \frac{1}{3}$.

(a) Show that $2y^2 - 2(r+1)y + r - r^2 = 0$. [4]

(b) By solving the equation in part (a), show that $y = \frac{(r+1) + \sqrt{3r^2+1}}{2}$. [4]

(c) Find two pairs of values for r and y that satisfy the condition $\mathrm{P}(YY) = \frac{1}{3}$. [4]

Now consider a similar game of chance that involves drawing **three** balls out of a box without replacement. The box initially contains 10 red balls and y yellow balls.

Let $\mathrm{P}(YYY)$ represent the probability of drawing three yellow balls from the box without replacement.

(d) Find an expression for $\mathrm{P}(YYY)$ in terms of y. [3]

A yellow ball is added so that the box now contains 10 red balls and $(y+1)$ yellow balls. The probability of drawing three yellow balls from the box without replacement is now twice the probability expressed in part (d).

(e) Find the initial number of yellow balls in the box. [5]

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12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$, where $x > 0$, $y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.1$. [2]

(b) By solving the differential equation, show that $y = x\sqrt{\frac{9x^4 - 1}{2}}$. [8]

(c) Find the value of y when $x = 1.1$. [1]

(d) With reference to the concavity of the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $1 \le x \le 1.1$, explain why the value of y found in part (c) is greater than the approximate value of y found in part (a). [2]

The graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $\frac{\sqrt{3}}{3} < x < 1$ has a point of inflexion at the point P.

(e) By sketching the graph of an appropriate derivative of y, determine the x-coordinate of P. [2]

It can be shown that $\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$, where $x > 0$, $y > 0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point P lies on the straight line $y = mx$ where the exact value of m is to be determined. [6]

References:

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Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
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- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

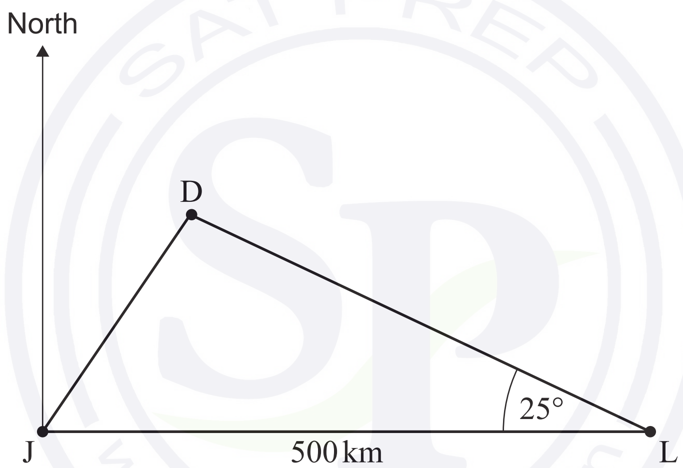
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The cities Lucknow (L), Jaipur (J) and Delhi (D) are represented in the following diagram. Lucknow lies $500\,\text{km}$ directly east of Jaipur, and $\text{JLD} = 25°$.

diagram not to scale



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The bearing of D from J is $034°$.

(a) Find $\text{J}\hat{\text{D}}\text{L}$. [2]

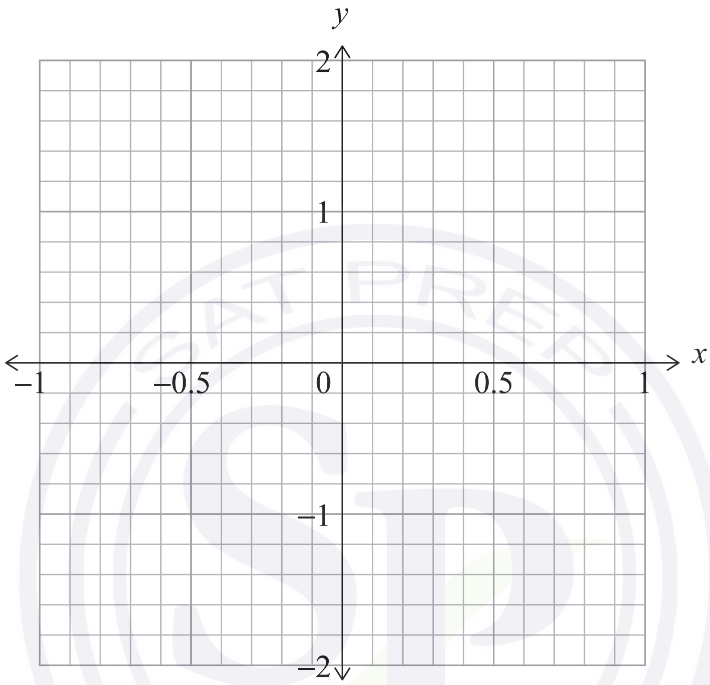
(b) Find the distance between Lucknow and Delhi. [3]

2. [Maximum mark: 5]

The functions f and g are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \le x \le 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]



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3. [Maximum mark: 7]

The total number of children, y, visiting a park depends on the highest temperature, T, in degrees Celsius (°C). A park official predicts the total number of children visiting his park on any given day using the model $y = -0.6T^2 + 23T + 110$, where $10 \le T \le 35$.

(a) Use this model to estimate the number of children in the park on a day when the highest temperature is $25\,°\text{C}$. [2]

An ice cream vendor investigates the relationship between the total number of children visiting the park and the number of ice creams sold, x. The following table shows the data collected on five different days.

Total number of children (y)	81	175	202	346	360
Ice creams sold (x)	15	27	23	35	46

(b) Find an appropriate regression equation that will allow the vendor to predict the number of ice creams sold on a day when there are y children in the park. [3]

(c) Hence, use your regression equation to predict the number of ice creams that the vendor sells on a day when the highest temperature is $25\,°\text{C}$. [2]

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4. [Maximum mark: 5]

A company manufactures metal tubes for bicycle frames. The diameters of the tubes, $D\,\text{mm}$, are normally distributed with mean 32 and standard deviation σ. The interquartile range of the diameters is 0.28.

Find the value of σ.

5. [Maximum mark: 7]

The coefficient of x^6 in the expansion of $(ax^3 + b)^8$ is 448.

The coefficient of x^6 in the expansion of $(ax^3 + b)^{10}$ is 2880.

Find the value of a and the value of b, where a, $b > 0$.

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6. [Maximum mark: 7]

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of n that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

(b) Hence or otherwise, describe a single geometric transformation applied to z on the Argand diagram that results in z^{10}. [3]

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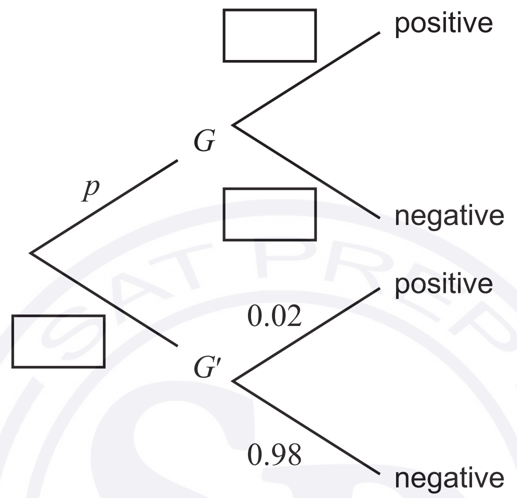
7. [Maximum mark: 6]

A new test has been developed to identify whether a particular gene, G, is present in a population of parrots. The test returns a correct positive result 95% of the time for parrots with the gene, and a false positive result 2% of the time for parrots without the gene.

The proportion of the population with the gene is p.

(a) Complete the tree diagram below. [2]



(b) A random sample of the population was tested. It was found that 150 tests returned a positive result. Out of the 150 parrots with a positive test result, 18 did not actually have the gene. Find an estimate for p. [4]

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8. [Maximum mark: 7]

The angle between a line and a plane is α, where $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

Find the value of α.

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9. [Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

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Do **not** write solutions on this page.

Section B

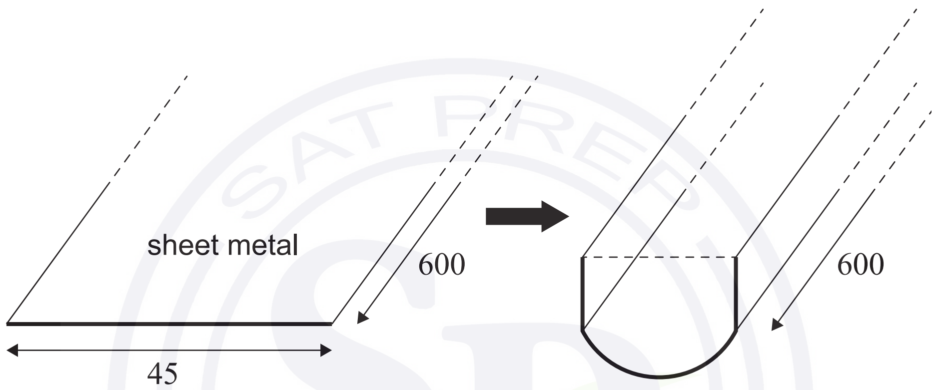
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

An engineer is designing a gutter to catch rainwater from the roof of a house.

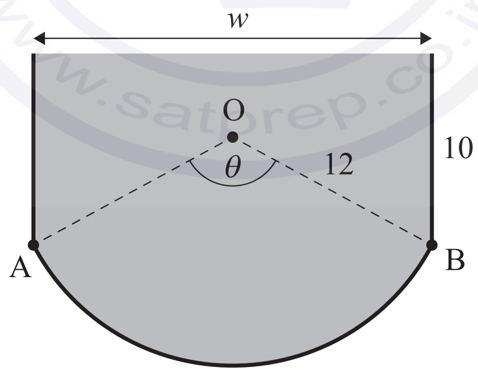
The gutter will be open at the top and is made by folding a piece of sheet metal 45 cm wide and 600 cm long.

diagram not to scale



The cross-section of the gutter is shaded in the following diagram.

diagram not to scale



The height of both vertical sides is 10 cm. The width of the gutter is $w\text{ cm}$.

Arc AB lies on the circumference of a circle with centre O and radius 12 cm.

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

Let $\text{A}\hat{\text{O}}\text{B} = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in cm^3, of rainwater that enters the gutter can be modelled by a function $R(t)$, where t is the time, in seconds, since the start of the storm.

It was determined that the **rate** at which rainwater entered the gutter could be modelled by

$$R'(t) = 50\cos\left(\frac{2\pi t}{5}\right) + 3000,\ t \geq 0.$$

During any 60-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

A continuous random variable, X, has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{6}{\pi\sqrt{16 - x^2}}, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the exact value of $\mathrm{E}(X)$. [5]

(b) Find $\mathrm{P}(X < 0.5)$. [2]

A laboratory trial may require up to 2 millilitres of reagent. The amount of reagent used has been found to have a probability distribution that can be modelled by $f(x)$, where X is the amount of reagent in millilitres.

Each laboratory trial is independent. A trial is considered a success when $X < 0.5$.

(c) Determine the least number of trials required to be 99% sure of at least one success. [3]

Ten trials were conducted.

(d) Find the probability that exactly three trials were successful. [2]

(e) Write down the number of ways these three successful trials could have occurred consecutively. [1]

Now consider n trials where it is given that exactly three successes have occurred.

(f) (i) Write down an expression for the number of ways these three successful trials could have occurred consecutively.

(ii) Find the greatest value of n such that the probability of three consecutive successful trials is more than 0.05. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

(a) Use Euler's method, with a step length of 0.03, to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [4]

(b) (i) Write down the value of $\frac{dy}{dx}$ when $x = 0$.

(ii) Show that $\frac{d^2 y}{dx^2} = 3$ when $x = 0$. [5]

(c) (i) Given that $\frac{d^3 y}{dx^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for y.

(ii) Use the Maclaurin series to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [3]

(d) (i) It is given that $\frac{x^2 - 1}{x^2 + 1} \equiv 1 - \frac{2}{x^2 + 1}$.

Solve the differential equation $\frac{dy}{dx} = \frac{x^2 y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(ii) Hence, find the value of y when $x = 0.15$. Give your answer correct to six significant figures. [7]

(e) For $0 \le x < 1$, explain why the approximate value for y obtained using Euler's method will always be less than the actual value for y. [2]

References:

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Mathematics: analysis and approaches
Higher level
Paper 2

Tuesday 1 November 2022 (morning)

Candidate session number

2 hours

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The following table shows the Mathematics test scores (x) and the Science test scores (y) for a group of eight students.

Mathematics scores (x)	64	68	72	75	80	82	85	86
Science scores (y)	67	72	77	76	84	83	89	91

The regression line of y on x for this data can be written in the form $y = ax + b$.

(a) Find the value of a and the value of b. [2]

(b) Write down the value of the Pearson's product-moment correlation coefficient, r. [1]

(c) Use the equation of your regression line to predict the Science test score for a student who has a score of 78 on the Mathematics test. Express your answer to the nearest integer. [2]

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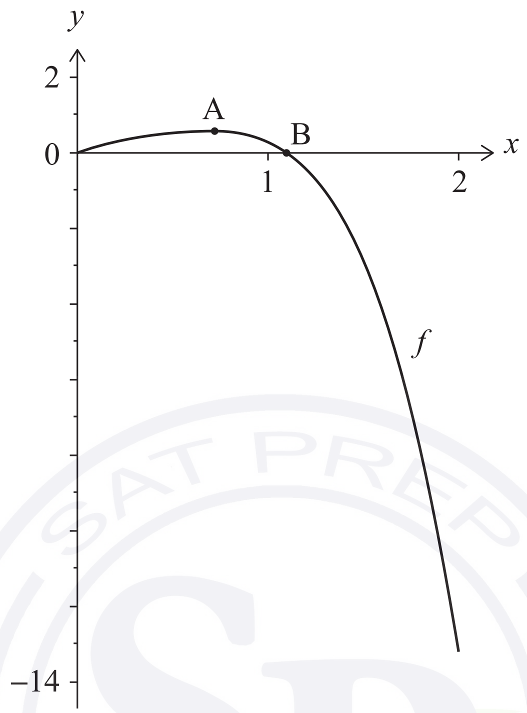
2. [Maximum mark: 6]

The function f is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 \le x \le 2$. The graph of f is shown in the following diagram.



The graph of f has a local maximum at point A. The graph intersects the x-axis at the origin and at point B.

(a) Find the coordinates of A. [2]

(b) Find the x-coordinate of B. [1]

(c) Find the total area enclosed by the graph of f, the x-axis and the line $x = 2$. [3]

3. [Maximum mark: 5]

A geometric sequence has a first term of 50 and a fourth term of 86.4.

The sum of the first n terms of the sequence is S_n.

Find the smallest value of n such that $S_n > 33\,500$.

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4. [Maximum mark: 7]

The population of a town t years after 1 January 2014 can be modelled by the function

$$P(t) = 15\,000\mathrm{e}^{kt}, \text{ where } k < 0 \text{ and } t \geq 0.$$

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041.

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5. [Maximum mark: 6]

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in x^4 is $\frac{8}{7}a^5$.

Find the value of a.

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6. [Maximum mark: 8]

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} axe^{x}, & 0 \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where a, $b \in \mathbb{R}^{+}$.

(a) Find an expression for a in terms of b. [5]

(b) In the case where $a = b = 1$, find the median of X. [3]

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7. [Maximum mark: 6]

Consider the vectors $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = \left(\cos\frac{1}{n}\right)\boldsymbol{i} + \left(\sin\frac{1}{n}\right)\boldsymbol{j}$, where $n \in \mathbb{Z}^+$.

Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Find an expression for $\cos\theta$ in terms of n. [3]

(b) Find the exact value of the limit approached by θ as $n \to \infty$. [3]

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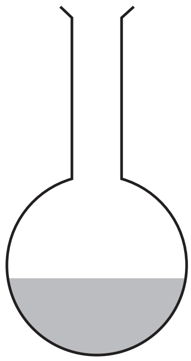
8. [Maximum mark: 6]

The following diagram shows liquid in a round-bottomed glass flask, which is made of a sphere and a cylindrical neck.



Initially, the flask is empty. Liquid is poured into the flask at a rate of $2\,\text{cm}^3\text{s}^{-1}$. You may assume that the liquid does not reach the cylindrical neck.

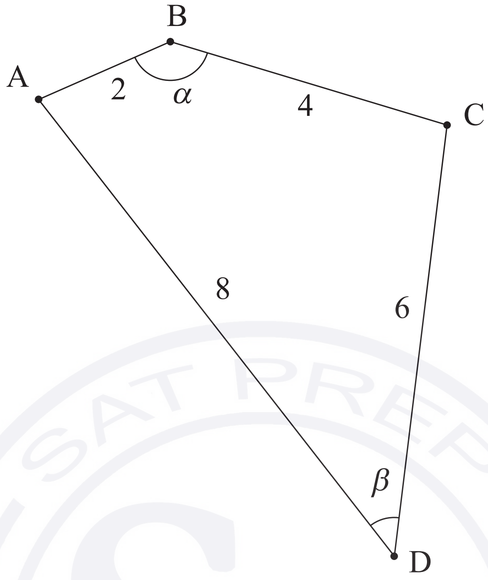
The volume $V\,\text{cm}^3$ and the height $h\,\text{cm}$ of the liquid in the flask satisfy the equation $V = 5\pi h^2 - \frac{1}{3}\pi h^3$.

Find the rate of change of the height of the liquid in the flask at the instant when the volume of the liquid is $200\,\text{cm}^3$.

9. [Maximum mark: 8]

Consider a quadrilateral ABCD such that $\mathrm{AB} = 2$, $\mathrm{BC} = 4$, $\mathrm{CD} = 6$ and $\mathrm{DA} = 8$, as shown in the following diagram. Let $\alpha = \mathrm{A\hat{B}C}$ and $\beta = \mathrm{A\hat{D}C}$.

diagram not to scale



(a) (i) Find AC in terms of α.

(ii) Find AC in terms of β.

(iii) Hence or otherwise, find an expression for α in terms of β. [4]

(b) Find the maximum area of the quadrilateral ABCD. [4]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

The time worked, T, in hours per week by employees of a large company is normally distributed with a mean of 42 and standard deviation 10.7.

(a) Find the probability that an employee selected at random works more than 40 hours per week. [2]

(b) A group of four employees is selected at random. Each employee is asked in turn whether they work more than 40 hours per week. Find the probability that the fourth employee is the only one in the group who works more than 40 hours per week. [3]

(c) A large group of employees work more than 40 hours per week.

(i) An employee is selected at random from this large group.

Find the probability that this employee works less than 55 hours per week.

(ii) Ten employees are selected at random from this large group.

Find the probability that exactly five of them work less than 55 hours per week. [7]

It is known that $\mathrm{P}(a \le T \le b) = 0.904$ and that $\mathrm{P}(T > b) = 2\mathrm{P}(T < a)$, where a and b are numbers of hours worked per week. An employee who works fewer than a hours per week is considered to be a part-time employee.

(d) Find the maximum time, in hours per week, that an employee can work and still be considered part-time. [4]

Do **not** write solutions on this page.

11. [Maximum mark: 15]

The function f is defined by $f(x) = e^{2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

12. [Maximum mark: 22]

Consider the points $A(1, 2, 3)$, $B(k, -2, 1)$ and $C(5, 0, 2)$, where $k \in \mathbb{R}$.

(a) Write down $\vec{AB}$ and $\vec{AC}$. [2]

(b) Given that the points A, B and C lie on a straight line, show that $k = 9$. [1]

(c) For $k = 9$, let L_1 be the line passing through A, B and C.

(i) Find a vector equation of the line L_1.

(ii) Line L_2 has the equation $\frac{x-1}{2} = \frac{y}{3} = 1 - z$. Show that the lines L_1 and L_2 are skew. [10]

(d) For $k \neq 9$, let Π be the plane containing A, B and C.

(i) Find the Cartesian equation of the plane Π.

(ii) Find the coordinates of the point on the plane Π which is closest to the origin $(0, 0, 0)$. [9]

References:

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Mathematics: analysis and approaches
Higher level
Paper 2

Monday 9 May 2022 (morning)

Candidate session number

2 hours

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Section A

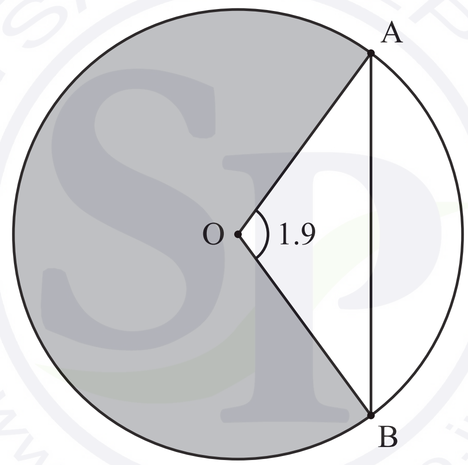
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius 5 metres.

Points A and B lie on the circle and $\text{A}\hat{\text{O}}\text{B} = 1.9$ radians.

diagram not to scale



(a) Find the length of the chord $[\text{AB}]$. [3]

(b) Find the area of the shaded sector. [3]

2. [Maximum mark: 5]

The derivative of a function g is given by $g'(x) = 3x^2 + 5e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0, 4)$. Find $g(x)$.

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3. [Maximum mark: 6]

Events A and B are independent and $\mathrm{P}(A) = 3\mathrm{P}(B)$.

Given that $\mathrm{P}(A \cup B) = 0.68$, find $\mathrm{P}(B)$.

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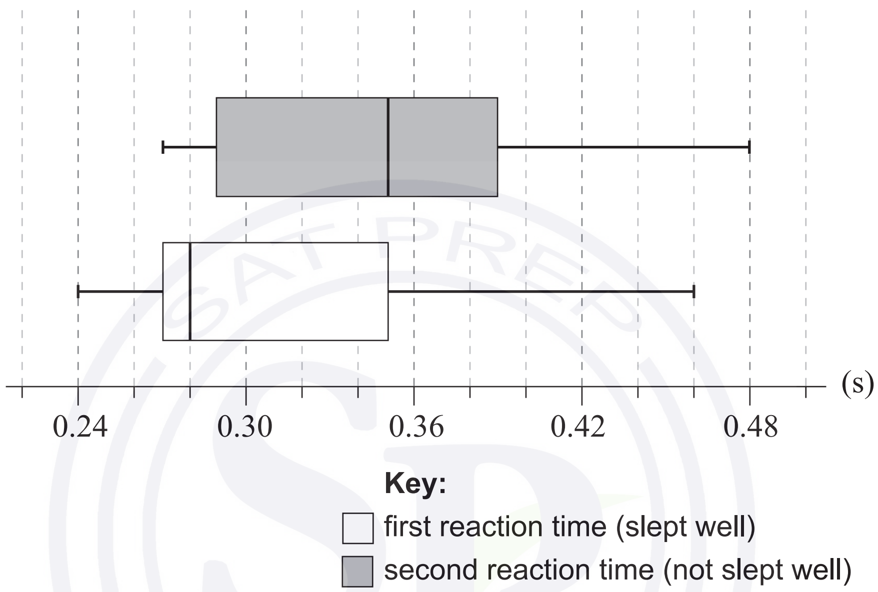
4. [Maximum mark: 6]

A random sample of nine adults were selected to see whether sleeping well affected their reaction times to a visual stimulus. Each adult's reaction time was measured twice.

The first measurement for reaction time was taken on a morning after the adult had slept well. The second measurement was taken on a morning after the same adult had not slept well.

The box and whisker diagrams for the reaction times, measured in seconds, are shown below.



Consider the box and whisker diagram representing the reaction times after sleeping well.

(a) State the median reaction time after sleeping well. [1]

(b) Verify that the measurement of 0.46 seconds is not an outlier. [3]

(c) State why it appears that the mean reaction time is greater than the median reaction time. [1]

Now consider the two box and whisker diagrams.

(d) Comment on whether these box and whisker diagrams provide any evidence that might suggest that not sleeping well causes an increase in reaction time. [1]

(This question continues on the following page)

(Question 4 continued)

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5. [Maximum mark: 7]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v = \frac{(t^2+1)\cos t}{4}$, $0 \le t \le 3$.

(a) Determine when the particle changes its direction of motion. [2]

(b) Find the times when the particle's acceleration is $-1.9\,\mathrm{m\,s^{-2}}$. [3]

(c) Find the particle's acceleration when its speed is at its greatest. [2]

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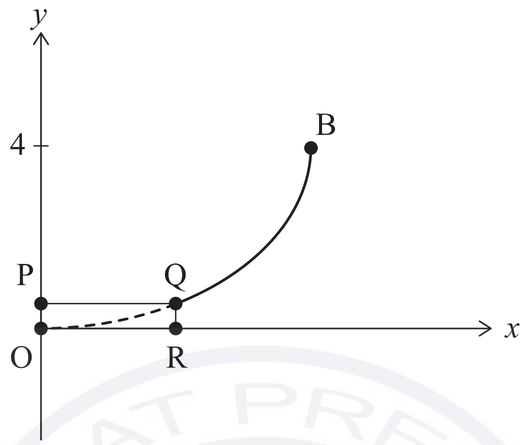
6. [Maximum mark: 5]

The following diagram shows the curve $\frac{x^2}{36}+\frac{(y-4)^2}{16}=1$, where $h \le y \le 4$.

diagram not to scale



The curve from point Q to point B is rotated $360°$ about the y-axis to form the interior surface of a bowl. The rectangle OPQR, of height h cm, is rotated $360°$ about the y-axis to form a solid base.

The bowl is assumed to have negligible thickness.

Given that the interior volume of the bowl is to be $285\,\text{cm}^3$, determine the height of the base.

7. [Maximum mark: 8]

Consider $\lim_{x\to 0}\frac{\arctan(\cos x)-k}{x^2}$, where $k \in \mathbb{R}$.

(a) Show that a finite limit only exists for $k=\frac{\pi}{4}$. [2]

(b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

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8. [Maximum mark: 7]

Rachel and Sophia are competing in a javelin-throwing competition.

The distances, R metres, thrown by Rachel can be modelled by a normal distribution with mean 56.5 and standard deviation 3.

The distances, S metres, thrown by Sophia can be modelled by a normal distribution with mean 57.5 and standard deviation 1.8.

In the first round of competition, each competitor must have five throws. To qualify for the next round of competition, a competitor must record at least one throw of 60 metres or greater in the first round.

Find the probability that only one of Rachel or Sophia qualifies for the next round of competition.

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9. [Maximum mark: 4]

Consider the set of six-digit positive integers that can be formed from the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

Find the total number of six-digit positive integers that can be formed such that

(a) the digits are distinct; [2]

(b) the digits are distinct and are in increasing order. [2]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A, h_A cm, at time t weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \le t \le 9$.

The scientist found that the height of Plant B, h_B cm, at time t weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 \le t \le 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant B;

(ii) Plant A correct to three significant figures. [3]

(b) Find the values of t when $h_A(t) = h_B(t)$. [3]

(c) For $t > 6$, prove that Plant A was always taller than Plant B. [3]

(d) For $0 \le t \le 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A. [6]

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11. [Maximum mark: 20]

Two airplanes, A and B, have position vectors with respect to an origin O given respectively by

$$\boldsymbol{r}_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$\boldsymbol{r}_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \le t \le 2.5$.

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

(a) Find the three-figure bearing on which airplane B is travelling. [2]

(b) Show that airplane A travels at a greater speed than airplane B. [2]

(c) Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. [4]

The two airplanes' lines of flight cross at point P.

(d) (i) Find the coordinates of P.

(ii) Determine the length of time between the first airplane arriving at P and the second airplane arriving at P. [7]

Let $D(t)$ represent the distance between airplane A and airplane B for $0 \le t \le 2.5$.

(e) Find the minimum value of $D(t)$. [5]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

The population, P, of a particular species of marsupial on a small remote island can be modelled by the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

where t is the time measured in years and k, N are positive constants.

The constant N represents the maximum population of this species of marsupial that the island can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{dP}{dt}$. [1]

(b) Show that $\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. [4]

(c) Hence show that the population of marsupials will increase at its maximum rate when $P = \frac{N}{2}$. Justify your answer. [5]

(d) Hence determine the maximum value of $\frac{dP}{dt}$ in terms of k and N. [2]

Let P_0 be the initial population of marsupials.

(e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right).$$ [7]

After 10 years, the population of marsupials is $3P_0$. It is known that $N = 4P_0$.

(f) Find the value of k for this population model. [2]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

Monday 9 May 2022 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The number of hours spent exercising each week by a group of students is shown in the following table.

Exercising time (in hours)	**Number of students**
2	5
3	1
4	4
5	3
6	x

The median is 4.5 hours.

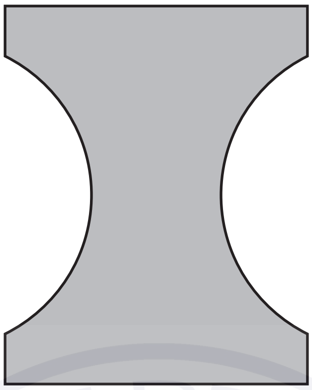
(a) Find the value of x. [2]

(b) Find the standard deviation. [2]

2. [Maximum mark: 6]

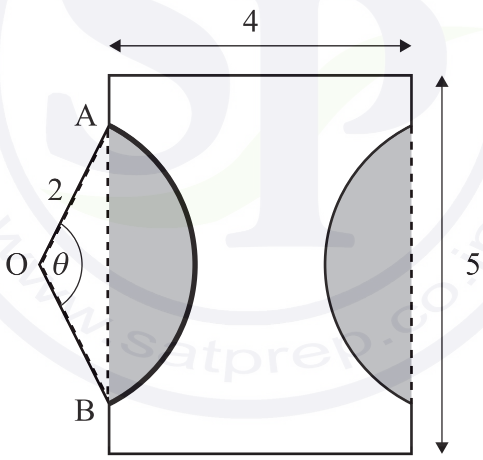
A company is designing a new logo. The logo is created by removing two equal segments from a rectangle, as shown in the following diagram.

diagram not to scale



The rectangle measures $5\,\text{cm}$ by $4\,\text{cm}$. The points A and B lie on a circle, with centre O and radius $2\,\text{cm}$, such that $\text{A}\hat{\text{O}}\text{B} = \theta$, where $0 < \theta < \pi$. This information is shown in the following diagram.

diagram not to scale



(a) Find the area of one of the shaded segments in terms of θ. [3]

(b) Given that the area of the logo is $13.4\,\text{cm}^2$, find the value of θ. [3]

(This question continues on the following page)

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(Question 2 continued)

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3. [Maximum mark: 6]

A discrete random variable, X, has the following probability distribution:

x	0	1	2	3
$\mathrm{P}(X=x)$	0.41	$k-0.28$	0.46	$0.29-2k^2$

(a) Show that $2k^2-k+0.12=0$. [1]

(b) Find the value of k, giving a reason for your answer. [3]

(c) Hence, find $\mathrm{E}(X)$. [2]

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4. [Maximum mark: 7]

A particle moves along a straight line so that its velocity, $v\,\mathrm{m\,s^{-1}}$, after t seconds is given by $v(t) = \mathrm{e}^{\sin t} + 4\sin t$ for $0 \le t \le 6$.

(a) Find the value of t when the particle is at rest. [2]

(b) Find the acceleration of the particle when it changes direction. [3]

(c) Find the total distance travelled by the particle. [2]

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5. [Maximum mark: 6]

Let A and B be two independent events such that $\mathrm{P}(A \cap B') = 0.16$ and $\mathrm{P}(A' \cap B) = 0.36$.

(a) Given that $\mathrm{P}(A \cap B) = x$, find the value of x. [4]

(b) Find $\mathrm{P}(A' \mid B')$. [2]

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6. [Maximum mark: 6]

Consider the function $f(x) = 2^x - \frac{1}{2^x}$, $x \in \mathbb{R}$.

(a) Show that f is an odd function. [2]

The function g is given by $g(x) = \frac{x-1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(b) Solve the inequality $f(x) \geq g(x)$. [4]

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7. [Maximum mark: 9]

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|\boldsymbol{b}| = 15$.

(a) Find the possible range of values for $|\boldsymbol{a} + \boldsymbol{b}|$. [2]

Consider the vector $\boldsymbol{p}$ such that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$.

(b) Given that $|\boldsymbol{a} + \boldsymbol{b}|$ is a minimum, find $\boldsymbol{p}$. [2]

Consider the vector $\boldsymbol{q}$ such that $\boldsymbol{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^+$.

(c) Find $\boldsymbol{q}$ such that $|\boldsymbol{q}| = |\boldsymbol{b}|$ and $\boldsymbol{q}$ is perpendicular to $\boldsymbol{a}$. [5]

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8. [Maximum mark: 4]

Consider the equation $kx^2 - (k+3)x + 2k + 9 = 0$, where $k \in \mathbb{R}$.

(a) Write down an expression for the product of the roots, in terms of k. [1]

(b) Hence or otherwise, determine the values of k such that the equation has one positive and one negative real root. [3]

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9. [Maximum mark: 7]

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

(a) Find the number of ways these five people can be seated in this row. [3]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

(b) Find the number of ways these five people can now be seated in this row. [4]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 20]

Consider the function $f(x)=\sqrt{x^2-1}$, where $1 \le x \le 2$.

(a) Sketch the curve $y=f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x)=\sqrt{x^2+1}$.

(ii) State the domain and range of f^{-1}. [5]

The curve $y=f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\text{m}^3$, of water in the container when it is filled to a height of h metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t=0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\text{m}^3\,\text{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

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11. [Maximum mark: 16]

A bakery makes two types of muffins: chocolate muffins and banana muffins.

The weights, C grams, of the chocolate muffins are normally distributed with a mean of $62\,\text{g}$ and standard deviation of $2.9\,\text{g}$.

(a) Find the probability that a randomly selected chocolate muffin weighs less than $61\,\text{g}$. [2]

(b) In a random selection of 12 chocolate muffins, find the probability that exactly 5 weigh less than $61\,\text{g}$. [2]

The weights, B grams, of the banana muffins are normally distributed with a mean of $68\,\text{g}$ and standard deviation of $3.4\,\text{g}$.

Each day 60% of the muffins made are chocolate.

On a particular day, a muffin is randomly selected from all those made at the bakery.

(c) (i) Find the probability that the randomly selected muffin weighs less than $61\,\text{g}$.

(ii) Given that a randomly selected muffin weighs less than $61\,\text{g}$, find the probability that it is chocolate. [7]

The machine that makes the chocolate muffins is adjusted so that the mean weight of the chocolate muffins remains the same but their standard deviation changes to $\sigma\,\text{g}$. The machine that makes the banana muffins is not adjusted. The probability that the weight of a randomly selected muffin from these machines is less than $61\,\text{g}$ is now 0.157.

(d) Find the value of σ. [5]

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12. [Maximum mark: 19]

Consider the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-2x^2$ for $x>0$ and $y>2x$. It is given that $y=3$ when $x=1$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x=1.5$. [4]

(b) Use the substitution $y=vx$ to show that $x\frac{\mathrm{d}v}{\mathrm{d}x}=v^2-v-2$. [3]

(c) (i) By solving the differential equation, show that $y=\frac{8x+x^4}{4-x^3}$.

(ii) Find the actual value of y when $x=1.5$.

(iii) Using the graph of $y=\frac{8x+x^4}{4-x^3}$, suggest a reason why the approximation given by Euler's method in part (a) is not a good estimate to the actual value of y at $x=1.5$. [12]

References:

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Mathematics: analysis and approaches
Higher level
Paper 2

Tuesday 2 November 2021 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

In Lucy's music academy, eight students took their piano diploma examination and achieved scores out of 150. For her records, Lucy decided to record the average number of hours per week each student reported practising in the weeks prior to their examination. These results are summarized in the table below.

Average weekly practice time (h)	28	13	45	33	17	29	39	36
Diploma score (D)	115	82	120	116	79	101	110	121

(a) Find Pearson's product-moment correlation coefficient, r, for these data. [2]

(b) The relationship between the variables can be modelled by the regression equation $D = ah + b$. Write down the value of a and the value of b. [1]

(c) One of these eight students was disappointed with her result and wished she had practised more. Based on the given data, determine how her score could have been expected to alter had she practised an extra five hours per week. [2]

(d) Lucy asserts that the number of hours a student practises has a direct effect on their final diploma result. Comment on the validity of Lucy's assertion. [1]

Lucy suspected that each student had not been practising as much as they reported. In order to compensate for this, Lucy deducted a fixed number of hours per week from each of the students' recorded hours.

(e) State how, if at all, the value of r would be affected. [1]

(This question continues on the following page)

(Question 1 continued)

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2. [Maximum mark: 5]

Consider a triangle ABC, where $\mathrm{AC} = 12$, $\mathrm{CB} = 7$ and $\mathrm{B\hat{A}C} = 25^{\circ}$.

Find the smallest possible perimeter of triangle ABC.

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3. [Maximum mark: 7]

A factory manufactures lamps. It is known that the probability that a lamp is found to be defective is 0.05. A random sample of 30 lamps is tested.

(a) Find the probability that there is at least one defective lamp in the sample. [3]

(b) Given that there is at least one defective lamp in the sample, find the probability that there are at most two defective lamps. [4]

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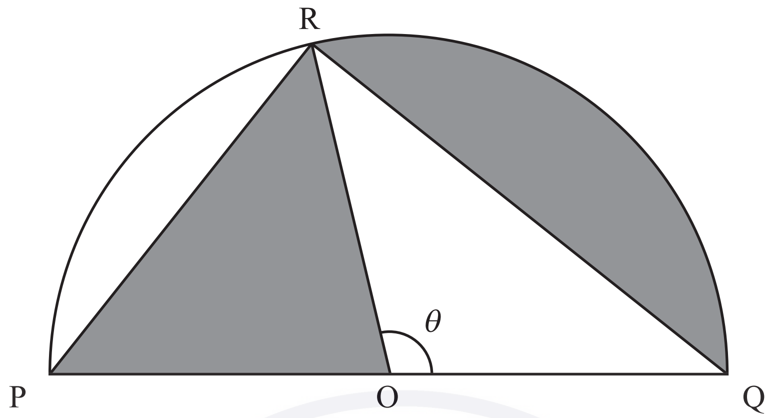
4. [Maximum mark: 6]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. [1]

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5. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1. [2]

(b) Find S_∞. [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

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6. [Maximum mark: 8]

(a) Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. [2]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β.

Consider the equation $x^2 + mx + n = 0$, where $m, n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n. [6]

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7. [Maximum mark: 6]

A continuous random variable X has a probability density function given by

$$f\left(x\right)=\begin{cases}\arccos x & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$

The median of this distribution is m.

(a) Determine the value of m. [2]

(b) Given that $P(|X-m|\le a)=0.3$, determine the value of a. [4]

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8. [Maximum mark: 8]

Consider the curve C given by $y = x - xy\ln(xy)$ where $x > 0$, $y > 0$.

(a) Show that $\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\left(1 + \ln(xy)\right) = 1$. [3]

(b) Hence find the equation of the tangent to C at the point where $x = 1$. [5]

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Do **not** write solutions on this page.

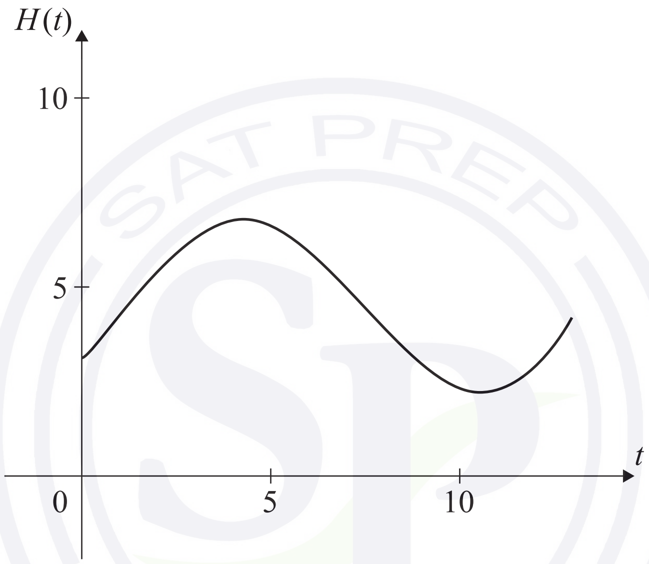
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin\left(b(t-c)\right) + d$, where t is the number of hours after midnight, and a, b, c and d are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between 2.2 m and 6.8 m.

All heights are given correct to one decimal place.

(a) Show that $b = \dfrac{\pi}{6}$. [1]

(b) Find the value of a. [2]

(c) Find the value of d. [2]

(d) Find the smallest possible value of c. [3]

(e) Find the height of the water at 12:00. [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 9 continued)

A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

(g) Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

10. [Maximum mark: 18]

Consider the function $f(x) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where a, $b \in \mathbb{Q}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,dx$, expressing your answer as a single logarithm. [7]

Do **not** write solutions on this page.

11. [Maximum mark: 21]

Three points $A(3, 0, 0)$, $B(0, -2, 0)$ and $C(1, 1, -7)$ lie on the plane Π_1.

(a) (i) Find the vector $\overrightarrow{AB}$ and the vector $\overrightarrow{AC}$.

(ii) Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane Π_2 has equation $3x - y + 2z = 2$.

(b) The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

(i) Show that at the point P, $\lambda = \frac{3}{4}$.

(ii) Hence find the coordinates of P. [3]

(d) The point $B(0, -2, 0)$ lies on L.

(i) Find the reflection of the point B in the plane Π_3.

(ii) Hence find the vector equation of the line formed when L is reflected in the plane Π_3. [9]

References:

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Mathematics: analysis and approaches
Higher level
Paper 2

Friday 7 May 2021 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2221 – 7112

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

At a café, the waiting time between ordering and receiving a cup of coffee is dependent upon the number of customers who have already ordered their coffee and are waiting to receive it.

Sarah, a regular customer, visited the café on five consecutive days. The following table shows the number of customers, x, ahead of Sarah who have already ordered and are waiting to receive their coffee and Sarah's waiting time, y minutes.

Number of customers (x)	3	9	11	10	5
Sarah's waiting time (y)	6	10	12	11	6

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$.

(a) (i) Find the value of a and the value of b.

(ii) Write down the value of Pearson's product-moment correlation coefficient, r. [3]

(b) Interpret, in context, the value of a found in part (a)(i). [1]

On another day, Sarah visits the café to order a coffee. Seven customers have already ordered their coffee and are waiting to receive it.

(c) Use the result from part (a)(i) to estimate Sarah's waiting time to receive her coffee. [2]

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2. [Maximum mark: 5]

An arithmetic sequence has first term 60 and common difference -2.5.

(a) Given that the kth term of the sequence is zero, find the value of k. [2]

Let S_n denote the sum of the first n terms of the sequence.

(b) Find the maximum value of S_n. [3]

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3. [Maximum mark: 8]

At a school, 70% of the students play a sport and 20% of the students are involved in theatre. 18% of the students do neither activity.

A student is selected at random.

(a) Find the probability that the student plays a sport and is involved in theatre. [2]

(b) Find the probability that the student is involved in theatre, but does not play a sport. [2]

At the school 48% of the students are girls, and 25% of the girls are involved in theatre.

A student is selected at random. Let G be the event "the student is a girl" and let T be the event "the student is involved in theatre".

(c) Find $\mathrm{P}(G \cap T)$. [2]

(d) Determine if the events G and T are independent. Justify your answer. [2]

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4. [Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of f. [2]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. [4]

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5. [Maximum mark: 7]

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

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6. [Maximum mark: 6]

A continuous random variable X has the probability density function f_n given by

$$f_n(x) = \begin{cases} (n+1)x^n, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $n \in \mathbb{R}$, $n \ge 0$.

(a) Show that $\mathrm{E}(X) = \dfrac{n+1}{n+2}$. [2]

(b) Show that $\mathrm{Var}(X) = \dfrac{n+1}{(n+2)^2(n+3)}$. [4]

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7. [Maximum mark: 5]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

(a) in the position immediately after Andrea; [2]

(b) in any position after Andrea. [3]

8. [Maximum mark: 5]

Consider $z = \cos\theta + \mathrm{i}\sin\theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $\mathrm{Re}\left(\dfrac{1+z}{1-z}\right) = 0$.

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9. [Maximum mark: 8]

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of t. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\frac{x^2}{2}+\frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x\arctan 2x}{\sec x-1}\right)$. [3]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The flight times, T minutes, between two cities can be modelled by a normal distribution with a mean of 75 minutes and a standard deviation of σ minutes.

(a) Given that 2% of the flight times are longer than 82 minutes, find the value of σ. [3]

(b) Find the probability that a randomly selected flight will have a flight time of more than 80 minutes. [2]

(c) Given that a flight between the two cities takes longer than 80 minutes, find the probability that it takes less than 82 minutes. [4]

On a particular day, there are 64 flights scheduled between these two cities.

(d) Find the expected number of flights that will have a flight time of more than 80 minutes. [3]

(e) Find the probability that more than 6 of the flights on this particular day will have a flight time of more than 80 minutes. [3]

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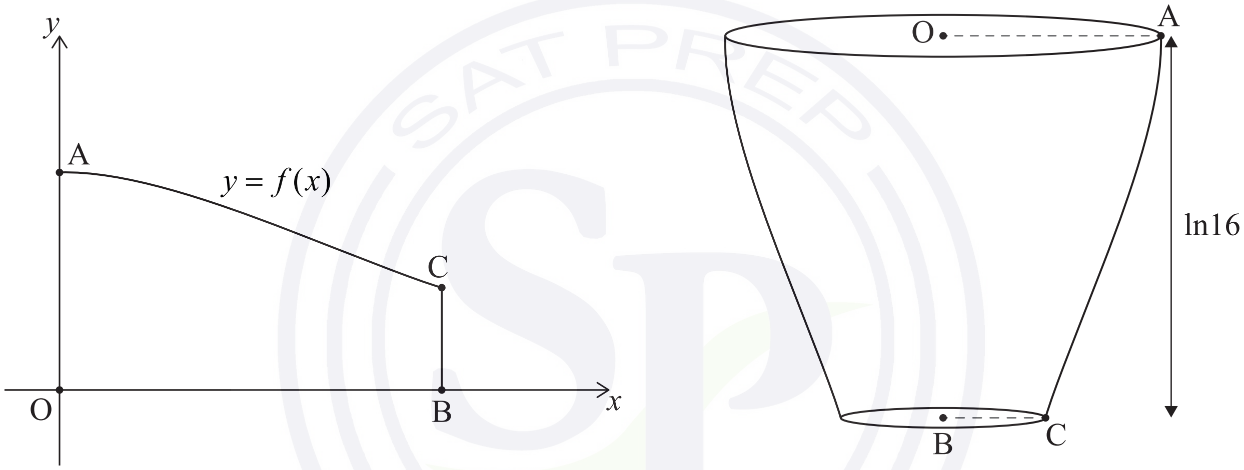
11. [Maximum mark: 18]

A function f is defined by $f\left(x\right)=\dfrac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^{x}}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y=f(x)$, the x-axis, the y-axis and the line $x=\ln 16$ is rotated $360°$ about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\dfrac{15k^{2}\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of $300\,\mathrm{cm}^{3}$ based on the result from part (a). Pedro's design is shown in the following diagrams.



The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of k that satisfies the requirements of Pedro's design. [2]

(c) Find

(i) OA;

(ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly.

(ii) State the cross-sectional radius of the bowl at this point. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2 + 1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

Friday 7 May 2021 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

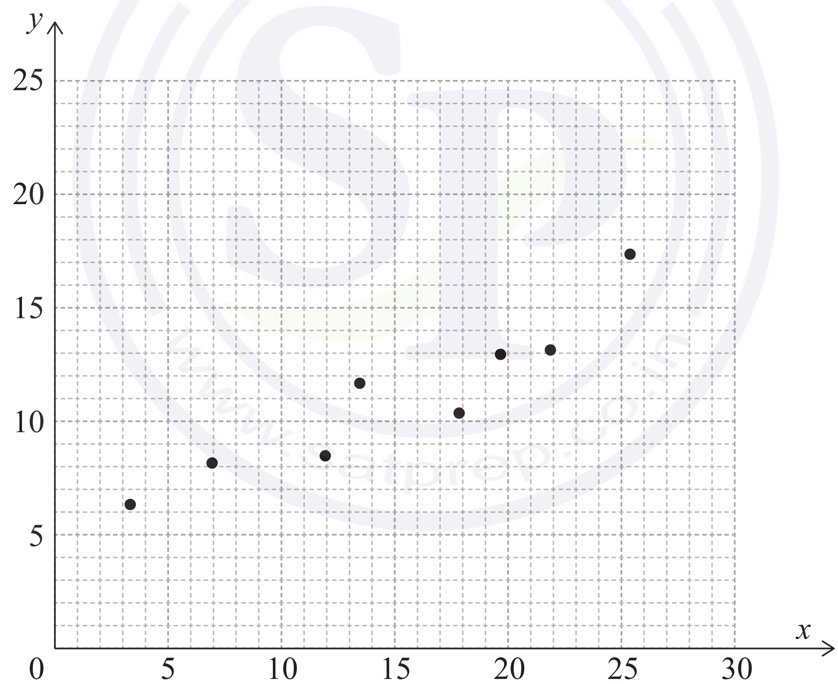
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

The following table shows the data collected from an experiment.

x	3.3	6.9	11.9	13.4	17.8	19.6	21.8	25.3
y	6.3	8.1	8.4	11.6	10.3	12.9	13.1	17.3

The data is also represented on the following scatter diagram.



(This question continues on the following page)

(Question 1 continued)

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$, where a, $b \in \mathbb{R}$.

(a) Write down the value of a and the value of b. [2]

(b) Use this model to predict the value of y when $x = 18$. [2]

(c) Write down the value of $\overline{x}$ and the value of $\overline{y}$. [1]

(d) Draw the line of best fit on the scatter diagram. [2]

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2. [Maximum mark: 6]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

(a) Find the probability that a bag selected at random is rejected. [2]

(b) Estimate the number of bags which will be rejected from a random sample of 100 bags. [1]

(c) Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. [3]

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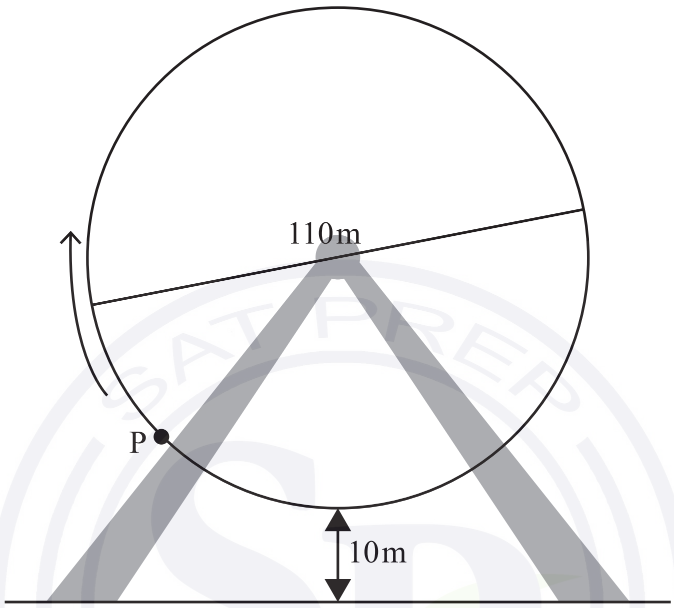
3. [Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale



The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where $a, b, c \in \mathbb{R}$.

Find the values of a, b and c.

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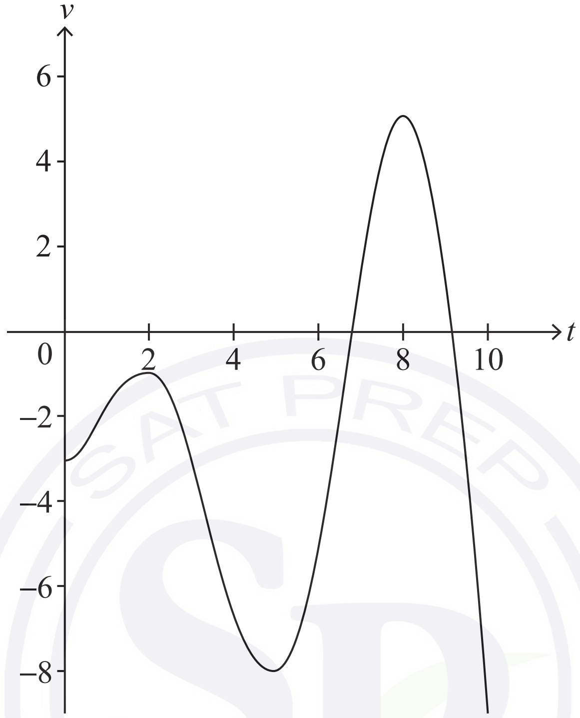
4. [Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\,\text{ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t\sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of v.



(a) Find the smallest value of t for which the particle is at rest. [2]

(b) Find the total distance travelled by the particle. [2]

(c) Find the acceleration of the particle when $t = 7$. [2]

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5. [Maximum mark: 5]

Consider the expansion of $(3+x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is 20412, find the value of n.

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6. [Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$\Pi_1: 3x + 2y + z = 6$

$\Pi_2: x - 2y + z = 4$

(a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]

(b) Find the coordinates of the point where Π_1, Π_2 and Π_3 intersect. [2]

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7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x)=\begin{cases}\dfrac{x}{\sqrt{\left(x^2+k\right)^3}} & 0\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

where $k \in \mathbb{R}^+$.

(a) Show that $\sqrt{16+k}-\sqrt{k}=\sqrt{k}\sqrt{16+k}$. [5]

(b) Find the value of k. [2]

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8. [Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - i\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of zw. [1]

(b) Find the argument of zw in terms of k. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of k.

(ii) For the value of k found in part (i), find the value of zw. [4]

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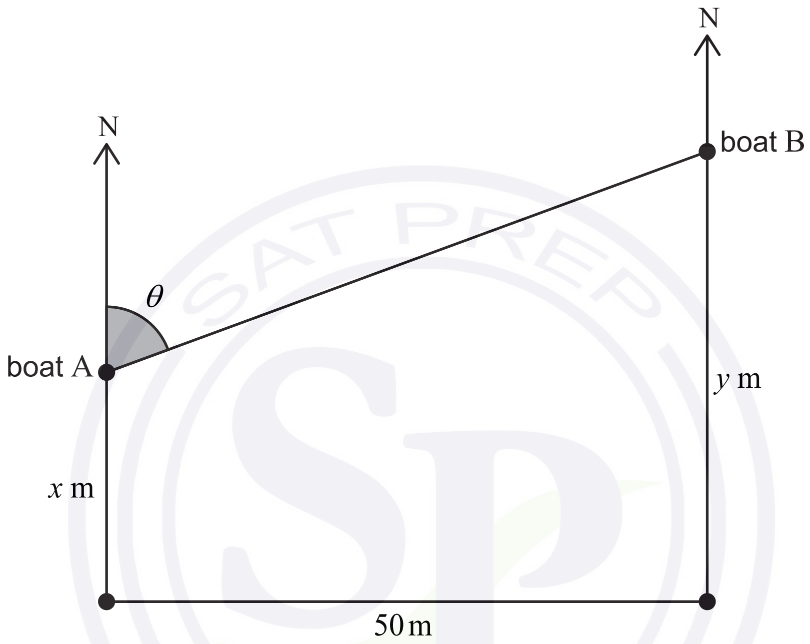
9. [Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.

diagram not to scale



(a) Show that $y = x + 50\cot\theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T. [6]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,e^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

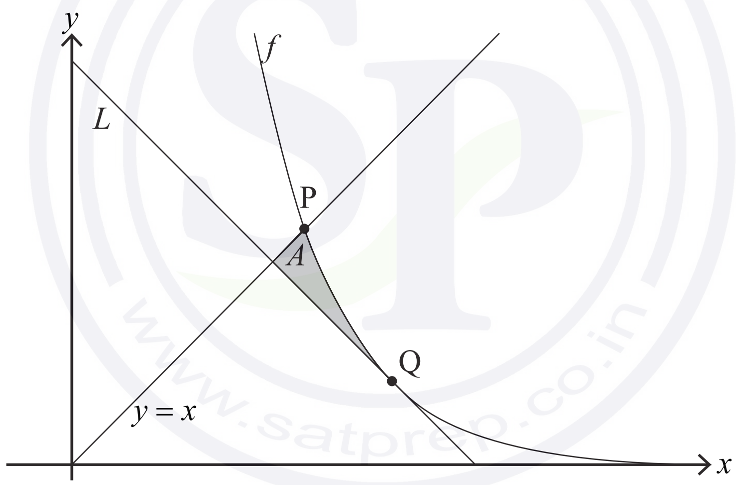
(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.



(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

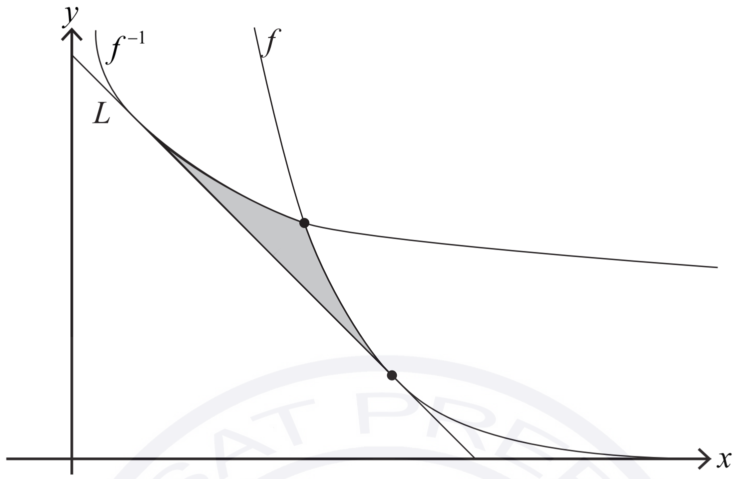
(ii) Hence, find the area of A. [5]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The line L is tangent to the graphs of both f and the inverse function f^{-1}.



(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

11. [Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 20]

The function f has a derivative given by $f'(x) = \frac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{x} + \frac{b}{k-x}$, where a, $b \in \mathbb{R}$. Find a and b in terms of k. [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\frac{dP}{dt} = \frac{P(k-P)}{5k}$, where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

(c) By solving the differential equation, show that $P = \frac{1200k}{(k-1200)e^{-\frac{t}{5}} + 1200}$. [8]

At $t = 10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of k, giving your answer correct to four significant figures. [3]

(e) Find the value of t when the rate of change of the population is at its maximum. [3]

References:

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Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 2

Specimen

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
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- The maximum mark for this examination paper is **[110 marks]**.

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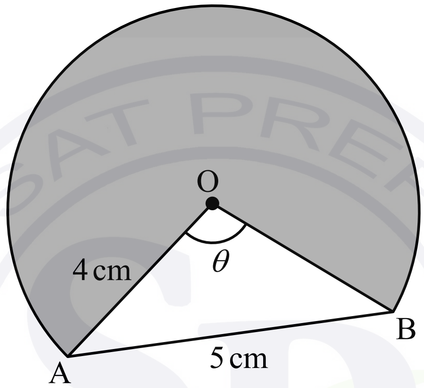
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius 4 cm.



Chord AB has a length of 5 cm and $\text{A}\hat{\text{O}}\text{B} = \theta$.

(a) Find the value of θ, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

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2. [Maximum mark: 6]

On 1st January 2020, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of 5.5%, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, r.

(a) Find the value of r, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

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3. [Maximum mark: 6]

A six-sided biased die is weighted in such a way that the probability of obtaining a "six" is $\frac{7}{10}$.

The die is tossed five times. Find the probability of obtaining

(a) at most three "sixes". [3]

(b) the third "six" on the fifth toss. [3]

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4. [Maximum mark: 7]

The following table below shows the marks scored by seven students on two different mathematics tests.

Test 1 (x)	15	23	25	30	34	34	40
Test 2 (y)	20	26	27	32	35	37	35

Let L_1 be the regression line of x on y. The equation of the line L_1 can be written in the form $x = ay + b$.

(a) Find the value of a and the value of b. [2]

Let L_2 be the regression line of y on x. The lines L_1 and L_2 pass through the same point with coordinates (p, q).

(b) Find the value of p and the value of q. [3]

(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's mark on the first test. [2]

5. [Maximum mark: 7]

The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t + 2t \sin t$, $0 \le t \le 5$.

(a) Find the maximum distance of the particle from O. [3]

(b) Find the acceleration of the particle at the instant it first changes direction. [4]

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6. [Maximum mark: 6]

In a city, the number of passengers, X, who ride in a taxi has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X=x)$	0.60	0.30	0.03	0.05	0.02

After the opening of a new highway that charges a toll, a taxi company introduces a charge for passengers who use the highway. The charge is $\$2.40$ per taxi plus $\$1.20$ per passenger. Let T represent the amount, in dollars, that is charged by the taxi company per ride.

(a) Find $\mathrm{E}(T)$. [4]

(b) Given that $\mathrm{Var}(X) = 0.8419$, find $\mathrm{Var}(T)$. [2]

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7. [Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

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8. [Maximum mark: 7]

The complex numbers w and z satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

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9. [Maximum mark: 5]

Consider the graphs of $y = \frac{x^2}{x - 3}$ and $y = m(x + 3)$, $m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points.

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The length, $X\,\text{mm}$, of a certain species of seashell is normally distributed with mean 25 and variance, σ^2.

The probability that X is less than 24.15 is 0.1446.

(a) Find $P(24.15 < X < 25)$. [2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a seashell selected at random has a length greater than $26\,\text{mm}$. [5]

A random sample of 10 seashells is collected on a beach. Let Y represent the number of seashells with lengths greater than $26\,\text{mm}$.

(c) Find $E(Y)$. [3]

(d) Find the probability that exactly three of these seashells have a length greater than $26\,\text{mm}$. [2]

A seashell selected at random has a length less than $26\,\text{mm}$.

(e) Find the probability that its length is between $24.15\,\text{mm}$ and $25\,\text{mm}$. [3]

Do **not** write solutions on this page.

11. [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let x grams represent the amount of salt in the tank and let t minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{dx}{dt}$, is described by the differential equation $\frac{dx}{dt} = 10e^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t+1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40e^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of x versus t for $0 \le t \le 60$ and hence find the maximum amount of salt in the tank and the value of t at which this occurs. [5]

(d) Find the value of t at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first 60 minutes. [4]

12. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1 - \tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]