8823 – 7112M

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 2

32 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.

- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.

- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.

- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***. If ***A1*** marks are on separate lines, they are assumed to be dependent and hence ***A0A1*** is unlikely to be awarded. However, where such marks are *independent* (e.g. the markscheme is presenting them in sequence, but in the solution one does not lead directly to the other) this should be communicated via a note, and hence ***A0A1*** (for example) can be awarded.

- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.

- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.

- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.

- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

Final answers will generally not need to restate the variable and/or units to be considered correct. To help examiners, the markscheme will include variables and units, where appropriate. However, their omission from a candidate's final answer should only be penalized if explicitly instructed in a markscheme note.

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, , the general rule applies to final answers: unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to a "correct" level of accuracy (e.g 3 sf) in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) $\mathrm{BV}=\sqrt{(6-3)^2+(8-4)^2+(0-9)^2}$ ***(A1)***

$=10.2956...$

$=10.3\ (=\sqrt{106})$ ***A1***

> **Note:** Award ***SC(A0)A1*** for $\mathrm{BV}=\begin{pmatrix}-3\\-4\\9\end{pmatrix}$ where a candidate has misinterpreted notation.

[2 marks]

(b) **METHOD 1**

$\mathrm{BV}=\mathrm{VC}$ AND $\mathrm{BC}=8$ (seen anywhere) ***(A1)***

attempt to use the cosine rule on triangle BVC for any angle ***(M1)***

> **Note:** Recognition must be shown in context either in terms of labelled sides or in side lengths.

$\cos\mathrm{B\hat{V}C}=\dfrac{10.2...^2+10.2...^2-8^2}{2\times10.2...\times10.2...}$ OR

$8^2=10.2....^2+10.2...^2-2\times10.2...\times10.2...\cos\mathrm{B\hat{V}C}$ ***(A1)***

$\mathrm{B\hat{V}C}=0.798037...$

$\mathrm{B\hat{V}C}=0.798$ (accept 45.7°) ***A1***

> **Note:** If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C}=0.80$ (or 46°) (2sf).

METHOD 2

let M be the midpoint of BC

$\mathrm{BM}=4$ (seen anywhere) ***(A1)***

attempt to use sine or cosine in triangle BMV or CMV ***(M1)***

$\arcsin\dfrac{4}{\sqrt{106}}$ OR $\dfrac{\pi}{2}-\arccos\dfrac{4}{\sqrt{106}}$ OR 0.399018 ***(A1)***

$\mathrm{B\hat{V}C}=0.798037...$

$\mathrm{B\hat{V}C}=0.798$ (accept 45.7°) ***A1***

> **Note:** If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C}=0.80$ (or 46°) (2sf).

METHOD 3

$$\overrightarrow{\mathrm{VC}} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix} \text{ and } \overrightarrow{\mathrm{VB}} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$$ ***(A1)***

attempt to use the cosine formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$$\cos \mathrm{B\hat{V}C} = \frac{3(3) - 4(4) - 9(-9)}{\sqrt{3^2 + (-4)^2 + (-9)^2}\sqrt{3^2 + 4^2 + (-9)^2}} \left(= \frac{74}{106} \right)$$ ***(A1)***

$\mathrm{B\hat{V}C} = 0.798037...$

$\mathrm{B\hat{V}C} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C} = 0.80$ (or 46°) (2sf).

METHOD 4

$$\overrightarrow{\mathrm{VC}} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix} \text{ and } \overrightarrow{\mathrm{VB}} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$$ ***(A1)***

attempt to use the cross product formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$$\sin \mathrm{B\hat{V}C} = \frac{\left| \begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix} \right|}{\sqrt{3^2 + (-4)^2 + (-9)^2}\sqrt{3^2 + 4^2 + (-9)^2}} \left(= \frac{\sqrt{5760}}{106} \right)$$ ***(A1)***

$\mathrm{B\hat{V}C} = 0.798037...$

$\mathrm{B\hat{V}C} = 0.798$ (accept 45.7°) ***A1***

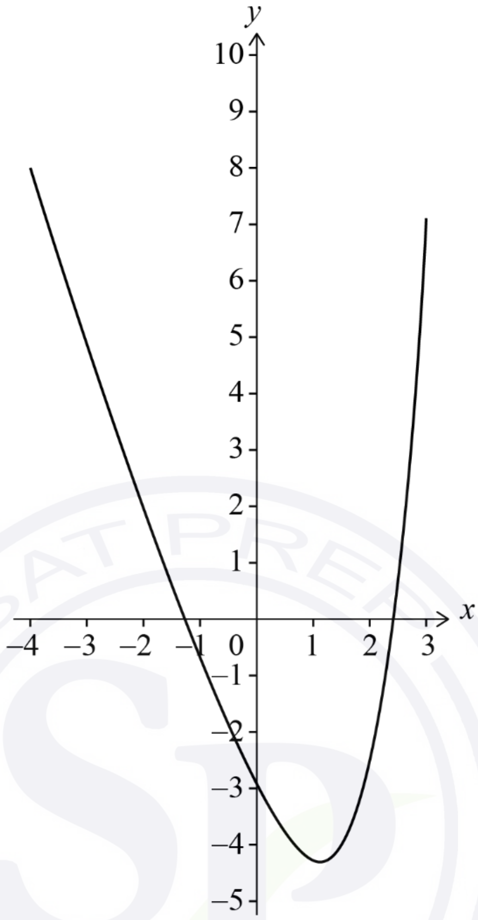
Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C} = 0.80$ (or 46°) (2sf).

Award ***SC(A1)(M1)(A0)A0*** for $\text{area} = \frac{1}{2}\left| \begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix} \right| = \frac{\sqrt{5760}}{2} (= 37.9)$ where a candidate has misinterpreted notation.

[4 marks]

Total [6 marks]

2. (a)



A1A1A1

Note:	Award marks as follows: ***A1*** for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$. ***A1*** for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$. ***A1*** for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5,\ 7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5,\ 6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$ ***A1***

$c = -3$ (accept translate/shift 3 (units) down) ***A1***

[2 marks]

Total [5 marks]

3. (a) use of sector area formula to find area of at least one sector ***(M1)***

$\frac{1}{2}\times 5.2\times 100-\frac{1}{2}\times 5.2\times r^2$ OR $10^2\pi-\frac{1}{2}10^2\times(2\pi-5.2)-\left(\pi r^2-\frac{1}{2}\times(2\pi-5.2)\times r^2\right)$ ***A1***

$(\text{area})=260-2.6r^2$ ***AG***

Note: There are many different ways to find the area of the "C". In all methods, the ***A*** mark is awarded for working which leads directly to the ***AG***. Many candidates are working with rounded intermediate values. Award the ***A*** mark to correct work with values that round to the 3sf value of 260 and the 2sf value of 2.6 eg $259.99-2.6015r^2$.

[2 marks]

(b) (i) $260-2.6r^2=64$ ***(A1)***

$r=8.68243...$

$=8.68\,(\text{cm})\ \left(\frac{14\sqrt{65}}{13}\text{ exact}\right)$ ***A1***

(ii) 10×5.2 OR $8.68...\times 5.2$ ***(A1)***

substituting their value of r into $10\times 5.2+r\times 5.2+2(10-r)$ (or equivalent) ***(M1)***

Perimeter = $10\times 5.2+8.68...\times 5.2+2(10-8.68...)$ $(=52+45.1486...+2.63513...)$

= 99.7837…

= 99.8 (cm) ***A1***

[5 marks]

Total [7 marks]

4. (a) recognizing at rest when $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ OR s is a minimum ***(M1)***

$q = 5.73553...$

$= 5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q = 5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...} |v(t)|\,\mathrm{d}t$ OR $\int_0^{5.73...} \left|\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right|\mathrm{d}t$ OR $\left|\int_0^{5.73...} v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...} v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of dt. Only accept $\left|\int_0^{q} v(t)\,\mathrm{d}t\right|$ if their value of q does not result in the particle changing direction in the first q seconds.

$= 7.68302...$

$= 7.68$ (m) ***A1***

Note: Special Cases: Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q = 1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ ($3.37644...$ from 3sf). Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...} v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302... - (-4.3)$

$= 7.68302...$

$= 7.68$ (m) ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

5. $\mathrm{E}(X) = k + 2k^2 + 3a + 4k^3 = 2.3$ ***(A1)***

$k + k^2 + a + k^3 = 1$ ***(A1)***

Note: The first two ***A*** marks are independent of each other.

EITHER (finding intersections of functions)

attempt to make a the subject in both of their equations ***(M1)***

$a = 1 - k - k^2 - k^3$ and $a = \frac{1}{3}\left(2.3 - k - 2k^2 - 4k^3\right)$

use of graph or table to attempt to find intersection ***(M1)***

OR (solving algebraically)

attempt to solve their equations algebraically to find a cubic in k ***(M1)***

$k^3 - k^2 - 2k + 0.7 = 0$ OR $3\left(1 - k - k^2 - k^3\right) = 2.3 - k - 2k^2 - 4k^3$ (or equivalent)

attempt to solve their cubic in k ***(M1)***

THEN

$a = 0.552839...$ OR $k = 0.315870...$ (other solutions to cubic are $k = -1.18538..., 1.86951...$)

$a = 0.553$ ***A1***

Note: If no working shown, award ***(A1)(A1)(M1)(M1)A0*** for $a = 2.44587...$ OR $a = -10.8987...$ and award ***(A0)(A0)(M1)(M1)A0*** for $a = 0.55$ (2sf) .

Total [5 marks]

6. (a) $5.75 = 25p(1-p)$ ***(A1)***

$p = 0.641421..., 0.358578...$

$p = 0.641, 0.359 \left(= \frac{5 \pm \sqrt{2}}{10} \right)$ ***A1A1***

[3 marks]

(b) $\mathrm{Var}(Y) = (-2)^2 \mathrm{Var}(X) (= 4\mathrm{Var}(X))$ ***(A1)***

$= 23$ ***A1***

[2 marks]

Total [5 marks]

7. (a) (i) $(9! \ =)\ 362880$ ***A1***

Note: Accept 9! or 363000.

(ii) attempt to consider girls as a single object ***(M1)***

$(3! \times 7! \ =)\ 30240$ ***A1***

Note: Accept 30200.

[3 marks]

(b) **METHOD 1**

recognition of the two different cases for 2 girls and 3 girls ***(M1)***

exactly 2 girls is ${}^{6}C_{3} \times {}^{3}C_{2} = 60$ and exactly 3 girls $\left({}^{3}C_{3} \times\right){}^{6}C_{2} = 15$ ***(A1)***

total $(= 60 + 15) = 75$ ***A1***

METHOD 2

recognition of the three different cases: total choices, 1 girl and no girls ***(M1)***

total choices ${}^{9}C_{5} = 126$, one girl case ${}^{3}C_{1} \times {}^{6}C_{4} = 45$, no girl case ${}^{6}C_{5} = 6$ ***(A1)***

total $(= 126 - 45 - 6) = 75$ ***A1***

[3 marks]

Total [6 marks]

8. (a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1-p \\ -1 \end{pmatrix}$ ***A1***

$\overrightarrow{AC} = \begin{pmatrix} p \\ -p \\ 2 \end{pmatrix}$ ***A1***

attempt to evaluate their $\overrightarrow{AB} \times \overrightarrow{AC}$ by use of formula or determinant ***M1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2(1-p)-p \\ -(2+p) \\ -p-p(1-p) \end{pmatrix}$ OR $(2(1-p)-p)\boldsymbol{i} - (2+p)\boldsymbol{j} + (-p-p(1-p))\boldsymbol{k}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$ ***AG***

[4 marks]

(b) $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$

$= (2-3p)^2 + (-2-p)^2 + (p^2-2p)^2 \left(= p^4 - 4p^3 + 14p^2 - 8p + 8\right)$ ***(A1)***

attempt to find minimum of their $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$ ***(M1)***

$6.75257...$ OR $p = 0.3264...$

min value is 6.75 ***A1***

[3 marks]

continued…

Question 8 continued

(c) **METHOD 1**

valid attempt to find $\text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ using their answer to part b) ***(M1)***

area $= \frac{1}{2}\sqrt{6.75257...}$

$= 1.299285...$

$= 1.30$ (units^2) ***A1***

[2 marks]

Total [9 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x=0, y=2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x=0$, $y=2$ to find the value of A ***M1***

$2 = A\mathrm{e}^{0} \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $|y-4| = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{10}y=\frac{4}{10}$ so IF $\mathrm{e}^{\int\frac{1}{10}\mathrm{d}x}=\mathrm{e}^{\frac{1}{10}x}$

$\mathrm{e}^{\frac{1}{10}x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{10}\mathrm{e}^{\frac{1}{10}x}y=\frac{4}{10}\mathrm{e}^{\frac{1}{10}x}$

$\mathrm{e}^{\frac{1}{10}x}y=4\mathrm{e}^{\frac{1}{10}x}\left(+c\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS***.***

substituting initial conditions $x=0, y=2$ to find the value of c ***M1***

$\left(2\mathrm{e}^{0}=4\mathrm{e}^{0}+c\Rightarrow\right)c=-2$ ***A1***

$\mathrm{e}^{\frac{1}{10}x}y=4\mathrm{e}^{\frac{1}{10}x}-2$

$y=4-2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $=2.0980199...-\left(4-2e^{-0.05}\right)=0.000478749...$

$=0.000479\left(=4.79\times10^{-4}\right)$ ***A1***

Note: Accept $0.000459\left(=4.59\times10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

10. (a) recognizing probabilities sum to 1 ***(M1)***

$0.288 + \mathrm{P}(94.6 < X < 98.1) + 0.434 = 1$

$\mathrm{P}(94.6 < X < 98.1) = 0.278$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(94.6 < X < 98.1) = 0.28$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing the need to use inverse normal with 0.288, $(1-0.434)$ or 0.434 ***(M1)***

Note: Accept use of calculator notation eg $\mathrm{invNorm}\left(0.288\right)\left(= -0.559236...\right)$.

$\mu + \mathrm{invNorm}(0.288)\sigma = 94.6, \mu + \mathrm{invNorm}(1-0.434)\sigma = 98.1$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations in two variables using the GDC (that involve either z-values or 'invNorm' rather than probabilities) ***(M1)***

$\mu = 97.2981...,\ \sigma = 4.82468...$

$\mu = 97.3,\ \sigma = 4.82$ ***A1***

Note: Condone use of different variables throughout, but do not award the final ***A1*** if they do not clearly identify which variable is their mean and standard deviation.

METHOD 2

use of inverse normal to find at least one z-score for $\mathrm{P}(Z < z) = 0.288$ or $\mathrm{P}(Z < z) = 1 - 0.434$ ***(M1)***

$z_1 = -0.559236...$ OR $z_2 = 0.166199...$

$\frac{94.6-\mu}{\sigma} = -0.559236...,\ \frac{98.1-\mu}{\sigma} = 0.166199...$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations (that involve z-values rather than probabilities) ***(M1)***

$\mu = 97.2981...,\ \sigma = 4.82468...$

$\mu = 97.3,\ \sigma = 4.82$ ***A1***

Note: Award marks as appropriate for work seen in part (a).

Note: If no working shown, award ***(M1)(A0)(A0)(M1)A0*** for $\mu = 97,\ \sigma = 4.8$ (2sf).

[5 marks]

Question 10 continued

(c) (i) recognition of Binomial distribution **(M1)**

$X \sim \mathrm{B}(100, 0.434)$

$\mathrm{P}(X = 34) = 0.0133198...$

$= 0.0133$ **A1**

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(X = 34) = 0.013$ (2sf).

(ii) $\mathrm{P}(X < 49) = 0.848218...$ (seen anywhere) **(A1)**

recognition of conditional probability **(M1)**

Note: recognition must be shown in context, either in symbols eg $\mathrm{P}\left(X = 34 \mid X < 49\right)$, or in words eg $\mathrm{P}\left(34 \text{ plants} \mid \text{less than 49 plants}\right)$, not only as $\mathrm{P}(A \mid B)$.

$\left(\mathrm{P}\left(X = 34 \mid X < 49\right) =\right) \dfrac{\mathrm{P}\left(X = 34\right)}{\mathrm{P}\left(X < 49\right)}$ OR $\dfrac{\mathrm{P}\left(X = 34\right)}{\mathrm{P}\left(X \le 48\right)} \left(= \dfrac{0.0133198...}{0.848218...}\right)$ **(A1)**

$= 0.0157033...$

$\mathrm{P}\left(X = 34 \mid X < 49\right) = 0.0157$ **A1**

Note: Exception to ***FT***: If the candidate finds $\mathrm{P}(X \le 49) \left(= 0.890474...\right)$ and uses that to calculate $\mathrm{P}\left(X = 34 \mid X \le 49\right) = 0.0149581...$ award ***(A0)(M1)(A1)A0***.

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{P}\left(X = 34 \mid X < 49\right) = 0.016$ (2sf).

[6 marks]

continued…

Question 10 continued

(d) $Q_1 = 96.19$ OR $Q_3 = 101.01$ (may be seen on a labelled diagram with areas indicated) ***(A1)***

$P(96.19 < F < 101.01) = 0.5$ OR $P(F < 96.19) = 0.25$ OR $P(F < 101.01) = 0.75$ (or equivalent)

EITHER

attempt to find d using graph or table ***(M1)***

OR

$1 - 2P\left(Z < -\frac{2.41}{d}\right) = 0.5$ OR $P(Z < -\frac{2.41}{d}) = 0.25$ OR $P(Z < \frac{2.41}{d}) = 0.75$

OR $P(-\frac{2.41}{d} < Z < \frac{2.41}{d}) = 0.5$ (or equivalent) ***(M1)***

$-\frac{2.41}{d} = -0.674489...$ OR $\frac{2.41}{d} = 0.674489...$

THEN

$3.57307...$

$d = 3.57$ ***A1***

Note: Accept 3.56 using 96.2 or 101.

Note: If no working shown, award ***(A0)(M1)A0*** for $d = 3.6$ (2sf).

[3 marks]

Total [16 marks]

11. (a) (vertical asymptote equation) $x = -3$ ***A1***

Note: Accept $2x+6=0$ or equivalent.

[1 mark]

(b) $(2,0)$ and $(12,0)$ ***A1A1***

Note: Award ***A1*** for $(2,0)$ and ***A1*** for $(12,0)$. Award ***A1A0*** if only x values are given.

[2 marks]

(c) **METHOD 1**

$a = \frac{1}{2}$ ***A1***

attempt at 'long division' on $\frac{x^2-14x+24}{2x+6}$ ***(M1)***

$\frac{x^2-14x+24}{2x+6}$

$= \frac{1}{2}x - \frac{17}{2}\left(+\frac{\dots}{2x+6}\right)$ ***(A1)***

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

continued…

Question 11 continued

METHOD 2

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6}$ ***(A1)***

$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$

attempt to equate coefficients of x: ***(M1)***

$-14 = 3 + 2b$

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6}$ ***(A1)***

attempt to find the limit of $f(x) - ax$ as $x \longrightarrow \infty$ ***(M1)***

$b = \lim_{x \to \infty} \frac{-17x + 24}{2x + 6}$

$= -\frac{17}{2}$ ***A1***

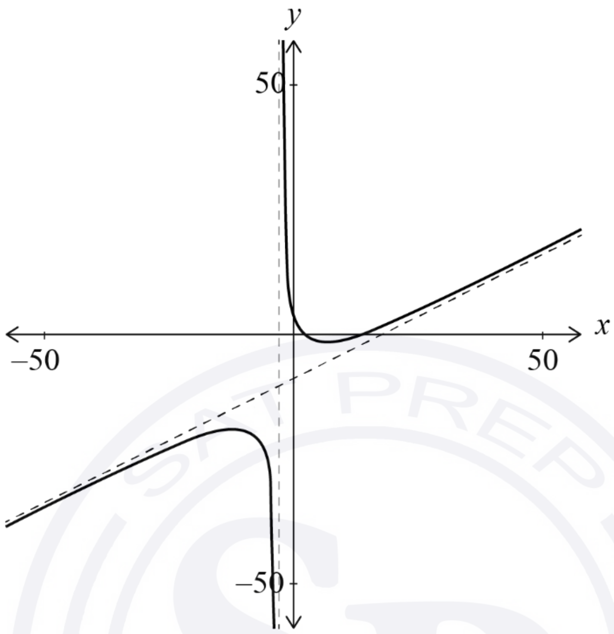
Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

continued…

Question 11 continued

(d)



two branches with approximately correct shape (for $-50 \le x \le 50$) ***A1***

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1A1***

Note: Award ***A1*** for vertical asymptote and behaviour and ***A1*** for oblique asymptote and behaviour. If only top half of the graph seen only award ***A1A0*** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $\left(-10-5\sqrt{3}=\right)-18.6602....$ OR $\left(-10+5\sqrt{3}=\right)-1.33974....$ seen anywhere ***(A1)***

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$y \le -18.7,\ y \ge -1.34$ ***A1A1***

Note: Award ***A1*** for each inequality**.** Award ***A1A0*** for strict inequalities in both. Do not award FT from (d). Accept equivalent set notation.

[4 marks]

(f) $\left(-10-2\sqrt{31}=\right)-21.1355....$ OR $\left(-10+2\sqrt{31}=\right)1.13552....$ seen anywhere ***(A1)***

$x < -21.1,\ -3 < x < 1.14$ ***A1A1A1***

Note: Award ***A1*** for $x < -21.1$, ***A1*** for correct endpoints of a single interval -3 and 1.14 and ***A1*** for $-3 < x < 1.14$. Do not award FT from (d). Accept equivalent set notation.

[4 marks]

Total [19 marks]

12. (a) attempt to set at least two components of L and M equal ***M1***

$1+2s=9+4t$

$2+3s=9+t$

$-3+6s=11+2t$

attempt to solve two of their equations simultaneously ***(M1)***

$s=2$ OR $t=-1$ ***A1***

EITHER

substitute $s=2$ and $t=-1$ into remaining component e.g. $-3+6(2)=11+2(-1)$ ***R1***

OR

recognition that 2nd and 3rd equations are equivalent ***R1***

THEN

position vector of A is $\begin{pmatrix}5\\8\\9\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.
The final ***A1*** is independent of ***R1***.

[5 marks]

(b) **METHOD 1**

attempt to substitute at least one line into the equation of the plane ***(M1)***

$$\begin{pmatrix}1+2s\\2+3s\\-3+6s\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(2+3s)-1(-3+6s)=7$$ ***A1***

$$\begin{pmatrix}9+4t\\9+t\\11+2t\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(9+t)-1(11+2t)=7$$ ***A1***

continued…

Question 12 continued

METHOD 2

consideration the direction of one line and a point on that line ***(M1)***

direction $\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

direction $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

METHOD 3

consideration of direction of both lines ***(M1)***

EITHER

$\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ and $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ (hence L and M are parallel to the plane) ***A1***

OR

$\begin{pmatrix}2\\3\\6\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}0\\20\\-10\end{pmatrix}=k\begin{pmatrix}0\\2\\-1\end{pmatrix}$ (hence L and M are parallel to the plane) ***A1***

THEN

$\begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ ***A1***

[3 marks]

continued…

Question 12 continued

(c) (i) position vector of point on the line is $\left(\boldsymbol{r}=\right)\begin{pmatrix}-3\\12\\2\end{pmatrix}+\lambda\begin{pmatrix}0\\2\\-1\end{pmatrix}\left(=\begin{pmatrix}-3\\12+2\lambda\\2-\lambda\end{pmatrix}\right)$ ***(A1)***

attempt to substitute position vector into equation of plane Π ***(M1)***

meets Π when $\begin{pmatrix}-3\\12+2\lambda\\2-\lambda\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$

$2(12+2\lambda)-(2-\lambda)=7$

$22+5\lambda=7$

$\lambda=-3$ ***(A1)***

position vector of $\left(\boldsymbol{r}=\begin{pmatrix}-3\\12\\2\end{pmatrix}-3\begin{pmatrix}0\\2\\-1\end{pmatrix}=\right)\begin{pmatrix}-3\\6\\5\end{pmatrix}=\begin{pmatrix}-3\\6\\5\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.

(ii) **METHOD 1**

attempt to find $\overrightarrow{BC}$ using $\overrightarrow{OC}-\overrightarrow{OB}$ ***(M1)***

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}-3\\6\\5\end{pmatrix}-\begin{pmatrix}-3\\12\\2\end{pmatrix}=\begin{pmatrix}0\\-6\\3\end{pmatrix}$

attempt to use distance formula to find $\left|\overrightarrow{BC}\right|$ ***(M1)***

$\left|\overrightarrow{BC}\right|=\sqrt{(-6)^2+3^2}$

$=6.71\left(=\sqrt{45}=3\sqrt{5}\right)$ ***A1***

METHOD 2

recognition that $\left|\overrightarrow{\mathrm{BC}}\right| = 3 \times \left|\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\right|$ ***(M1)***

attempt to use distance formula to find $\left|\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\right|$ ***(M1)***

$\left|\overrightarrow{\mathrm{BC}}\right| = 3\sqrt{2^2 + (-1)^2}$

$= 6.71\left(= \sqrt{45} = 3\sqrt{5}\right)$ ***A1***

[7 marks]

continued…

Question 12 continued

(d) let B' be the image of B

METHOD 1

$$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$ ***(A1)***

recognition that $\mu = 2\lambda\left(=-6\right)$ OR $\left|\mathrm{BC}\right| = \left|\mathrm{CB}'\right|$ (may be seen in a diagram) ***(M1)***

$$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$$

so coordinates are $\mathrm{B}'\left(-3,0,8\right)$ ***A1***

METHOD 2

$$\overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{CB}'} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$$ ***(A1)***

Note: This may come from $\overrightarrow{\mathrm{BC}} = -3\sqrt{5}\boldsymbol{n}$ using the unit normal vector $\boldsymbol{n} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{CB}'}$ OR $\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OB}} + 2\overrightarrow{\mathrm{BC}}$ OR $\overrightarrow{\mathrm{OB}'} = 2\overrightarrow{\mathrm{OC}} - \overrightarrow{\mathrm{OB}}$ ***(M1)***

$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -6 \\ 12 \\ 10 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$ (so coordinates are $\mathrm{B}'\left(-3,0,8\right)$) ***A1***

[3 marks]

Total [18 marks]

8823 – 7107M

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 2

32 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.

- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.

- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.

- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***. If ***A1*** marks are on separate lines, they are assumed to be dependent and hence ***A0A1*** is unlikely to be awarded. However, where such marks are *independent* (e.g. the markscheme is presenting them in sequence, but in the solution one does not lead directly to the other) this should be communicated via a note, and hence ***A0A1*** (for example) can be awarded.

- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.

- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.

- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.

- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

Final answers will generally not need to restate the variable and/or units to be considered correct. To help examiners, the markscheme will include variables and units, where appropriate. However, their omission from a candidate's final answer should only be penalized if explicitly instructed in a markscheme note.

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, , the general rule applies to final answers: unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to a "correct" level of accuracy (e.g 3 sf) in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) $\text{BV} = \sqrt{(6-3)^2 + (8-4)^2 + (0-9)^2}$ ***(A1)***

$= 10.2956...$

$= 10.3\ (= \sqrt{106})$ ***A1***

Note: Award ***SC(A0)A1*** for $\text{BV} = \begin{pmatrix} -3 \\ -4 \\ 9 \end{pmatrix}$ where a candidate has misinterpreted notation.

[2 marks]

(b) **METHOD 1**

$\text{BV} = \text{VC}$ AND $\text{BC} = 8$ (seen anywhere) ***(A1)***

attempt to use the cosine rule on triangle BVC for any angle ***(M1)***

Note: Recognition must be shown in context either in terms of labelled sides or in side lengths.

$\cos \text{B}\hat{\text{V}}\text{C} = \dfrac{10.2...^2 + 10.2...^2 - 8^2}{2 \times 10.2... \times 10.2...}$ OR

$8^2 = 10.2....^2 + 10.2...^2 - 2 \times 10.2... \times 10.2... \cos \text{B}\hat{\text{V}}\text{C}$ ***(A1)***

$\text{B}\hat{\text{V}}\text{C} = 0.798037...$

$\text{B}\hat{\text{V}}\text{C} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\text{B}\hat{\text{V}}\text{C} = 0.80$ (or 46°) (2sf).

METHOD 2

let M be the midpoint of BC

$\text{BM} = 4$ (seen anywhere) ***(A1)***

attempt to use sine or cosine in triangle BMV or CMV ***(M1)***

$\arcsin \dfrac{4}{\sqrt{106}}$ OR $\dfrac{\pi}{2} - \arccos \dfrac{4}{\sqrt{106}}$ OR 0.399018 ***(A1)***

$\text{B}\hat{\text{V}}\text{C} = 0.798037...$

$\text{B}\hat{\text{V}}\text{C} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\text{B}\hat{\text{V}}\text{C} = 0.80$ (or 46°) (2sf).

METHOD 3

$\overrightarrow{VC}=\begin{pmatrix}3\\-4\\-9\end{pmatrix}$ and $\overrightarrow{VB}=\begin{pmatrix}3\\4\\-9\end{pmatrix}$ ***(A1)***

attempt to use the cosine formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$\cos \mathrm{B\hat{V}C}=\dfrac{3(3)-4(4)-9(-9)}{\sqrt{3^2+(-4)^2+(-9)^2}\sqrt{3^2+4^2+(-9)^2}}\left(=\dfrac{74}{106}\right)$ ***(A1)***

$\mathrm{B\hat{V}C}=0.798037...$

$\mathrm{B\hat{V}C}=0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C}=0.80$ (or 46°) (2sf).

METHOD 4

$\overrightarrow{VC}=\begin{pmatrix}3\\-4\\-9\end{pmatrix}$ and $\overrightarrow{VB}=\begin{pmatrix}3\\4\\-9\end{pmatrix}$ ***(A1)***

attempt to use the cross product formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$\sin \mathrm{B\hat{V}C}=\dfrac{\left|\begin{pmatrix}72\\0\\24\end{pmatrix}\right|}{\sqrt{3^2+(-4)^2+(-9)^2}\sqrt{3^2+4^2+(-9)^2}}\left(=\dfrac{\sqrt{5760}}{106}\right)$ ***(A1)***

$\mathrm{B\hat{V}C}=0.798037...$

$\mathrm{B\hat{V}C}=0.798$ (accept 45.7°) ***A1***

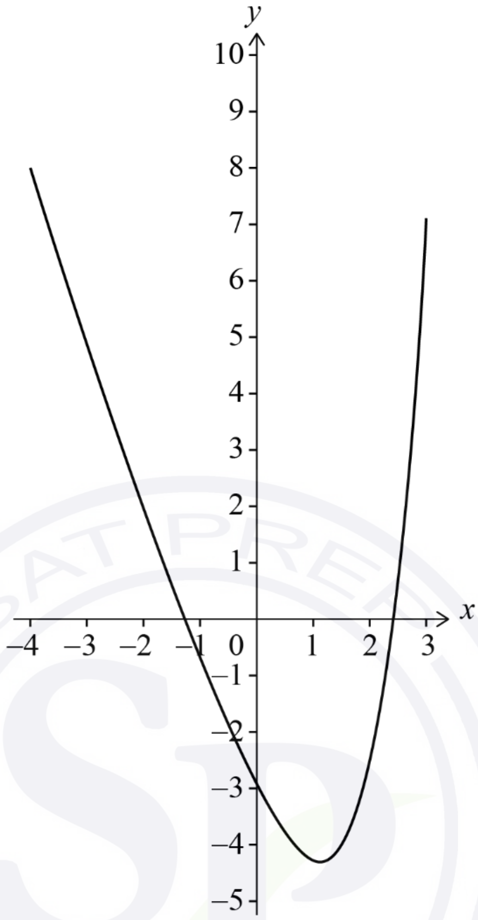
Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C}=0.80$ (or 46°) (2sf).

Award ***SC(A1)(M1)(A0)A0*** for $\text{area}=\dfrac{1}{2}\left|\begin{pmatrix}72\\0\\24\end{pmatrix}\right|=\dfrac{\sqrt{5760}}{2}(=37.9)$ where a candidate has misinterpreted notation.

[4 marks]

Total [6 marks]

2. (a)



A1A1A1

Note:	Award marks as follows: ***A1*** for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$. ***A1*** for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$. ***A1*** for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5,\ 7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5,\ 6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$ ***A1***

$c = -3$ (accept translate/shift 3 (units) down) ***A1***

[2 marks]

Total [5 marks]

3. (a) use of sector area formula to find area of at least one sector ***(M1)***

$\frac{1}{2}\times5.2\times100-\frac{1}{2}\times5.2\times r^2$ OR $10^2\pi-\frac{1}{2}10^2\times(2\pi-5.2)-\left(\pi r^2-\frac{1}{2}\times(2\pi-5.2)\times r^2\right)$ ***A1***

$(\text{area})=260-2.6r^2$ ***AG***

Note: There are many different ways to find the area of the "C". In all methods, the ***A*** mark is awarded for working which leads directly to the ***AG***.
Many candidates are working with rounded intermediate values. Award the ***A*** mark to correct work with values that round to the 3sf value of 260 and the 2sf value of 2.6 eg $259.99-2.6015r^2$.

[2 marks]

(b) (i) $260-2.6r^2=64$ ***(A1)***

$r=8.68243...$

$=8.68\,(\text{cm})\ \left(\frac{14\sqrt{65}}{13}\text{ exact}\right)$ ***A1***

(ii) 10×5.2 OR $8.68...\times5.2$ ***(A1)***

substituting their value of *r* into $10\times5.2+r\times5.2+2(10-r)$ (or equivalent) ***(M1)***

Perimeter = $10\times5.2+8.68...\times5.2+2(10-8.68...)$ $(=52+45.1486...+2.63513...)$

= 99.7837…

= 99.8 (cm) ***A1***

[5 marks]

Total [7 marks]

4. (a) recognizing at rest when $\frac{ds}{dt} = 0$ OR s is a minimum ***(M1)***

$q = 5.73553...$

$= 5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q = 5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...} |v(t)| \mathrm{d}t$ OR $\int_0^{5.73...} \left|\frac{\mathrm{d}}{\mathrm{d}t} s(t)\right| \mathrm{d}t$ OR $\left|\int_0^{5.73...} v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...} v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of dt. Only accept $\left|\int_0^q v(t)\,\mathrm{d}t\right|$ if their value of q does not result in the particle changing direction in the first q seconds.

$= 7.68302...$

$= 7.68$ (m) ***A1***

Note: Special Cases: Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q = 1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ ($3.37644...$ from 3sf). Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...} v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302... - (-4.3)$

$= 7.68302...$

$= 7.68$ (m) ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

5. $\mathrm{E}(X) = k + 2k^2 + 3a + 4k^3 = 2.3$ ***(A1)***

$k + k^2 + a + k^3 = 1$ ***(A1)***

Note: The first two ***A*** marks are independent of each other.

EITHER (finding intersections of functions)

attempt to make a the subject in both of their equations ***(M1)***

$a = 1 - k - k^2 - k^3$ and $a = \frac{1}{3}\left(2.3 - k - 2k^2 - 4k^3\right)$

use of graph or table to attempt to find intersection ***(M1)***

OR (solving algebraically)

attempt to solve their equations algebraically to find a cubic in k ***(M1)***

$k^3 - k^2 - 2k + 0.7 = 0$ OR $3\left(1 - k - k^2 - k^3\right) = 2.3 - k - 2k^2 - 4k^3$ (or equivalent)

attempt to solve their cubic in k ***(M1)***

THEN

$a = 0.552839...$ OR $k = 0.315870...$ (other solutions to cubic are $k = -1.18538..., 1.86951...$)

$a = 0.553$ ***A1***

Note: If no working shown, award ***(A1)(A1)(M1)(M1)A0*** for $a = 2.44587...$ OR $a = -10.8987...$ and award ***(A0)(A0)(M1)(M1)A0*** for $a = 0.55$ (2sf) .

Total [5 marks]

6. (a) $5.75 = 25p(1-p)$ ***(A1)***

$p = 0.641421..., 0.358578...$

$p = 0.641, 0.359 \left(= \dfrac{5 \pm \sqrt{2}}{10}\right)$ ***A1A1***

[3 marks]

(b) $\mathrm{Var}(Y) = (-2)^2\, \mathrm{Var}(X) \left(= 4\mathrm{Var}(X)\right)$ ***(A1)***

$= 23$ ***A1***

[2 marks]

Total [5 marks]

7. (a) (i) $(9!\ =)\ 362880$ ***A1***

Note: Accept 9! or 363000.

(ii) attempt to consider girls as a single object ***(M1)***

$(3!\ \times\ 7!\ =)\ 30240$ ***A1***

Note: Accept 30200.

[3 marks]

(b) **METHOD 1**

recognition of the two different cases for 2 girls and 3 girls ***(M1)***

exactly 2 girls is ${}^{6}C_{3}\times{}^{3}C_{2}=60$ and exactly 3 girls $\left({}^{3}C_{3}\times\right){}^{6}C_{2}=15$ ***(A1)***

total $(=60+15)=75$ ***A1***

METHOD 2

recognition of the three different cases: total choices, 1 girl and no girls ***(M1)***

total choices ${}^{9}C_{5}=126$, one girl case ${}^{3}C_{1}\times{}^{6}C_{4}=45$, no girl case ${}^{6}C_{5}=6$ ***(A1)***

total $(=126-45-6)=75$ ***A1***

[3 marks]

Total [6 marks]

8. (a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1-p \\ -1 \end{pmatrix}$ ***A1***

$\overrightarrow{AC} = \begin{pmatrix} p \\ -p \\ 2 \end{pmatrix}$ ***A1***

attempt to evaluate their $\overrightarrow{AB} \times \overrightarrow{AC}$ by use of formula or determinant ***M1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2(1-p)-p \\ -(2+p) \\ -p-p(1-p) \end{pmatrix}$ OR $\left(2(1-p)-p\right)\boldsymbol{i} - (2+p)\,\boldsymbol{j} + \left(-p-p(1-p)\right)\boldsymbol{k}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$ ***AG***

[4 marks]

(b) $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$

$= (2-3p)^2 + (-2-p)^2 + \left(p^2-2p\right)^2 \left(= p^4 - 4p^3 + 14p^2 - 8p + 8\right)$ ***(A1)***

attempt to find minimum of their $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$ ***(M1)***

$6.75257...$ OR $p = 0.3264...$

min value is 6.75 ***A1***

[3 marks]

continued…

Question 8 continued

(c) **METHOD 1**

valid attempt to find $\text{area} = \frac{1}{2}\left|\overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}}\right|$ using their answer to part b) ***(M1)***

area $= \frac{1}{2}\sqrt{6.75257...}$

$= 1.299285...$

$= 1.30$ (units^2) ***A1***

[2 marks]

Total [9 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x = 0$, $y = 2$ to find the value of A ***M1***

$2 = A\mathrm{e}^{0} \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $|y-4| = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{10}y=\frac{4}{10}$ so IF $\mathrm{e}^{\int\frac{1}{10}\mathrm{d}x}=\mathrm{e}^{\frac{1}{10}x}$

$\mathrm{e}^{\frac{1}{10}x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{10}\mathrm{e}^{\frac{1}{10}x}y=\frac{4}{10}\mathrm{e}^{\frac{1}{10}x}$

$\mathrm{e}^{\frac{1}{10}x}y=4\mathrm{e}^{\frac{1}{10}x}\left(+c\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS***.***

substituting initial conditions $x=0, y=2$ to find the value of c ***M1***

$\left(2\mathrm{e}^{0}=4\mathrm{e}^{0}+c\Rightarrow\right)c=-2$ ***A1***

$\mathrm{e}^{\frac{1}{10}x}y=4\mathrm{e}^{\frac{1}{10}x}-2$

$y=4-2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $=2.0980199...-\left(4-2e^{-0.05}\right)=0.000478749...$

$=0.000479\left(=4.79\times10^{-4}\right)$ ***A1***

Note: Accept $0.000459\left(=4.59\times10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

10. (a) recognizing probabilities sum to 1 ***(M1)***

$0.288 + \mathrm{P}(94.6 < X < 98.1) + 0.434 = 1$

$\mathrm{P}(94.6 < X < 98.1) = 0.278$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(94.6 < X < 98.1) = 0.28$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing the need to use inverse normal with 0.288, $(1-0.434)$ or 0.434 ***(M1)***

Note: Accept use of calculator notation eg $\mathrm{invNorm}(0.288)(= -0.559236...)$.

$\mu + \mathrm{invNorm}(0.288)\sigma = 94.6$, $\mu + \mathrm{invNorm}(1-0.434)\sigma = 98.1$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations in two variables using the GDC (that involve either z-values or 'invNorm' rather than probabilities) ***(M1)***

$\mu = 97.2981...$, $\sigma = 4.82468...$

$\mu = 97.3$, $\sigma = 4.82$ ***A1***

Note: Condone use of different variables throughout, but do not award the final ***A1*** if they do not clearly identify which variable is their mean and standard deviation.

METHOD 2

use of inverse normal to find at least one z-score for $\mathrm{P}(Z < z) = 0.288$ or $\mathrm{P}(Z < z) = 1 - 0.434$ ***(M1)***

$z_1 = -0.559236...$ OR $z_2 = 0.166199...$

$\frac{94.6-\mu}{\sigma} = -0.559236...$, $\frac{98.1-\mu}{\sigma} = 0.166199...$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations (that involve z-values rather than probabilities) ***(M1)***

$\mu = 97.2981...$, $\sigma = 4.82468...$

$\mu = 97.3$, $\sigma = 4.82$ ***A1***

Note: Award marks as appropriate for work seen in part (a).

Note: If no working shown, award ***(M1)(A0)(A0)(M1)A0*** for $\mu = 97$, $\sigma = 4.8$ (2sf).

[5 marks]

Question 10 continued

(c) (i) recognition of Binomial distribution ***(M1)***

$X \sim \mathrm{B}(100, 0.434)$

$\mathrm{P}(X = 34) = 0.0133198...$

$= 0.0133$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(X = 34) = 0.013$ (2sf).

(ii) $\mathrm{P}(X < 49) = 0.848218...$ (seen anywhere) ***(A1)***

recognition of conditional probability ***(M1)***

Note: recognition must be shown in context, either in symbols eg $\mathrm{P}\left(X = 34 \mid X < 49\right)$, or in words eg $\mathrm{P}\left(34 \text{ plants} \mid \text{less than 49 plants}\right)$, not only as $\mathrm{P}(A \mid B)$.

$\left(\mathrm{P}\left(X = 34 \mid X < 49\right) =\right) \dfrac{\mathrm{P}\left(X = 34\right)}{\mathrm{P}\left(X < 49\right)}$ OR $\dfrac{\mathrm{P}\left(X = 34\right)}{\mathrm{P}\left(X \le 48\right)}$ $\left(= \dfrac{0.0133198...}{0.848218...}\right)$ ***(A1)***

$= 0.0157033...$

$\mathrm{P}\left(X = 34 \mid X < 49\right) = 0.0157$ ***A1***

Note: Exception to ***FT***: If the candidate finds $\mathrm{P}(X \le 49)$ $(= 0.890474...)$ and uses that to calculate $\mathrm{P}\left(X = 34 \mid X \le 49\right) = 0.0149581...$ award ***(A0)(M1)(A1)A0***.

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{P}\left(X = 34 \mid X < 49\right) = 0.016$ (2sf).

[6 marks]

continued…

Question 10 continued

(d) $Q_1 = 96.19$ OR $Q_3 = 101.01$ (may be seen on a labelled diagram with areas indicated) ***(A1)***

$P(96.19 < F < 101.01) = 0.5$ OR $P(F < 96.19) = 0.25$ OR $P(F < 101.01) = 0.75$ (or equivalent)

EITHER

attempt to find d using graph or table ***(M1)***

OR

$1 - 2P\left(Z < -\dfrac{2.41}{d}\right) = 0.5$ OR $P(Z < -\dfrac{2.41}{d}) = 0.25$ OR $P(Z < \dfrac{2.41}{d}) = 0.75$

OR $P(-\dfrac{2.41}{d} < Z < \dfrac{2.41}{d}) = 0.5$ (or equivalent) ***(M1)***

$-\dfrac{2.41}{d} = -0.674489...$ OR $\dfrac{2.41}{d} = 0.674489...$

THEN

$3.57307...$

$d = 3.57$ ***A1***

Note: Accept 3.56 using 96.2 or 101.

Note: If no working shown, award ***(A0)(M1)A0*** for $d = 3.6$ (2sf).

[3 marks]

Total [16 marks]

11. (a) (vertical asymptote equation) $x = -3$ ***A1***

Note: Accept $2x + 6 = 0$ or equivalent.

[1 mark]

(b) $(2, 0)$ and $(12, 0)$ ***A1A1***

Note: Award ***A1*** for $(2, 0)$ and ***A1*** for $(12, 0)$. Award ***A1A0*** if only x values are given.

[2 marks]

(c) **METHOD 1**

$a = \frac{1}{2}$ ***A1***

attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ ***(M1)***

$\frac{x^2 - 14x + 24}{2x + 6}$

$= \frac{1}{2}x - \frac{17}{2}\left(+\frac{\dots}{2x + 6}\right)$ ***(A1)***

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

continued…

Question 11 continued

METHOD 2

$a = \frac{1}{2}$ ***A1***

$$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6}$$ ***(A1)***

$$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$$

attempt to equate coefficients of x: ***(M1)***

$-14 = 3 + 2b$

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$a = \frac{1}{2}$ ***A1***

$$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6}$$ ***(A1)***

attempt to find the limit of $f(x) - ax$ as $x \longrightarrow \infty$ ***(M1)***

$$b = \lim_{x \to \infty} \frac{-17x + 24}{2x + 6}$$

$= -\frac{17}{2}$ ***A1***

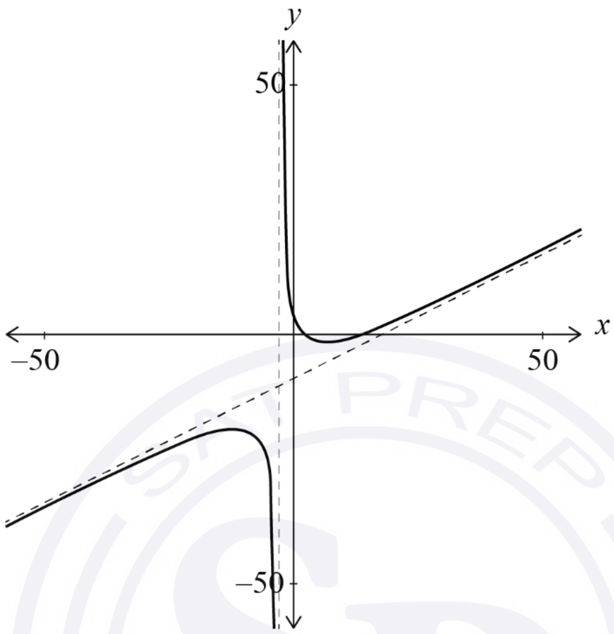
Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

continued…

Question 11 continued

(d)



two branches with approximately correct shape (for $-50 \le x \le 50$) ***A1***

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1A1***

Note: Award ***A1*** for vertical asymptote and behaviour and ***A1*** for oblique asymptote and behaviour. If only top half of the graph seen only award ***A1A0*** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $\left(-10-5\sqrt{3}=\right)-18.6602....$ OR $\left(-10+5\sqrt{3}=\right)-1.33974....$ seen anywhere ***(A1)***

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$ ***(M1)***

$y \le -18.7,\ y \ge -1.34$ ***A1A1***

Note: Award ***A1*** for each inequality**.** Award ***A1A0*** for strict inequalities in both.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

(f) $\left(-10-2\sqrt{31}=\right)-21.1355....$ OR $\left(-10+2\sqrt{31}=\right)1.13552....$ seen anywhere ***(A1)***

$x < -21.1,\ -3 < x < 1.14$ ***A1A1A1***

Note: Award ***A1*** for $x < -21.1$, ***A1*** for correct endpoints of a single interval -3 and 1.14 and ***A1*** for $-3 < x < 1.14$.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

Total [19 marks]

12. (a) attempt to set at least two components of L and M equal ***M1***

$1+2s=9+4t$

$2+3s=9+t$

$-3+6s=11+2t$

attempt to solve two of their equations simultaneously ***(M1)***

$s=2$ OR $t=-1$ ***A1***

EITHER

substitute $s=2$ and $t=-1$ into remaining component e.g. $-3+6(2)=11+2(-1)$ ***R1***

OR

recognition that 2nd and 3rd equations are equivalent ***R1***

THEN

position vector of A is $\begin{pmatrix}5\\8\\9\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates. The final ***A1*** is independent of ***R1***.

[5 marks]

(b) **METHOD 1**

attempt to substitute at least one line into the equation of the plane ***(M1)***

$$\begin{pmatrix}1+2s\\2+3s\\-3+6s\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(2+3s)-1(-3+6s)=7$$ ***A1***

$$\begin{pmatrix}9+4t\\9+t\\11+2t\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(9+t)-1(11+2t)=7$$ ***A1***

continued…

Question 12 continued

METHOD 2

consideration the direction of one line and a point on that line **(M1)**

direction $\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

direction $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

METHOD 3

consideration of direction of both lines **(M1)**

EITHER

$\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ and $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ (hence L and M are parallel to the plane) ***A1***

OR

$\begin{pmatrix}2\\3\\6\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}0\\20\\-10\end{pmatrix}=k\begin{pmatrix}0\\2\\-1\end{pmatrix}$ (hence L and M are parallel to the plane) ***A1***

THEN

$\begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ ***A1***

[3 marks]

continued…

Question 12 continued

(c) (i) position vector of point on the line is $(\boldsymbol{r}=)\begin{pmatrix}-3\\12\\2\end{pmatrix}+\lambda\begin{pmatrix}0\\2\\-1\end{pmatrix}\left(=\begin{pmatrix}-3\\12+2\lambda\\2-\lambda\end{pmatrix}\right)$ ***(A1)***

attempt to substitute position vector into equation of plane Π ***(M1)***

meets Π when $\begin{pmatrix}-3\\12+2\lambda\\2-\lambda\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$

$2(12+2\lambda)-(2-\lambda)=7$

$22+5\lambda=7$

$\lambda=-3$ ***(A1)***

position vector of $\left(\boldsymbol{r}=\begin{pmatrix}-3\\12\\2\end{pmatrix}-3\begin{pmatrix}0\\2\\-1\end{pmatrix}=\right)\begin{pmatrix}-3\\6\\5\end{pmatrix}=\begin{pmatrix}-3\\6\\5\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.

(ii) **METHOD 1**

attempt to find $\overrightarrow{\mathrm{BC}}$ using $\overrightarrow{\mathrm{OC}}-\overrightarrow{\mathrm{OB}}$ ***(M1)***

$\overrightarrow{\mathrm{BC}}=\overrightarrow{\mathrm{OC}}-\overrightarrow{\mathrm{OB}}=\begin{pmatrix}-3\\6\\5\end{pmatrix}-\begin{pmatrix}-3\\12\\2\end{pmatrix}=\begin{pmatrix}0\\-6\\3\end{pmatrix}$

attempt to use distance formula to find $\left|\overrightarrow{\mathrm{BC}}\right|$ ***(M1)***

$\left|\overrightarrow{\mathrm{BC}}\right|=\sqrt{(-6)^2+3^2}$

$=6.71\left(=\sqrt{45}=3\sqrt{5}\right)$ ***A1***

METHOD 2

recognition that $\left|\overrightarrow{BC}\right| = 3\times\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|$ ***(M1)***

attempt to use distance formula to find $\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|$ ***(M1)***

$\left|\overrightarrow{BC}\right| = 3\sqrt{2^2+(-1)^2}$

$= 6.71\left(=\sqrt{45}=3\sqrt{5}\right)$ ***A1***

[7 marks]

continued…

Question 12 continued

(d) let B' be the image of B

METHOD 1

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ ***(A1)***

recognition that $\mu = 2\lambda\,(= -6)$ OR $|\mathrm{BC}| = |\mathrm{CB}'|$ (may be seen in a diagram) ***(M1)***

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$

so coordinates are $\mathrm{B}'(-3, 0, 8)$ ***A1***

METHOD 2

$\overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{CB}'} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ ***(A1)***

Note: This may come from $\overrightarrow{\mathrm{BC}} = -3\sqrt{5}\boldsymbol{n}$ using the unit normal vector $\boldsymbol{n} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{CB}'}$ OR $\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OB}} + 2\overrightarrow{\mathrm{BC}}$ OR $\overrightarrow{\mathrm{OB}'} = 2\overrightarrow{\mathrm{OC}} - \overrightarrow{\mathrm{OB}}$ ***(M1)***

$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -6 \\ 12 \\ 10 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$ (so coordinates are $\mathrm{B}'(-3, 0, 8)$) ***A1***

[3 marks]

Total [18 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 2

29 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) $a = 1.93258...$, $b = 7.21662...$

$a = 1.93$, $b = 7.22$ ***A1A1***

[2 marks]

(b) attempt to substitute $d = 20$ into their equation ***(M1)***

$\text{height} = 45.8683...$

$\text{height} = 45.9$ (cm) ***A1***

[2 marks]

Total [4 marks]

2. METHOD 1

attempt to substitute into cosine rule **(M1)**

$154^2 = 150^2 + 90^2 - 2(150)(90)\cos A\hat{P}B$ OR $\cos A\hat{P}B = \dfrac{150^2 + 90^2 - 154^2}{2(150)(90)}$ **(A1)**

$A\hat{P}B = 75.2286...°$ OR $1.31298...$ radians

$A\hat{P}B = 75.2°$ OR 1.31 radians **(A1)**

valid approach to find θ **(M1)**

$\theta = \dfrac{180° - A\hat{P}B}{2}$ OR $\theta = \dfrac{180° - 75.2286...°}{2}$ $(= 52.3856...)$ OR

$\theta = \dfrac{\pi - 1.31298...}{2}(= 0.914302...)$

valid approach to express h in terms of θ **(M1)**

$\sin\theta = \dfrac{h}{150}$ OR $h = 150\sin 52.3856...°$

$h = 118.820...$

$h = 119$ (m) **A1**

[6 marks]

METHOD 2

attempts to find either the distance between the buildings or the difference in height between the buildings in terms of θ **(M1)**

distance between the buildings is $(150 + 90)\cos\theta$ and the difference in height between the buildings is $(150 - 90)\sin\theta$ **(A1)**

uses Pythagoras and attempts to solve for θ **(M1)**

$(60\sin\theta)^2 + (240\cos\theta)^2 = 154^2$

$\theta = 0.914302...$ $(= 52.3856...°)$ **(A1)**

$\dfrac{h}{150} = \sin(0.914302...)$ **(M1)**

$h = 118.820...$

$h = 119$ (m) **A1**

[6 marks]

3. (a) evidence of attempting to find correct area under normal curve ***(M1)***

$\mathrm{P}(W > 210)$ OR sketch

$\mathrm{P}(W > 210) = 0.115069...$

$\mathrm{P}(W > 210) = 0.115$ ***A1***

[2 marks]

(b) recognizing $\mathrm{P}(W < w) = 1 - \mathrm{P}(w < W < 210) - \mathrm{P}(W > 210)$ ***(M1)***

$\mathrm{P}(W < w) = 1 - 0.8 - 0.115069...$

$\mathrm{P}(W < w) = 0.084930...$

$\mathrm{P}(W < w) = 0.0849$ ***A1***

[2 marks]

(c) evidence of attempting to use inverse normal function ***(M1)***

$w = 197.136...$

$w = 197$ (grams) ***A1***

[2 marks]

(d) recognition of binomial distribution ***(M1)***

$X \sim \mathrm{B}(10, 0.0849302...)$

$\mathrm{P}(X = 1) = 0.382076...$

$\mathrm{P}(X = 1) = 0.382$ ***A1***

[2 marks]

Total [8 marks]

4. attempt to use the binomial expansion of $(x+h)^8$ ***(M1)***

$^8C_0x^8h^0 + {}^8C_1x^7h^1 + {}^8C_2x^6h^2 + \ldots$

$a = 8h$ (accept $^8C_1\,h$) ***A1***

$b = 28h^2$ (accept $^8C_2\,h^2$) ***A1***

$d = 70h^4$ (accept $^8C_4\,h^4$) ***A1***

recognition that there is a common ratio between their terms ***(M1)***

$8h \times r = 28h^2$ OR $28h^2 \times r = 70h^4$ OR $8h \times r^2 = 70h^4$

correct equation in terms of h ***(A1)***

$\frac{28h^2}{8h} = \frac{70h^4}{28h^2}$ (or equivalent)

$h = 1.4$ ***A1***

[7 marks]

5. (a) recognize that acceleration is zero when $v'(t)=0$ OR at a local maximum on the graph of v ***(M1)***

$t_1 = 0.394791...$

$t_1 = 0.395\left(=\arctan\left(\frac{5}{12}\right)\right)$ (seconds) ***A1***

[2 marks]

(b) recognition that $v=0$ ***(M1)***

sketch OR $t=4.71238...$ OR $t=10.9955...$

$t_2 = 10.9955...$

$t_2 = 11.0\left(=\frac{7\pi}{2}\right)$ ***A1***

[2 marks]

(c) $\int_{t_1}^{t_2} |v|\,\mathrm{d}t$ OR $\int_{0.394791...}^{10.9955...} |v|\,\mathrm{d}t$ OR $\int_{0.394791...}^{4.71238...} v\,\mathrm{d}t + \int_{4.71238...}^{10.9955...} |v|\,\mathrm{d}t\,(=6.53806...+1.29313...)$

OR $\int_{0.394791...}^{4.71238...} v\,\mathrm{d}t - \int_{4.71238...}^{10.9955...} v\,\mathrm{d}t\,(=6.53806...-(-1.29313...))$ ***(A1)***

distance = $7.83118\ldots$

= 7.83 (m) ***A1***

[2 marks]

Total [6 marks]

6. (a) **METHOD 1**

the general point on L has coordinates $(\lambda, 2-2\lambda, 4-2\lambda)$

substitutes this general point into both Π_1 and Π_2 ***(M1)***

$2\lambda-(2-2\lambda)+2(4-2\lambda)(=2\lambda-2+2\lambda+8-4\lambda)$ ***A1***

$=6$ ***AG***

$4\lambda+3(2-2\lambda)-(4-2\lambda)(=4\lambda+6-6\lambda-4+2\lambda)$ ***A1***

$=2$ ***AG***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Award ***(M1)A0A0*** for correct verification using a specific value of λ.

continued…

Question 6 continued

METHOD 2

substitutes $(0,2,4)$ into both Π_1 and Π_2 and shows that

$0-2+8=6$ and $0+6-4=2$ ***A1***

hence $(0,2,4)$ lies in both Π_1 and Π_2 ***AG***

EITHER

attempts to find $\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}4\\3\\-1\end{pmatrix}$ ***M1***

$=\begin{pmatrix}-5\\10\\10\end{pmatrix}$ ***A1***

OR

attempts to find $\begin{pmatrix}1\\-2\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-2\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-1\end{pmatrix}$ ***M1***

$(2+2-4)=0$ and $(4-6+2)=0$ ***A1***

THEN

(so $\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ is perpendicular to both normal vectors)

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

> **Note:** Award ***M1*** for substituting $x=0$ (or $y=2$ or $z=4$) into Π_1 and Π_2 and solving simultaneously, for example, solving $-y+2z=6$ and $3y-z=2$.
> Award ***A1*** for $y=2$ and $z=4$, for example.

continued…

Question 6 continued

METHOD 3

attempts row reduction to obtain *eg*,

$x+\dfrac{z}{2}=2$ and $y-z=-2$ ***(M1)***

substitutes $x=\lambda$ into $x+\dfrac{z}{2}=2$, solves for z and obtains $z=4-2\lambda$ ***A1***

substitutes $z=4-2\lambda$ into $y-z=-2$, solves for y and obtains $y=2-2\lambda$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

METHOD 4

attempts to solve $2x-y+2z=6$ and $4x+3y-z=2$ ***(M1)***

for example, $x=\lambda, y=2-2\lambda, z=4-2\lambda$ ***A2***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Only award marks for convincing use of a GDC.

[3 marks]

continued…

Question 6 continued

(b) **EITHER**

the position vector for point P nearest to the origin is perpendicular to the direction of L

$$\begin{pmatrix} \lambda \\ 2-2\lambda \\ 4-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 0$$ ***(M1)***

$$\lambda - 2(2-2\lambda) - 2(4-2\lambda) = 0$$ ***(A1)***

$$9\lambda - 12 = 0$$ ***(A1)***

OR

let s be the distance from the origin to a point P on L, then

$$s^2 = \lambda^2 + (2-2\lambda)^2 + (4-2\lambda)^2$$ ***(A1)***

attempts to find λ such that $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 0$ ***(M1)***

either $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 18\lambda - 24 (= 0)$ or a graph of s^2 versus λ ***(A1)***

Note: Award as above for use of $s = \sqrt{\lambda^2 + (2-2\lambda)^2 + (4-2\lambda)^2}$.

THEN

$$\lambda = \frac{4}{3}$$ ***A1***

$$\mathrm{P}\left(\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \left(\mathrm{P}(1.33, -0.667, 1.33)\right)$$ ***A1***

[5 marks]

Total [8 marks]

7. attempts to express x in terms of $\tan y$ ***(M1)***

$x = \tan y + 2$ ***(A1)***

let V be the volume of the solid

correctly uses $V = \pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

Note: Award ***M0*** for $V = \pi\int_a^b \left(\arctan(x-2)\right)^2 dy$

$$V = \pi\int_0^{\frac{\pi}{3}} \left(\tan y + 2\right)^2 \mathrm{d}y \left(= \pi\int_0^{1.0472...} \left(\tan y + 2\right)^2 \mathrm{d}y\right)$$ ***(A1)***

$= 24.0213...$

$= 24.0 \ \left(= \pi\left(4\ln 2 + \pi + \sqrt{3}\right)\right)$ ***A1***

Note: GDC in degrees gives13.3

[5 marks]

8. (a) **EITHER**

attempts to find the y- coordinate of either the local minimum point or the local maximum point ***(M1)***

OR

attempts to find the discriminant of $2x-5=y\left(x^2-3\right)$ $\left(yx^2-2x+(5-3y)=0\right)$ ***(M1)***

$\Delta=4-4y\left(5-3y\right)\left(=4-20y+12y^2\right)$

THEN

$y=1.43425...$ (local min.) and $y=0.232408...$ (local max.) ***(A1)(A1)***

$g\left(x\right)\le 0.232$ OR $g\left(x\right)\ge 1.43$ ($g\left(x\right)\le\dfrac{-\sqrt{13}+5}{6}$ OR $g\left(x\right)\ge\dfrac{\sqrt{13}+5}{6}$) ***A1***

Note: Accept other valid notations such as interval notation.

[4 marks]

(b) $\dfrac{2|x|-5}{x^2-3}\ge 0$ (since $\cos t<0$ for $\dfrac{\pi}{2}<t\le\pi$) ***(R1)***

attempts to solve graphically or algebraically ***(M1)***

$x\le-\dfrac{5}{2}$ OR $-\sqrt{3}<x<\sqrt{3}$ $\left(-1.73<x<1.73\right)$ OR $x\ge\dfrac{5}{2}$ ***A1***

[3 marks]

Total [7 marks]

9. METHOD 1

10 numbers of the form $3n$, 10 numbers of the form $(3n-1)$ and 10 numbers of the form $(3n-2)$ (may be seen anywhere) ***(M1)***

considers one of the following two cases of forming a sum divisible by 3 ***(M1)***

case 1:

chooses 3 numbers of the form $3n$ or chooses 3 numbers of the form $(3n-1)$ or chooses 3 numbers of the form $(3n-2)$

${}^{10}C_3 + {}^{10}C_3 + {}^{10}C_3 \ \left(= 3\times {}^{10}C_3 = 3\times 120 = 360\right)$ ways ***A1***

case 2:

chooses 1 number of the form $3n$ and chooses 1 number of the form $(3n-1)$ and chooses 1 number of the form $(3n-2)$

${}^{10}C_1 \times {}^{10}C_1 \times {}^{10}C_1 \ \left(= \left({}^{10}C_1\right)^3 = 10^3 = 1000\right)$ ways OR $\dfrac{{}^{30}C_1 \times {}^{20}C_1 \times {}^{10}C_1}{3!}(=1000)$ ways ***A1***

total number of ways is $3\times {}^{10}C_3 + {}^{10}C_1 \times {}^{10}C_1 \times {}^{10}C_1 \ (=360+1000)$

$=1360$ ***A1***

METHOD 2

total number of ways of choosing 3 numbers (without restriction) is ${}^{30}C_3 = 4060$ ***A1***

attempts to find the total number of ways of choosing 3 numbers whose sum is not divisible by 3 ***(M1)***

chooses 2 numbers from one group and chooses 1 number from another group

eg chooses 2 numbers of the form $3n$ and chooses 1 number of the form $3n-1$

$3!\times {}^{10}C_2 \times {}^{10}C_1 = 2700$ ***(M1)A1***

Note: Award ***(M1)*** for any integer multiple of ${}^{10}C_2 \times {}^{10}C_1$.

total number of ways is $4060-2700$

$=1360$ ***A1***

[5 marks]

Section B

10. (a) $7.8 = \dfrac{2\pi}{\text{period}}$ ***(M1)***

$\dfrac{2\pi}{7.8} = 0.805536...$

$\text{period} = 0.806\left(= \dfrac{20\pi}{78}\right)$ ***A1***

[2 marks]

(b) **METHOD 1**

(i) $\text{amplitude} = \dfrac{\max - \min}{2}$ ***(M1)***

$\dfrac{1.8-1}{2}$

$a = -0.4$ ***A1***

(ii) $b = 1.4$ ***A1***

METHOD 2

attempt to form two simultaneous equations in a and b ***(M1)***

$H(0) = 1 \Rightarrow a + b = 1$, $H\left(\dfrac{\pi}{7.8}\right) = 1.8 \Rightarrow -a + b = 1.8$

$a = -0.4, b = 1.4$ ***A1A1***

[3 marks]

continued…

Question 10 continued

(c) **EITHER**

$\frac{5}{\text{period}} = 6.207... < 6\frac{1}{2}$ ***(A1)***

OR

consideration of number of maximums on graph in first 5 seconds ***(A1)***

OR

maximums when $t = 0.403, 1.21, 2.01, 2.82, 3.62, 4.43$ ***(A1)***

THEN

6 times ***A1***

[2 marks]

(d) recognizing that $H(t) = 1.5$ ***(M1)***

$-0.4\cos(7.8t) + 1.4 = 1.5$

$0.233779...$

$t = 0.234$ (seconds) ***A1***

[2 marks]

continued…

Question 10 continued

(e) finding second time height is 1.5 metres ***(M1)***

$t = 0.571757...$

in each period, height is greater than 1.5 metres for 0.337978… seconds ***(A1)***

Note: Award ***(M1)(A1)*** for total time $2.02787...$ seen.

multiplying their value by 6 and divide by 5 ***(M1)***

$\frac{0.337978...\times 6}{5}$ OR $\frac{2.02787...}{5}$

$= 0.405574...$

$\text{P}(\text{height is greater than } 1.5 \text{ m}) = 0.406$ ***A1***

[4 marks]

Total [13 marks]

11. (a) attempts to form a numerator involving a product of two terms involving y and a denominator involving a product of two terms involving $r+y$ ***(M1)***

$$\frac{y(y-1)}{(r+y)(r+y-1)}=\frac{1}{3}$$ ***A1***

attempts to remove the fractions and expand the brackets ***(M1)***

$$3y^2-3y=y^2+2ry-y+r^2-r$$ ***A1***

$$2y^2-2ry-2y+r-r^2=0$$

$$2y^2-2(r+1)y+r-r^2=0$$ ***AG***

[4 marks]

(b) attempts to solve for y ***(M1)***

$$y=\frac{2(r+1)\pm\sqrt{4(r+1)^2-8(r-r^2)}}{4}$$ ***A1***

$$y=\frac{2(r+1)\pm\sqrt{12r^2+4}}{4}$$ ***A1***

$$y=\frac{(r+1)\pm\sqrt{3r^2+1}}{2}$$

(since $r, y\in\mathbb{Z}^+$) and $\frac{(r+1)-\sqrt{3r^2+1}}{2}<0$ for $r>1$ ***R1***

> ***Note:*** Award the ***R1*** for stating that number of balls cannot be negative, or similar.
>
> ***Note:*** Accept $y>0$

so $y=\frac{(r+1)+\sqrt{3r^2+1}}{2}$ ***AG***

[4 marks]

continued…

Question 11 continued

(c) attempts to find a pair of positive integer values *eg* by using a table **(M1)**

Note: Award ***M0*** if numbers are not positive integers.

1 red ball and 2 yellow balls ($r = 1$ and $y = 2$) ***A1***

4 red balls and 6 yellow balls ($r = 4$ and $y = 6$) ***A2***

Note: Award ***A1*** for one solution and ***A2*** for another. 15 red balls and 21 yellow balls ($r = 15$ and $y = 21$) is the next solution.

[4 marks]

(d) attempts to form a numerator involving a product of three terms involving y and a denominator involving a product of three terms that includes a $(y+10)$ term ***(M1)***

$$\mathrm{P}(YYY) = \frac{y(y-1)(y-2)}{(y+10)(y-1+10)(y-2+10)} \left(= \frac{y(y-1)(y-2)}{(y+10)(y+9)(y+8)} \right)$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

[3 marks]

(e) $$\mathrm{P}(\text{new } YYY) = \frac{(y+1)(y)(y-1)}{(y+1+10)(y+10)(y-1+10)} \left(= \frac{(y+1)(y)(y-1)}{(y+11)(y+10)(y+9)} \right)$$ ***(A1)***

equates their answer for $\mathrm{P}(\text{new } YYY)$ to $2\times$ their answer for part (d) ***M1***

$$\frac{2y(y-1)(y-2)}{(y+10)(y+9)(y+8)} = \frac{(y+1)(y)(y-1)}{(y+11)(y+10)(y+9)}$$

attempts to solve for y ***(M1)***

Note: Award ***(M1)*** for attempting to write the above expression as $\frac{2(y-2)}{y+8} = \frac{y+1}{y+11}$.

$y = 4$ ***A2***

[5 marks]

Total [20 marks]

12. (a) attempts to use $y_1 = y_0 + h \times f(x_0, y_0)$ ***(M1)***

$$y_1 = 2 + 0.1 \times \frac{1^2 + 3(2)^2}{2}$$

$= 2.65$ ***A1***

Note: Award ***(M1)A0*** for 2.35.

[2 marks]

(b) let $y = vx$ ***M1***

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$ ***(A1)***

$$v + x\frac{dv}{dx} = \frac{x^2 + 3v^2x^2}{vx^2}$$ ***(M1)***

$$x\frac{dv}{dx} = \frac{1 + 2v^2}{v}$$ ***A1***

attempt to separate variables x and v ***M1***

$$\int \frac{v}{2v^2 + 1}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{4}\ln\left(2v^2 + 1\right) = \ln x + C$$ ***A1***

Note: Condone the absence of C to this stage.

continued…

Question 12 continued

EITHER

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+C$$

when $x=1$, $y=2 \Rightarrow C=\frac{1}{4}\ln 9$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+\frac{1}{4}\ln 9$$

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=\sqrt{3}x$$

OR

$$\ln\left(\frac{2y^2}{x^2+1}\right)=\ln\left(x^4\right)+\ln C$$

$$\frac{2y^2}{x^2}+1=Cx^4$$

when $x=1$, $y=2 \Rightarrow C=9$ ***M1***

THEN

$$\frac{2y^2}{x^2}+1=9x^4$$ ***A1***

$$y=x\sqrt{\frac{9x^4-1}{2}}$$ ***AG***

[8 marks]

continued…

Question 12 continued

(c) $y = 2.71422...$

$y = 2.71$ ***A1***

[1 mark]

(d) **EITHER**

the graph of $y = x\sqrt{\dfrac{9x^4 - 1}{2}}$ is concave up (for $1 \le x \le 1.1$) ***A1***

OR

$\dfrac{d^2y}{dx^2} > 0$ (for $1 \le x \le 1.1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

THEN

hence the tangent drawn using Euler's method gives an underestimate of the true value, so the value of y when $x = 1.1$ is greater than the approximate value found in part (a) ***R1***

Note: Only award ***R1*** if there is reference to tangent (in words or in a diagram).

[2 marks]

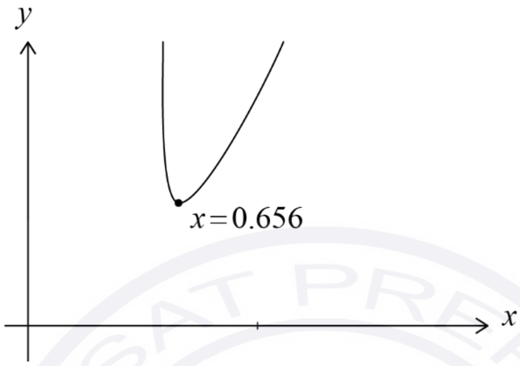
continued…

Question 12 continued

(e) **EITHER**

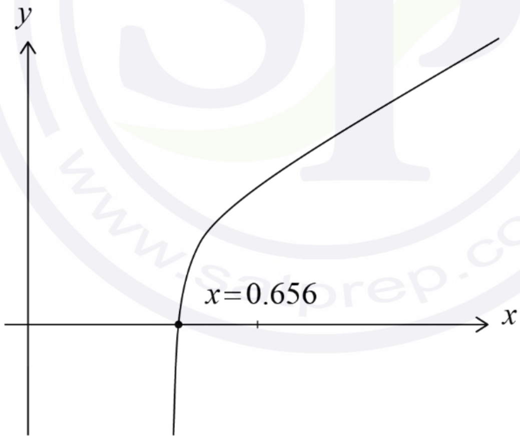
Note: Award the first ***A*** mark for a correct graph seen in part (d).

correct graph of $\frac{dy}{dx}$ ***A1***



OR

correct graph of $\frac{d^2y}{dx^2}$ ***A1***



THEN

$x = 0.655996...$

$x = 0.656$ ***A1***

[2 marks]

continued…

Question 12 continued

(f) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ (seen anywhere) ***(A1)***

Note: Award ***(A1)*** for equivalent answers (seen anywhere) such as $\frac{-x^4+x^2y^2+6y^4}{x^2y^3}=0$ or $-x^4+x^2y^2+6y^4=0$.

EITHER

divides $-x^4+x^2y^2+6y^4(=0)$ through by y^4 ***(M1)***

$$-\frac{x^4}{y^4}+\frac{x^2}{y^2}+6(=0)$$

$$m=\frac{y}{x}\Rightarrow-\frac{1}{m^4}+\frac{1}{m^2}+6(=0)$$ ***(M1)***

$$6m^4+m^2-1(=0)$$

$$(3m^2-1)(2m^2+1)=0$$ ***A1***

$$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$$

OR

divides $-x^4+x^2y^2+6y^4(=0)$ through by x^2y^2 ***(M1)***

$$-\frac{x^2}{y^2}+1+6\frac{y^2}{x^2}(=0)$$

$$m=\frac{y}{x}\Rightarrow-\frac{1}{m^2}+1+6m^2(=0)$$ ***(M1)***

$$6m^4+m^2-1(=0)$$

$$(3m^2-1)(2m^2+1)=0$$ ***A1***

$$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$$

continued…

Question 12 continued

OR

attempts to factorize $-x^4+x^2y^2+6y^4(=0)$ ***(M1)***

$-(x^2-3y^2)(x^2+2y^2)(=0)$ ***A1***

attempts to solve their factorized equation ***(M1)***

$\Rightarrow y=\pm\frac{1}{\sqrt{3}}x \left(y^2=-\frac{1}{2}x^2\right)$

THEN

$y=\frac{1}{\sqrt{3}}x$ (and so $m=\frac{1}{\sqrt{3}}$) ***A1***

as $x>0,\ y>0$ (or equivalent reasoning/justification) ***R1***

Note: Award ***R1*** for $y=\frac{1}{\sqrt{3}}x$ (and so $m=\frac{1}{\sqrt{3}}$) as $y\neq-\frac{1}{\sqrt{3}}x$ and $x^2+2y^2=0$ for $x=0$ and $y=0$ only.

[6 marks]

Total [21 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1**, **A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) Let N be North

$\mathrm{N\hat{J}D} = 34°$ OR $\mathrm{D\hat{J}L} = 56°$ (must be labelled or indicated in diagram): ***(A1)***

$\mathrm{J\hat{D}L} = 99\,(°)$ ***A1***

Note: Accept $\dfrac{11\pi}{20}, 1.73$ (radians).

[2 marks]

(b) attempt to apply the sine rule ***(M1)***

$\dfrac{\mathrm{DL}}{\sin 56°} = \dfrac{500}{\sin 99°}$ OR $\dfrac{\mathrm{DL}}{\sin 0.977384...} = \dfrac{500}{\sin 1.72787...}$ ***(A1)***

$419.685\ldots$

$\mathrm{DL} = 420$ (km) ***A1***

Note: Award ***M1A1A0*** for 261 (km) from use of degrees with GDC set in radians (with or without working).

[3 marks]

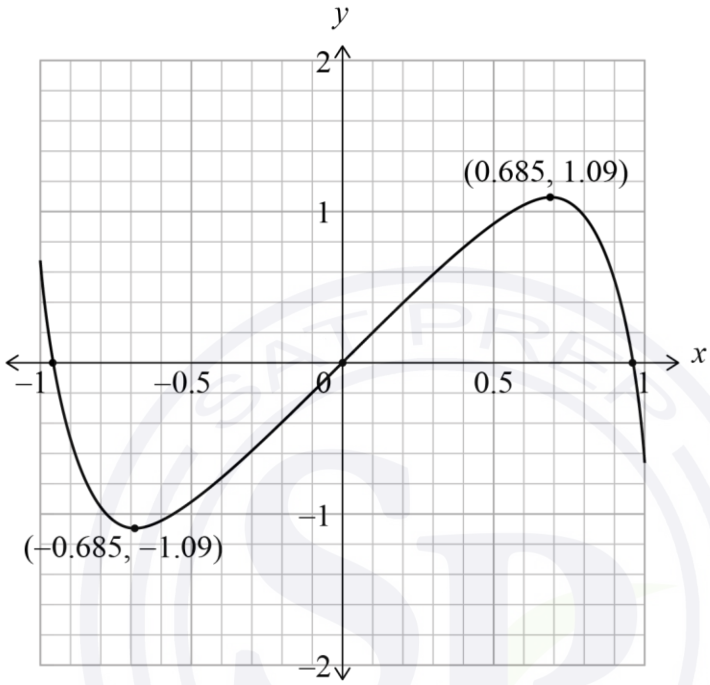
Total [5 marks]

2. (a) attempt to substitute g into f ***(M1)***

$(f \circ g)(x) = 2\tan x - \tan^3 x$ ***A1***

[2 marks]

(b)



A1A1A1

Note: ***A1*** for approximately correct odd function passing through the origin with a maximum above $y = 1$ and a minimum below $y = -1$.

A1 for endpoints at $x = \pm 1$ and y in the intervals $[0.6, 0.8]$ and $[-0.8, -0.6]$

A1 for maximum in approximately correct position and labelled $(0.685, 1.09)$ AND minimum in approximately correct position and labelled $(-0.685, -1.09)$. For approximate position, allow $-0.8 \le x \le -0.6$, $-1.2 \le y \le -1$ for minimum and $0.6 \le x \le 0.8$, $1 \le y \le 1.2$ for maximum. If the candidate gives the coordinates of extrema below their sketch, only award this mark if extrema are marked in the correct interval (eg by a dot).

[3 marks]

Total [5 marks]

3. (a) recognising to find $y(25)$ ***(M1)***

$y(25) = -0.6 \times 25^2 + 23 \times 25 + 110$

$= 310$ (children) ***A1***

[2 marks]

(b) recognizing x on y is required ***(M1)***

$0.0935114\ldots$ and $7.43053\ldots$ ***(A1)***

$x = 0.0935y + 7.43$ ***A1***

[3 marks]

(c) attempt to substitute their answer to part (a) into their regression equation for either x or y ***(M1)***

$x = 0.0935114... \times 310 + 7.43053... (= 36.4190...)$

36 (accept 37 or 36.4) ***A1***

Note: Award ***(M1)A1FT*** for $x = 37$ found from using $y = 9.39x - 41.5$.

Award ***(M1)A0FT*** for a correct ***FT*** answer that lies outside $[15, 46]$.

[2 marks]

Total [7 marks]

4. METHOD 1

$Q_1 = 31.86$ OR $Q_3 = 32.14$ ***(A1)***

recognition that the area under the normal curve below Q_1 or above Q_3 is 0.25 OR the area between Q_1 and Q_3 is 0.5 (seen anywhere including on a diagram) ***(M1)***

EITHER

equating an appropriate correct normal CDF function to its correct probability (0.25 or 0.5 or 0.75) ***(A2)***

OR

$z = -0.674489...$ OR $z = 0.674489...$ (seen anywhere) ***(A1)***

$-0.674489... = \dfrac{31.86-32}{\sigma}$ OR $0.674489... = \dfrac{32.14-32}{\sigma}$ ***(A1)***

THEN

$0.207564...$

$\sigma = 0.208$ (mm) ***A1***

METHOD 2

recognition that the area under the normal curve below Q_1 or above Q_3 is 0.25 OR the area between Q_1 and Q_3 is 0.5 (seen anywhere including on a diagram) ***(M1)***

$z = -0.674489...$ OR $z = 0.674489...$ ***(A1)***

$(Q_1 =)32 - 0.674489...\sigma$ OR $(Q_3 =)32 + 0.674489...\sigma$ ***(A1)***

$(Q_3 - Q_1 =)2 \times 0.674489...\sigma$

$2 \times 0.674489...\sigma = 0.28$ ***(A1)***

$0.207564...$

$\sigma = 0.208$ (mm) ***A1***

Total [5 marks]

5. product of a binomial coefficient, a power of ax^3 and a power of b seen ***(M1)***

evidence of correct term chosen

for $n=8$: $r=2$ (or $r=6$) OR for $n=10$: $r=2$ (or $r=8$) ***(A1)***

correct equations (may include powers of x) ***A1A1***

$${}^{8}C_2 a^2b^6 = 448 \ \left(28a^2b^6 = 448 \Rightarrow a^2b^6 = 16\right), \ {}^{10}C_2 a^2b^8 = 2880 \ \left(45a^2b^8 = 2880 \Rightarrow a^2b^8 = 64\right)$$

attempt to solve their system in a and b algebraically or graphically ***(M1)***

$b=2$; $a=\frac{1}{2}$ ***A1A1***

Note: Award a maximum of ***(M1)(A1)A1A1(M1)A1A0*** for $b=\pm 2$ and/or $a=\pm\frac{1}{2}$.

[7 marks]

6. (a) attempt to use De Moivre's theorem *(M1)*

$$\left(\cos\frac{11\pi}{18}+\mathrm{i}\sin\frac{11\pi}{18}\right)^n=\cos\frac{11\pi n}{18}+\mathrm{i}\sin\frac{11\pi n}{18}\left(=\mathrm{e}^{\frac{11\pi n}{18}\mathrm{i}}\right) \text{ OR } \cos\left(110^\circ n\right)+\mathrm{i}\sin\left(110^\circ n\right)$$

EITHER

attempt to consider imaginary part *(M1)*

$$\sin\frac{11\pi n}{18}=-1 \text{ OR } \sin\left(110^\circ n\right)=-1$$

OR

attempt to consider argument of $-\mathrm{i}$ *(M1)*

$$\mathrm{e}^{\frac{11\pi n}{18}\mathrm{i}}=\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$$

THEN

$$\frac{11\pi n}{18}=\frac{3\pi}{2},\frac{7\pi}{2}\left(,\frac{11\pi}{2}\right)\ldots\left(=\frac{3\pi}{2}+2\pi k,\ k\in\mathbb{Z}\right) \text{ OR}$$

$$110^\circ n=270^\circ,630^\circ\left(,990^\circ\right)\ldots\left(=270^\circ+360^\circ k,\ k\in\mathbb{Z}\right)$$ *(A1)*

$$11n=27,63,99,\ldots$$

$$n=9$$ *A1*

[4 marks]

continued…

Question 6 continued

(b) **EITHER**

$z^{10} = e^{10\left(\frac{11\pi}{18}i\right)} \left(= e^{\frac{55\pi}{9}i} = e^{\frac{\pi}{9}i}\right)$ OR $\arg(z^{10}) = \dfrac{\pi}{9}$ OR $\arg(z^{10}) = 20^{\circ}$ ***(A1)***

Note: Accept equivalent arguments given in any interval, in degrees or radians.

recognising that the difference between $\arg(z^{10})$ and $\arg(z)$ is needed ***(M1)***

$\arg(z^{10}) - \arg(z) = \dfrac{\pi}{9} - \dfrac{11\pi}{18} = -\dfrac{\pi}{2}$

OR

recognising that $z^{10} = z^9 \times z$ ***(M1)***

$z^9 = e^{9\left(\frac{11\pi}{18}i\right)} \left(= e^{\frac{11\pi}{2}i} = e^{\frac{3\pi}{2}i}\right)$ OR $\arg(z^9) = \dfrac{3\pi}{2}$ or $-\dfrac{\pi}{2}$ OR $\arg(z^9) = 270^{\circ}$ or -90° ***(A1)***

Note: Accept equivalent arguments given in any interval, in degrees or radians.

THEN

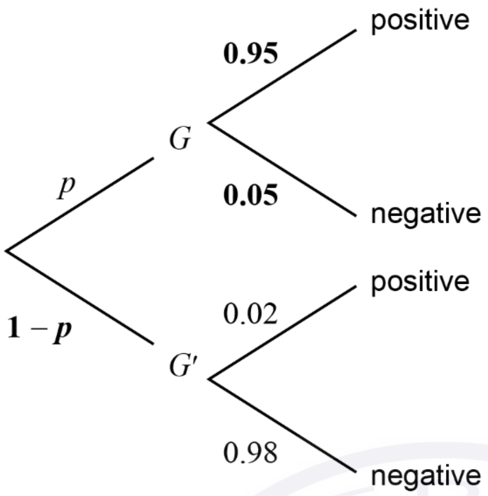
a rotation $\dfrac{3\pi}{2}$ OR $-\dfrac{\pi}{2}$ OR equivalent angle about the origin. ***A1***

Note: Accept correct answer given in degrees.
Accept $\dfrac{\pi}{2}$ clockwise or $\dfrac{11\pi}{2}$ or $\dfrac{(4k-1)\pi}{2}$ for $k \in \mathbb{Z}$.
The centre must be stated to gain the final ***A1***.

[3 marks]

Total [7 marks]

7. (a)



A1A1

Note: award ***A1*** for branch correctly labelled $1-p$ award ***A1*** for branches correctly labelled 0.95 and 0.05 award ***A0*** for G' branch labelled p' award ***A0*** for G' branch labelled q unless explicitly defined as $1-p$

[2 marks]

continued…

Question 7 continued

(b) **METHOD 1**

recognizing conditional probability ***(M1)***

$\mathrm{P}\left(G'\,|\,pos\right)$ OR $\mathrm{P}\left(G\,|\,pos\right)$

$$\frac{0.02(1-p)}{0.95p+0.02(1-p)}\left(=\frac{18}{150}\right) \text{ OR } \frac{0.95p}{0.95p+0.02(1-p)}\left(=\frac{132}{150}\right)$$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$p = 0.133738$

$p = 0.134$ ***A1***

METHOD 2

attempt to set up a system of equations (S = sample size) ***(M1)***

$p(0.95S) = 132$ and $(1-p)(0.02S) = 18$ ***(A1)***

attempt to solve for p or S ***(M1)***

$$\frac{0.95p}{0.02(1-p)} = \frac{132}{18}$$

OR $S = pS + (1-p)S = \frac{132}{0.95} + \frac{18}{0.02} = 138.947... + 900 = 1038.94...$

$p = 0.133738...$

$p = 0.134$ ***A1***

METHOD 3

attempt to find the number of parrots with the gene and the number without ***(M1)***

number of parrots with the gene $\approx \frac{132}{0.95} = 138.947...$ AND

number of parrots without the gene $\approx \frac{18}{0.02} = 900$ ***(A1)***

number of parrots in the sample $\approx 138.947... + 900 = 1038.94...$

attempt to find proportion of sample with the gene ***(M1)***

$p \approx \frac{138.947...}{1038.94...} = 0.133738...$

$p = 0.134$ ***A1***

[4 marks]

Total [6 marks]

8. direction vector of the line is $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (seen anywhere) ***(A1)***

normal vector of the plane is $\begin{pmatrix} 4 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$ (seen anywhere) ***(A1)***

EITHER

correct scalar product $12+2\cos\alpha-\sin\alpha$ (seen anywhere) ***(A1)***

one correct magnitude (seen anywhere) ***(A1)***

$\sqrt{16+\cos^2\alpha+\sin^2\alpha}\left(=\sqrt{17}\right),\ \sqrt{9+4+1}\left(=\sqrt{14}\right)$

recognizing angle between normal and direction vector is $\frac{\pi}{2}-\alpha$ (seen anywhere) ***(M1)***

Note: angle $\frac{\pi}{2}-\alpha$ may be implied by use of $\sin\alpha$ on the RHS of the step below

attempt to substitute into the formula for the angle between two vectors to form an equation in α ***(M1)***

$12+2\cos\alpha-\sin\alpha=\sqrt{17}\sqrt{14}\cos\left(\frac{\pi}{2}-\alpha\right)$ OR $12+2\cos\alpha-\sin\alpha=\sqrt{17}\sqrt{14}\sin\alpha$

continued…

Question 8 continued

OR

correct expression for the magnitude of the vector product

$\left|\begin{pmatrix} 2\sin\alpha+\cos\alpha \\ -4-3\sin\alpha \\ 3\cos\alpha-8 \end{pmatrix}\right| \left(=\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}\right)$ (seen anywhere) ***(A1)***

one correct magnitude (seen anywhere) ***(A1)***

$\sqrt{16+\cos^2\alpha+\sin^2\alpha}\ \left(=\sqrt{17}\right),\ \sqrt{9+4+1}\ \left(=\sqrt{14}\right)$

recognizing angle between normal and direction vector is $\frac{\pi}{2}-\alpha$ (seen anywhere) ***(M1)***

Note: angle $\frac{\pi}{2}-\alpha$ may be implied by use of $\cos\alpha$ on the RHS of the step below

attempt to substitute into the formula for the angle between two vectors to form an equation in α ***(M1)***

$\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}=\sqrt{17}\sqrt{14}\sin\left(\frac{\pi}{2}-\alpha\right)$ OR

$\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}=\sqrt{17}\sqrt{14}\cos\alpha$

THEN

$\alpha=0.932389...$

$\alpha=0.932$ ***A1***

Note: Award maximum ***(A1)(A1)(A1)(A1)(M1)(M1)A0*** for a correct answer given in degrees $\alpha=54.4219...^\circ$.

[7 marks]

9. Assume $p^2 - 8q - 11 = 0$, $(p, q \in \mathbb{Z})$ ***M1***

Note: This ***M1*** is dependent on the assumption of truth (implied by "assume" or "suppose that … is true".)
Subsequent marks should be awarded independently.

EITHER

$p^2 = 8q + 11 (= 2(4q+5)+1)$ so p^2 odd $\Rightarrow p$ odd ***R1***

OR

p even $\Rightarrow p^2 - 8q = 11$ even which is a contradiction so p is odd ***R1***

Note: This ***R1*** should be awarded for any valid reason to conclude that p must be odd.

THEN

$p = 2k + 1 (, k \in \mathbb{Z})$ ***M1***

$(2k+1)^2 = 8q + 11$

$4k^2 + 4k + 1 = 8q + 11$ ***(A1)***

$4k^2 + 4k = 8q + 10$

$2k^2 + 2k = 4q + 5$ or equivalent with one side odd and one side even ***A1***

a contradiction as LHS is even and RHS is odd ***R1***

Note: This ***R1*** is dependent on all previous marks.
Accept correct variations such as work based on $p = 2k - 1$.

therefore, if $p, q \in \mathbb{Z}$ then $p^2 - 8q - 11 \neq 0$ ***AG***

Total [6 marks]

Section B

10. (a) recognition that $45 = 10 + 10 + \text{arc length}$ ***(M1)***

arc length $= 25$ (cm) ***(A1)***

$25 = 12\theta$ ***A1***

$\theta = 2.08$ correct to 3 significant figures ***AG***

[3 marks]

(b)

Note:	There are many different ways to dissect the cross-section to determine its area. In all approaches, candidates will need to find w or $\frac{w}{2}$. Award the first three marks for work seen anywhere.

EITHER

evidence of using the cosine rule OR sine rule ***(M1)***

$w^2 = 12^2 + 12^2 - 2 \cdot 12 \cdot 12 \cos(2.08)$ OR $\frac{w}{\sin(2.08)} = \frac{12}{\sin(0.530796...)}$ ***(A1)***

$w = 20.6977\ldots$ or $\frac{w}{2} = 10.3488...$ ***(A1)***

OR

using trig ratios in a right triangle with angle $\frac{2.08}{2}$ and side length $\frac{w}{2}$ ***(M1)***

$\sin\left(\frac{2.08}{2}\right) = \frac{\frac{w}{2}}{12}$ ***(A1)***

$w = 20.6977\ldots$ or $\frac{w}{2} = 10.3488...$ ***(A1)***

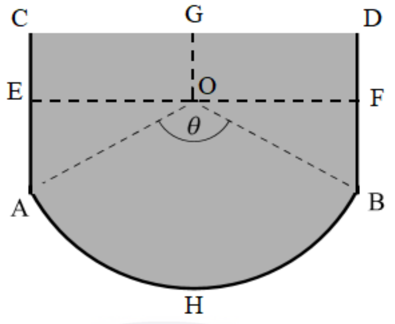
Note: Accept $w = 20.7179\ldots$ from use of $\frac{\theta}{2} = \frac{25}{24}$.

continued…

Question 10 continued

THEN

Let the points A, B, C, D, E, F, G, H lie on the figure as follows:



EITHER

(segment AHB =) sector OAB – triangle OAB ***(M1)***

$= \frac{1}{2}\times 12^2 \times 2.08 - \frac{1}{2}\times 12^2 \times \sin 2.08 \left(= 149.76 - 62.8655... = 86.8944...\right)$ ***(A1)***

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

sector OAB – triangle OAB + rectangle CDBA

$= 86.8944... + 10w \left(= 86.8944... + 206.977...\right)$

Note:	Use of $\theta = \frac{25}{12}$ throughout leads to segment OAB $= 87.2517...$ and cross-sectional area $= 87.2517... + 207.179...$.

continued…

Question 10 continued

OR

trapezium CGOA (= rectangle CGOE + triangle EOA) ***(M1)***

$=\frac{1}{2}\times\left(10+\left(10-12\cos(1.04)\right)\right)\times\frac{20.6977...}{2}$ $(=72.0557)$ ***(A1)***

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

2 × trapezium CGOA + sector OAB

$= 2(72.0557...)+\frac{1}{2}\times12^2\times2.08\left(=144.111...+149.76\right)$

Note:	Use of $\theta=\frac{25}{12}$ leads to area of trapezium CGOA $=72.2154...$ and cross-sectional area $=144.430...+150$.

OR

2 x area of trapezium CGOA (= area of rectangle CDFE + 2 x triangle EOA) ***(M1)***

$20.6977...\times\left(10-12\cos(1.04)\right)+2\times\frac{1}{2}\times12\cos(1.04)\times12\sin(1.04)$ ***(A1)***

$\left(=81.2458...+62.8655...\right)$

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

2 x trapezium CGOA + sector OAB

$=144.111...+\frac{1}{2}\times12^2\times2.08$ $\left(=144.111...+149.76\right)$

Note:	Use of $\theta=\frac{25}{12}$ leads to 2 x area of trapezium CGOA $=144.430...$ and cross-sectional area $=144.430...+150$.

THEN

area of cross-section $=293.871...$ ($294.430...$ from exact answer)

$=294$ (cm^2) ***A1***

[7 marks]

continued…

Question 10 continued

(c) **METHOD 1**

$-4.71976...$ volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognising rainfall can be represented by an integral ***(M1)***

$\int_0^{60} R'(t)\,\mathrm{d}t \quad \left(=\frac{250}{2\pi}\sin\left(\frac{2\pi\times 60}{5}\right)+3000\times 60\right)$ ***(A1)***

Note:	Accept any 60 second interval or any interval which is a multiple of 5 seconds (one period) scaled up to 60 seconds e.g. $12\int_0^{5} R'(t)\,\mathrm{d}t$.

rainfall over 60 seconds $=180000$ (cm^3) ***A1***

the gutter will overflow because the rainfall > gutter volume ***A1***

METHOD 2

volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognition that cosine has a minimum value of -1 ***(M1)***

$R'(t)\geq -1\times 50+3000\left(\mathrm{cm}^3\mathrm{s}^{-1}\right)$ ***(A1)***

rainfall over 60 seconds ≥ 177000 ***(A1)***

the gutter will overflow because the rainfall > gutter volume ***A1***

METHOD 3

volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognising rainfall can be represented by an integral ***(M1)***

attempt to solve $60>58.8$ OR $\int_0^{T} R'(t)\,\mathrm{d}t=176658$ ***(M1)***

time to reach overflow point = $58.7875...$ OR $58.8990...$ ***A1***

the gutter will overflow because $60>58.8$ OR $60>58.9$ ***A1***

[5 marks]

Total [15 marks]

11. (a) $\mathrm{E}(X) = \int_0^2 \frac{6x}{\pi\sqrt{16-x^2}}\,\mathrm{d}x$ ***(A1)***

> **Note:** Condone the absence of $\mathrm{d}x$.
> Accept $\int_0^2 xf(x)\,\mathrm{d}x$

attempt to integrate $\frac{6x}{\pi\sqrt{16-x^2}}$ using inspection/substitution ***(M1)***

$-\frac{6}{2\pi}\int -2x\left(16-x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$ or let $u = 16 - x^2$

$-\frac{6}{2\pi}\left[2\left(16-x^2\right)^{\frac{1}{2}}\right]_0^2$ OR $\frac{6}{\pi}\left[-u^{\frac{1}{2}}\right]_{16}^{12}$ ***A1***

> **Note:** For this ***A1*** condone absent or incorrect limits.

attempt to substitute their limits and evaluate ***(M1)***

$\frac{24}{\pi} - \frac{6}{\pi}\sqrt{12}\ \left(= \frac{12}{\pi}\left(2-\sqrt{3}\right)\right)$ ***A1***

> **Note:** The substitution $\sin\theta = \frac{x}{4}$ may also be used, leading to
> $\frac{24}{\pi}\int_0^{\frac{\pi}{6}} \sin\theta\,\mathrm{d}\theta = \frac{24}{\pi}\left[-\cos\theta\right]_0^{\frac{\pi}{6}} = \frac{24}{\pi}\left(1-\cos\frac{\pi}{6}\right)$ **.** Award marks as
> appropriate and accept $\frac{24}{\pi}\left(1-\cos\frac{\pi}{6}\right)$ for the final ***A1***.

[5 marks]

(b) $\int_0^{0.5} f(x)\,\mathrm{d}x \left(= \int_0^{0.5} \frac{6}{\pi\sqrt{16-x^2}}\,\mathrm{d}x\right)$ ***(M1)***

$P(X < 0.5) = 0.239358...$

$= 0.239$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) **EITHER**

recognition P(at least one success after n trials) = 1 – P(no successes after n trials) ***(M1)***

$1-(1-0.239...)^n \geq 0.99$ ***(A1)***

$n = 16.8321...$

Note: Use of 0.239 results in $n = 16.8612...$

OR

recognition that $Y \sim \mathrm{B}(n, 0.239...)$ ***(M1)***

If $n=16$ P(at least one success after n trials) $= 0.987443...$

and if $n=17$ P(at least one success after n trials) $= 0.990448...$ ***(A1)***

Note: Use of 0.239 results in the values $0.987348...$ and $0.990371...$

THEN

17 trials ***A1***

[3 marks]

(d) recognition that $Y \sim \mathrm{B}(10, \text{their part b})$ ***(M1)***

$\mathrm{B}(10, 0.239...)$

$\mathrm{P}(X=3) = 0.242430...$

$= 0.242$ ***A1***

[2 marks]

continued…

Question 11 continued

(e) 8 ***A1***

[1 mark]

(f) (i) $n-2$ ***A1***

(ii) ${}^{n}C_{3}$ (ways of 3 successes in n trials) ***(A1)***

$\frac{n-2}{{}^{n}C_{3}}$ ***(A1)***

Attempt to solve their $\frac{n-2}{{}^{n}C_{3}} > 0.05$ OR $\frac{6}{n(n-1)} > 0.05$ (or equivalent) ***(M1)***

Note: Accept an equation.

$n = 11.4658\ldots$ OR

table values $n = 11, \frac{n-2}{{}^{n}C_{3}} = 0.0545454\ldots$ and $n = 12, \frac{n-2}{{}^{n}C_{3}} = 0.0454545\ldots$ ***(A1)***

$n = 11$ ***A1***

[6 marks]

Total [19 marks]

12.

Note: Penalise only once for an answer not given to six significant figures in parts (a), c(ii) and (d)(ii).

(a) attempt to use Euler's method ***(M1)***

$$y_{n+1} = y_n + 0.03\left(\frac{x_n^{\ 2} y_n - y_n}{x_n^{\ 2} + 1}\right), y_1 = 3 + 0.03\left(\frac{0-3}{0+1}\right)$$

$y_1 = 2.91$ ***(A1)***

at least one **correct** further intermediate value given to at least 3 significant figures ***(A1)***

y_0	3
y_1	2.91
y_2	2.82285...
y_3	2.73877...
y_4	2.65793...
y_5	2.58046...

$y(0.15) \approx y_5 = 2.58046160...$

$= 2.58046$ ***A1***

Note: Award final ***A1*** for the correct answer seen as the last line in a table. If the table goes beyond this value and the correct answer is not explicitly identified award maximum ***(M1)(A1)(A1)A0***

[4 marks]

continued…

Question 12 continued

(b) (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = -3$ ***A1***

(ii) **METHOD 1**

attempt to use quotient (or product) rule on

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 y - y}{x^2+1} \left(= \left(x^2+1\right)^{-1}\left(x^2 y - y\right) \right)$ ***(M1)***

attempt to use product rule and implicit differentiation on $x^2 y$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y - y\right) = x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + (2x)y - \frac{\mathrm{d}y}{\mathrm{d}x}$ (seen anywhere) ***A1***

$= 3$ (when $x = 0, y = 3, \frac{\mathrm{d}y}{\mathrm{d}x} = -3$)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\left(x^2+1\right)\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + (2x)y - \frac{\mathrm{d}y}{\mathrm{d}x}\right) - \left(x^2 y - y\right)(2x)}{\left(x^2+1\right)^2}$$

OR $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(x^2+1\right)^{-1}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + (2x)y - \frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2x\left(x^2+1\right)^{-2}\left(x^2 y - y\right)$ ***A1***

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3$ (when $x = 0, y = 3, \frac{\mathrm{d}y}{\mathrm{d}x} = -3$) ***AG***

METHOD 2

$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y - y$ ***A1***

attempt to use product rule and implicit differentiation ***(M1)***

$2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2+1\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + (2x)y - \frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3$ (when $x = 0, y = 3, \frac{\mathrm{d}y}{\mathrm{d}x} = -3$) ***AG***

[5 marks]

continued…

Question 12 continued

(c) (i) $3-3x+\frac{x^2}{2!}(3)+\frac{x^3}{3!}(9)+\ldots\left(=3-3x+\frac{3}{2}x^2+\frac{3}{2}x^3+\ldots\right)$ ***A1A1***

Note: Award ***A1*** for first three terms, ***A1*** for fourth term

(ii) $y(0.15)=2.58881125\ldots$

$=2.58881$ ***A1***

[3 marks]

continued…

Question 12 continued

(d) (i) **EITHER**

attempt to separate variables **_M1_**

$$\int \frac{1}{y}\mathrm{d}y = \int \frac{x^2-1}{x^2+1}\mathrm{d}x \quad \text{OR} \quad \int \frac{1}{y}\mathrm{d}y = \int\left(1-\frac{2}{x^2+1}\right)\mathrm{d}x$$

$\ln y = x - 2\arctan x + c$ **_A1A1_**

> **Note:** Award ***A1*** for $\ln y$ or $\ln|y|$, ***A1*** for $x-2\arctan x$.
> Condone missing $+c$ at this stage.

OR

attempt to use integrating factor **_M1_**

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(1-\frac{2}{x^2+1}\right) = 0$$

IF $= \mathrm{e}^{\int -\left(1-\frac{2}{x^2+1}\right)\mathrm{d}x} = \mathrm{e}^{-x+2\arctan x}$ **_A1_**

$$\mathrm{e}^{-x+2\arctan x}\frac{\mathrm{d}y}{\mathrm{d}x} - y\mathrm{e}^{-x+2\arctan x}\left(1-\frac{2}{x^2+1}\right) = 0$$

$\mathrm{e}^{-x+2\arctan x}y = A$ **_A1_**

THEN

$\ln y = x - 2\arctan x + c$ OR $y = A\mathrm{e}^{x-2\arctan x}$

attempt to find c or A using $x=0, y=3$ ***(M1)***

$\ln 3 = 0 - 2\arctan 0 + c$ OR $3 = A\mathrm{e}^{0-2\arctan 0}$

$c = \ln 3$ OR $A = 3$ ***(A1)***

> **Note:** This ***A1*** should not be awarded if a correct value of c or A is preceded by incorrect working.

$y = \mathrm{e}^{x-2\arctan x+\ln 3}\left(=3\mathrm{e}^{x-2\arctan x}\right)$ **_A1_**

continued…

Question 12 continued

(ii) $y(0.15) = 2.58786288...$

$= 2.58786$ ***A1***

[7 marks]

(e) the graph of $y = f(x)$ is concave up OR $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$ (for $0 \le x < 1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

hence tangents used (in Euler's method) give an underestimate,

so the approximate value for y when $x = 1.5$ is less than the actual value. ***R1***

[2 marks]

Note: ***R1*** is dependent on ***A1***, as well as reference to tangents, in words or on a diagram.

Total [21 marks]

N22/5/MATHX/HP2/ENG/TZ0/XX/M

Markscheme

November 2022

Mathematics: analysis and approaches

Higher level

Paper 2

32 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates

fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first"

Section A

1. (a) $1.01206\ldots, 2.45230\ldots$

$a = 1.01$, $b = 2.45$ $(1.01x + 2.45)$ ***A1A1***

[2 marks]

(b) $0.981464\ldots$

$r = 0.981$ ***A1***

Note: A common error is to enter the data incorrectly into the GDC, and obtain the answers $a = 1.01700...,\ b = 2.09814...$ and $r = 0.980888...$ Some candidates may write the 3 sf answers, ie. $a = 1.02,\ b = 2.10$ and $r = 0.981$ or 2 sf answers, ie. $a = 1.0,\ b = 2.1$ and $r = 0.98$. In these cases award ***A0A0*** for part (a) and ***A0*** for part (b). Even though some values round to an accepted answer, they come from incorrect working.

[1 mark]

(c) correct substitution of 78 into **their** regression equation ***(M1)***

$81.3930\ldots$, 81.23 from 3 sf answer

81 ***A1***

[2 marks]

Total [5 marks]

2. (a) $(0.708519...,0.639580...)$

$(0.709, 0.640)$ ($x = 0.709,\ y = 0.640$) ***A1A1***

[2 marks]

(b) $1.09885...$

$x = 1.10$ (accept $(1.10, 0)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 |f(x)|dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^{2} f(x)dx$ OR $\int_{1.09885...}^{2} |f(x)|dx$ OR $4.17527\ldots$ ***(A1)***

$\int_0^{1.09885...} f(x)dx - \int_{1.09885...}^{2} f(x)dx$ OR $0.435901\ldots + 4.17527\ldots$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

Total [6 marks]

3. $86.4 = 50r^3$ ***(A1)***

$r = 1.2 \left(= \sqrt[3]{\frac{86.4}{50}} \right)$ seen anywhere ***(A1)***

$\frac{50\left(1.2^n - 1\right)}{0.2} > 33500$ OR $250\left(1.2^n - 1\right) = 33500$ ***(A1)***

attempt to solve their geometric S_n inequality or equation ***(M1)***

sketch OR $n > 26.9045,\ n = 26.9$ OR $S_{26} = 28368.8$ OR $S_{27} = 34092.6$ OR algebraic manipulation involving logarithms

$n = 27$ (accept $n \geq 27$) ***A1***

Total [5 marks]

4. recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0) = 15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000 \times 0.89 (= 13350)$ ***(A1)***

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000 \times 0.89 = 15000\mathrm{e}^{8k}$ OR $13350 = 15000\mathrm{e}^{8k}$

$k = \dfrac{\ln 0.89}{8}$ (-0.014566) ***(A1)***

substitution of $t = 2041 - 2014 (= 27)$ and their value for k into the model ***(M1)***

$P(27) = 15000e^{-0.0145... \times 27}$

$10122.3\ldots$

$P(27) = 10100$ (10122) ***A1***

Total [7 marks]

5.

> **Note:** Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n = 9$ and a power of ax) ***(M1)***

${}^9C_r (ax)^{9-r} (1)^r$ OR ${}^9C_{9-r} (ax)^r (1)^{9-r}$ OR ${}^9C_0 (ax)^0 (1)^9 + {}^9C_1 (ax)^1 (1)^8 + \dots$

recognizing that the term in x^6 is needed ***(M1)***

$\dfrac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r = 6$ OR $r = 3$ OR $9 - r = 6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6 (ax)^6 (1)^3$ OR ${}^9C_3\, a^6 x^6$ OR $84\left(a^6 x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\dfrac{{}^9C_6}{21} a^6 x^4$ OR $4a^6 x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\dfrac{8}{7} a^5 x^4$ or $\dfrac{8}{7} a^5$ (do not accept other powers of *x*) ***(M1)***

$\dfrac{{}^9C_3}{21} a^6 x^4 = \dfrac{8}{7} a^5 x^4$ OR $4a^6 = \dfrac{8}{7} a^5$

continued…

Question 5 continued

OR

correct term in x^6 or coefficient of x^6 (may be seen in equation) ***(A1)***

$84a^6x^6$ OR $84a^6$

Set their term in x^6 or coefficient of x^6 equal to $24a^5x^6$ or $24a^5$
(do not accept other powers of *x*) ***(M1)***

$84a^6x^6 = 24a^5x^6$ OR $84a = 24$

THEN

$a = \frac{2}{7} \approx 0.286(0.285714...)$ ***A1***

Note: Award ***A0*** for the final mark for $a = \frac{2}{7}$ and $a = 0$.

Total [6 marks]

6. (a) $\int_0^b axe^x \mathrm{d}x = 1$ (seen anywhere) ***M1***

attempt to use integration by parts (either way around) (***M1***)

$\left[axe^x\right]_0^b - \int_0^b ae^x \mathrm{d}x \left(=1\right)$ ***(A1)***

$\left[axe^x\right]_0^b - \left[ae^x\right]_0^b \left(=1\right)$ ***A1***

Note: Condone incorrect or absent limits up to this point.

$abe^b - ae^b + a = 1$

$a = \dfrac{1}{be^b - e^b + 1}$ ***A1***

[5 marks]

(b) $\int_0^m xe^x \mathrm{d}x = \dfrac{1}{2}$ ***(M1)***

$\left[xe^x\right]_0^m - \left[e^x\right]_0^m = \dfrac{1}{2}$

$me^m - e^m + 1 = \dfrac{1}{2}$ ***(A1)***

$m = 0.768039...$

$m = 0.768$ ***A1***

[3 marks]

Total [8 marks]

7. (a) **METHOD 1**

attempt to use scalar product or formula for angle between two vectors *(M1)*

$\boldsymbol{u}\cdot\boldsymbol{v}=\cos\frac{1}{n}+\sin\frac{1}{n}$ (seen anywhere) *(A1)*

$$\cos\theta=\frac{\cos\frac{1}{n}+\sin\frac{1}{n}}{\sqrt{2}\sqrt{\left(\cos^2\frac{1}{n}+\sin^2\frac{1}{n}\right)}}\left(=\frac{\cos\frac{1}{n}+\sin\frac{1}{n}}{\sqrt{2}}\right)$$ *A1*

METHOD 2

attempt to use an Argand diagram showing two complex numbers in the first quadrant with the angle between them marked as θ *(M1)*

$\arg(\boldsymbol{u})=\frac{\pi}{4}$ (accept 45° or $\arctan(1)$) **and** $\arg(\boldsymbol{v})=\frac{1}{n}$ *(A1)*

$\cos\theta=\cos\left|\frac{\pi}{4}-\frac{1}{n}\right|$ *A1*

continued…

Question 7 continued

(b) use of $\frac{1}{n} \to 0$ as $n \to \infty$ ***(M1)***

EITHER

$(\cos\theta \to)\frac{1}{\sqrt{2}}$ ***(A1)***

OR

$(\boldsymbol{v} \to)\boldsymbol{i}$ ***(A1)***

THEN

the limit is $\frac{\pi}{4}$ ***A1***

Note: Accept $45°$. Do not accept rounded values such as 0.785.

[3 marks]

Total [6 marks]

8. EITHER

$\left(\frac{\mathrm{d}V}{\mathrm{d}h}=\right)10\pi h-\pi h^2$ ***(A1)***

Note: This ***A1*** may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h}=76.5616...$.

attempt to use chain rule to find a relationship between $\frac{\mathrm{d}h}{\mathrm{d}t}$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}\left(=\frac{1}{\left(\frac{\mathrm{d}V}{\mathrm{d}h}\right)}\times\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

OR

attempt to differentiate $V=5\pi h^2-\frac{1}{3}\pi h^3$ throughout with respect to t ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t}=10\pi h\frac{\mathrm{d}h}{\mathrm{d}t}-\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

continued…

Question 8 continued

THEN

$\left(10\pi h-\pi h^{2}\right)\dfrac{\mathrm{d}h}{\mathrm{d}t}=2$ OR $\dfrac{\mathrm{d}h}{\mathrm{d}t}=\dfrac{2}{10\pi h-\pi h^{2}}$ ***(A1)***

Note: Award this ***A1*** if the correct expression is seen with their h already substituted.

attempt to solve $200=5\pi h^{2}-\dfrac{1}{3}\pi h^{3}$ ***(M1)***

$h=4.20648...$ ***(A1)***

Note: This ***(M1)(A1)*** can be awarded independently of all previous marks, and may be implied by the value $\dfrac{\mathrm{d}V}{\mathrm{d}h}=76.5616...$
Ignore extra values of h -3.24 and 14.0.

$\dfrac{\mathrm{d}h}{\mathrm{d}t}=0.0261227...$

$\dfrac{\mathrm{d}h}{\mathrm{d}t}=0.0261\left(\mathrm{cm\,s}^{-1}\right)$ ***A1***

[6 marks]

9. (a) (i) attempt to use the cosine rule ***(M1)***

$$\mathrm{AC}=\sqrt{2^2+4^2-2(2)(4)\cos\alpha}\left(=\sqrt{20-16\cos\alpha}=2\sqrt{5-4\cos\alpha}\right)$$ ***A1***

(ii) $$\mathrm{AC}=\sqrt{6^2+8^2-2(6)(8)\cos\beta}\left(=\sqrt{100-96\cos\beta}=2\sqrt{25-24\cos\beta}\right)$$ ***A1***

(iii) $5-4\cos\alpha=25-24\cos\beta$

$\alpha=\arccos\left(6\cos\beta-5\right)$ ***A1***

[4 marks]

(b) attempt to find the sum of two triangle areas using $A=\frac{1}{2}ab\sin C$ ***(M1)***

Note: Do not award this ***M1*** if the triangle is assumed to be right angled.

$\text{Area}=\frac{1}{2}(8)\sin\alpha+\frac{1}{2}(48)\sin\beta$ ***(A1)***

attempt to express the area in terms of one variable only ***(M1)***

$=4\sqrt{1-\left(6\cos\beta-5\right)^2}+24\sin\beta$ or $4\sin\left(\arccos\left(6\cos\beta-5\right)\right)+24\sin\beta$ OR

$4\sin\alpha+24\sqrt{1-\left(\frac{5+\cos\alpha}{6}\right)^2}$ or $4\sin\alpha+24\sin\left(\arccos\left(\frac{5+\cos\alpha}{6}\right)\right)$

Max area = $19.5959\ldots$

= 19.6 ***A1***

[4 marks]

Total [8 marks]

Section B

10.

Note: Do not penalize for inclusion or non-inclusion of endpoints for probabilities using a normal distribution. For example, for $\mathrm{P}\left(T < 55 \mid T > 40\right)$ accept $\mathrm{P}\left(T \le 55 \mid T > 40\right)$, $\mathrm{P}\left(T \le 55 \mid T \ge 40\right)$, etc.

(a) recognising to find $\mathrm{P}(T > 40)$ ***(M1)***

$\mathrm{P}(T > 40) = 0.574136...$

$\mathrm{P}(T > 40) = 0.574$ ***A1***

[2 marks]

(b) attempt to multiply four independent probabilities using their $\mathrm{P}(T > 40)$ and $\mathrm{P}(T < 40)$ ***(M1)***

$(1-p)^3 \cdot p$ OR $(1-0.574136...)^3 \cdot 0.574136...$ OR $(0.425863...)^3 \cdot 0.574136...$ ***(A1)***

$0.0443430...$

0.0443 , 0.0444 from 3 sf values ***A1***

[3 marks]

continued…

Question 10 continued

(c) (i) recognizing conditional probability ***(M1)***

$\mathrm{P}(T < 55 \mid T > 40)$

> **Note:** Award ***(M1)*** for an expression or description in context. Accept $\mathrm{P}(T > 40 \mid T < 55)$ but do not accept just $\mathrm{P}(A \mid B)$.

$\dfrac{\mathrm{P}(40 < T < 55)}{\mathrm{P}(T > 40)}$ ***(A1)***

$\dfrac{0.461944...}{0.574136...}$ ***(A1)***

$\mathrm{P}(T < 55 \mid T > 40) = 0.804590...$

$= 0.805$ ***A1***

(ii) recognizing binomial probability ***(M1)***

$X \sim \mathrm{B}(n,\ p)$

$n = 10$ and $p = 0.804589...$ ***(A1)***

$0.0242111...,0.0240188...\text{using } p = 0.805$

$\mathrm{P}(X = 5) = 0.0242$ ***A1***

[7 marks]

continued…

Question 10 continued

(d) Let $\mathrm{P}(T < a) = x$

recognition that probabilities sum to 1 (seen anywhere) ***(M1)***

EITHER

expressing the three regions in one variable ***(M1)***

$x + 0.904 + 2x$ OR $\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a)$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b)$

OR x and $2x$ correctly indicated on labelled bell diagram

$\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a) = 1$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b) = 1$ (or equivalent) ***(A1)***

OR

expressing either $\mathrm{P}(T < a)$ or $\mathrm{P}(T > b)$ only in terms of $\mathrm{P}(a \le T \le b)$ ***(M1)***

$\left(\mathrm{P}(T < a) =\right)\frac{1}{3}\left(1 - \mathrm{P}(a \le T \le b)\right)$ OR $\left(\mathrm{P}(T > b) =\right)\frac{2}{3}\cdot\left(1 - \mathrm{P}(a \le T \le b)\right)$

$x = \frac{1}{3}(1 - 0.904)(= 0.032)$ OR $\mathrm{P}(T > b) = \frac{2}{3}(1 - 0.904)(= 0.064)$ ***(A1)***

THEN

$\mathrm{P}(T < a) = 0.032$

$a = 22.18167...$

$a = 22.2$ (accept 22.1) ***A1***

[4 marks]

Total [16 marks]

11. (a) attempt to use product rule ***(M1)***

$f'(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ ***(M1)***

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$$\pi\int_0^{\frac{4}{3}}\left(e^{2x}(3x-4)\right)^2 dx$$ ***(A1)***

Note: This ***(A1)*** can be awarded if the dx is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

continued…

Question 11 continued

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 \left(= -\mathrm{e}^2\right)$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1)(= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 \left(= 2\mathrm{e}^2\right)$ ***A1***

[5 marks]

Total [15 marks]

12.

(a) $\vec{AB} = \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix}, \vec{AC} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ ***A1A1***

[2 marks]

(b) **METHOD 1**

$k-1 = 2\times 4$ ***M1***

$k = 9$ ***AG***

METHOD 2

in order by y or z-ordinate, the points are $(k, -2, 1), (5, 0, 2)\ (1, 2, 3)$

$k-5 = 5-1$ ***M1***

$k = 9$ ***AG***

[1 mark]

continued…

Question 12 continued

(c) (i) attempt to set up a vector equation using a point, a parameter and a direction vector ***(M1)***

$$\boldsymbol{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\-2\\-1\end{pmatrix}$$ (or equivalent) ***A1***

Note: "$\boldsymbol{r}=$" or equivalent must be seen for ***A1***.

continued…

Question 12 continued

(ii) **METHOD 1**

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use a different parameter for L_2 ***(M1)***

$\frac{x-1}{2}=\frac{y}{3}=1-z=\mu$ or $\boldsymbol{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-1\end{pmatrix}$

point on line L_2 has coordinates $(1+2\mu, 3\mu, 1-\mu)$ ***(A1)***

Note: This ***A1*** may be implied by $\boldsymbol{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-1\end{pmatrix}$.

$1+4\lambda=1+2\mu$

$2-2\lambda=3\mu$

$3-\lambda=1-\mu$

any two of the above equations ***A1***

attempt to solve two simultaneous equations with two parameters ***(M1)***

eg $\lambda=0.25$, $\mu=0.5$ or $\lambda=1.6$, $\mu=-0.4$ or $\lambda=-2$, $\mu=-4$ ***A1***

substitute into third equation or solve a different pair of simultaneous equations ***M1***

obtain contradiction *eg* $3-0.25\neq 1-0.5$ or $1+4(1.6)\neq 1+2(-0.4)$ or $2-2(-2)\neq 3(-4)$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

continued…

Question 12 continued

METHOD 2

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use the equation of L_2 to generate at least two equations in λ ***(M1)***

if the two lines intersect,

$$\frac{(1+4\lambda)-1}{2}=\frac{2-2\lambda}{3}\left(\Rightarrow 2\lambda=\frac{2-2\lambda}{3}\right)$$

$$\frac{(1+4\lambda)-1}{2}=1-(3-\lambda)(\Rightarrow 2\lambda=\lambda-2)$$

$$\frac{2-2\lambda}{3}=1-(3-\lambda)\Rightarrow\left(\frac{2-2\lambda}{3}=\lambda-2\right)$$

any two of the above equations ***A1A1***

attempt to solve at least one equation in λ ***(M1)***

one of $\lambda=\frac{1}{4}$, $\lambda=-2$, $\lambda=\frac{8}{5}$ seen ***A1***

substitute into second equation or solve second equation ***M1***

obtain contradiction *eg* $\lambda=\frac{1}{4}\neq-2$ or $2\left(\frac{1}{4}\right)\neq\frac{1}{4}-2$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

continued…

Question 12 continued

METHOD 3

attempt to use a find Cartesian equation for L_1 ***(M1)***

$$\frac{x-1}{4}=\frac{y-2}{-2}=\frac{z-3}{-1}$$ ***A1***

attempt to isolate one variable in both equations ***(M1)***

$L_1: z=\dfrac{1-x}{4}+3=\dfrac{y-2}{2}+3 \quad L_2: z=\dfrac{1-x}{2}+1=\dfrac{-y}{3}+1$ OR

$L_1: y=\dfrac{1-x}{2}+2=2(z-3)+2 \quad L_2: y=\dfrac{3(x-1)}{2}=3(1-z)$ OR

$L_1: x=1-2(y-2)=1-4(z-3) \quad L_2: x=\dfrac{2y}{3}+1=1-2(z-1)$ ***A1***

attempt to solve for each of the other two variables ***(M1)***

e.g. $\dfrac{1-x}{2}+1=\dfrac{1-x}{4}+3$ and $\dfrac{-y}{3}+1=\dfrac{y-2}{2}+3$

$x=-7$, $y=-1.2$ OR $x=2$, $z=1.4$ OR $y=1.5$, $z=5$ ***A1***

obtain contradiction *eg* $z=5\neq 1.4$ OR $y=1.5\neq -1.2$ OR $x=2\neq -7$

(so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

[10 marks]

continued…

Question 12 continued

(d) (i) **METHOD 1**

attempt to find cross product of two of $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ or their opposites **M1**

eg $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ k-9 \\ 18-2k \end{pmatrix} \left(= (k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$ **A1**

attempt to substitute their cross product and a point into the equation of a plane **(M1)**

$(k-9)y + 2(9-k)z = 2(k-9) + 6(9-k)$

$(k-9)y + 2(9-k)z = 36-4k \ (\Rightarrow y-2z=-4 \text{ since } k \neq 9)$ **A1**

METHOD 2

attempt to find vector equation of Π and write x, y and z in parametric form **M1**

$\left(\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \Rightarrow \right) x = 1 + \lambda(k-1) + 4\mu, \ y = 2 - 4\lambda - 2\mu,$

$z = 3 - 2\lambda - \mu$ or equivalent **A1**

attempt to eliminate both parameters to work towards Cartesian form **M1**

$(k-9)y + 2(9-k)z = 36-4k \ (\Rightarrow y-2z=-4 \text{ since } k \neq 9)$ **A1**

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to find the equation of the line through (0, 0, 0) perpendicular to the plane ***(M1)***

EITHER

$$(\boldsymbol{r} =)t\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$$t + 4t + 4 = 0$$

$$t = -\frac{4}{5}$$ ***(A1)***

OR

$$(\boldsymbol{r} =)t(k-9)\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$$t(k-9)^2 + 4t(k-9)^2 + 4(k-9) = 0$$

$$t = -\frac{4}{5(k-9)}$$ ***(A1)***

THEN

so the point on the plane closest to the origin is $(0\,,\ -0.8\,,\ 1.6)$ ***A1***

continued…

Question 12 continued

METHOD 2

choose a point on the plane (p , q , r)

$q-2r+4=0$ OR $q(k-9)-2r(k-9)+4(k-9)=0 \Rightarrow q=2r-4$

distance to the origin is $\sqrt{p^2+(2r-4)^2+r^2}$ ***(A1)***

since *p* is independent of *r*, distance is minimised when $p=0$ ***(R1)***

attempt to find the value of *r* for which their $\sqrt{(2r-4)^2+r^2}$ is minimised ***(M1)***

$r=1.6$ ***(A1)***

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

continued…

Question 12 continued

METHOD 3

attempt to find a vector from the origin to the closest point on the plane ***(M1)***

EITHER

$(\boldsymbol{r}=)t\begin{pmatrix}0\\1\\-2\end{pmatrix}$ ***(A1)***

distance to the origin $=\left(\dfrac{4}{\sqrt{1^2+(-2)^2}}=\dfrac{4}{\sqrt{5}}\right)=\dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t=\pm\dfrac{4}{5}$

check in equation of plane $y-2z=-4$ to get $t=-\dfrac{4}{5}$ ***(R1)***

OR

$(\boldsymbol{r}=)t(k-9)\begin{pmatrix}0\\1\\-2\end{pmatrix}$ ***(A1)***

distance to the origin $=\left(\dfrac{4}{\sqrt{1^2+(-2)^2}}=\dfrac{4}{\sqrt{5}}\right)=\dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t=\pm\dfrac{4}{5(k-9)}$

check in equation of plane $y-2z=-4$ to get $t=-\dfrac{4}{5(k-9)}$ ***(R1)***

THEN

so the point on the plane closest to the origin is $(0,-0.8,1.6)$ ***A1***

[9 marks]

Total [22 marks]

Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 2

35 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2****.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) **EITHER**

uses the cosine rule ***(M1)***

$\mathrm{AB}^2 = 5^2 + 5^2 - 2\times5\times5\times\cos1.9$ ***(A1)***

OR

uses right-angled trigonometry ***(M1)***

$\dfrac{\frac{\mathrm{AB}}{2}}{5} = \sin 0.95$ ***(A1)***

OR

uses the sine rule ***(M1)***

$\alpha = \dfrac{1}{2}(\pi - 1.9)(=0.6207\ldots)$

$\dfrac{\mathrm{AB}}{\sin1.9} = \dfrac{5}{\sin0.6207\ldots}$ ***(A1)***

THEN

$\mathrm{AB} = 8.1341\ldots$

$\mathrm{AB} = 8.13$ (m) ***A1***

[3 marks]

continued…

Question 1 continued

(b) let the shaded area be A

METHOD 1

Attempt at finding reflex angle **(M1)**

$\text{A}\hat{\text{O}}\text{B} = 2\pi - 1.9 \ (= 4.3831...)$

substitution into area formula **(A1)**

$A = \frac{1}{2} \times 5^2 \times 4.3831... \text{ OR } \left(\frac{2\pi - 1.9}{2\pi}\right) \times \pi\left(5^2\right)$

$= 54.7898...$

$= 54.8$ (m^2) **A1**

METHOD 2

let the area of the circle be A_C and the area of the unshaded sector be A_U

$A = A_C - A_U$ **(M1)**

$A = \pi \times 5^2 - \frac{1}{2} \times 5^2 \times 1.9 \ (= 78.5398... - 23.75)$ **(A1)**

$= 54.7898...$

$= 54.8$ (m^2) **A1**

[3 marks]

Total [6 marks]

2. **METHOD 1**

recognises that $g(x)=\int\left(3x^2+5\mathrm{e}^x\right)\,\mathrm{d}x$ ***(M1)***

$g(x)=x^3+5\mathrm{e}^x\,(+C)$ ***(A1)(A1)***

Note: Award ***A1*** for each integrated term.

substitutes $x=0$ and $y=4$ into their integrated function (must involve $+C$) ***(M1)***

$4=0+5+C\Rightarrow C=-1$

$g(x)=x^3+5\mathrm{e}^x-1$ ***A1***

METHOD 2

attempts to write both sides in the form of a definite integral ***(M1)***

$\int_0^x \mathrm{g}'(t)\,\mathrm{d}t=\int_0^x\left(3t^2+5\mathrm{e}^t\right)\,\mathrm{d}t$ ***(A1)***

$g(x)-4=x^3+5\mathrm{e}^x-5\mathrm{e}^0$ ***(A1)(A1)***

Note: Award ***A1*** for $g(x)-4$ and ***A1*** for $x^3+5\mathrm{e}^x-5\mathrm{e}^0$.

$g(x)=x^3+5\mathrm{e}^x-1$ ***A1***

[5 marks]

3. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.68$

substitution of $\mathrm{P}(A) \cdot \mathrm{P}(B)$ for $\mathrm{P}(A \cap B)$ in $\mathrm{P}(A \cup B)$ ***(M1)***

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B) \; (= 0.68)$

substitution of $3\mathrm{P}(B)$ for $\mathrm{P}(A)$ ***(M1)***

$3\mathrm{P}(B) + \mathrm{P}(B) - 3\mathrm{P}(B)\mathrm{P}(B) = 0.68$ (or equivalent) ***(A1)***

Note: The first two marks are independent of each other.

attempts to solve their quadratic equation ***(M1)***

$\mathrm{P}(B) = 0.2, 1.133... \left(\frac{1}{5}, \frac{17}{15}\right)$

$\mathrm{P}(B) = 0.2 \left(= \frac{1}{5}\right)$ ***A2***

Note: Award ***A1*** if both answers are given as final answers for $\mathrm{P}(B)$.

[6 marks]

4. (a) 0.28 (s) ***A1***

[1 mark]

(b) $\text{IQR} = 0.35 - 0.27\ (= 0.08)$ (s) ***(A1)***

substituting **their** IQR into correct expression for upper fence ***(A1)***

$0.35 + 1.5 \times 0.08\ (= 0.47)$ (s)

$0.46 < 0.47$ ***R1***

so 0.46 (s) is not an outlier ***AG***

[3 marks]

(c) **EITHER**

the median is closer to the lower quartile (positively skewed) ***R1***

OR

the distribution is positively skewed ***R1***

OR

the range of reaction times below the median is smaller than the range of reaction times above the median ***R1***

Note: These are sample answers from a range of acceptable correct answers. Award ***R1*** for any correct statement that explains this.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct. Accept a correctly and clearly labelled diagram.

[1 mark]

continued…

Question 4 continued

(d) **EITHER**

the distribution for ‘not sleeping well’ is centred at a higher reaction time ***R1***

OR

the median reaction time after not sleeping well is equal to the upper quartile reaction time after sleeping well ***R1***

OR

75% of reaction times are <0.35 seconds after sleeping well, compared with 50% after not sleeping well ***R1***

OR

the sample size of 9 is too small to draw any conclusions ***R1***

Note: These are sample answers from a range of acceptable correct answers. Accept any relevant correct statement t**hat relates to the median and/or quartiles shown in the box plots**. Do not accept a comparison of means.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct

Award ***R0*** to “correlation does not imply causation”.

[1 mark]

Total [6 marks]

5. (a) recognises the need to find the value of t when $v=0$ ***(M1)***

$t=1.5707...\left(=\frac{\pi}{2}\right)$

$t=1.57\left(=\frac{\pi}{2}\right)$ (s) ***A1***

[2 marks]

(b) recognises that $a(t)=v'(t)$ ***(M1)***

$t_1=2.2627...,\ t_2=2.9573...$

$t_1=2.26,\ t_2=2.96$ (s) ***A1A1***

Note:	Award ***M1A1A0*** if the two correct answers are given with additional values outside $0\le t\le 3$.

[3 marks]

(c) speed is greatest at $t=3$ ***(A1)***

$a=-1.8377...$

$a=-1.84$ (m s^{-2}) ***A1***

[2 marks]

Total [7 marks]

6. attempts to express x^2 in terms of y ***(M1)***

$$V = \pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y$$ ***A1***

Note: Correct limits are required.

Attempts to solve $\pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y = 285$ for h ***(M1)***

Note: Award ***M1*** for attempting to solve $36\pi\left(\frac{h^3}{48}-\frac{h^2}{4}+\frac{8}{3}\right)=285$ or equivalent for h.

$h = 0.7926...$

$h = 0.793$ (cm) ***A2***

[5 marks]

7. (a) (as $\lim_{x\to 0} x^2 = 0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$\Rightarrow \lim_{x\to 0}\left(\arctan\left(\cos x\right) - k\right) = 0$ ***M1***

$\arctan 1 - k = 0 \ \left(k = \arctan 1\right)$ ***A1***

so $k = \frac{\pi}{4}$ ***AG***

Note: Award ***M1A0*** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

continued…

Question 7 continued

(b) $\lim_{x\to 0}\dfrac{\arctan(\cos x)-\frac{\pi}{4}}{x^2}\ \left(=\dfrac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{\frac{-\sin x}{1+\cos^2 x}}{2x}$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

recognises to apply l'Hôpital's rule again ***(M1)***

$$=\lim_{x\to 0}\frac{\frac{-\sin x}{1+\cos^2 x}}{2x}\ \left(=\frac{0}{0}\right)$$

Note: Award ***M0*** if their limit is not the indeterminate form $\frac{0}{0}$.

EITHER

$$=\lim_{x\to 0}\frac{\frac{-\cos x\left(1+\cos^2 x\right)-2\sin^2 x\cos x}{\left(1+\cos^2 x\right)^2}}{2}$$ ***A1A1***

Note: Award ***A1*** for a correct first term in the numerator and ***A1*** for a correct second term in the numerator.

OR

$$\lim_{x\to 0}\frac{-\cos x}{2\left(1+\cos^2 x\right)-4x\sin x\cos x}$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

THEN

substitutes $x=0$ into the correct expression to evaluate the limit ***A1***

Note: The final ***A1*** is dependent on all previous marks.

$$=-\frac{1}{4}$$ ***AG***

[6 marks]

Total [8 marks]

8. Rachel: $R \sim \mathrm{N}\left(56.5, 3^2\right)$

$\mathrm{P}(R \geq 60) = 0.1216\ldots$ ***(A1)***

Sophia: $S \sim \mathrm{N}\left(57.5, 1.8^2\right)$

$\mathrm{P}(S \geq 60) = 0.0824\ldots$ ***(A1)***

recognises binomial distribution with $n = 5$ ***(M1)***

let N_R represent the number of Rachel's throws that are longer than 60 metres

$N_R \sim \mathrm{B}(5, 0.1216\ldots)$

either $\mathrm{P}(N_R \geq 1) = 0.4772\ldots$ or $\mathrm{P}(N_R = 0) = 0.5227\ldots$ ***(A1)***

let N_S represent the number of Sophia's throws that are longer than 60 metres

$N_S \sim \mathrm{B}(5, 0.0824\ldots)$

either $\mathrm{P}(N_S \geq 1) = 0.3495\ldots$ or $\mathrm{P}(N_S = 0) = 0.6504\ldots$ ***(A1)***

EITHER

uses $\mathrm{P}(N_R \geq 1)\mathrm{P}(N_S = 0) + \mathrm{P}(N_S \geq 1)\mathrm{P}(N_R = 0)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = (0.4772\ldots \times 0.6504\ldots) + (0.3495\ldots \times 0.5227\ldots)$

OR

uses $\mathrm{P}(N_R \geq 1) + \mathrm{P}(N_S \geq 1) - 2 \times \mathrm{P}(N_R \geq 1) \times \mathrm{P}(N_S \geq 1)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = 0.4772\ldots + 0.3495\ldots - 2 \times 0.4772\ldots \times 0.3495\ldots$

THEN

$= 0.4931\ldots$

$= 0.493$ ***A1***

[7 marks]

Note: ***M*** marks are not dependent on the previous ***A*** marks.

9. (a) $9\times 9\times 8\times 7\times 6\times 5 \left(=9\times {}^{9}P_{5}\right)$ ***(M1)***

$=136080 \left(=9\times \dfrac{9!}{4!}\right)$ ***A1***

Note: Award ***M1A0*** for $10\times 9\times 8\times 7\times 6\times 5 \left(={}^{10}P_{6}=151200 \ =\dfrac{10!}{4!}\right)$.

Note: Award ***M1A0*** for ${}^{9}P_{6}=60480$

[2 marks]

(b) **METHOD 1**

EITHER

every unordered subset of 6 digits from the set of 9 non-zero digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{6}\left(\times 1\right)$ ***A1***

THEN

$=84$ ***A1***

METHOD 2

EITHER

removes 3 digits from the set of 9 non-zero digits and these 6 remaining digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{3}\left(\times 1\right)$ ***A1***

THEN

$=84$ ***A1***

[2 marks]

Total [4 marks]

Section B

10. (a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t+6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t+6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t+6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

continued…

Question 10 continued

(d) recognises that $h_A{}'(t)$ and $h_B{}'(t)$ are required **(M1)**

attempts to solve $h_A{}'(t) = h_B{}'(t)$ for t **(M1)**

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ **(A1)**

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ **(A1)**

attempts to calculate the total amount of time **(M1)**

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) **A1**

[6 marks]

Total [15 marks]

11. (a) let ϕ be the required angle (bearing)

EITHER

$\phi = 90° - \arctan\frac{1}{2}\ \left(= \arctan 2\right)$ ***(M1)***

Note: Award ***M1*** for a labelled sketch.

OR

$\cos\phi = \frac{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\2\end{pmatrix}}{\sqrt{1}\times\sqrt{20}}\ \left(= 0.4472..., = \frac{1}{\sqrt{5}}\right)$ ***(M1)***

$\phi = \arccos(0.4472...)$

THEN

$063°$ ***A1***

Note: Do not accept $063.4°$ or $63.4°$ or 1.10^c.

[2 marks]

(b) **Method 1**

let $|\boldsymbol{b_A}|$ be the speed of A and let $|\boldsymbol{b_B}|$ be the speed of B

attempts to find the speed of one of A or B ***(M1)***

$|\boldsymbol{b_A}| = \sqrt{(-6)^2 + 2^2 + 4^2}$ or $|\boldsymbol{b_B}| = \sqrt{4^2 + 2^2 + (-2)^2}$

Note: Award ***M0*** for $|\boldsymbol{b_A}| = \sqrt{19^2 + (-1)^2 + 1^2}$ and $|\boldsymbol{b_B}| = \sqrt{1^2 + 0^2 + 12^2}$.

$|\boldsymbol{b_A}| = 7.48...\ \left(= \sqrt{56}\right)$ (km min^{-1}) and $|\boldsymbol{b_B}| = 4.89...\ \left(= \sqrt{24}\right)$ (km min^{-1}) ***A1***

$|\boldsymbol{b_A}| > |\boldsymbol{b_B}|$ so A travels at a greater speed than B ***AG***

[2 marks]

continued…

Question 11 continued

Method 2

attempts to use $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$\text{speed}_A = \dfrac{|r_A(t_2) - r_A(t_1)|}{t_2 - t_1}$ and $\text{speed}_B = \dfrac{|r_B(t_2) - r_B(t_1)|}{t_2 - t_1}$ ***(M1)***

for example:

$\text{speed}_A = \dfrac{|r_A(1) - r_A(0)|}{1}$ and $\text{speed}_B = \dfrac{|r_B(1) - r_B(0)|}{1}$

$\text{speed}_A = \dfrac{\sqrt{(-6)^2 + 2^2 + 4^2}}{1}$ and $\text{speed}_B = \dfrac{\sqrt{4^2 + 2^2 + 2^2}}{1}$

$\text{speed}_A = 7.48\ldots\left(2\sqrt{14}\right)$ and $\text{speed}_B = 4.89\ldots\left(\sqrt{24}\right)$ ***A1***

$\text{speed}_A > \text{speed}_B$ so A travels at a greater speed than B ***AG***

[2 marks]

continued…

Question 11 continued

(c) attempts to use the angle between two direction vectors formula ***(M1)***

$$\cos\theta = \frac{(-6)(4)+(2)(2)+(4)(-2)}{\sqrt{(-6)^2+2^2+4^2}\sqrt{4^2+2^2+(-2)^2}}$$ ***(A1)***

$\cos\theta = -0.7637\ldots \left(= -\frac{7}{\sqrt{84}}\right)$ or $\theta = \arccos(-0.7637\ldots)$ $(= 2.4399\ldots)$

attempts to find the acute angle $180° - \theta$ using their value of θ ***(M1)***

$= 40.2°$ ***A1***

[4 marks]

continued…

Question 11 continued

(d) (i) for example, sets $\boldsymbol{r_A}(t_1)=\boldsymbol{r_B}(t_2)$ and forms at least two equations ***(M1)***

$19-6t_1=1+4t_2$

$-1+2t_1=2t_2$

$1+4t_1=12-2t_2$

Note: Award ***M0*** for equations involving t only.

EITHER

attempts to solve the system of equations for one of t_1 or t_2 ***(M1)***

$t_1=2$ or $t_2=\frac{3}{2}$ ***A1***

OR

attempts to solve the system of equations for t_1 and t_2 ***(M1)***

$t_1=2$ and $t_2=\frac{3}{2}$ ***A1***

THEN

substitutes their t_1 or t_2 value into the corresponding $\boldsymbol{r_A}$ or $\boldsymbol{r_B}$ ***(M1)***

$\mathrm{P}(7,3,9)$ ***A1***

Note: Accept $\overrightarrow{\mathrm{OP}}=\begin{pmatrix}7\\3\\9\end{pmatrix}$. Accept 7 km east of O, 3 km north of O and 9 km above sea level.

continued…

Question 11 continued

(ii) attempts to find the value of $t_1 - t_2$ ***(M1)***

$$t_1 - t_2 = 2 - \frac{3}{2}$$

0.5 minutes (30 seconds) ***A1***

[7 marks]

continued…

Question 11 continued

(e) **EITHER**

attempts to find $\boldsymbol{r}_B - \boldsymbol{r}_A$ ***(M1)***

$$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix} -18 \\ 1 \\ 11 \end{pmatrix} + t\begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$D(t) = \sqrt{(10t-18)^2 + 1 + (11-6t)^2}$ ***A1***

OR

attempts to find $\boldsymbol{r}_A - \boldsymbol{r}_B$ ***(M1)***

$$\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t\begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$D(t) = \sqrt{(18-10t)^2 + (-1)^2 + (-11+6t)^2}$ ***A1***

Note: Award ***M0M0A0*** for expressions using two different time parameters.

THEN

either attempts to find the local minimum point of $D(t)$ or attempts to find the value of t such that $D'(t) = 0$ (or equivalent) ***(M1)***

$t = 1.8088... \left(= \frac{123}{68}\right)$

$D(t) = 1.01459...$

minimum value of $D(t)$ is $1.01 \left(= \frac{\sqrt{1190}}{34}\right)$ (km) ***A1***

[5 marks]

Note: Award ***M0*** for attempts at the shortest distance between two lines.

Total [20 marks]

12. (a) rate of growth (change) of the (marsupial) population (with respect to time) ***A1***

[1 mark]

Note: Do not accept growth (change) in the (marsupials) population per year.

(b) **METHOD 1**

attempts implicit differentiation on $\frac{\mathrm{d}P}{\mathrm{d}t}=kP-\frac{kP^2}{N}$ by expanding $kP\left(1-\frac{P}{N}\right)$ ***(M1)***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=k\frac{\mathrm{d}P}{\mathrm{d}t}-2\frac{kP}{N}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***A1A1***

$=k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1-\frac{2P}{N}\right)$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{N}\right)$ and so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ ***AG***

METHOD 2

attempts implicit differentiation (product rule) on $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{N}\right)$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1-\frac{P}{N}\right)+kP\left(-\left(\frac{1}{N}\right)\frac{\mathrm{d}P}{\mathrm{d}t}\right)$ ***A1***

substitutes $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{N}\right)$ into their $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=k\left(kP\left(1-\frac{P}{N}\right)\right)\left(1-\frac{P}{N}\right)+kP\left(-\left(\frac{1}{N}\right)kP\left(1-\frac{P}{N}\right)\right)$

$=k^2P\left(1-\frac{P}{N}\right)^2-k^2P\left(1-\frac{P}{N}\right)\left(\frac{P}{N}\right)$

$=k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{P}{N}-\frac{P}{N}\right)$ ***A1***

so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = 0 \Rightarrow k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right) = 0$ ***(M1)***

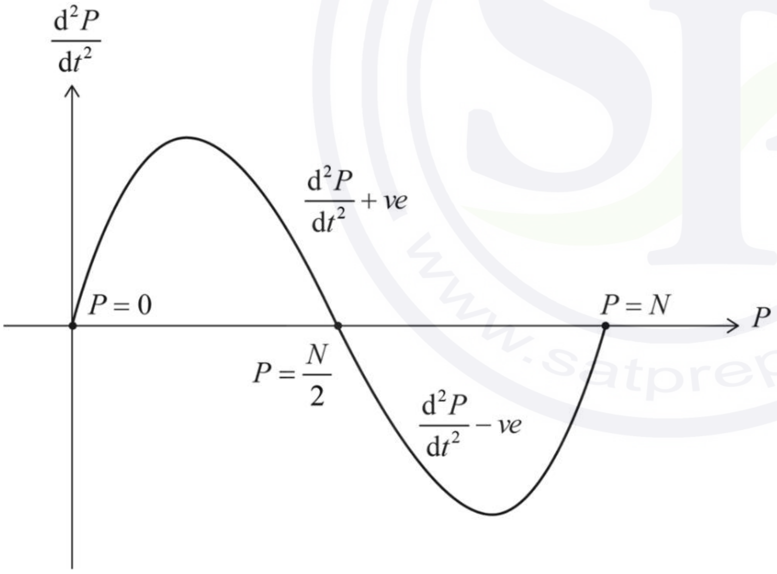
$P = 0, \frac{N}{2}, N$ ***A2***

Note: Award ***A1*** for $P = \frac{N}{2}$ only.

uses the second derivative to show that concavity changes at $P = \frac{N}{2}$ or the first derivative to show a local maximum at $P = \frac{N}{2}$ ***M1***

EITHER

a clearly labelled correct sketch of $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ versus P showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***



OR

a correct and clearly labelled sign diagram (table) showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

continued…

Question 12 continued

OR

for example, $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = \frac{3k^2N}{32}(>0)$ with $P = \frac{N}{4}$ and $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = -\frac{3k^2N}{32}(<0)$ with $P = \frac{3N}{4}$

showing $P = \frac{N}{2}$ corresponds to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

so the population is increasing at its maximum rate when $P = \frac{N}{2}$ ***AG***

[5 marks]

(d) substitutes $P = \frac{N}{2}$ into $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***(M1)***

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{N}{2}\right)\left(1 - \frac{\frac{N}{2}}{N}\right)$$

the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ is $\frac{kN}{4}$ ***A1***

[2 marks]

continued…

Question 12 continued

(e) **METHOD 1**

attempts to separate variables ***M1***

$$\int \frac{N}{P(N-P)} \mathrm{d}P = \int k \ \mathrm{d}t$$

attempts to write $\frac{N}{P(N-P)}$ in partial fractions form ***M1***

$$\frac{N}{P(N-P)} \equiv \frac{A}{P} + \frac{B}{(N-P)} \Rightarrow N \equiv A(N-P) + BP$$

$A=1,\ B=1$ ***A1***

$$\frac{N}{P(N-P)} \equiv \frac{1}{P} + \frac{1}{(N-P)}$$

$$\int\left(\frac{1}{P} + \frac{1}{(N-P)}\right)\mathrm{d}P = \int k \ \mathrm{d}t$$

$$\Rightarrow \ln P - \ln(N-P) = kt(+C)$$ ***A1A1***

Note: Award ***A1*** for $-\ln(N-P)$ and ***A1*** for $\ln P$ and $kt(+C)$. Absolute value signs are not required.

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$ and so $C = \ln P_0 - \ln(N-P_0)$

$$kt = \ln\left(\frac{P}{N-P}\right) - \ln\left(\frac{P_0}{N-P_0}\right) \left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

so $kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 2

attempts to separate variables **_M1_**

$$\int \frac{1}{P\left(1-\frac{P}{N}\right)} \mathrm{d}P = \int k \ \mathrm{d}t$$

attempts to write $\frac{1}{P\left(1-\frac{P}{N}\right)}$ in partial fractions form **_M1_**

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{A}{P} + \frac{B}{1-\frac{P}{N}} \Rightarrow 1 \equiv A\left(1-\frac{P}{N}\right) + BP$$

$A=1,\ B=\frac{1}{N}$ **_A1_**

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)}$$

$$\int \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)} \mathrm{d}P = \int k \ \mathrm{d}t$$

$\Rightarrow \ln P - \ln\left(1-\frac{P}{N}\right) = kt\,(+C)$ **_A1A1_**

Note: Award ***A1*** for $-\ln\left(1-\frac{P}{N}\right)$ and ***A1*** for $\ln P$ and $kt\,(+C)$. Absolute value signs are not required.

continued…

Question 12 continued

$$\ln\left(\frac{P}{1-\frac{P}{N}}\right)=kt+C \Rightarrow \ln\left(\frac{NP}{N-P}\right)=kt+C$$

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$ and so $C=\ln\left(\frac{NP_0}{N-P_0}\right)$

$$kt=\ln\left(\frac{NP}{N-P}\right)-\ln\left(\frac{NP_0}{N-P_0}\right)\left(=\ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

$$kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 3

lets $u=\frac{1}{P}$ and forms $\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***M1***

multiplies both sides of the differential equation by $-\frac{1}{P^2}$ and makes the above substitutions ***M1***

$$-\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t}=k\left(\frac{1}{N}-\frac{1}{P}\right)\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}t}=k\left(\frac{1}{N}-u\right)$$

$\frac{\mathrm{d}u}{\mathrm{d}t}+ku=\frac{k}{N}$ (linear first-order DE) ***A1***

$\mathrm{IF}=\mathrm{e}^{\int k\ \mathrm{d}t}=\mathrm{e}^{kt}\Rightarrow\mathrm{e}^{kt}\frac{\mathrm{d}u}{\mathrm{d}t}+k\mathrm{e}^{kt}u=\frac{k}{N}\mathrm{e}^{kt}$ ***(M1)***

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\mathrm{e}^{kt}\right)=\frac{k}{N}\mathrm{e}^{kt}$$

$u\mathrm{e}^{kt}=\frac{1}{N}\mathrm{e}^{kt}\left(+C\right)\ \left(\frac{1}{P}\mathrm{e}^{kt}=\frac{1}{N}\mathrm{e}^{kt}\left(+C\right)\right)$ ***A1***

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$, $u=\frac{1}{P_0}$ and so $C=\frac{1}{P_0}-\frac{1}{N}\left(=\frac{N-P_0}{NP_0}\right)$

$$\mathrm{e}^{kt}\left(\frac{N-P}{NP}\right)=\frac{N-P_0}{NP_0}$$

$\mathrm{e}^{kt}=\left(\frac{P}{N-P}\right)\left(\frac{N-P_0}{P_0}\right)$ ***A1***

$kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***AG***

[7 marks]

continued…

Question 12 continued

(f) substitutes $t=10$, $P=3P_0$ and $N=4P_0$ into $kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***M1***

$10k=\ln 3\left(\frac{4P_0-P_0}{4P_0-3P_0}\right)\ (=\ln 9)$

$k=0.220\ \left(=\frac{1}{10}\ln 9,=\frac{1}{5}\ln 3\right)$ ***A1***

[2 marks]

Total [21 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 2

32 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) **EITHER**

recognising that half the total frequency is 10 (may be seen in an ordered list or indicated on the frequency table) ***(A1)***

OR

$5+1+4=3+x$ ***(A1)***

OR

$\sum f=20$ ***(A1)***

THEN

$x=7$ ***A1***

[2 marks]

continued…

Question 1 continued

(b) **METHOD 1**

$1.58429\ldots$

1.58 ***A2***

METHOD 2

EITHER

$$\sigma^2 = \frac{5\times(2-4.3)^2+1\times(3-4.3)^2+4\times(4-4.3)^2+3\times(5-4.3)^2+7\times(6-4.3)^2}{20} \quad (=2.51)$$ ***(A1)***

OR

$$\sigma^2 = \frac{5\times2^2+1\times3^2+4\times4^2+3\times5^2+7\times6^2}{20}-4.3^2 \quad (=2.51)$$ ***(A1)***

THEN

$\sigma = \sqrt{2.51} = 1.58429...$

$= 1.58$ ***A1***

[2 marks]

Total [4 marks]

2. (a) valid approach to find area of segment by finding area of sector – area of triangle ***(M1)***

$$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$\frac{1}{2}(2)^2\theta - \frac{1}{2}(2)^2\sin\theta$$ ***(A1)***

area = $2\theta - 2\sin\theta$ ***A1***

[3 marks]

(b) **EITHER**

area of logo = area of rectangle – area of segments ***(M1)***

$5\times4 - 2\times(2\theta - 2\sin\theta) = 13.4$ ***(A1)***

OR

area of one segment = $\dfrac{20-13.4}{2}$ $(=3.3)$ ***(M1)***

$2\theta - 2\sin\theta = 3.3$ ***(A1)***

THEN

$\theta = 2.35672...$

$\theta = 2.36$ (do not accept an answer in degrees) ***A1***

Note: Award ***(M1)(A1)A0*** if there is more than one solution.
Award ***(M1)(A1FT)A0*** if the candidate works in degrees and obtains a final answer of $135.030\ldots$

[3 marks]

Total [6 marks]

3. (a) $0.41 + k - 0.28 + 0.46 + 0.29 - 2k^2 = 1$ OR $k - 2k^2 + 0.01 = 0.13$ (or equivalent) ***A1***

$2k^2 - k + 0.12 = 0$ ***AG***

[1 mark]

(b) one of 0.2 OR 0.3 ***(M1)***

$k = 0.3$ ***A1***

reasoning to reject $k = 0.2$ eg $\mathrm{P}(1) = k - 0.28 \geq 0$ therefore $k \neq 0.2$ ***R1***

[3 marks]

(c) attempting to use the expected value formula ***(M1)***

$\mathrm{E}(X) = 0 \times 0.41 + 1 \times (0.3 - 0.28) + 2 \times 0.46 + 3 \times (0.29 - 2 \times 0.3^2)$

$= 1.27$ ***A1***

Note: Award ***M1A0*** if additional values are given.

[2 marks]

Total [6 marks]

4. (a) recognizing at rest $v = 0$ ***(M1)***

$t = 3.34692...$

$t = 3.35$ (seconds) ***A1***

Note: Award ***(M1)A0*** for any other solution to $v = 0$ eg $t = -0.205$ or $t = 6.08$.

[2 marks]

(b) recognizing particle changes direction when $v = 0$ OR when $t = 3.34692...$ ***(M1)***

$a = -4.71439...$

$a = -4.71\,(\mathrm{ms}^{-2})$ ***A2***

[3 marks]

(c) distance travelled = $\int_0^6 |v|\,\mathrm{d}t$ OR

$$\int_0^{3.34...}\left(\mathrm{e}^{\sin(t)} + 4\sin(t)\right)\mathrm{d}t - \int_{3.34...}^{6}\left(\mathrm{e}^{\sin(t)} + 4\sin(t)\right)\mathrm{d}t \quad (= 14.3104... + 6.44300...)$$ ***(A1)***

$= 20.7534...$

= 20.8 (metres) ***A1***

[2 marks]

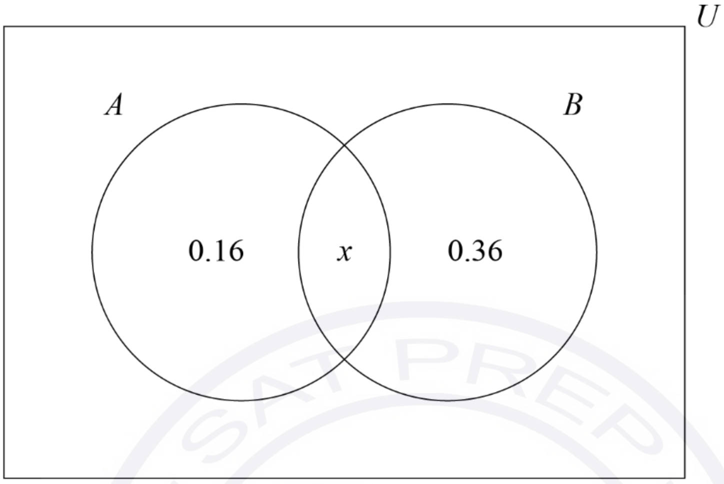
Total [7 marks]

5. (a) **METHOD 1**

EITHER

one of $\mathrm{P}(A) = x + 0.16$ OR $\mathrm{P}(B) = x + 0.36$ ***A1***

OR



A1

THEN

attempt to equate their $\mathrm{P}(A \cap B)$ with their expression for $\mathrm{P}(A) \times \mathrm{P}(B)$ ***M1***

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) \Rightarrow x = (x + 0.16) \times (x + 0.36)$ ***A1***

$x = 0.24$ ***A1***

METHOD 2

attempt to form at least one equation in $\mathrm{P}(A)$ and $\mathrm{P}(B)$ using independence ***M1***

$\left(\mathrm{P}(A \cap B') = \mathrm{P}(A) \times \mathrm{P}(B') \Rightarrow\right)$ $\mathrm{P}(A) \times \left(1 - \mathrm{P}(B)\right) = 0.16$ OR

$\left(\mathrm{P}(A' \cap B) = \mathrm{P}(A') \times \mathrm{P}(B) \Rightarrow\right)$ $\left(1 - \mathrm{P}(A)\right) \times \mathrm{P}(B) = 0.36$

$\mathrm{P}(A) = 0.4$ AND $\mathrm{P}(B) = 0.6$ ***A1***

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) = 0.4 \times 0.6$ ***(A1)***

$x = 0.24$ ***A1***

[4 marks]

continued…

Question 5 continued

(b) **METHOD 1**

recognising $\mathrm{P}(A' \mid B') = \mathrm{P}(A')$ ***(M1)***

$= 1 - 0.16 - 0.24$

$= 0.6$ ***A1***

METHOD 2

$\mathrm{P}(B) = 0.36 + 0.24 \left(= 0.6\right)$

$\mathrm{P}(A' \mid B') = \dfrac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} \quad \left(= \dfrac{0.24}{0.4}\right)$ ***(A1)***

$= 0.6$ ***A1***

[2 marks]

Total [6 marks]

6. (a) attempt to replace x with $-x$ ***M1***

$$f(-x) = 2^{-x} - \frac{1}{2^{-x}}$$

EITHER

$$= \frac{1}{2^{x}} - 2^{x} = -f(x)$$ ***A1***

OR

$$= -\left(2^{x} - \frac{1}{2^{x}}\right)\left(= -f(x)\right)$$ ***A1***

> **Note:** Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180°about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempt to find at least one intersection point ***(M1)***

$x = -1.26686..., x = 0.177935..., x = 3.06167...$

$x = -1.27, x = 0.178, x = 3.06$

$-1.27 \le x < -1$, ***A1***

$0.178 \le x < 3$, ***A1***

$x \ge 3.06$ ***A1***

[4 marks]

Total [6 marks]

7. (a) $|\boldsymbol{a}| = \sqrt{12^2 + (-5)^2}\ (=13)$ ***(A1)***

$2 \le |\boldsymbol{a}+\boldsymbol{b}| \le 28$ (accept min 2 and max 28) ***A1***

Note: Award ***(A1)A0*** for 2 and 28 seen with no indication that they are the endpoints of an interval.

[2 marks]

(b) recognition that $\boldsymbol{p}$ or $\boldsymbol{b}$ is a negative multiple of $\boldsymbol{a}$ ***(M1)***

$\boldsymbol{p} = -2\hat{\boldsymbol{a}}$ OR $\boldsymbol{b} = -\frac{15}{13}\boldsymbol{a}\left(= -\frac{15}{13}\begin{pmatrix}12\\-5\end{pmatrix}\right)$

$\boldsymbol{p} = -\frac{2}{13}\begin{pmatrix}12\\-5\end{pmatrix}\left(=\begin{pmatrix}-1.85\\0.769\end{pmatrix}\right)$ ***A1***

[2 marks]

(c) **METHOD 1**

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix}12\\-5\end{pmatrix}$

$\Rightarrow \boldsymbol{q}$ is in the direction $\begin{pmatrix}5\\12\end{pmatrix}$ ***(M1)***

$\boldsymbol{q} = k\begin{pmatrix}5\\12\end{pmatrix}$ ***(A1)***

$(|\boldsymbol{q}| =)\sqrt{(5k)^2 + (12k)^2} = 15$ ***(M1)***

$k = \frac{15}{13}$ ***(A1)***

$\boldsymbol{q} = \frac{15}{13}\begin{pmatrix}5\\12\end{pmatrix}\left(=\begin{pmatrix}\frac{75}{13}\\\frac{180}{13}\end{pmatrix}=\begin{pmatrix}5.77\\13.8\end{pmatrix}\right)$ ***A1***

[5 marks]

continued…

Question 7 continued

METHOD 2

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

attempt to set scalar product $\boldsymbol{q.a} = 0$ OR product of gradients $= -1$ ***(M1)***

$12x - 5y = 0$ ***(A1)***

$\left(|\boldsymbol{q}| =\right)\sqrt{x^2 + y^2} = 15$

attempt to solve simultaneously to find a quadratic in x or in y ***(M1)***

$x^2 + \left(\frac{12x}{5}\right)^2 = 15^2$ OR $\left(\frac{5y}{12}\right)^2 + y^2 = 15^2$

$\boldsymbol{q} = \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix} \left(= \begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix} \right)$ ***A1A1***

> **Note:** Award ***A1*** independently for each value. Accept values given as $x = \frac{75}{13}$ and $y = \frac{180}{13}$ or equivalent.

[5 marks]

Total [9 marks]

8. (a) product of roots $= \dfrac{2k+9}{k}$ ***A1***

[1 mark]

(b) recognition that the product of the roots will be negative ***(M1)***

$$\frac{2k+9}{k} < 0$$

critical values $k = 0, -\dfrac{9}{2}$ seen ***(A1)***

$$-\frac{9}{2} < k < 0$$ ***A1***

[3 marks]

Total [4 marks]

9. (a) $6\times 5!$ *(A1)(A1)*

$=720$ **(** accept $6!$) *A1*

[3 marks]

(b) **METHOD 1**

(Peter apart from girls, in an end seat) $^{8}P_{4}\left(=1680\right)$ OR

(Peter apart from girls, not in end seat) $^{7}P_{4}\left(=840\right)$ *(A1)*

case 1: Peter at either end

$2\times {}^{8}P_{4}\left(=3360\right)$ OR $2\times {}^{8}C_{4}\times 4!\left(=3360\right)$ *(A1)*

case 2: Peter not at the end

$8\times {}^{7}P_{4}\left(=6720\right)$ OR $8\times {}^{7}C_{4}\times 4!\left(=6720\right)$ *(A1)*

Total number of ways $=3360+6720$

$=10080$ *A1*

METHOD 2

(Peter next to girl, in an end seat) $4\times {}^{8}P_{3}\left(=1344\right)$ OR

(Peter next to one girl, not in end seat) $2\times 4\times {}^{7}P_{3}\left(=1680\right)$ OR

(Peter next to two girls, not in end seat) $4\times 3\times {}^{7}P_{2}\left(=504\right)$ *(A1)*

case 1: Peter at either end

$2\times 4\times {}^{8}P_{3}\left(=2688\right)$ *(A1)*

case 2: Peter not at the end

$8\left(2\times 4\times {}^{7}P_{3}+4\times 3\times {}^{7}P_{2}\right)\left(=17472\right)$ *(A1)*

Total number of ways $={}^{10}P_{5}-\left(2688+17472\right)$

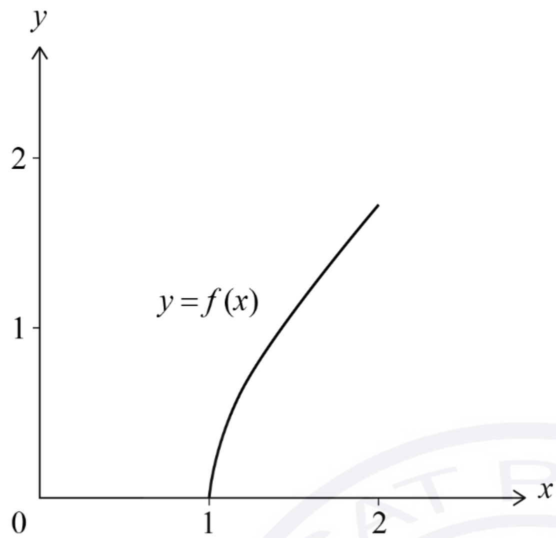
$=10080$ *A1*

[4 marks]

Total [7 marks]

Section B

10. (a)



correct shape (concave down) within the given domain $1 \le x \le 2$ ***A1***

$(1,0)$ and $\left(2,\sqrt{3}\right)\left(=(2,1.73)\right)$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

continued…

Question 10 continued

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $\left[0, \sqrt{3}\right]\left(= [0, 1.73]\right)$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1, 2]$ ***A1***

[5 marks]

continued…

Question 10 continued

(c) (i) attempt to substitute $x=\sqrt{y^2+1}$ into the correct volume formula ***(M1)***

$V=\pi\int_0^h\left(\sqrt{y^2+1}\right)^2 \mathrm{d}y \left(=\pi\int_0^h\left(y^2+1\right)\mathrm{d}y\right)$ ***A1***

$=\pi\left[\frac{1}{3}y^3+y\right]_0^h$ ***A1***

$=\pi\left(\frac{1}{3}h^3+h\right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g. $\pi\int_0^h\left(\sqrt{x^2+1}\right)^2 \mathrm{d}x$

(ii) attempt to substitute $h=\sqrt{3}\ (=1.732...)$ into V ***(M1)***

$V=10.8828...$

$V=10.9$ (m^3) $\left(=2\sqrt{3}\pi\right)$ (m^3) ***A1***

[5 marks]

continued…

Question 10 continued

(d) **METHOD 1**

time $= \dfrac{10.8828...}{0.4}\left(= \dfrac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$= 27.207...$

$= 27.2\left(= 5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

continued…

Question 10 continued

(e) attempt to find the height of the tank when $V = 5.4414... \left(= \sqrt{3}\pi\right)$ ***(M1)***

$$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414... \left(= \sqrt{3}\pi\right)$$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi(h^2+1)} \times \frac{\mathrm{d}V}{\mathrm{d}t} \text{ OR } \frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.4$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.4}{\pi\left(1.1818...^2 + 1\right)} = 0.053124...$$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

Total [20 marks]

11. (a) $\mathrm{P}(C < 61)$ ***(M1)***

$= 0.365112...$

$= 0.365$ ***A1***

[2 marks]

(b) recognition of binomial eg $X \sim \mathrm{B}(12, 0.365...)$ ***(M1)***

$\mathrm{P}(X = 5) = 0.213666...$

$= 0.214$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) (i) Let CM represent 'chocolate muffin' and BM represent 'banana muffin'

$\mathrm{P}(B < 61) = 0.0197555...$ ***(A1)***

EITHER

$\mathrm{P}(CM) \times \mathrm{P}(C < 61 | CM) + \mathrm{P}(BM) \times \mathrm{P}(B < 61 | BM)$ (or equivalent in words) ***(M1)***

OR

tree diagram showing two ways to have a muffin weigh < 61 ***(M1)***

THEN

$(0.6 \times 0.365...) + (0.4 \times 0.0197...)$ ***(A1)***

$= 0.226969...$

$= 0.227$ ***A1***

(ii) recognizing conditional probability ***(M1)***

Note:	Recognition must be shown in context either in words or symbols, not just $\mathrm{P}(A \| B)$.

$\dfrac{0.6 \times 0.365112...}{0.226969...}$ ***(A1)***

$= 0.965183...$

$= 0.965$ ***A1***

[7 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$\mathrm{P}(CM)\times\mathrm{P}(C<61\,|\,CM)+\mathrm{P}(BM)\times\mathrm{P}(B<61\,|\,BM)=0.157$ ***(M1)***

$\left(0.6\times\mathrm{P}(C<61)\right)+(0.4\times0.0197555...)=0.157$

$\mathrm{P}(C<61)=0.248496...$ ***(A1)***

attempt to solve for σ using GDC ***(M1)***

Note:	Award ***(M1)*** for a graph or table of values to show their $\mathrm{P}\left(C<61\right)$ with a variable standard deviation.

$\sigma=1.47225...$

$\sigma=1.47$ (g) ***A2***

continued…

Question 11 continued

METHOD 2

$\mathrm{P}(CM) \times \mathrm{P}(C < 61 \mid CM) + \mathrm{P}(BM) \times \mathrm{P}(B < 61 \mid BM) = 0.157$ ***(M1)***

$\left(0.6 \times \mathrm{P}(C < 61)\right) + (0.4 \times 0.0197555...) = 0.157$

$\mathrm{P}(C < 61) = 0.248496...$ ***(A1)***

use of inverse normal to find z score of their $\mathrm{P}(C < 61)$ ***(M1)***

$z = -0.679229...$

correct substitution ***(A1)***

$\frac{61-62}{\sigma} = -0.679229...$

$\sigma = 1.47225...$

$\sigma = 1.47$ (g) ***A1***

[5 marks]

Total [16 marks]

12. (a) attempt to use Euler's method ***(M1)***

$x_{n+1} = x_n + 0.1;\ \ y_{n+1} = y_n + 0.1 \times \frac{\mathrm{d}y}{\mathrm{d}x}$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{x^2}$

correct intermediate y-values ***(A1)(A1)***

$3.7, 4.63140..., 5.92098..., 7.79542...$

Note: ***A1*** for any two correct y-values seen

$y = 10.6958...$

$y = 10.7$ ***A1***

Note: For the final ***A1,*** the value 10.7 must be the last value in a table or a list, or be given as a final answer, not just embedded in a table which has further lines.

[4 marks]

(b) $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

replacing y with vx and $\frac{\mathrm{d}y}{\mathrm{d}x}$ with $v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x^2 \Rightarrow x^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = v^2x^2 - 2x^2$ ***A1***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - 2$ (since $x > 0$)

$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - v - 2$ ***AG***

[3 marks]

continued…

Question 12 continued

(c) (i) attempt to separate variables v and x ***(M1)***

$$\int \frac{\mathrm{d}v}{v^2 - v - 2} = \int \frac{\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}v}{(v-2)(v+1)} = \int \frac{\mathrm{d}x}{x}$$ ***(A1)***

attempt to express in partial fraction form ***M1***

$$\frac{1}{(v-2)(v+1)} \equiv \frac{A}{v-2} + \frac{B}{v+1}$$

$$\frac{1}{(v-2)(v+1)} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)$$ ***A1***

$$\frac{1}{3}\int\left(\frac{1}{v-2} - \frac{1}{v+1}\right)\mathrm{d}v = \int \frac{\mathrm{d}x}{x}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| (+c)$$ ***A1***

Note: Condone absence of modulus signs throughout.

EITHER

attempt to find c using $x = 1, y = 3, v = 3$ ***M1***

$$c = \frac{1}{3}\ln\frac{1}{4}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| + \frac{1}{3}\ln\frac{1}{4}$$

expressing both sides as a single logarithm ***(M1)***

$$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(\frac{|x|^3}{4}\right)$$

continued…

Question 12 continued

OR

expressing both sides as a single logarithm **(M1)**

$$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(A|x|^3\right)$$

attempt to find A using $x=1, y=3, v=3$ **M1**

$$A = \frac{1}{4}$$

THEN

$$\left|\frac{v-2}{v+1}\right| = \frac{1}{4}x^3 \text{ (since } x>0\text{)}$$

substitute $v = \frac{y}{x}$ (seen anywhere) **M1**

$$\frac{\frac{y}{x}-2}{\frac{y}{x}+1} = \frac{1}{4}x^3 \text{ (since } y>2x\text{)}$$

$$\left(\Rightarrow \frac{y-2x}{y+x} = \frac{1}{4}x^3\right)$$

attempt to make y the subject **M1**

$$y - \frac{x^3y}{4} = 2x + \frac{x^4}{4}$$ **A1**

$$y = \frac{8x+x^4}{4-x^3}$$ **AG**

continued…

Question 12 continued

(ii) actual value at $y(1.5) = 27.3$ ***A1***

(iii) gradient changes rapidly (during the interval considered) OR

the curve has a vertical asymptote at $x = \sqrt[3]{4}\left(= 1.5874...\right)$ ***R1***

[12 marks]

Total [19 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2021

Mathematics: analysis and approaches

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) use of GDC to give ***(M1)***

$r = 0.883529...$

$r = 0.884$ ***A1***

Note: Award the ***(M1)*** for any correct value of r, a, b or $r^2 = 0.780624...$ seen in part (a) or part (b).

[2 marks]

(b) $a = 1.36609...$, $b = 64.5171...$

$a = 1.37$, $b = 64.5$ ***A1***

[1 mark]

(c) attempt to find their difference ***(M1)***

$5 \times 1.36609...$ OR $1.36609...(h+5) + 64.5171... - (1.36609...h + 64.5171...)$

$6.83045...$

$= 6.83\,(6.85 \text{ from } 1.37)$

the student could have expected her score to increase by 7 marks. ***A1***

Note: Accept an increase of 6, 6.83 or 6.85.

[2 marks]

(d) Lucy is incorrect in suggesting there is a causal relationship.
This might be true, but the data can only indicate a correlation. ***R1***

Note: Accept 'Lucy is incorrect as correlation does not imply causation' or equivalent.

[1 mark]

(e) no effect ***A1***

[1 mark]

Total [7 marks]

2. **EITHER**

attempt to use cosine rule **(M1)**

$12^2 + \mathrm{AB}^2 - 2 \times 12 \times \cos 25^\circ \times \mathrm{AB} = 7^2$ OR $\mathrm{AB}^2 - 21.7513...\mathrm{AB} + 95 = 0$ **(A1)**

at least one correct value for AB **(A1)**

$\mathrm{AB} = 6.05068...$ OR $\mathrm{AB} = 15.7007...$

using their smaller value for AB to find minimum perimeter **(M1)**

$12 + 7 + 6.05068...$

OR

attempt to use sine rule **(M1)**

$\frac{\sin \mathrm{B}}{12} = \frac{\sin 25^\circ}{7}$ OR $\sin \mathrm{B} = 0.724488...$ OR $\hat{\mathrm{B}} = 133.573...^\circ$ OR $\hat{\mathrm{B}} = 46.4263...^\circ$ **(A1)**

at least one correct value for $\hat{C}$

$\hat{C} = 21.4263...^\circ$ OR $\hat{C} = 108.573...^\circ$ **(A1)**

using their acute value for $\hat{C}$ to find minimum perimeter **(M1)**

$12 + 7 + \sqrt{12^2 + 7^2 - 2 \times 12 \times 7 \cos 21.4263...^\circ}$ OR $12 + 7 + \frac{7 \sin 21.4263...^\circ}{\sin 25^\circ}$

THEN

$25.0506\ldots$

minimum perimeter $= 25.1$. **A1**

Total [5 marks]

3. (a) recognize that the variable has a Binomial distribution ***(M1)***

$X \sim \mathrm{B}(30, 0.05)$

attempt to find $\mathrm{P}(X \geq 1)$ ***(M1)***

$1 - \mathrm{P}(X = 0)$ OR $1 - 0.95^{30}$ OR $1 - 0.214638\ldots$ OR $0.785361\ldots$

Note: The two ***M*** marks are independent of each other.

$\mathrm{P}(X \geq 1) = 0.785$ ***A1***

[3 marks]

(b) recognition of conditional probability ***(M1)***

$\mathrm{P}(X \leq 2 \mid X \geq 1)$ OR $\mathrm{P}(\text{at most 2 defective} \mid \text{at least 1 defective})$

Note: Recognition must be shown in context either in words or symbols but not just $\mathrm{P}(A \mid B)$.

$\dfrac{\mathrm{P}(1 \leq X \leq 2)}{\mathrm{P}(X \geq 1)}$ OR $\dfrac{\mathrm{P}(X = 1) + \mathrm{P}(X = 2)}{\mathrm{P}(X \geq 1)}$ ***(A1)***

$\dfrac{0.597540\ldots}{0.785361\ldots}$ OR $\dfrac{0.812178\ldots - 0.214638\ldots}{0.785361\ldots}$ OR $\dfrac{0.338903\ldots + 0.258636\ldots}{0.785361\ldots}$ ***(A1)***

$= 0.760847\ldots$

$\mathrm{P}(X \leq 2 \mid X \geq 1) = 0.761$ ***A1***

[4 marks]

Total [7 marks]

4. (a) attempt to find the area of either shaded region in terms of r and θ ***(M1)***

Note*:* Do not award ***M1*** if they have only copied from the booklet and not applied to the shaded area.

Area of segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$ ***A1***

Area of triangle $= \frac{1}{2}r^2\sin(\pi-\theta)$ ***A1***

correct equation in terms of θ only ***(A1)***

$\theta - \sin\theta = \sin(\pi-\theta)$

$\theta - \sin\theta = \sin\theta$ ***A1***

$\theta = 2\sin\theta$ ***AG***

Note*:* Award a maximum of ***M1A1A0A0A0*** if a candidate uses degrees (i.e., $\frac{1}{2}r^2\sin(180°-\theta)$), even if later work is correct.

Note*:* If a candidate directly states that the area of the triangle is $\frac{1}{2}r^2\sin\theta$, award a maximum of ***M1A1A0A1A1***.

[5 marks]

(b) $\theta = 1.89549...$

$\theta = 1.90$ ***A1***

Note: Award ***A0*** if there is more than one solution. Award ***A0*** for an answer in degrees.

[1 mark]

Total [6 marks]

5. (a) $u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ ***(M1)***

$= \frac{14}{24} \left(= \frac{7}{12} = 0.583333\ldots \right)$ ***A1***

[2 marks]

(b) $r = \frac{7}{8} (= 0.875)$ ***(A1)***

substituting their values for u_1 and r into $S_\infty = \frac{u_1}{1-r}$ ***(M1)***

$= \frac{14}{3} (= 4.66666\ldots)$ ***A1***

[3 marks]

(c) attempt to substitute their values into the inequality or formula for S_n ***(M1)***

$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3} \left(\frac{7}{8} \right)^r < 0.001$ OR $S_n = \frac{\frac{7}{12}\left(1 - \left(\frac{7}{8}\right)^n\right)}{\left(1 - \frac{7}{8}\right)}$

attempt to solve their inequality using a table, graph or logarithms
(must be exponential) ***(M1)***

Note: Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$.

correct critical value or at least one correct crossover value ***(A1)***

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036...$ OR $S_\infty - S_{64} = 0.000906...$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683...$ OR $S_\infty - S_{64} - 0.001 = -0.0000931777...$

least value is $n = 64$ ***A1***

[4 marks]

Total [9 marks]

6. (a) **METHOD 1**

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to expand $(p+q)^3$ ***M1***

$p^3 + 3p^2q + 3pq^2 + q^3$

$(p+q)^3 - 3pq(p+q) \equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3pq(p+q)$

$\equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3p^2q - 3pq^2$ ***A1***

$\equiv p^3 + q^3$ ***AG***

Note*:* Condone the use of equals signs throughout.

METHOD 2

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to factorise $(p+q)^3 - 3pq(p+q)$ ***M1***

$\equiv (p+q)\left((p+q)^2 - 3pq\right) \ \left(\equiv (p+q)\left(p^2 - pq + q^2\right)\right)$

$\equiv p^3 - p^2q + pq^2 + p^2q - pq^2 + q^3$ ***A1***

$\equiv p^3 + q^3$ ***AG***

Note*:* Condone the use of equals signs throughout.

METHOD 3

$p^3 + q^3 \equiv (p+q)^3 - 3pq(p+q)$

attempts to factorise $p^3 + q^3$ ***M1***

$\equiv (p+q)\left(p^2 - pq + q^2\right)$

$\equiv (p+q)\left((p+q)^2 - 3pq\right)$ ***A1***

$\equiv (p+q)^3 - 3pq(p+q)$ ***AG***

Note*:* Condone the use of the equals sign throughout.

[2 marks]

(b)

> **Note*:*** Award a maximum of ***A1M0A0A1M0A0*** for $m=-95$ and $n=8$ found by using $\alpha,\beta=\dfrac{5\pm\sqrt{17}}{4}$ $(\alpha,\beta=0.219...,2.28...)$.
> Condone, as appropriate, solutions that state but clearly do not use the values of α and β.
> Special case: Award a maximum of ***A1M1A0A1M0A0*** for $m=-95$ and $n=8$ obtained by solving simultaneously for α and β from product of roots and sum of roots equations.

product of roots of $x^2-\dfrac{5}{2}x+\dfrac{1}{2}=0$

$\alpha\beta=\dfrac{1}{2}$ (seen anywhere) ***A1***

considers $\left(\dfrac{1}{\alpha^3}\right)\left(\dfrac{1}{\beta^3}\right)$ by stating $\dfrac{1}{(\alpha\beta)^3}(=n)$ ***M1***

> **Note*:*** Award ***M1*** for attempting to substitute their value of $\alpha\beta$ into $\dfrac{1}{(\alpha\beta)^3}$.

$$\frac{1}{(\alpha\beta)^3}=\frac{1}{\left(\frac{1}{2}\right)^3}$$

$n=8$ ***A1***

sum of roots of $x^2-\dfrac{5}{2}x+\dfrac{1}{2}=0$

$\alpha+\beta=\dfrac{5}{2}$ (seen anywhere) ***A1***

considers $\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3}$ by stating $\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$ $\left(\left(\dfrac{\alpha+\beta}{\alpha\beta}\right)^3-\dfrac{3(\alpha+\beta)}{(\alpha\beta)^2}\right)(=-m)$ ***M1***

> **Note*:*** Award ***M1*** for attempting to substitute their values of $\alpha+\beta$ and $\alpha\beta$ into their expression. Award ***M0*** for use of $(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$ only.

$$=\frac{\left(\frac{5}{2}\right)^3-\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\frac{1}{8}}\ (=125-30=95)$$

$m=-95$ ***A1***

$(x^2-95x+8=0)$

[6 marks]
Total [8 marks]

7. (a) recognises that $\int_0^m \arccos x \, dx = 0.5$ ***(M1)***

$m \arccos m - \sqrt{1-m^2} - \left(0 - \sqrt{1}\right) = 0.5$

$m = 0.360034...$

$m = 0.360$ ***A1***

[2 marks]

(b) **METHOD 1**

attempts to find at least one endpoint (limit) both in terms of m (or their m) and a ***(M1)***

$\mathrm{P}\left(m - a \le X \le m + a\right) = 0.3$

$\int_{0.360034...-a}^{0.360034...+a} \arccos x \, dx = 0.3$ ***(A1)***

Note: Award ***(A1)*** for $\int_{m-a}^{m+a} \arccos x \, dx = 0.3$.

$\left[x \arccos x - \sqrt{1-x^2}\right]_{0.360034...-a}^{0.360034...+a}$

attempts to solve their equation for a ***(M1)***

Note: The above ***(M1)*** is dependent on the first ***(M1)***.

$a = 0.124861...$

$a = 0.125$ ***A1***

METHOD 2

$\int_{-a}^{a} \arccos\left|x - 0.360034...\right| dx \ \left(= 0.3\right)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{-a}^{a} \arccos\left|x - m\right| dx \ \left(= 0.3\right)$.

attempts to solve their equation for a ***(M1)***

$a = 0.124861...$

$a = 0.125$ ***A1***

METHOD 3

EITHER

$$\int_{-a}^{a} \arccos\left(x + 0.360034...\right) \, dx \ \left(= 0.3\right)$$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{-a}^{a} \arccos\left(x + m\right) \, dx \ \left(= 0.3\right)$.

OR

$$\int_{2(0.360034...)-a}^{2(0.360034...)+a} \arccos\left(x - 0.360034...\right) \, dx \ \left(= 0.3\right)$$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{2m-a}^{2m+a} \arccos\left(x - m\right) \, dx \ \left(= 0.3\right)$.

THEN

attempts to solve their equation for a ***(M1)***

Note*:* The above ***(M1)*** is dependent on the first ***(M1)***.

$a = 0.124861...$

$a = 0.125$ ***A1***

[4 marks]

Total [6 marks]

8. (a) **METHOD 1**

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$u = xy$, $v = \ln(xy)$, $\frac{\mathrm{d}u}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\frac{1}{xy}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left[\frac{xy}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right]$ ***A1***

Note: Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left[\frac{xy}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + x\frac{\mathrm{d}y}{\mathrm{d}x}\ln(xy) + y\ln(xy)\right].$$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left[\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right]$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy))$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 2

$y = x - xy\ln x - xy\ln y$

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(\frac{xy}{x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln y\right)$ ***A1***

or equivalent to the above, for example

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\ln x\frac{\mathrm{d}y}{\mathrm{d}x} + (1 + \ln x)y\right) - \left(y\ln y + x\left(\ln y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x\frac{\mathrm{d}y}{\mathrm{d}x}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$ ***A1***

or equivalent to the above, for example

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x\frac{\mathrm{d}y}{\mathrm{d}x}(\ln(xy) + 1) - y(\ln(xy) + 1)$

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 3

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$u = x\ln(xy),\ v = y,\ \frac{\mathrm{d}u}{\mathrm{d}x} = \ln(xy) + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\frac{x}{xy},\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x}\ln(xy) + y\ln(xy) + \frac{xy}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x\frac{\mathrm{d}y}{\mathrm{d}x}\left(\ln(xy) + 1\right) - y\left(\ln(xy) + 1\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln(xy)\right) = 1$ ***AG***

METHOD 4

lets $w = xy$ and attempts to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = x - w\ln w$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x}\ln w\right)\ \left(= 1 - \frac{\mathrm{d}w}{\mathrm{d}x}(1 + \ln w)\right)$ ***A1***

$\frac{\mathrm{d}w}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right)\ \left(= 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln(xy)\right)\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln(xy)\right) = 1$ ***AG***

[3 marks]

(b) **METHOD 1**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$ ***A1***

substitutes $x=1$ and their non-zero value of y into $\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\right)\left(1+\ln(xy)\right)=1$ ***(M1)***

$2\frac{dy}{dx}=0 \left(\frac{dy}{dx}=0\right)$ ***A1***

equation of the tangent is $y=1$ ***A1***

METHOD 2

substitutes $x=1$ into $\frac{dy}{dx}+\left(x\frac{dy}{dx}+y\right)\left(1+\ln(xy)\right)=1$ ***(M1)***

$\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)\left(1+\ln(y)\right)=1$

EITHER

correctly substitutes $\ln y=\frac{1-y}{y}$ into $\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)\left(1+\ln(y)\right)=1$ ***A1***

$\frac{dy}{dx}\left(1+\frac{1}{y}\right)=0 \Rightarrow \frac{dy}{dx}=0 \ (y=1)$ ***A1***

OR

correctly substitutes $y+y\ln y=1$ into $\frac{dy}{dx}+\left(\frac{dy}{dx}+y\right)\left(1+\ln(y)\right)=1$ ***A1***

$\frac{dy}{dx}\left(2+\ln y\right)=0 \Rightarrow \frac{dy}{dx}=0 \ (y=1)$ ***A1***

THEN

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$

equation of the tangent is $y=1$ ***A1***

[5 marks]

Total [8 marks]

Section B

9. (a) $12 = \frac{2\pi}{b}$ OR $b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG***

[1 mark]

(b) $a = \frac{6.8 - 2.2}{2}$ OR $a = \frac{\max - \min}{2}$ ***(M1)***

$= 2.3$ (m) ***A1***

[2 marks]

(c) $d = \frac{6.8 + 2.2}{2}$ OR $d = \frac{\max + \min}{2}$ ***(M1)***

$= 4.5$ (m) ***A1***

[2 marks]

continued…

Question 9 continued.

(d) **METHOD 1**

substituting $t=4.5$ and $H=6.8$ for example into their equation for H ***(A1)***

$6.8=2.3\sin\left(\frac{\pi}{6}(4.5-c)\right)+4.5$

attempt to solve their equation ***(M1)***

$c=1.5$ ***A1***

METHOD 2

using horizontal translation of $\frac{12}{4}$ ***(M1)***

$4.5-c=3$ ***(A1)***

$c=1.5$ ***A1***

METHOD 3

$H'(t)=(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(t-c)\right)$ ***(A1)***

attempts to solve their $H'(4.5)=0$ for c ***(M1)***

$(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(4.5-c)\right)=0$

$c=1.5$ ***A1***

[3 marks]

(e) attempt to find H when $t=12$ or $t=0$, graphically or algebraically ***(M1)***

$H=2.87365...$

$H=2.87(\text{m})$ ***A1***

[2 marks]

continue…

Question 9 continued.

(f) attempt to solve $5 = 2.3\sin\left(\frac{\pi}{6}(t-1.5)\right)+4.5$ ***(M1)***

times are $t = 1.91852...$ and $t = 7.08147...$, $(t = 13.9185..., t = 19.0814...)$ ***(A1)***

total time is $2\times(7.081...-1.919...)$

$10.3258...$

$= 10.3$ (hours) ***A1***

Note: Accept 10.

[3 marks]

(g) **METHOD 1**

substitutes $t = \frac{11}{3}$ and $H = 6.8$ into their equation for H and attempts to solve for c ***(M1)***

$6.8 = 2.3\sin\left(\frac{\pi}{6}\left(\frac{11}{3}-c\right)\right)+4.5 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

METHOD 2

uses their horizontal translation $\left(\frac{12}{4}=3\right)$ ***(M1)***

$\frac{11}{3}-c = 3 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

[2 marks]
Total [15 marks]

10. (a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3,0)$ and $(4,0)$ ***A1***

(ii) $\left(0,\frac{4}{5}\right)$ ***A1***

[3 marks]

(b) $x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax+b)(2x-15) \equiv x^2 - x - 12$

attempts to expand $(ax+b)(2x-15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+\ldots$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of x: ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$$

$a=\frac{1}{2}$ ***A1***

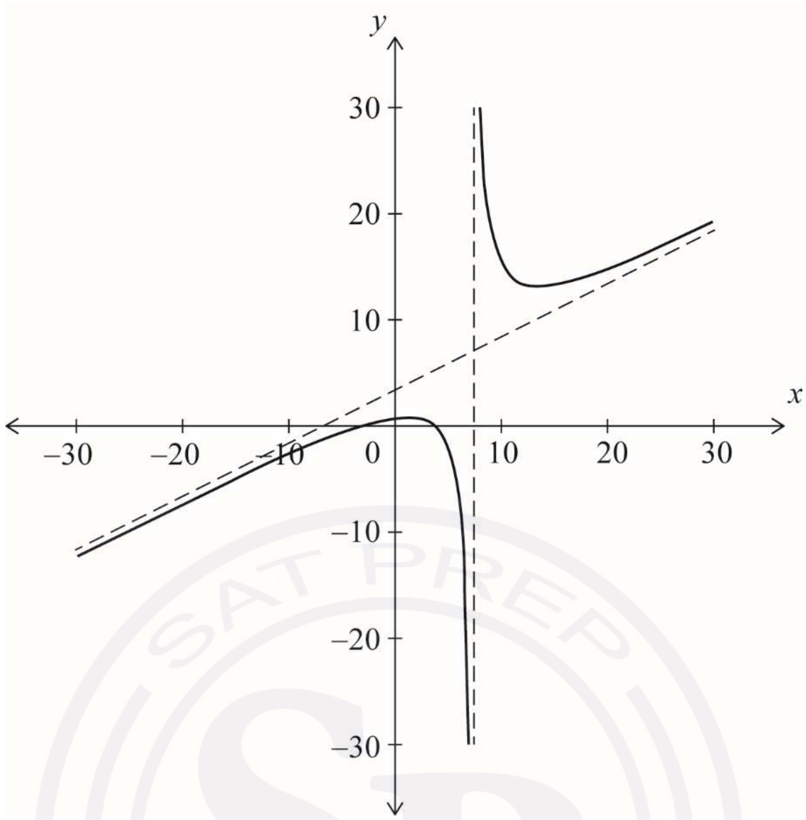
$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+\ldots$ ***M1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \le x \le 30$) ***A1***
their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***
their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4) + B(x+3)$$

$A = 3$ ***A1***

$B = -1$ ***A1***

$$\left(\frac{3}{x+3} - \frac{1}{x-4}\right)$$

(ii) $$\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4}\right) \mathrm{d}x$$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$= \left[3\ln|x+3| - \ln|x-4|\right]_0^3$ ***A1***

$= 3\ln 6 - \ln 1 - 3\ln 3 + \ln 4$ ***A1***

$= 3\ln 2 + \ln 4 \ (= \ln 8 + \ln 4)$

$= \ln 32 \ (= 5\ln 2)$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]
Total [18 marks]

11. (a) (i) attempts to find either $\vec{AB}$ or $\vec{AC}$ ***(M1)***

$\vec{AB} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

(ii) **METHOD 1**

attempts to find $\vec{AB} \times \vec{AC}$ ***(M1)***

$\vec{AB} \times \vec{AC} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ ***A1***

EITHER

equation of plane is of the form $14x - 21y - 7z = d$ $(2x - 3y - z = d)$ ***(A1)***

substitutes a valid point e.g $(3,0,0)$ to obtain a value of d ***M1***

$d = 42$ $(d = 6)$

OR

attempts to use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ ***(M1)***

$\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} \left(\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = 42 \right)$ ***A1***

$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \left(\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6 \right)$

THEN

$14x - 21y - 7z = 42$ $(2x - 3y - z = 6)$ ***A1***

METHOD 2

equation of plane is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

attempts to form equations for x, y, z in terms of their parameters ***(M1)***

$x = 3 - 3s - 2t$, $y = -2s + t$, $z = -7t$

A1

eliminates at least one of their parameters ***(M1)***

for example, $2x - 3y = 6 - 7t (\Rightarrow 2x - 3y = 6 + z)$

$2x - 3y - z = 6$ ***A1***

[7 marks]

(b) **METHOD 1**

substitutes $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ into their Π_1 and Π_2 (given) ***(M1)***

$\Pi_1:\ 2\lambda-3(-2+\lambda)-(-\lambda)=6$ and $\Pi_2:\ 3\lambda-(-2+\lambda)+2(-\lambda)=2$ ***A1***

Note: Award ***(M1)A0*** for correct verification using a specific value of λ.

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

METHOD 2
EITHER

attempts to find $\begin{pmatrix}2\\-3\\-1\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}$ ***M1***

$=\begin{pmatrix}-7\\-7\\7\end{pmatrix}$

OR

$\begin{pmatrix}2\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=(2-3+1)=0$ and $\begin{pmatrix}3\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=(3-1-2)=0$ ***M1***

THEN

substitutes $(0,-2,0)$ into Π_1 and Π_2

$\Pi_1:\ 2(0)-3(-2)-(0)=6$ and $\Pi_2:\ 3(0)-(-2)+2(0)=2$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

METHOD 3

attempts to solve $2x-3y-z=6$ and $3x-y+2z=2$ ***(M1)***

for example, $x=-\lambda, y=-2-\lambda, z=\lambda$ ***A1***

Note: Award ***A1*** for substituting $x=0$ (or $y=-2$ or $z=0$) into Π_1 and Π_2 and solving simultaneously. For example, solving $-3y-z=6$ and $-y+2z=2$ to obtain $y=-2$ and $z=0$.

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

[2 marks]

(c) (i) substitutes the equation of L into the equation of Π_3 ***(M1)***

$2\lambda+2\lambda=3\Rightarrow 4\lambda=3$ ***A1***

$\lambda=\frac{3}{4}$ ***AG***

(ii) P has coordinates $\left(\frac{3}{4},-\frac{5}{4},-\frac{3}{4}\right)$ ***A1***

[3 marks]

(d) (i) normal to Π_3 is $\boldsymbol{n}=\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***(A1)***

Note: May be seen or used anywhere.

considers the line normal to Π_3 passing through $\mathrm{B}(0,-2,0)$ ***(M1)***

$\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***A1***

EITHER

finding the point on the normal line that intersects Π_3

attempts to solve simultaneously with plane $2x-2z=3$ ***(M1)***

$4\mu+4\mu=3$

$\mu=\frac{3}{8}$ ***A1***

point is $\left(\frac{3}{4},-2,-\frac{3}{4}\right)$

OR

$$\left(\begin{pmatrix} 2\mu \\ -2 \\ -2\mu \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 0$$ ***(M1)***

$$4\mu - \frac{3}{2} + 4\mu - \frac{3}{2} = 0$$

$$\mu = \frac{3}{8}$$ ***A1***

OR

attempts to find the equation of the plane parallel to Π_3 containing B' $(x - z = 3)$ and solve simultaneously with L ***(M1)***

$$2\mu' + 2\mu' = 3$$

$$\mu' = \frac{3}{4}$$ ***A1***

THEN

so, another point on the reflected line is given by

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$ ***(A1)***

$$\Rightarrow \mathrm{B}'\left(\frac{3}{2}, -2, -\frac{3}{2}\right)$$ ***A1***

(ii) **EITHER**

attempts to find the direction vector of the reflected line using their P and B' ***(M1)***

$$\overrightarrow{\mathrm{PB}'} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

OR

attempts to find their direction vector of the reflected line using a vector approach ***(M1)***

$$\overrightarrow{\mathrm{PB}'} = \overrightarrow{\mathrm{PB}} + \overrightarrow{\mathrm{BB}'} = -\frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

THEN

$$\boldsymbol{r} = \begin{pmatrix} \frac{3}{2} \\ -2 \\ -\frac{3}{2} \end{pmatrix} + \lambda\begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$ (or equivalent) ***A1***

Note: Award ***A0*** for either '$\boldsymbol{r} =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'.

[9 marks]
Total [21 marks]

M21/5/MATHX/HP2/ENG/TZ2/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 2

28 pages

International Baccalaureate®
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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) (i) $a = 0.805084...$ and $b = 2.88135...$
$a = 0.805$ and $b = 2.88$ ***A1A1***

(ii) $r = 0.97777...$
$r = 0.978$ ***A1***

[3 marks]

(b) a represents the (average) increase in waiting time (0.805 mins) per additional customer (waiting to receive their coffee) ***R1***

[1 mark]

(c) attempt to substitute $x = 7$ into their equation ***(M1)***

$8.51693...$
8.52 (mins) ***A1***

[2 marks]
Total [6 marks]

2. (a) attempt to use $u_1 + (n-1)d = 0$ *(M1)*

$60 - 2.5(k-1) = 0$

$k = 25$ ***A1***

[2 marks]

(b) **METHOD 1**

attempting to express S_n in terms of n *(M1)*

use of a graph or a table to attempt to find the maximum sum *(M1)*

$= 750$ ***A1***

METHOD 2

EITHER

recognizing maximum occurs at $n = 25$ *(M1)*

$S_{25} = \frac{25}{2}(60+0)$, $S_{25} = \frac{25}{2}(2\times 60 + 24 \times -2.5)$ *(A1)*

OR

attempting to calculate S_{24} *(M1)*

$S_{24} = \frac{24}{2}(2\times 60 + 23 \times -2.5)$ *(A1)*

THEN

$= 750$ ***A1***

[3 marks]

Total [5 marks]

3. (a) **EITHER**

$\mathrm{P}(S)+\mathrm{P}(T)+\mathrm{P}(S'\cap T')-\mathrm{P}(S\cap T)=1$ OR $\mathrm{P}(S\cup T)=\mathrm{P}((S'\cap T')')$ ***(M1)***

$0.7+0.2+0.18-\mathrm{P}(S\cap T)=1$ ***OR*** $\mathrm{P}(S\cup T)=1-0.18$

OR

a clearly labelled Venn diagram ***(M1)***

THEN

$\mathrm{P}(S\cap T)=0.08$ (accept 8%) ***A1***

Note: To obtain the ***M1*** for the Venn diagram all labels must be correct and in the correct sections. For example, do not accept 0.7 in the area corresponding to $S\cap T'$.

[2 marks]

(b) **EITHER**

$\mathrm{P}(T\cap S')=\mathrm{P}(T)-\mathrm{P}(T\cap S)(=0.2-0.08)$ OR

$\mathrm{P}(T\cap S')=\mathrm{P}(T\cup S)-\mathrm{P}(S)(=0.82-0.7)$ ***(M1)***

OR

a clearly labelled Venn diagram including $\mathrm{P}(S)$, $\mathrm{P}(T)$ and $\mathrm{P}(S\cap T)$ ***(M1)***

THEN

$=0.12$ (accept 12%) ***A1***

[2 marks]

(c) $\mathrm{P}(G\cap T)=\mathrm{P}(T|G)\mathrm{P}(G)$ (0.25×0.48) ***(M1)***

$=0.12$ ***A1***

[2 marks]

continued…

Question 3 continued

(d) **METHOD 1**

$\mathrm{P}(G)\times\mathrm{P}(T)(=0.48\times 0.2)=0.096$ ***A1***

$\mathrm{P}(G)\times\mathrm{P}(T)\neq\mathrm{P}(G\cap T)\Rightarrow$ G and T are not independent ***R1***

METHOD 2

$\mathrm{P}(T\mid G)=0.25$ ***A1***

$\mathrm{P}(T\mid G)\neq\mathrm{P}(T)\Rightarrow$ G and T are not independent ***R1***

Note: Do not award ***A0R1.***

[2 marks]
Total [8 marks]

4. (a) attempting to find the vertex *(M1)*

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ *A1*

[2 marks]

(b) **METHOD 1**

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \quad \left(= -\left(6(x-1)^2 - 5\right) + c\right)$ *(A1)*

EITHER

relating to the range of f OR attempting to find $g(-5)$ *(M1)*

$5 + c \leq 0$ *(A1)*

OR

attempting to find the discriminant of $(g \circ f)(x)$ *(M1)*

$144 + 24(c-1) \leq 0 \quad (120 + 24c \leq 0)$ *(A1)*

THEN

$c \leq -5$ *A1*

[4 marks]

METHOD 2

vertical reflection followed by vertical shift *(M1)*

new vertex is $(1, 5+c)$ *(A1)*

$5 + c \leq 0$ *(A1)*

$c \leq -5$ *A1*

[4 marks]

Total [6 marks]

5. (a) $100 = A_0 e^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

(b) correct substitution of values into exponential equation ***(M1)***

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln\frac{1}{2}$ ***A1***

$\ln\frac{1}{2} = -\ln 2$ OR $-\ln\frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if 25% of the carbon-14 has decayed, 75% remains ie, 75 units remain

$75 = 100e^{-\frac{\ln 2}{5730}t}$ ***(A1)***

EITHER

using an appropriate graph to attempt to solve for t ***(M1)***

OR

manipulating logs to attempt to solve for t ***(M1)***

$\ln 0.75 = -\frac{\ln 2}{5730}t$

$t = 2378.164...$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]

Total [7 marks]

6. (a) $\mathrm{E}(X)=(n+1)\int_0^1 x^{n+1}\mathrm{d}x$ ***M1***

$=(n+1)\left[\frac{x^{n+2}}{n+2}\right]_0^1$ ***A1***

leading to $\mathrm{E}(X)=\frac{n+1}{n+2}$ ***AG***

[2 marks]

(b) **METHOD 1**

use of $\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ ***M1***

$$\mathrm{Var}(X)=(n+1)\int_0^1 x^{n+2}\mathrm{d}x-\left(\frac{n+1}{n+2}\right)^2$$

$$=(n+1)\left[\frac{1}{n+3}x^{n+3}\right]_0^1-\left(\frac{n+1}{n+2}\right)^2$$

$$=\frac{n+1}{n+3}-\left(\frac{n+1}{n+2}\right)^2$$ ***A1***

$$=\frac{(n+1)(n+2)^2-(n+1)^2(n+3)}{(n+2)^2(n+3)}$$ ***M1***

EITHER

$$=\frac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$$ ***A1***

OR

$$=\frac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$$ ***A1***

THEN

so $\mathrm{Var}(X)=\frac{n+1}{(n+2)^2(n+3)}$ ***AG***

continued…

Question 6 continued

METHOD 2

use of $\operatorname{Var}(X)=\operatorname{E}(X-\operatorname{E}(X))^2$ ***M1***

$$\operatorname{Var}(X)=(n+1)\int_0^1\left(x-\frac{n+1}{n+2}\right)^2 x^n\,\mathrm{d}x$$

$$=(n+1)\left[\frac{1}{n+3}x^{n+3}-\frac{2(n+1)}{(n+2)^2}x^{n+2}+\frac{n+1}{(n+2)^2}x^{n+1}\right]_0^1$$

$$=\frac{n+1}{n+3}-\left(\frac{n+1}{n+2}\right)^2$$ ***A1***

$$=\frac{(n+1)\left((n+2)^2-(n+1)(n+3)\right)}{(n+2)^2(n+3)}$$ ***M1***

EITHER

$$=\frac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$$ ***A1***

OR

$$=\frac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$$ ***A1***

THEN

so $\operatorname{Var}(X)=\dfrac{n+1}{(n+2)^2(n+3)}$ ***AG***

[4 marks]
Total [6 marks]

7. (a) Jack and Andrea finish in that order (as a unit) so we are considering the arrangement of 7 objects ***(M1)***

$7!\ (=5040)$ ways ***A1***

[2 marks]

(b) **METHOD 1**

the number of ways that Andrea finishes in front of Jack is equal to the number of ways that Jack finishes in front of Andrea ***(M1)***

total number of ways is $8!$ ***(A1)***

$\frac{8!}{2}\ (=20160)$ ways ***A1***

[3 marks]

METHOD 2

the other six runners can finish in $6!\ (=720)$ ways ***(A1)***

when Andrea finishes first, Jack can finish in 7 different positions

when Andrea finishes second, Jack can finish in 6 different positions etc

$7+6+5+4+3+2+1\ (=28)$ ways ***(A1)***

hence there are $(7+6+5+4+3+2+1)\times 6!$ ways

$28\times 6!\ (=20160)$ ways ***A1***

[3 marks]
Total [5 marks]

8. $\frac{1+z}{1-z}=\frac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta}$

attempt to use the complex conjugate of their denominator ***M1***

$$=\frac{(1+\cos\theta+\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}{(1-\cos\theta-\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}$$ ***A1***

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right)=\frac{1-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta}\quad\left(=\frac{1-\cos^2\theta-\sin^2\theta}{2-2\cos\theta}\right)$$ ***M1A1***

Note:	Award ***M1*** for expanding the numerator and ***A1*** for a correct numerator. Condone either an incorrect denominator or the absence of a denominator.

using $\cos^2\theta+\sin^2\theta=1$ to simplify the numerator ***(M1)***

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right)=0$$ ***AG***

[5 marks]

9. (a) $1-t+t^2$ ***A1***

Note: Accept $1, -t$ and t^2.

[1 mark]

(b) $\sec x = \dfrac{1}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-\ldots)}$ $\left(=\left(1-\dfrac{x^2}{2!}+\left(\dfrac{x^4}{4!}(-\ldots)\right)\right)^{-1}\right)$ ***(M1)***

$t=\cos x-1$ or $\sec x = 1-(\cos x-1)+(\cos x-1)^2$ ***(M1)***

$=1-\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-\ldots)\right)+\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-\ldots)\right)^2$ ***A1***

$=1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}$ ***A1***

so the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}$ ***AG***

[4 marks]

Note: Condone the absence of '…'

(c) $\arctan 2x = 2x - \dfrac{(2x)^3}{3}+\ldots$

$\lim\limits_{x\to 0}\left(\dfrac{x\arctan 2x}{\sec x-1}\right)=\lim\limits_{x\to 0}\left(\dfrac{x\left(2x-\dfrac{(2x)^3}{3}+\ldots\right)}{\left(1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}\right)-1}\right)$ ***M1***

$=\lim\limits_{x\to 0}\left(\dfrac{2x^2-\dfrac{8x^4}{3}+\ldots}{\dfrac{x^2}{2}+\dfrac{5x^4}{24}}\right)$ ***A1***

$=\lim\limits_{x\to 0}\left(\dfrac{2x^2\left(1-\dfrac{4x^2}{3}\right)}{\dfrac{x^2}{2}\left(1+\dfrac{5x^2}{12}\right)}\right)$

$=4$ ***A1***

Note: Condone missing 'lim' and errors in higher derivatives. Do not award ***M1*** unless x is replaced by $2x$ in arctan.

[3 marks]
Total [8 marks]

Section B

10. (a) use of inverse normal to find z-score *(M1)*

$z = 2.0537...$

$2.0537... = \frac{82-75}{\sigma}$ *(A1)*

$\sigma = 3.408401...$

$\sigma = 3.41$ ***A1***

[3 marks]

(b) evidence of identifying the correct area under the normal curve *(M1)*

$\mathrm{P}(T > 80) = 0.071193...$

$\mathrm{P}(T > 80) = 0.0712$ ***A1***

[2 marks]

(c) recognition that $\mathrm{P}(80 < T < 82)$ is required *(M1)*

$P(T < 82 | T > 80) = \frac{\mathrm{P}(80 < T < 82)}{\mathrm{P}(T > 80)} = \left(\frac{0.051193...}{0.071193...}\right)$ *(M1)(A1)*

$= 0.719075...$

$= 0.719$ ***A1***

[4 marks]

continued…

Question 10 continued

(d) recognition of binomial probability ***(M1)***

$X \sim \mathrm{B}(64, 0.071193...)$ or $\mathrm{E}(X) = 64 \times 0.071193...$ ***(A1)***

$\mathrm{E}(X) = 4.556353...$

$\mathrm{E}(X) = 4.56$ (flights) ***A1***

[3 marks]

(e) $\mathrm{P}(X > 6) = P(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 1 - 0.83088...$ ***(A1)***

$= 0.1691196...$

$= 0.169$ ***A1***

[3 marks]

Total [15 marks]

11. (a) attempt to use $V = \pi\int_a^b \left(f(x)\right)^2 \mathrm{d}x$ ***(M1)***

$$V = \pi\int_0^{\ln 16}\left(\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}\right)^2 \mathrm{d}x \quad \left(V = k^2\pi\int_0^{\ln 16}\frac{\mathrm{e}^x}{\left(1+\mathrm{e}^x\right)^2}\mathrm{d}x\right)$$

EITHER

applying integration by recognition ***(M1)***

$$= k^2\pi\left[-\frac{1}{1+\mathrm{e}^x}\right]_0^{\ln 16}$$ ***A3***

OR

$u = 1+\mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 2$ and when $x = \ln 16$, $u = 17$

$$V = k^2\pi\int_2^{17}\frac{1}{u^2}\mathrm{d}u$$ ***(A1)***

$$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$$ ***A1***

OR

$u = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 1$ and when $x = \ln 16$, $u = 16$

$$V = k^2\pi\int_1^{16}\frac{1}{(1+u)^2}\mathrm{d}u$$ ***(A1)***

$$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

$$= k^2\pi\left(\frac{1}{2}-\frac{1}{17}\right)$$ ***A1***

so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

continued…

Question 11 continued

(b) a valid algebraic or graphical attempt to find k ***(M1)***

$$k^2 = \frac{300 \times 34}{15\pi}$$

$$k = 14.7 \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$$ (as $k \in \mathbb{R}^+$) ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

(c) (i) attempting to find $\text{OA} = f(0) = \frac{k}{2}$

with $k = 14.712\ldots \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$$\text{OA} = 7.36 \left(= \sqrt{\frac{170}{\pi}} \right)$$ ***A1***

(ii) attempting to find $\text{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712\ldots \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$$\text{BC} = 3.46 \left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}} \right)$$ ***A1***

[4 marks]

continued…

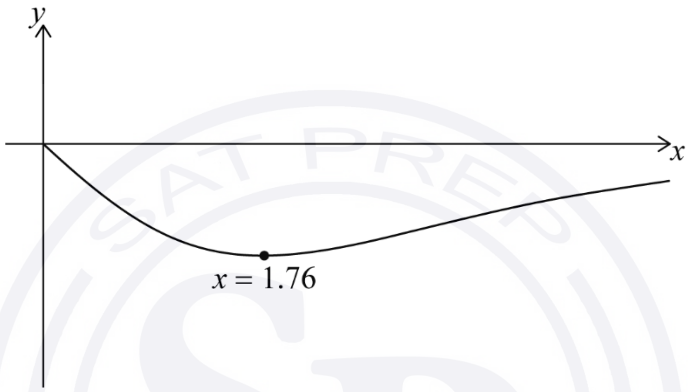
Question 11 continued

(d) (i) **EITHER**

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f'(x) = \frac{ke^{\frac{x}{2}}(1-e^x)}{2(1+e^x)^2}$$



for $x > 0$ graph decreasing to the local minimum ***A1***

before increasing towards the x-axis ***A1***

continued…

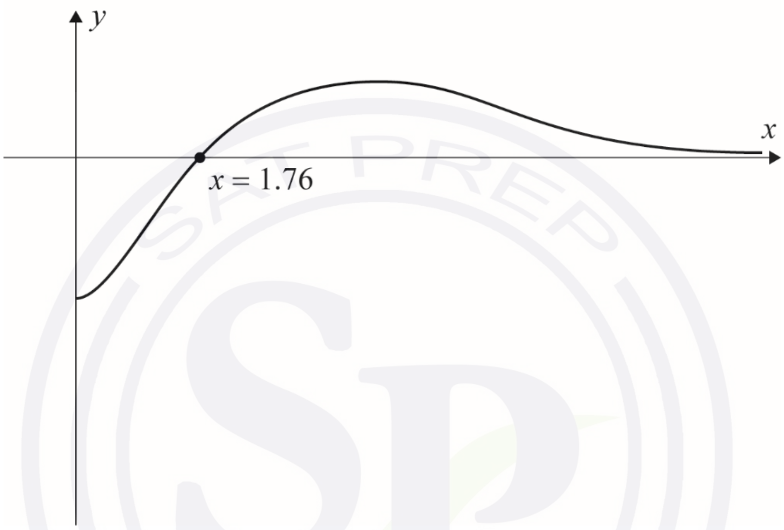
Question 11 continued

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f''(x) = \frac{ke^{\frac{x}{2}}\left(e^{2x} - 6e^{x} + 1\right)}{4\left(1 + e^{x}\right)^{3}}$$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76 \left(= \ln\left(2\sqrt{2} + 3\right)\right)$ ***A1***

Note: Only award ***A*** marks if either graph is seen.

(ii) attempting to find $f(1.76...)$ ***(M1)***

the cross-sectional radius at this point is $5.20 \left(\sqrt{\frac{85}{\pi}}\right)$ (cm) ***A1***

[6 marks]
Total [18 marks]

12. (a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

continued…

Question 12 continued

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\ \left(=\frac{4x}{\left(x^2+1\right)^2}\right)$$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$$ ***M1***

$$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

Question 12 continued

(d) $x = \arcsin\left(\dfrac{y^2-1}{y^2+1}\right)$ **M1**

$\sin x = \dfrac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ **A1**

$y^2 = \dfrac{1+\sin x}{1-\sin x}$ **A1**

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) **R1**

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ **A1**

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

(e) domain is $-\dfrac{\pi}{2} \leq x < \dfrac{\pi}{2}$ **A1**

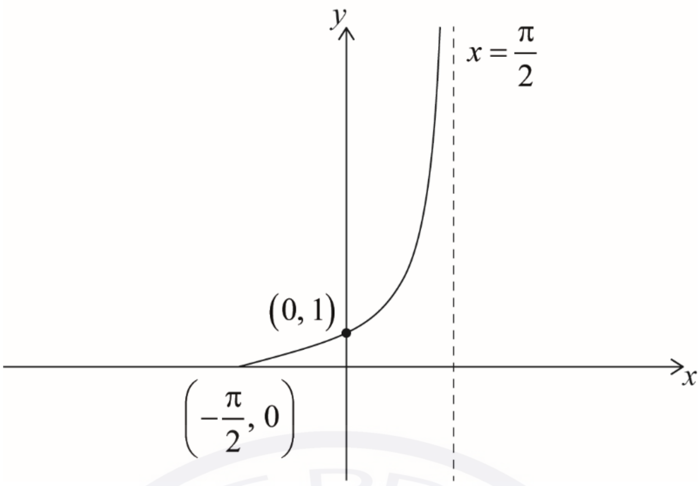
Note: Accept correct alternative notations, for example, $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$ or $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

continued…

Question 12 continued

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$ ***A1*** for asymptotic behaviour $x \to \frac{\pi}{2}$ ***A1*** for $x = \frac{\pi}{2}$ Coordinates are not required. Do not accept $x = 1.57$ or other inexact values.

[3 marks]
Total [21 marks]

M21/5/MATHX/HP2/ENG/TZ1/XX/M

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 2

22 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1**, **A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) $a = 0.433156\ldots$, $b = 4.50265\ldots$

$a = 0.433$, $b = 4.50$ ***A1A1***

[2 marks]

(b) attempt to substitute $x = 18$ into their equation ***(M1)***

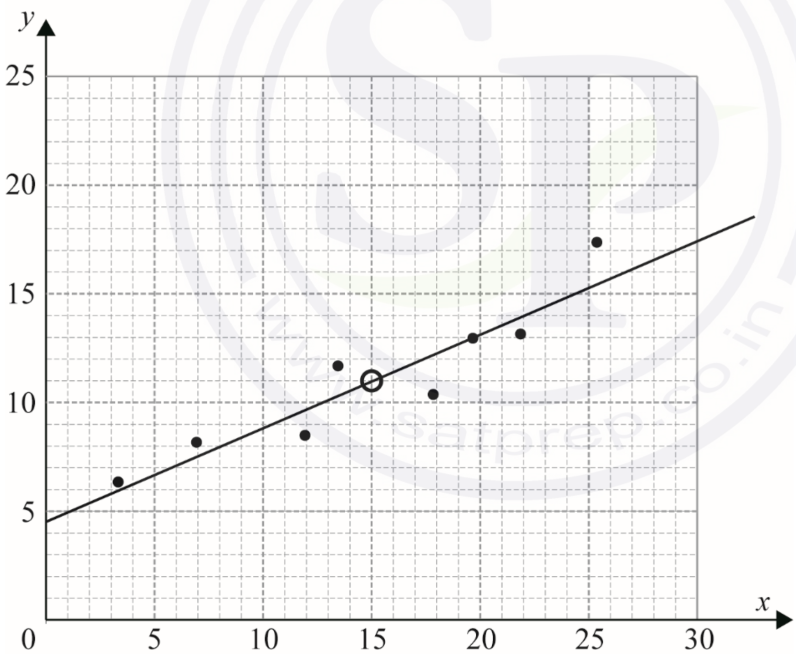
$y = 0.433 \times 18 + 4.50$
$= 12.2994\ldots$
$= 12.3$ ***A1***

[2 marks]

(c) $\bar{x} = 15$, $\bar{y} = 11$ ***A1***

[1 mark]

(d)



A1A1

Note: Award marks as follows:

A1 for a straight line going through $(15, 11)$

A1 for intercepting the y-axis between their $b \pm 1.5$ (when their line is extended), which includes all the data for $3.3 \le x \le 25.3$.

If the candidate does not use a ruler, award ***A0A1*** where appropriate.

[2 marks]
Total [7 marks]

2.

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let X = mass of a bag of sugar

(a) evidence of identifying the correct area ***(M1)***

$$\mathrm{P}(X < 995) = 0.0765637...$$

$$= 0.0766$$ ***A1***

[2 marks]

(b) 0.0766×100

≈ 8 ***A1***

[1 mark]

Note: Accept 7.66 .

(c) recognition that $\mathrm{P}\left(X > 1005 | X \geq 995\right)$ is required ***(M1)***

$$\frac{\mathrm{P}\left(X \geq 995 \cap X > 1005\right)}{\mathrm{P}\left(X \geq 995\right)}$$

$$\frac{\mathrm{P}(X > 1005)}{\mathrm{P}(X \geq 995)}$$ ***(A1)***

$$\frac{0.07656\ldots}{1 - 0.07656\ldots}\left(= \frac{0.07656\ldots}{0.9234\ldots}\right)$$

$$= 0.0829$$ ***A1***

[3 marks]

Total [6 marks]

3. Amplitude is $\frac{110}{2}=55$ ***(A1)***

$a=-55$ ***A1***

$c=65$ ***A1***

$\frac{2\pi}{b}=20$ OR $-55\cos(20b)+65=10$ ***(M1)***

$b=\frac{\pi}{10}\left(=0.314\right)$ ***A1***

Total [5 marks]

4. (a) recognising $v=0$ ***(M1)***

$t=6.74416...$

$=6.74$ (sec) ***A1***

Note: Do not award ***A1*** if additional values are given.

[2 marks]

(b) $\int_0^{10}\left|v\left(t\right)\right|\mathrm{d}t$ **OR** $-\int_0^{6.74416...}v(t)\,\mathrm{d}t+\int_{6.74416...}^{9.08837...}v(t)\,\mathrm{d}t-\int_{9.08837...}^{10}v(t)\,\mathrm{d}t$ ***(A1)***

$=37.0968...$

$=37.1$ (m) ***A1***

[2 marks]

(c) recognising acceleration at $t=7$ is given by $v'(7)$ ***(M1)***

acceleration $=5.93430...$

$=5.93$ (ms^{-2}) ***A1***

[2 marks]

Total [6 marks]

5. METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen *(M1)*
evidence of correct term chosen *(A1)*

$^{n+1}C_2 \times 3^{n+1-2} \times \left(x^2\right)^2 \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right)$ OR $n - r = 1$

equating their coefficient to 20412 or their term to $20412x^4$ *(M1)*

EITHER

$^{n+1}C_2 \times 3^{n-1} = 20412$ *(A1)*

OR

$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$ *(A1)*

THEN

$n = 7$ *A1*

METHOD 2

$3^{n+1}\left(1 + \frac{x^2}{3}\right)^{n+1}$

product of a binomial coefficient and a power of $\frac{x^2}{3}$ OR $\frac{1}{3}$ seen *(M1)*

evidence of correct term chosen *(A1)*

$3^{n+1} \times {}^{n+1}C_2 \times \left(\frac{x^2}{3}\right)^2 \left(= 3^{n-1}\frac{n(n+1)}{2}x^4\right)$

equating their coefficient to 20412 or their term to $20412x^4$ *(M1)*

$3^{n-1} \times \frac{n(n+1)}{2} = 20412$ *(A1)*

$n = 7$ *A1*

Total [5 marks]

6 (a) attempt to find a vector perpendicular to Π_1 and Π_2 using a cross product ***(M1)***

$$\begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}=\left(2-(-2)\right)\boldsymbol{i}+(1-3)\boldsymbol{j}+(-6-2)\boldsymbol{k}$$

$$=\begin{pmatrix}4\\-2\\-8\end{pmatrix}\left(=2\begin{pmatrix}2\\-1\\-4\end{pmatrix}\right)$$ ***(A1)***

equation is $4x-2y-8z=0\left(\Rightarrow 2x-y-4z=0\right)$ ***A1***

[3 marks]

(b) attempt to solve 3 simultaneous equations in 3 variables ***(M1)***

$$\left(\frac{41}{21},-\frac{10}{21},\frac{23}{21}\right)\left(=\left(1.95,-0.476,1.10\right)\right)$$ ***A1***

[2 marks]
Total [5 marks]

7. (a) recognition of the need to integrate $\dfrac{x}{\sqrt{\left(x^2+k\right)^3}}$ ***(M1)***

$$\int\frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x\left(=1\right)$$

EITHER

$u=x^2+k\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=2x$ (or equivalent) ***(A1)***

$$\int\frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x=\frac{1}{2}\int u^{-\frac{3}{2}}\mathrm{d}u$$

$$=-u^{-\frac{1}{2}}(+c)\left(=-\left(x^2+k\right)^{-\frac{1}{2}}(+c)\right)$$ ***A1***

continued…

Question 7 continued

OR

$\int \frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x$ ***(A1)***

$= -\left(x^2+k\right)^{-\frac{1}{2}}(+c)$ ***A1***

THEN

attempt to use correct limits for their integrand and set equal to 1 ***M1***

$\left[-u^{-\frac{1}{2}}\right]_k^{16+k} = 1$ OR $\left[-\left(x^2+k\right)^{-\frac{1}{2}}\right]_0^4 = 1$

$-(16+k)^{-\frac{1}{2}} + k^{-\frac{1}{2}} = 1\left(\Rightarrow \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1\right)$ ***A1***

$\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***AG***

[5 marks]

(b) attempt to solve $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***(M1)***

$k = 0.645038...$

$= 0.645$ ***A1***

[2 marks]

Total [7 marks]

8. (a) $\left(|zw| =\right)16$ ***A1***

[1 mark]

(b) attempt to find $\arg(z) + \arg(w)$ ***(M1)***

$\arg(zw) = \arg(z) + \arg(w)$

$= \frac{\pi}{5} - \frac{2k\pi}{5}\left(= \frac{(1-2k)\pi}{5}\right)$ ***A1***

[2 marks]

continued…

Question 8 continued

(c) (i) $zw \in \mathbb{Z} \Rightarrow \arg(zw)$ is a multiple of π ***(M1)***

$\Rightarrow 1-2k$ is a multiple of 5 ***(M1)***

$k=3$ ***A1***

(ii) $zw = 16(\cos(-\pi) + \mathrm{i}\sin(-\pi))$

-16 ***A1***

[4 marks]
Total [7 marks]

9. (a) $\tan\theta = \dfrac{50}{y-x}$ OR $\cot\theta = \dfrac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to *t* ***(M1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2 \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

attempt to set speed of B equal to double the speed of A ***(M1)***

$2\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2 \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -50(\operatorname{cosec}\theta)^2 \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots°)$ OR $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\dfrac{1}{5}\right)^2 = \dfrac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\dfrac{26}{25}\right) \times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]
Total [7 marks]

Section B.

10. (a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $\left(2\ln 45, 2\right)$ (accept $(-2\ln\frac{1}{45}, 2)$ ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -\left(x - 2\ln 45\right)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

continued…

Question 10 continued

(d) (i) $x = \ln 45 + 1 (= 4.81)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$$\int_{4.806...}^{5.566...} \left(x - (-x + 2\ln 45 + 2)\right) dx + \int_{5.566...}^{7.613...} \left(90e^{-0.5x} - (-x + 2\ln 45 + 2)\right) dx$$ ***(A1)(A1)***

Note: Award ***A1*** for one correct integral expression including correct limits and integrand. Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]
Total [15 marks]

11. (a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ ***(M1)***

$p = \frac{1}{2}, q = -\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3(4x^2-1) - 8x(3x+2)}{(4x^2-1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2-1)^2} \right)$$ ***A1A1***

Note: Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x+2)(4x^2-1)^{-2} + 3(4x^2-1)^{-1}$$ ***A1A1***

Note: Award ***A1*** for each term.

[3 marks]

(c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ ***(M1)***

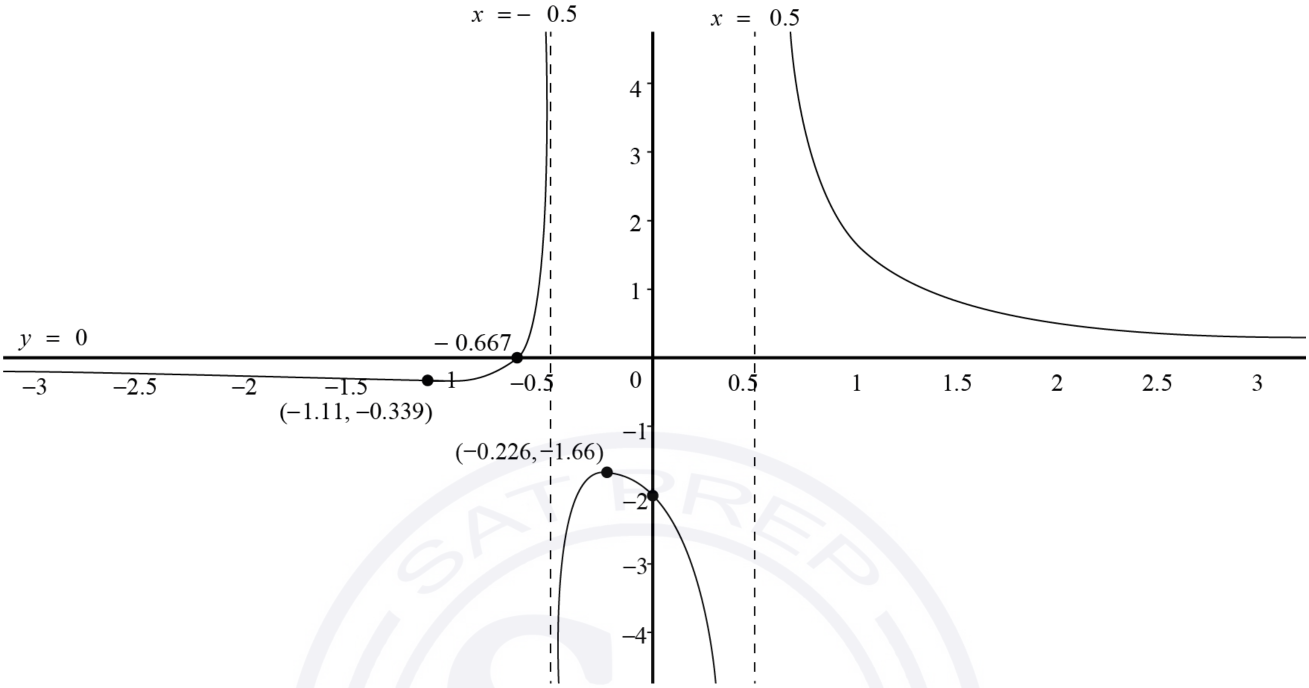
$x = -1.60$ ***A1***

[2 marks]

continued…

Question 11 continued

(d)



A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes with their equations, award ***A1*** for horizontal asymptote with equation, award ***A1*** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts.
If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum ***A0A1A0A1A1.***

[5 marks]

continued…

Question 11 continued

e) $x = -\frac{2}{3} (= -0.667)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3} (= 1.33)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$\begin{array}{r|l} & \frac{4}{3}x - \frac{8}{9} \\ \hline 3x+2 & 4x^2 + 0x - 1 \\ & 4x^2 + \frac{8}{3}x \\ \hline & -\frac{8}{3}x - 1 \\ & -\frac{8}{3}x - \frac{16}{9} \\ \hline & \frac{7}{9} \end{array}$$

$y = \frac{4}{3}x - \frac{8}{9} (= 1.33x - 0.889)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $(x = -0.568729\ldots, x = 1.31872\ldots)$ ***(M1)***

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ ***A1A1A1***

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]
Total [20 marks]

12. (a) $\frac{1}{x(k-x)} \equiv \frac{a}{x} + \frac{b}{k-x}$

$a(k-x) + bx = 1$ ***(A1)***

attempt to compare coefficients OR substitute $x = k$ and $x = 0$ and solve ***(M1)***

$a = \frac{1}{k}$ and $b = \frac{1}{k}$ ***A1***

$f'(x) = \frac{1}{kx} + \frac{1}{k(k-x)}$

[3 marks]

(b) attempt to integrate their $\frac{a}{x} + \frac{b}{k-x}$ ***(M1)***

$f(x) = \frac{1}{k}\int\left(\frac{1}{x} + \frac{1}{k-x}\right)\mathrm{d}x$

$= \frac{1}{k}\left(\ln|x| - \ln|k-x|\right)(+c)\left(= \frac{1}{k}\ln\left|\frac{x}{k-x}\right|(+c)\right)$ ***A1A1***

[3 marks]

Note: Award ***A1*** for each correct term. Award ***A1A0*** for a correct answer without modulus signs. Condone the absence of $+c$.

continued…

Question 12 continued

(c) attempt to separate variables and integrate both sides ***M1***

$$5k\int \frac{1}{P(k-P)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$$

$$5\left(\ln P - \ln(k-P)\right) = t + c$$ ***A1***

Note: There are variations on this which should be accepted, such as $\frac{1}{k}\left(\ln P - \ln(k-P)\right) = \frac{1}{5k}t + c$. Subsequent marks for these variations should be awarded as appropriate.

EITHER

attempt to substitute $t = 0$, $P = 1200$ into an equation involving c ***M1***

$$c = 5\left(\ln 1200 - \ln(k-1200)\right)\left(= 5\ln\left(\frac{1200}{k-1200}\right)\right)$$ ***A1***

$$5\left(\ln P - \ln(k-P)\right) = t + 5\left(\ln 1200 - \ln(k-1200)\right)$$ ***A1***

$$\ln\left(\frac{P(k-1200)}{1200(k-P)}\right) = \frac{t}{5}$$

$$\frac{P(k-1200)}{1200(k-P)} = \mathrm{e}^{\frac{t}{5}}$$ ***A1***

OR

$$\ln\left(\frac{P}{k-P}\right) = \frac{t+c}{5}$$

$$\frac{P}{k-P} = A\mathrm{e}^{\frac{t}{5}}$$ ***A1***

attempt to substitute $t = 0$, $P = 1200$ ***M1***

$$\frac{1200}{k-1200} = A$$ ***A1***

$$\frac{P}{k-P} = \frac{1200\mathrm{e}^{\frac{t}{5}}}{k-1200}$$ ***A1***

continued…

Question 12 continued

THEN

attempt to rearrange and isolate P ***M1***

$Pk - 1200P = 1200k\mathrm{e}^{\frac{t}{5}} - 1200P\mathrm{e}^{\frac{t}{5}}$ OR $Pk\mathrm{e}^{-\frac{t}{5}} - 1200P\mathrm{e}^{-\frac{t}{5}} = 1200k - 1200P$

OR $\dfrac{k}{P} - 1 = \dfrac{k-1200}{1200\mathrm{e}^{\frac{t}{5}}}$

$P\left(k - 1200 + 1200\mathrm{e}^{\frac{t}{5}}\right) = 1200k\mathrm{e}^{\frac{t}{5}}$ OR $P\left(k\mathrm{e}^{-\frac{t}{5}} - 1200\mathrm{e}^{-\frac{t}{5}} + 1200\right) = 1200k$ ***A1***

$P = \dfrac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$ ***AG***

[8 marks]

(d) attempt to substitute $t = 10$, $P = 2400$ ***(M1)***

$2400 = \dfrac{1200k}{(k-1200)\mathrm{e}^{-2} + 1200}$ ***(A1)***

$k = 2845.34...$

$k = 2845$ ***A1***

Note: Award ***(M1)(A1)A0*** for any other value of k which rounds to 2850

[3 marks]

(e) attempt to find the maximum of the first derivative graph OR zero of the second derivative graph OR that $P = \dfrac{k}{2}(= 1422.67...)$ ***(M1)***

$t = 1.57814...$

$= 1.58$ (days) ***A2***

Note: Accept any value which rounds to 1.6.

[3 marks]

Total [20 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 2

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1****,* ***A2****.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***A2***, *etc*., do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s) or subpart(s). Usually, to award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then* ***FT*** *marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features/ CAS functionality are not allowed.

Calculator notation
The subject guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

Section A

1. (a) ***METHOD 1***

attempt to use the cosine rule ***(M1)***

$\cos\theta = \dfrac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$ (or equivalent) ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

METHOD 2

attempt to split triangle AOB into two congruent right triangles ***(M1)***

$\sin\left(\dfrac{\theta}{2}\right) = \dfrac{2.5}{4}$ ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

(b) attempt to find the area of the shaded region ***(M1)***

$\dfrac{1}{2} \times 4 \times 4 \times (2\pi - 1.35...)$ ***A1***

$= 39.5 \text{ (cm}^2\text{)}$ ***A1***

[3 marks]

Total [6 marks]

2. (a) $\left(1 + \dfrac{5.5}{4 \times 100}\right)^4$ ***(M1)(A1)***

$= 1.056$ ***A1***

[3 marks]

continued…

Question 2 continued

(b) ***EITHER***

$2P = P \times \left(1 + \frac{5.5}{100 \times 4}\right)^{4n}$ **OR** $2P = P \times (\text{their } (a))^m$ ***(M1)(A1)***

Note: Award ***(M1)*** for substitution into loan payment formula. Award ***(A1)*** for correct substitution.

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 5.5$
$\text{P/Y} = 4$
$\text{C/Y} = 4$
$n = 50.756\ldots$ ***(M1)(A1)***

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 100(\text{their } (a) - 1)$
$\text{P/Y} = 1$
$\text{C/Y} = 1$ ***(M1)(A1)***

THEN

$\Rightarrow$ 12.7 years
Laurie will have double the amount she invested during 2032 ***A1***

[3 marks]

Total [6 marks]

3. (a) recognition of binomial ***(M1)***
$X \sim \text{B}(5, 0.7)$
attempt to find $\text{P}(X \le 3)$ ***M1***
$= 0.472 (= 0.47178)$ ***A1***

[3 marks]

(b) recognition of 2 sixes in 4 tosses ***(M1)***
$\text{P}(\text{3rd six on the 5th toss}) = \left[\binom{4}{2} \times (0.7)^2 \times (0.3)^2\right] \times 0.7 (= 0.2646 \times 0.7)$ ***A1***
$= 0.185 (= 0.18522)$ ***A1***

[3 marks]

Total [6 marks]

4. (a) $a = 1.29$ and $b = -10.4$ ***A1A1***

[2 marks]

(b) recognising both lines pass through the mean point ***(M1)***

$p = 28.7$, $q = 30.3$ ***A2***

[3 marks]

(c) substitution into **their** x on y equation ***(M1)***

$x = 1.29082(29) - 10.3793$

$x = 27.1$ ***A1***

Note: Accept 27.

[2 marks]

Total [7 marks]

5. (a) use of a graph to find the coordinates of the local minimum ***(M1)***

$s = -16.513...$ ***(A1)***

maximum distance is $16.5\,\text{cm}$ (to the left of O) ***A1***

[3 marks]

(b) attempt to find time when particle changes direction *eg* considering the first maximum on the graph of s or the first t – intercept on the graph of s'. ***(M1)***

$t = 1.51986...$ ***(A1)***

attempt to find the gradient of s' for **their** value of t, $s''(1.51986...)$ ***(M1)***

$= -8.92\ (\text{cm/s}^2)$ ***A1***

[4 marks]

Total [7 marks]

6. (a) ***METHOD 1***

attempting to use the expected value formula ***(M1)***

$\mathrm{E}(X) = (1\times0.60)+(2\times0.30)+(3\times0.03)+(4\times0.05)+(5\times0.02)$

$\mathrm{E}(X) = 1.59(\$)$ ***(A1)***

use of $\mathrm{E}(1.20X+2.40)=1.20\mathrm{E}(X)+2.40$ ***(M1)***

$\mathrm{E}(T)=1.20(1.59)+2.40$

$=4.31(\$)$ ***A1***

METHOD 2

attempting to find the probability distribution for T ***(M1)***

t	3.60	4.80	6.00	7.20	8.40
$\mathrm{P}(T=t)$	0.60	0.30	0.03	0.05	0.02

(A1)

attempting to use the expected value formula ***(M1)***

$\mathrm{E}(T)=(3.60\times0.60)+(4.80\times0.30)+(6.00\times0.03)+(7.20\times0.05)+(8.40\times0.02)$

$=4.31(\$)$ ***A1***

[4 marks]

(b) ***METHOD 1***

using $\mathrm{Var}(1.20X+2.40)=(1.20)^2\,\mathrm{Var}(X)$ with $\mathrm{Var}(X)=0.8419$ ***(M1)***

$\mathrm{Var}(T)=1.21$ ***A1***

METHOD 2

finding the standard deviation for **their** probability distribution found in part (a) ***(M1)***

$\mathrm{Var}(T)=(1.101...)^2$

$=1.21$ ***A1***

Note: Award ***M1A1*** for $\mathrm{Var}(T)=(1.093...)^2=1.20$.

[2 marks]

Total [6 marks]

7. attempting to find $\boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{A}}$ for example ***(M1)***

$\boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + t\begin{pmatrix} -5 \\ 4 \end{pmatrix}$

attempting to find $\left|\boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{A}}\right|$ ***M1***

distance $d(t) = \sqrt{(3-5t)^2 + (4t-6)^2} \left(= \sqrt{41t^2 - 78t + 45}\right)$ ***A1***

using a graph to find the d – coordinate of the local minimum ***M1***

the minimum distance between the ships is $2.81\ (\mathrm{km}) \left(= \frac{11\sqrt{41}}{41}\ (\mathrm{km})\right)$ ***A1***

Total [5 marks]

8. substituting $w = 2\mathrm{i}z$ into $z^* - 3w = 5 + 5\mathrm{i}$ ***M1***

$z^* - 6\mathrm{i}z = 5 + 5\mathrm{i}$ ***A1***

let $z = x + y\mathrm{i}$

comparing real and imaginary parts of $(x - y\mathrm{i}) - 6\mathrm{i}(x + y\mathrm{i}) = 5 + 5\mathrm{i}$ ***M1***

to obtain $x + 6y = 5$ and $-6x - y = 5$ ***A1***

attempting to solve for x and y ***M1***

$x = -1$ and $y = 1$ and so $z = -1 + \mathrm{i}$ ***A1***

hence $w = -2 - 2\mathrm{i}$ ***A1***

9. ***METHOD 1***

sketching the graph of $y=\frac{x^2}{x-3}$ ($y=x+3+\frac{9}{x-3}$) ***M1***

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining $(-3,0)$ and $(0,0)$ ***M1***

so $m>0$ ***R1***

hence $0<m\le 1$ ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2=m(x^2-9)$

$\Rightarrow (m-1)x^2-9m=0$ ***A1***

EITHER

attempting to solve $-4(m-1)(-9m)<0$ for m ***M1***

OR

attempting to solve $x^2<0$ *ie* $\frac{9m}{m-1}<0$ $(m\ne 1)$ for m ***M1***

THEN

$\Rightarrow 0<m<1$ ***A1***

a valid reason to explain why $m=1$ gives no solutions *eg* if $m=1$, $(m-1)x^2-9m=0\Rightarrow -9=0$ and so $0<m\le 1$ ***R1***

Total [5 marks]

Section B

10. (a) attempt to use the symmetry of the normal curve ***(M1)***
eg diagram, $0.5-0.1446$

$\mathrm{P}(24.15<X<25)=0.3554$ ***A1***

[2 marks]

(b) (i) use of inverse normal to find z score ***(M1)***
$z=-1.0598$

correct substitution $\dfrac{24.15-25}{\sigma}=-1.0598$ ***(A1)***

$\sigma=0.802$ ***A1***

(ii) $\mathrm{P}(X>26)=0.106$ ***(M1)A1***

[5 marks]

(c) recognizing binomial probability ***(M1)***
$\mathrm{E}(Y)=10\times 0.10621$ ***(A1)***
$=1.06$ ***A1***

[3 marks]

(d) $\mathrm{P}(Y=3)$ ***(M1)***
$=0.0655$ ***A1***

[2 marks]

(e) recognizing conditional probability ***(M1)***
correct substitution ***A1***

$$\frac{0.3554}{1-0.10621}$$

$=0.398$ ***A1***

[3 marks]

Total [15 marks]

11. (a) ***METHOD 1***

using $I(t) = \mathrm{e}^{\int P(t)\mathrm{d}t}$ ***M1***

$\mathrm{e}^{\int \frac{1}{t+1}\mathrm{d}t}$

$= \mathrm{e}^{\ln(t+1)}$ ***A1***

$= t+1$ ***AG***

METHOD 2

attempting product rule differentiation on $\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right) = \frac{\mathrm{d}x}{\mathrm{d}t}(t+1) + x$

$= (t+1)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{t+1}\right)$ ***A1***

so $t+1$ is an integrating factor for this differential equation ***AG***

[2 marks]

continued…

Question 11 continued

(b) attempting to multiply through by $(t+1)$ and rearrange to give ***(M1)***

$$(t+1)\frac{dx}{dt}+x=10(t+1)e^{-\frac{t}{4}}$$ ***A1***

$$\frac{d}{dt}\left(x(t+1)\right)=10(t+1)e^{-\frac{t}{4}}$$

$$x(t+1)=\int 10(t+1)e^{-\frac{t}{4}}\,dt$$ ***A1***

attempting to integrate the RHS by parts ***M1***

$$=-40(t+1)e^{-\frac{t}{4}}+40\int e^{-\frac{t}{4}}\,dt$$

$$=-40(t+1)e^{-\frac{t}{4}}-160e^{-\frac{t}{4}}+C$$ ***A1***

Note: Condone the absence of C .

EITHER

substituting $t=0, x=0 \Rightarrow C=200$ ***M1***

$$x=\frac{-40(t+1)e^{-\frac{t}{4}}-160e^{-\frac{t}{4}}+200}{t+1}$$ ***A1***

using $-40e^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)e^{-\frac{t}{4}}$ and $-160e^{-\frac{t}{4}}$ ***M1***

OR

using $-40e^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)e^{-\frac{t}{4}}$ and $-160e^{-\frac{t}{4}}$ giving

$x(t+1)=-40e^{-\frac{t}{4}}(t+5)+C$ (or equivalent) ***M1A1***

substituting $t=0,\ x=0 \Rightarrow C=200$ ***M1***

THEN

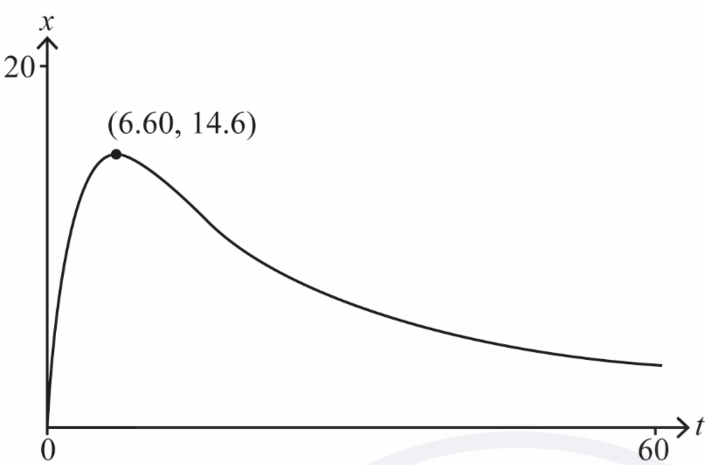
$$x(t)=\frac{200-40e^{-\frac{t}{4}}(t+5)}{t+1}$$ ***AG***

[8 marks]

continued…

Question 11 continued

(c)



graph starts at the origin and has a local maximum (coordinates not required) ***A1***
sketched for $0 \le t \le 60$ ***A1***
correct concavity for $0 \le t \le 60$ ***A1***
maximum amount of salt is 14.6 (grams) at $t = 6.60$ (minutes) ***A1A1***

[5 marks]

(d) using an appropriate graph or equation (first or second derivative) ***M1***
amount of salt is decreasing most rapidly at $t = 12.9$ (minutes) ***A1***

[2 marks]

(e) **EITHER**

attempting to form an integral representing the amount of salt that left the tank ***M1***

$$\int_0^{60} \frac{x(t)}{t+1}\,\mathrm{d}t$$

$$\int_0^{60} \frac{200 - 40\mathrm{e}^{-\frac{t}{4}}(t+5)}{(t+1)^2}\,\mathrm{d}t$$ ***A1***

OR

attempting to form an integral representing the amount of salt that entered the tank minus the amount of salt in the tank at $t = 60$ (minutes) ***M1***

amount of salt that left the tank is $\int_0^{60} 10\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t - x(60)$ ***A1***

THEN

$= 36.7$ (grams) ***A2***

[4 marks]

Total [21 marks]

12. (a) stating the relationship between cot and tan and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \dfrac{1}{\tan 2\theta}$ and $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b) ***METHOD 1***

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \tan^2\theta + 2\tan\theta\left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0 \, (= \text{RHS})$ ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \cot^2\theta - 2\cot\theta\left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \dfrac{1}{\tan^2\theta} - \left(\dfrac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\dfrac{1}{\tan^2\theta} - \dfrac{1}{\tan^2\theta} + 1 - 1 = 0 \, (= \text{RHS})$ ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \dfrac{1}{\tan\theta}$

$= \dfrac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

$= -1$ ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and -1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) ***METHOD 1***

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

> **Note:** Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}}$$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2 - \sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

Total [19 marks]