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Mathematics: analysis and approaches

Higher level

Paper 3

29 October 2024

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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1. [Maximum mark: 27]

This question asks you to investigate models for the population of trout in a lake.

Trout is a type of fish. At the start of a year, a lake is estimated to contain 6000 trout.

The owner of the lake estimates that the number of trout will increase by 10% per year.

At the end of each year, the owner proposes to remove 500 trout from the lake to prevent overpopulation.

Therefore, the relationship between T_n , the predicted number of trout at the start of year n , and T_{n+1} , the predicted number of trout at the start of year $n + 1$, is given by

$$T_{n+1} = 1.1T_n - 500 \text{ and } T_1 = 6000.$$

For example, the predicted number of trout at the start of the second year is given by

$$T_2 = 1.1T_1 - 500.$$

(a) Use this formula to verify that $T_2 = 6100$. [2]

(b) (i) Verify that $T_3 = 6210$. [1]

(ii) Find T_4 . [2]

It is also known that $T_n = 6000(1.1)^{n-1} - \frac{500((1.1)^{n-1} - 1)}{1.1 - 1}$.

(c) (i) Show that $T_n = 1000(1.1)^{n-1} + 5000$. [2]

(ii) Hence, or otherwise, find T_6 . Give your answer to the nearest whole number. [2]

(This question continues on the following page)

(Question 1 continued)

After deciding that the trout population would increase too quickly, the lake owner proposes instead to remove 750 trout at the end of each year.

The relationship between D_n , the predicted number of trout at the start of year n , and D_{n+1} , the predicted number of trout at the start of year $n + 1$, is now given by

$$D_{n+1} = 1.1D_n - 750 \text{ and } D_1 = 6000.$$

It is also known that $D_n = -1500(1.1)^{n-1} + 7500$.

(d) (i) Show that $D_{n+1} - D_n = -150(1.1)^{n-1}$. [3]

(ii) Use the result in part (d)(i) to deduce that the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year. [1]

(e) Determine the first year during which there will be no trout remaining in the lake. [4]

The lake owner now considers a more general approach where d trout are removed at the end of each year.

Let C_n denote the predicted number of trout in the lake at the start of the n th year where

$$C_n = 6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1).$$

(f) Find the value of d such that the predicted number of trout at the start of each year is constant. [3]

To model predicted numbers of trout, the lake owner has been using sequences generated by

$$u_{n+1} = ru_n - d, \text{ where } d, r \in \mathbb{R}^+ \text{ and } r \neq 1.$$

(g) Use mathematical induction to prove that $u_n = u_1 r^{n-1} - \frac{d(r^{n-1} - 1)}{r - 1}$, for $n \in \mathbb{Z}^+$. [7]

2. [Maximum mark: 28]

A polynomial is said to be palindromic if the sequence of its coefficients remains the same in reverse. This question asks you to investigate some properties and solutions of palindromic polynomial equations.

In parts (a) and (b), consider quadratic equations of the form $ax^2 + bx + a = 0$, where $a \neq 0$.

The sequence of coefficients, $\{a, b, a\}$, remains the same in reverse.

The following table shows three palindromic quadratic equations and their sequence of coefficients.

Palindromic quadratic equation	Sequence of coefficients
$2x^2 - 5x + 2 = 0$	$\{2, -5, 2\}$
$x^2 + 4x + 1 = 0$	$\{1, 4, 1\}$
$x^2 + 1 = 0$	$\{1, 0, 1\}$

The quadratic equation $2x^2 - 5x + 2 = 0$ has roots 2 and $\frac{1}{2}$.

These roots form a “reciprocal pair”, since one root is the reciprocal of the other.

(a) (i) Determine the roots of $x^2 + 4x + 1 = 0$.

Give these roots in the form $s \pm \sqrt{t}$, where $s \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. [3]

(ii) Hence, or otherwise, show that these roots form a reciprocal pair. [2]

(b) Show that the complex roots of $x^2 + 1 = 0$ form a reciprocal pair. [2]

Let $p(x) = ax^2 + bx + a$, where $a \neq 0$.

(c) Verify that $p(x) = x^2 p\left(\frac{1}{x}\right)$ where $x \neq 0$. [2]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), you may assume the result that a polynomial, $p(x)$, of degree n is palindromic if and only if $p(x) = x^n p\left(\frac{1}{x}\right)$.

- (d) Use $p(x) = x^n p\left(\frac{1}{x}\right)$ to show that if $\alpha \neq 0$ is a root of $p(x) = 0$, then $\frac{1}{\alpha}$ is also a root. [2]

Let $f(x) = p(x)q(x)$, where p and q are palindromic polynomials of degree n and m respectively.

- (e) Show that f is a palindromic polynomial. [4]

Consider the palindromic polynomial $f(x) = x^4 + 2x^3 - x^2 + 2x + 1$.

This polynomial can be expressed in the form

$$f(x) = (x^2 + ux + 1)(x^2 + vx + 1), \text{ where } u, v \in \mathbb{Z} \text{ and } u < v.$$

- (f) By forming and solving an appropriate system of equations in u and v , determine the value of u and the value of v . [6]
- (g) Hence, find all the exact complex and purely real roots of $x^4 + 2x^3 - x^2 + 2x + 1 = 0$. [3]

Consider the palindromic polynomial equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + 1 = 0, \text{ where } n \text{ is odd.}$$

- (h) Show that -1 is always a root of this equation. [4]

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Mathematics: analysis and approaches

Higher level

Paper 3

6 May 2024

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

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1. [Maximum mark: 27]

This question considers two possible models for the occurrence of random events in a computer game.

In a new computer game, each time a player performs an action, there is a random chance that the action will be *boosted*, meaning that it provides a benefit to the player.

The designer of this computer game is considering two possible models for when to boost an action.

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1 .

(i) Find the probability that the first boost occurs on the third action. [2]

(ii) Find the probability that at least one boost occurs in the first six actions. [3]

(b) Suppose the probability that an action will be boosted is p , where $0 < p < 1$.

(i) Explain why the probability that the first boost occurs on the x^{th} action is $p(1 - p)^{x-1}$. [1]

Let X be the number of actions until the first boost occurs.

(ii) Hence, write down an expression, using sigma notation, for $E(X)$ in terms of x and p . [1]

Consider the sum of an infinite geometric sequence, with first term a and common ratio r ($|r| < 1$),

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1 - r} .$$

(c) (i) By differentiating both sides of the above equation with respect to r , find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r . [4]

(ii) Hence, show that $E(X) = \frac{1}{p}$. [2]

(This question continues on the following page)

(Question 1 continued)

It can be shown that $\text{Var}(X) = \frac{1-p}{p^2}$.

- (d) Find $E(X)$ and $\text{Var}(X)$ when $p = 0.1$. [2]

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

- (e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Let Y be the number of actions until the first boost occurs.

- (f) Explain why $Y \leq 5$. [1]

The following table shows the probability distribution of Y .

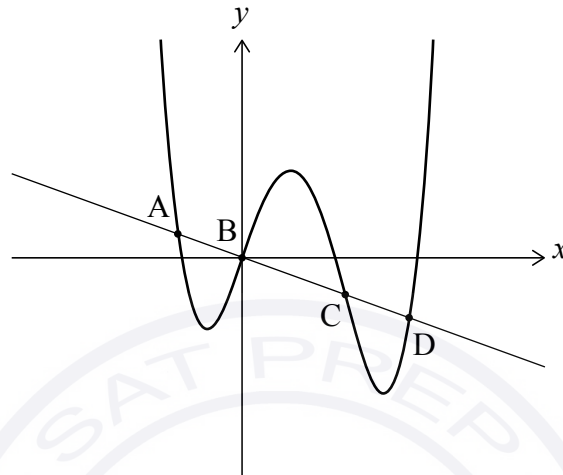
y	1	2	3	4	5
$P(Y = y)$	0.2	m	0.288	n	0.0384

- (g) (i) Find the value of m and the value of n . [2]
 (ii) Show that $E(Y) = 2.5104$. [2]
 (iii) Find $\text{Var}(Y)$. [2]
- (h) (i) Use the expression given in (c)(ii) to find the value of p for which $E(X) = E(Y)$. [1]
 (ii) Find $\text{Var}(X)$ for this value of p . [1]
 (iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]

2. [Maximum mark: 28]

This question investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.

The curve $y = x^4 - 3x^3 + 3x$ has points of inflexion at B and C. The line passing through B and C intersects the curve again at points A and D. This is shown in the following graph.



- (a) Find $\frac{d^2y}{dx^2}$. [3]
- (b) Find the coordinates of B and C. [4]
- (c) Show that the equation of the line through B and C is $y = -0.375x$. [2]
- (d) Find, correct to three decimal places, the x -coordinate of D. [2]

Now consider the general curve $y = x^4 - mx^3 + nx$, where $m, n \in \mathbb{R}$ and $m > 0$.

- (e) Find the x -coordinates of the two points of inflexion in terms of m . [3]

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x -coordinate of point A, and similarly for x_B , x_C and x_D . It is given that $x_A < x_B < x_C < x_D$.

- (f)
 - (i) Write down the coordinates of B. [1]
 - (ii) Find, in terms of m and n , the coordinates of C. [2]

(This question continues on the following page)

(Question 2 continued)

(g) Show that the equation of the line through B and C is $y = \left(-\frac{m^3}{8} + n\right)x$. [2]

(h) Show that $x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$. [7]

(i) Hence, find the exact value of $\frac{x_B - x_A}{x_C - x_B}$. [2]



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Mathematics: analysis and approaches

Higher level

Paper 3

6 May 2024

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour

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1. [Maximum mark: 24]

If two functions $f(x)$ and $g(x)$ are differentiable, then their product is differentiable and the two functions satisfy the product rule: $(f(x)g(x))' = f'(x)g(x) + g(x)f'(x)$.

In this question, you will meet examples of pairs of differentiable functions, $f(x)$ and $g(x)$, that also satisfy $(f(x)g(x))' = f'(x)g(x) + g(x)f'(x)$.

In part (a), consider $f(x) = \frac{1}{(2-x)^2}$, where $x \in \mathbb{R}$, $x \neq 2$, and $g(x) = x^2$, where $x \in \mathbb{R}$.

(a) (i) Find an expression for $f'(x)$. [2]

(ii) Show that $f'(x)g'(x) = \frac{4x}{(2-x)^3}$. [2]

(iii) Show that $f(x)g'(x) + g(x)f'(x) = \frac{4x}{(2-x)^3}$. [4]

In parts (b) and (c), consider two non-constant functions, $f(x)$ and $g(x)$, where $f(x) > 0$ and $g(x) \neq g'(x)$.

(b) By rearranging the equation $f(x)g'(x) + g(x)f'(x) = f'(x)g'(x)$, show that $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$. [2]

(This question continues on the following page)

(Question 1 continued)

- (c) Hence, by integrating both sides of $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$, show that $f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$, where A is an arbitrary positive constant. [2]

The result from part (c) can be used to find pairs of functions, $f(x)$ and $g(x)$, which satisfy **both** of the following:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \text{ and } (f(x)g(x))' = f'(x)g'(x).$$

In parts (d) and (e), use the result in part (c) with $A = 1$.

- (d) Consider $g(x) = xe^x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations. [5]

- (e) Consider $g(x) = \sin x + \cos x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations over the domain $0 < x < \pi$.

Give your answer in the form $f(x) = \sqrt{e^x h(x)}$, where $h(x)$ is a function to be determined. [7]

2. [Maximum mark: 31]

This question asks you to find the probability of graphs of randomly generated quadratic functions having a specified number of x -intercepts.

In parts (a) – (f), consider quadratic functions, $f(x) = ax^2 + bx + c$, whose coefficients, a , b and c , are randomly generated in turn by rolling an unbiased six-sided die three times and reading off the value shown on the uppermost face of the die.

For example, rolling a 2, 3 and 5 in turn generates the quadratic function $f(x) = 2x^2 + 3x + 5$.

- (a) Explain why there are 216 possible quadratic functions that can be generated using this method. [1]

- (b) The set of coefficients, $a = 1$, $b = 4$ and $c = 4$, is randomly generated to form the quadratic function $f(x) = x^2 + 4x + 4$.

Verify that this graph of f has only one x -intercept. [2]

- (c) By considering the discriminant, or otherwise, show that the probability of the graph of such a randomly generated quadratic function having only one x -intercept is $\frac{5}{216}$. [6]

Now consider randomly generated quadratic functions whose corresponding graphs have two **distinct** x -intercepts.

- (d) By considering the discriminant, determine the set of possible values of ac . [3]

- (e) (i) For the case where $ac = 1$, show that there are four quadratic functions whose corresponding graphs have two distinct x -intercepts. [1]

- (ii) For the case where $ac = 2$, show that there are eight quadratic functions whose corresponding graphs have two distinct x -intercepts. [2]

Let p be the probability of the graph of such a randomly generated quadratic function having two distinct x -intercepts.

- (f) Using the approach started in part (e), or otherwise, find the value of p . [6]

(This question continues on the following page)

(Question 2 continued)

In parts (g) and (h), consider a randomly generated quadratic function, $f(x) = x^2 + 2Zx + 1$, where the continuous random variable $Z \sim N(0, 1)$.

- (g) Find the probability that the graph of f has two x -intercepts. [3]

The continuous random variables, X_1 and X_2 , represent the x -intercepts of the graph of f where $X_1 = -Z - \sqrt{Z^2 - 1}$ and $X_2 = -Z + \sqrt{Z^2 - 1}$.

- (h) Given that the graph of f has two x -intercepts, X_1 and X_2 , find the probability that both X_1 and X_2 are greater than 0.5. [7]



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Mathematics: analysis and approaches

Higher level

Paper 3

31 October 2023

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

This question asks you to explore some properties of the family of curves

$y = x^3 + ax^2 + b$ **where** $x \in \mathbb{R}$ **and** a, b **are real parameters.**

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

- (a) By systematically varying the value of b , or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x -axis intercepts. [2]
- (b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly
 - (i) one x -axis intercept; [1]
 - (ii) three x -axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

- (c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly
 - (i) two x -axis intercepts; [1]
 - (ii) one x -axis intercept; [1]
 - (iii) three x -axis intercepts. [1]

(This question continues on the following page)

(Question 1 continued)

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

- (d) Consider the case where the curve has exactly three x -axis intercepts. State whether each point of zero gradient is located above or below the x -axis. [1]
- (e) Show that the curve has a point of zero gradient at $P(0, b)$ and a point of zero gradient at $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$. [5]
- (f) Consider the points P and Q for $a > 0$ and $b > 0$.
- (i) Find an expression for $\frac{d^2y}{dx^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]
- (ii) Determine whether each point is located above or below the x -axis. [1]
- (g) Consider the points P and Q for $a < 0$ and $b > 0$.
- (i) State whether P is a local maximum or a local minimum and whether it is above or below the x -axis. [1]
- (ii) State the conditions on a and b that determine when Q is below the x -axis. [1]
- (h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x -axis intercepts. [5]

2. [Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α .

- (a) Consider a family of straight lines, L , with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C , at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

- (i) Write down an expression for the gradient of L in terms of x and y . [1]

- (ii) Hence show that the gradient of C is given by $\frac{dy}{dx} = -\frac{x}{y}$. [1]

- (iii) By solving the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, show that the family of curves, C , has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

- (b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x -intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

- (c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $M(a-b, 2\sqrt{ab})$ and $N(a-b, -2\sqrt{ab})$. [6]

- (d) At point M , show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

(This question continues on the following page)

(Question 2 continued)

Consider two families of curves, F and G .

The gradient of F is denoted by $f(x, y)$.

The gradient of G is denoted by $g(x, y)$.

Each member of F intersects every member of G at an acute angle, α .

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan \alpha}{1 - f(x, y) \tan \alpha}.$$

In part (e), consider the specific case where $f(x, y) = -\frac{x}{y}$, for $x \neq 0$, $y \neq 0$ and $\alpha = \frac{\pi}{4}$.

- (e) (i) Show that $g(x, y) = \frac{y-x}{y+x}$. [2]

- (ii) Hence, by solving the homogeneous differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, find a general equation that represents this family of curves, G . Give your answer in the form $h(x, y) = d$ where d is a parameter. [9]

- (f) By considering $\lim_{\alpha \rightarrow \frac{\pi}{2}} \tan \alpha$, show that, for all finite $f(x, y)$,

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}. \quad [2]$$

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Mathematics: analysis and approaches

Higher level

Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

Instructions to candidates

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1. [Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a .

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator “logarithm to any base feature”.

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+, a \neq 1.$$

- (a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x -axis intercepts.

[4]

(This question continues on the following page)

(Question 1 continued)

In parts (b) and (c), consider the case where $a = e$. Note that $\ln x \equiv \log_e x$.

- (b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a minimum. [5]
- (c) Hence deduce that $x > \ln x$. [1]
- (d) There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a .

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a , write down the values of p , q and r . [4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

- (e) Find the exact coordinates of P and the exact value of a . [8]
- (f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have
- (i) two intersection points; [1]
- (ii) no intersection points. [1]

2. [Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where $m, c \in \mathbb{R}$ and $m, c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c , in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

- (a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

- (b) (i) Show that $r = -\frac{c}{m}$. [1]
 (ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \frac{m^2}{m+2}$. [4]
 (iii) State any further restrictions on the value of m . [1]

There are only three **integer** sets of values of m, r and c , that form an AS-linear function. One of these is $L(x) = -x - 1$.

- (c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c . [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}, a \neq 0$ and $b, c \in \mathbb{R}$.

Let $r_1, r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

- (d) Write down an expression for
 (i) the sum of roots, $r_1 + r_2$, in terms of a and b . [1]
 (ii) the product of roots, $r_1 r_2$, in terms of a and c . [1]

(This question continues on the following page)

(Question 2 continued)

If a , r_1 , b , r_2 and c , in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(i) write down an expression for $r_2 - r_1$ in terms of a and b ; [1]

(ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \frac{a^2 - ab - b}{2a}$; [2]

(iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\frac{1}{2}$. [3]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where $a = -\frac{1}{2}$.

(g) (i) Find an expression for r_1 in terms of b . [2]

(ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\frac{-p \pm q\sqrt{s}}{2}$ where $p, q, s \in \mathbb{Z}^+$. [5]

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Mathematics: analysis and approaches

Higher level

Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

Instructions to candidates

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1. [Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = xe^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x -axis and the line $x = b$, where $b > 0$, is given by $\frac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, A_n , of the region between the graph $y = f_n(x)$ and the x -axis can be written as $A_n = \int_0^\infty f_n(x) dx$ and is given by $\lim_{b \rightarrow \infty} \int_0^b f_n(x) dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b \rightarrow \infty} \frac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1 . [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4 ; [2]

(ii) A_5 . [1]

(e) Suggest an expression for A_n in terms of n , where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m , $\lim_{x \rightarrow \infty} x^m e^{-x} = 0$. [8]

2. [Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers $x_1, x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of x_1 and x_2 as a function, f , of x_1 only. [2]

(b) (i) Find the value of x_1 for which the function is maximum. [1]

(ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S , where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, x_1, x_2, \dots, x_n .

The geometric mean is defined as $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1 + x_2 + \dots + x_n)}{n}$.

(d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \dots = x_n$. [2]

(ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]

(e) Hence determine the value of

(i) $M_3(12)$; [1]

(ii) $M_4(12)$; [1]

(iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n .

(f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]

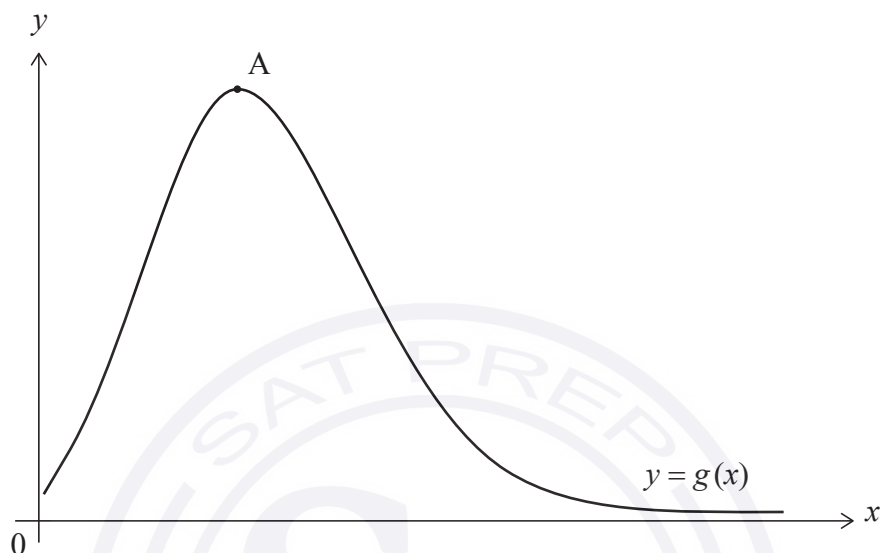
(g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the function g , defined by $\ln(g(x)) = x \ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



- (h) Find, in terms of S , the x -coordinate of point A. [6]
- (i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]
- (j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}^+$. [3]

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Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 8 November 2022 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^n i^q = 1^q + 2^q + 3^q + \dots + n^q \text{ where } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n , for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^n i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^n i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p . [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^n i^2 = a_1n + a_2n^2 + a_3n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1 , a_2 and a_3 . [3]

(iii) Hence, find the values of a_1 , a_2 and a_3 . [2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q .

Consider the function $f(x) = 1 + x + x^2 + \dots + x^n$, $n \in \mathbb{Z}^+$.

(c) Show that $xf'(x) = x + 2x^2 + 3x^3 + \dots + nx^n$. [1]

Let $f_1(x) = xf'(x)$ and consider the following family of functions:

$$f_2(x) = xf_1'(x)$$

$$f_3(x) = xf_2'(x)$$

$$f_4(x) = xf_3'(x)$$

...

$$f_q(x) = xf_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^n i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^n i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \dots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \frac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

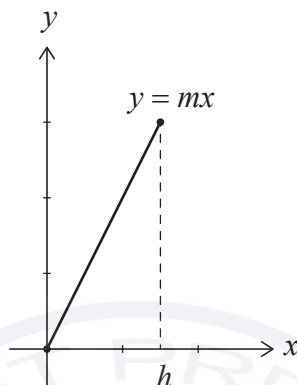
(g) (i) Show that $\lim_{x \rightarrow 1} f_1(x)$ is in indeterminate form. [1]

(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \rightarrow 1} f_1(x) = \frac{1}{2}n(n+1)$. [5]

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \leq x \leq h$ and m, h are positive constants.



When this line is rotated through 360° about the x -axis, a cone is formed with a curved surface area A given by:

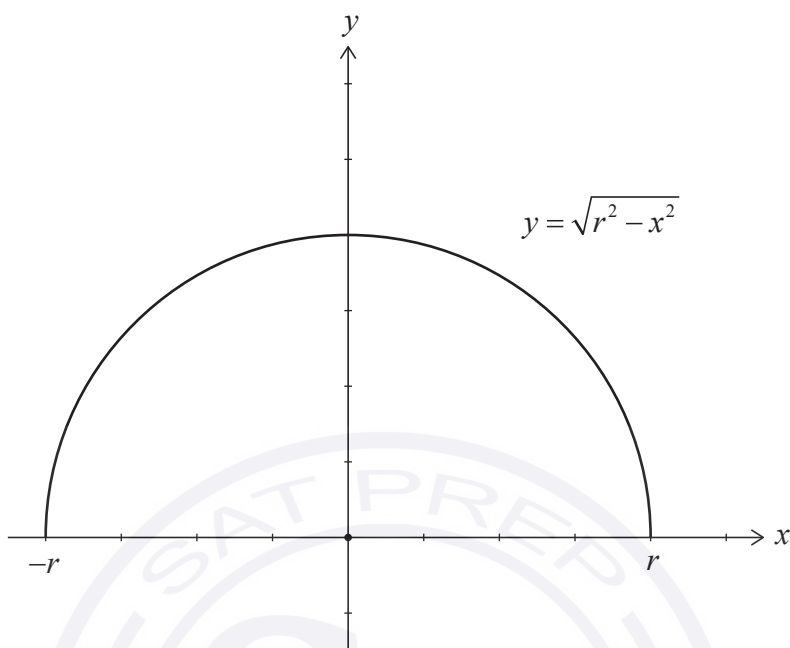
$$A = 2\pi \int_0^h y \sqrt{1 + m^2} \, dx.$$

- (a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\pi$. [2]
- (b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \leq x \leq h$ through 360° about the x -axis.
- (i) Deduce an expression for the radius of this cone r in terms of h and m . [1]
- (ii) Deduce an expression for the slant height l in terms of h and m . [2]
- (iii) Hence, by using the above integral, show that $A = \pi r l$. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the semi-circle, with radius r , defined by $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.



- (c) Find an expression for $\frac{dy}{dx}$. [2]

A differentiable curve $y = f(x)$ is defined for $x_1 \leq x \leq x_2$ and $y \geq 0$. When any such curve is rotated through 360° about the x -axis, the surface formed has an area A given by:

$$A = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

- (d) A sphere is formed by rotating the semi-circle $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$ through 360° about the x -axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(This question continues on the following page)

(Question 2 continued)

- (e) Let $f(x) = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.

The graph of $y = f(x)$ is transformed to the graph of $y = f(kx)$, $k > 0$. This forms a different curve, called a semi-ellipse.

- (i) Describe this geometric transformation. [2]
- (ii) Write down the x -intercepts of the graph $y = f(kx)$ in terms of r and k . [1]
- (iii) For $y = f(kx)$, find an expression for $\frac{dy}{dx}$ in terms of x , r and k . [2]
- (iv) The semi-ellipse $y = f(kx)$ is rotated 360° about the x -axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A , of the ellipsoid.

Give your answer in the form $2\pi \int_{x_1}^{x_2} \sqrt{p(x)} \, dx$, where $p(x)$ is a polynomial. [4]

- (v) Planet Earth can be modelled as an ellipsoid. In this model:
 - the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
 - the distance from the North Pole to the South Pole is 12 714 km.
 - the diameter of the equator is 12 756 km.

By choosing suitable values for r and k , find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a \times 10^q$ where $1 \leq a < 10$ and $q \in \mathbb{Z}^+$. [4]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

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1. [Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b , where $a, b \in \mathbb{N}$.

(a) On the same set of axes, sketch the following curves for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3, x \geq 0$ [2]

(ii) $y^2 = x^3 + 1, x \geq -1$ [2]

(b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

(ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \geq -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

(c) By varying the value of b , suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x, x \geq 0$.

(d) (i) Show that $\frac{dy}{dx} = \pm \frac{3x^2 + 1}{2\sqrt{x^3 + x}}$, for $x > 0$. [3]

(ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x -coordinate.

(e) Find the value of this x -coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3} + q}{r}}$, where $p, q, r \in \mathbb{Z}$. [7]

(This question continues on the following page)

(Question 1 continued)

$P(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q .

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $P(-1, -1)$ lies on C .

- (f) (i) Find the equation of the tangent to C at P . [2]
- (ii) Hence, find the coordinates of the rational point Q where this tangent intersects C , expressing each coordinate as a fraction. [2]
- (g) The point $S(-1, 1)$ also lies on C . The line $[QS]$ intersects C at a further point. Determine the coordinates of this point. [5]



2. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p, q, r \in \mathbb{R}$, has roots α, β and γ .

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma. \quad [3]$$

(b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

(e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

(ii) Explain why the equation will have at least one real root for all values of q . [1]

(This question continues on the following page)

(Question 2 continued)

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p, q, r, s \in \mathbb{R}$, has roots α, β, γ and δ .

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

- (f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q . [3]
- (ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]
- (g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

- (h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]
- (ii) Write down the integer root of this equation. [1]
- (iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

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1. [Maximum marks: 27]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are 1, 3, 6, 10 and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		1, 3, 6, 10, 15, ...
Square numbers		1, 4, 9, 16, 25, ...
Pentagonal numbers		1, 5, 12, 22, 35, ...

For an r -sided regular polygon, where $r \in \mathbb{Z}^+$, $r \geq 3$, the n th polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

(This question continues on the following page)

(Question 1 continued)

Hence, for square numbers, $P_4(n) = \frac{(4-2)n^2 - (4-4)n}{2} = n^2$.

- (a) (i) For triangular numbers, verify that $P_3(n) = \frac{n(n+1)}{2}$. [2]
- (ii) The number 351 is a triangular number. Determine which one it is. [2]
- (b) (i) Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. [2]
- (ii) State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. [1]
- (iii) For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). [1]
- (c) Show that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3]

The n th pentagonal number can be represented by the arithmetic series

$$P_5(n) = 1 + 4 + 7 + \dots + (3n - 2).$$

- (d) Hence show that $P_5(n) = \frac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. [3]
- (e) By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. [5]

A polygonal number, $P_r(n)$, can be represented by the series

$$\sum_{m=1}^n (1 + (m-1)(r-2)) \text{ where } r \in \mathbb{Z}^+, r \geq 3.$$

- (f) Use mathematical induction to prove that $P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}$ where $n \in \mathbb{Z}^+$. [8]

2. [Maximum marks: 28]

This question asks you to explore cubic polynomials of the form $(x - r)(x^2 - 2ax + a^2 + b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r = 1$, $a = 4$ and $b = 1$.

Consider the equation $(z - 1)(z^2 - 8z + 17) = 0$ for $z \in \mathbb{C}$.

- (a) (i) Given that 1 and $4 + i$ are roots of the equation, write down the third root. [1]
- (ii) Verify that the mean of the two complex roots is 4. [1]

Consider the function $f(x) = (x - 1)(x^2 - 8x + 17)$ for $x \in \mathbb{R}$.

- (b) Show that the line $y = x - 1$ is tangent to the curve $y = f(x)$ at the point $A(4, 3)$. [4]
- (c) Sketch the curve $y = f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x -axis. [2]

Consider the function $g(x) = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b > 0$.

- (d) (i) Show that $g'(x) = 2(x - r)(x - a) + x^2 - 2ax + a^2 + b^2$. [2]
- (ii) Hence, or otherwise, prove that the tangent to the curve $y = g(x)$ at the point $A(a, g(a))$ intersects the x -axis at the point $R(r, 0)$. [6]

The equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ for $z \in \mathbb{C}$ has roots r and $a \pm bi$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b > 0$.

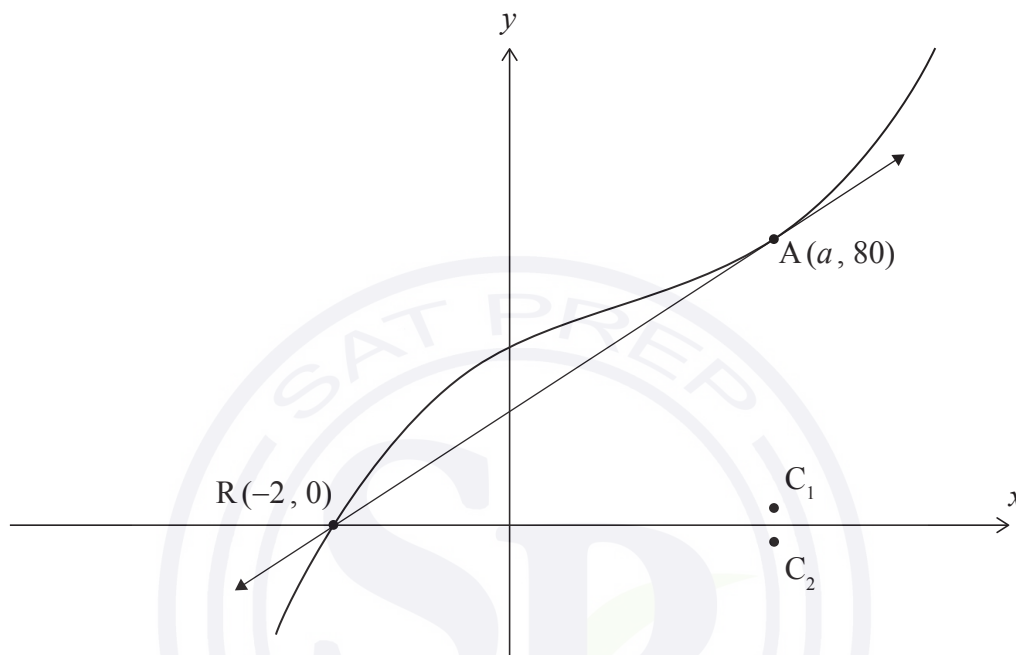
- (e) Deduce from part (d)(i) that the complex roots of the equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ can be expressed as $a \pm i\sqrt{g'(a)}$. [1]

(This question continues on the following page)

(Question 2 continued)

On the Cartesian plane, the points $C_1(a, \sqrt{g'(a)})$ and $C_2(a, -\sqrt{g'(a)})$ represent the real and imaginary parts of the complex roots of the equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$.

The following diagram shows a particular curve of the form $y = (x - r)(x^2 - 2ax + a^2 + 16)$ and the tangent to the curve at the point $A(a, 80)$. The curve and the tangent both intersect the x -axis at the point $R(-2, 0)$. The points C_1 and C_2 are also shown.



- (f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z - r)(z^2 - 2az + a^2 + 16) = 0$ for $z \in \mathbb{C}$. [4]

- (ii) State the coordinates of C_2 . [1]

Consider the curve $y = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $a \neq r$, $b > 0$. The points $A(a, g(a))$ and $R(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

- (g) (i) Show that the x -coordinate of P is $\frac{1}{3}(2a + r)$.
You are **not** required to demonstrate a change in concavity. [2]
- (ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A. [1]

Consider the special case where $a = r$ and $b > 0$.

- (h) (i) Sketch the curve $y = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $a = r = 1$ and $b = 2$. [2]
- (ii) For $a = r$ and $b > 0$, state in terms of r , the coordinates of points P and A. [1]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 9 November 2021 (morning)

1 hour

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1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z) = \frac{e^z + e^{-z}}{2}$ and $g(z) = \frac{e^z - e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u , such that $t, u \in \mathbb{R}$.

- (a) Verify that $u = f(t)$ satisfies the differential equation $\frac{d^2u}{dt^2} = u$. [2]
- (b) Show that $(f(t))^2 + (g(t))^2 = f(2t)$. [3]
- (c) Using $e^{iu} = \cos u + i \sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for
 - (i) $f(iu)$; [3]
 - (ii) $g(iu)$. [2]
- (d) Hence find, and simplify, an expression for $(f(iu))^2 + (g(iu))^2$. [2]
- (e) Show that $(f(t))^2 - (g(t))^2 = (f(iu))^2 - (g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos \theta, \sin \theta)$ defines points on the unit circle with equation $x^2 + y^2 = 1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2 - y^2 = 1$. This hyperbola has two asymptotes.

- (f) Sketch the graph of $x^2 - y^2 = 1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2 - y^2 = 1$ can be rotated to coincide with the curve defined by $xy = k$, $k \in \mathbb{R}$.

- (g) Find the possible values of k . [5]

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{dx}{dt} = x - y \text{ and } \frac{dy}{dt} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

- (a) (i) By solving the differential equation $\frac{dy}{dt} = y$, show that $y = Ae^t$ where A is a constant. [3]
- (ii) Show that $\frac{dx}{dt} - x = -Ae^t$. [1]
- (iii) Solve the differential equation in part (a)(ii) to find x as a function of t . [4]

Now consider the case where $a = -1$.

- (b) (i) By differentiating $\frac{dy}{dt} = -x + y$ with respect to t , show that $\frac{d^2y}{dt^2} = 2\frac{dy}{dt}$. [3]
- (ii) By substituting $Y = \frac{dy}{dt}$, show that $Y = Be^{2t}$ where B is a constant. [3]
- (iii) Hence find y as a function of t . [2]
- (iv) Hence show that $x = -\frac{B}{2}e^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

- (c) (i) Show that $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = Fe^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

- (ii) Find the two values for λ that satisfy $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2 .

- (iii) Verify that $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

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1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a - x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2 - x)$.

- (a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2 - x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

- (b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for
- the odd values $n = 3$ and $n = 5$;
 - the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	Number of local minimum points	Number of points of inflexion with zero gradient
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a - x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

- (c) Show that $f'_n(x) = nx^{n-1}(a - 2x)(a - x)^{n-1}$. [5]
- (d) State the three solutions to the equation $f'_n(x) = 0$. [2]
- (e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(This question continues on the following page)

(Question 1 continued)

(f) Hence, or otherwise, show that $f'_n\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f'_n(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n , where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n , where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a - x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a - x)^n = k$ has four solutions for x . [5]



2. [Maximum mark: 24]

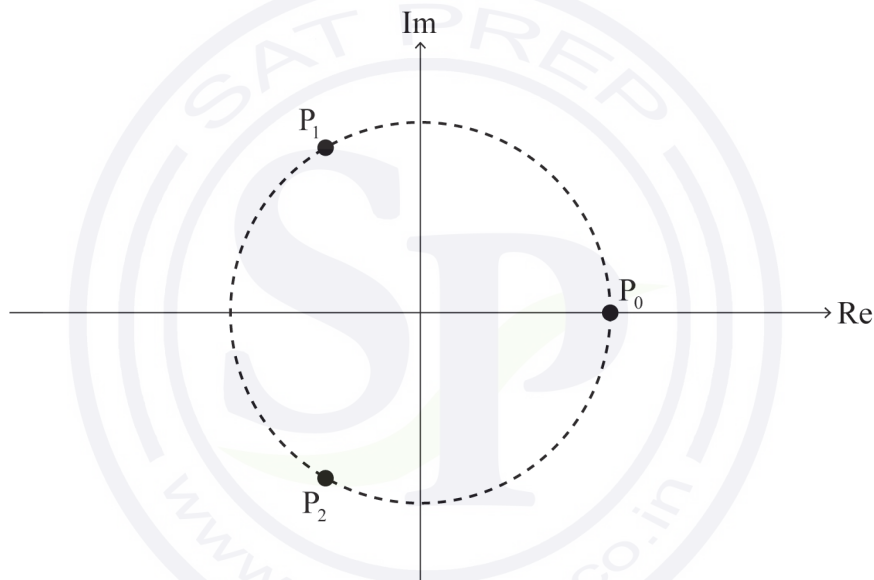
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \dots, \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$. Each root can be represented by a point $P_0, P_1, P_2, \dots, P_{n-1}$, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω . On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1 .

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are $1, \omega$ and ω^2 . On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $O(0, 0)$.



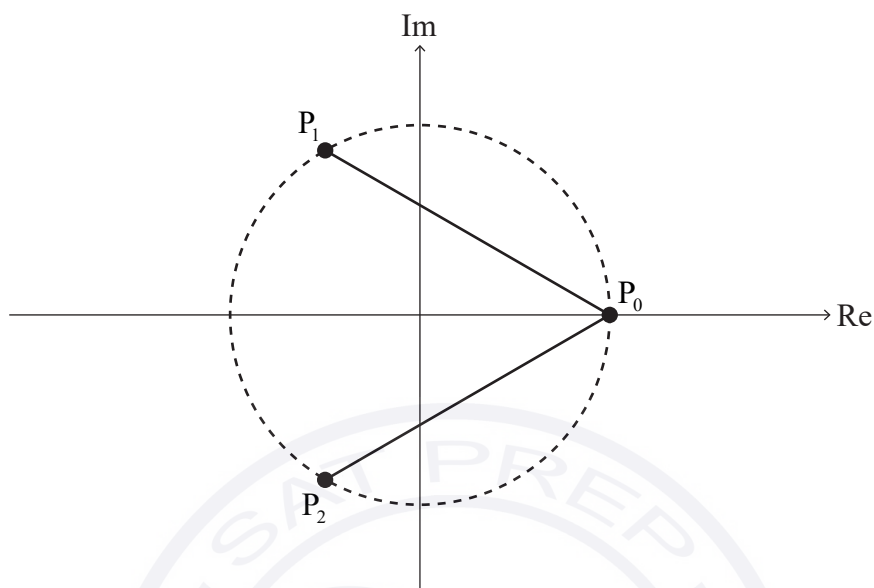
(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

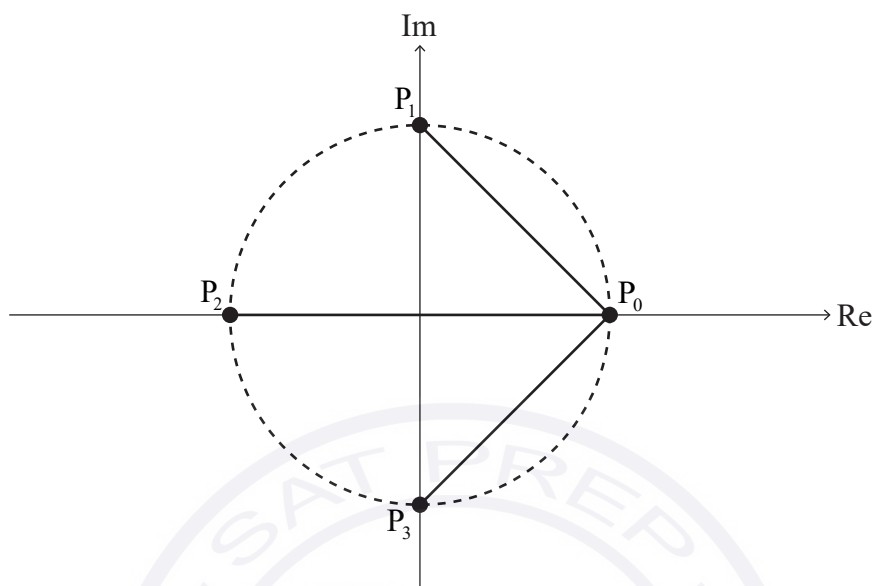
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2$ and ω^3 .

(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $O(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



- (d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots $1, \omega, \omega^2, \omega^3$ and ω^4 .

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \dots, \omega^{n-1}$. On an Argand diagram, these roots can be represented by the points $P_0, P_1, P_2, \dots, P_{n-1}$ respectively where $[P_0P_1], [P_0P_2], \dots, [P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $O(0, 0)$.

- (e) Suggest a value for $P_0P_1 \times P_0P_2 \times \dots \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

- (f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω . [2]

- (ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω . [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$ where $z \in \mathbb{C}$.

- (g) (i) Express $z^{n-1} + z^{n-2} + \dots + z + 1$ as a product of linear factors over the set \mathbb{C} . [3]

- (ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

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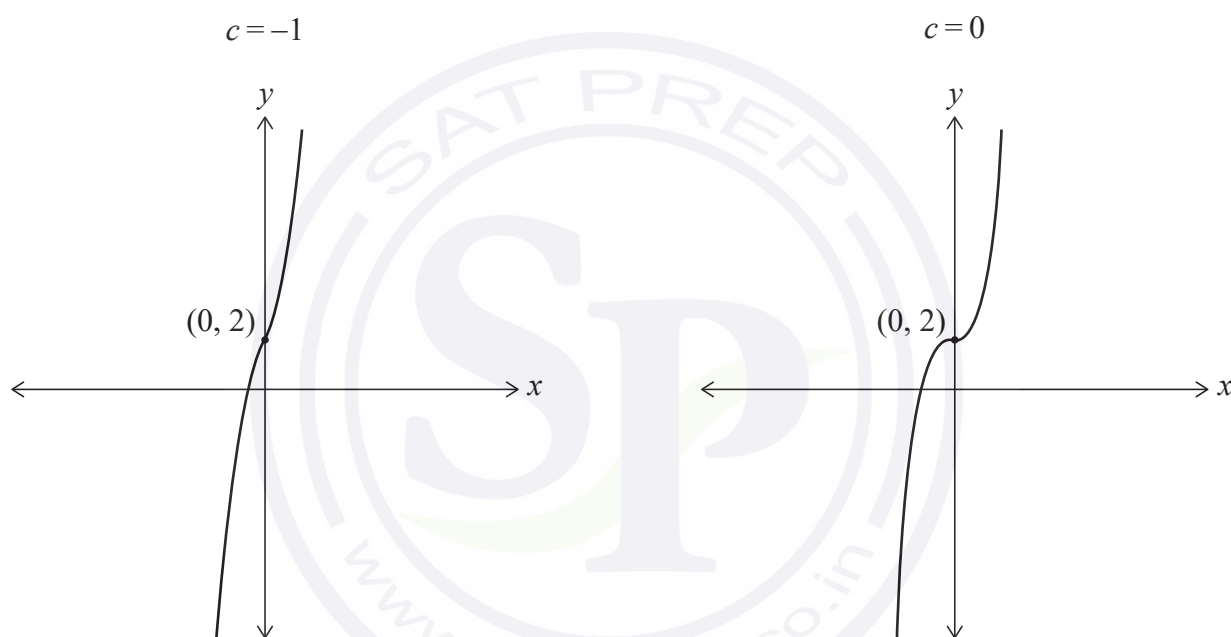
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1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y -intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

- (c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has
- (i) a point of inflexion with zero gradient; [1]
 - (ii) one local maximum point and one local minimum point; [2]
 - (iii) no points where the gradient is equal to zero. [1]
- (d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that
- (i) the y -coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]
 - (ii) the y -coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]
- (e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has
- (i) exactly one x -axis intercept; [2]
 - (ii) exactly two x -axis intercepts; [2]
 - (iii) exactly three x -axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c, d \in \mathbb{R}$.

- (f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x -axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

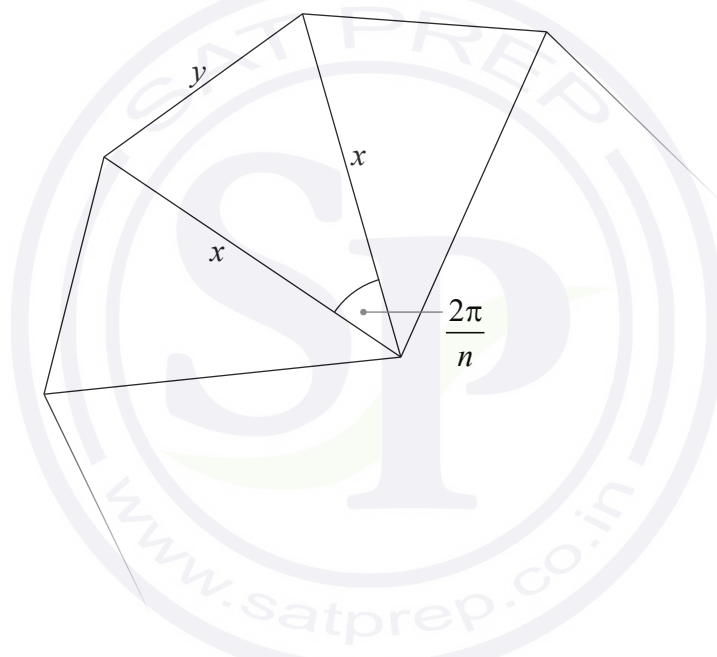
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P .

(a) Find the side length, s , where $s > 0$, of a square such that $A = P$. [3]

An n -sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n -sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n , an expression for the area, A_T , of one of these isosceles triangles. [1]

(c) Show that $y = 2x \sin \frac{\pi}{n}$. [2]

Consider a n -sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n \tan \frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n \rightarrow \infty} \left(4n \tan \frac{\pi}{n} \right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a , b and $\sqrt{a^2 + b^2}$, where $a \geq b$, such that $A = P$.

(f) Show that $a = \frac{8}{b-4} + 4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a, b, A, P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Specimen

1 hour

Instructions to candidates

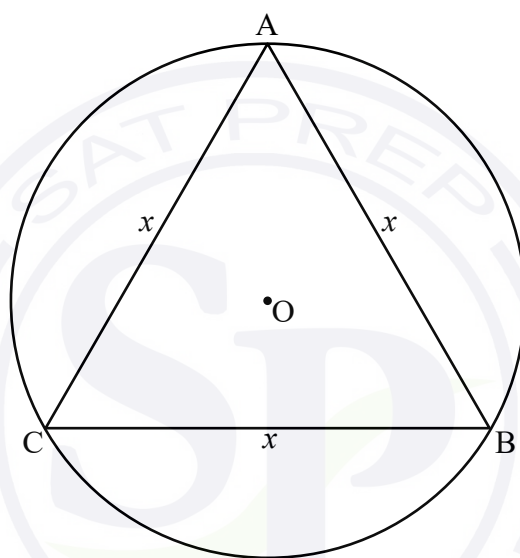
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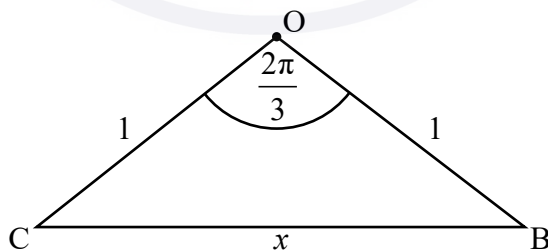
1. [Maximum mark: 30]

This question asks you to investigate regular n -sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π .

- (a) Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram.



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units. [3]

- (b) Consider a square of side length, x units, inscribed in a circle of radius 1 unit. By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. [3]

(This question continues on the following page)

(Question 1 continued)

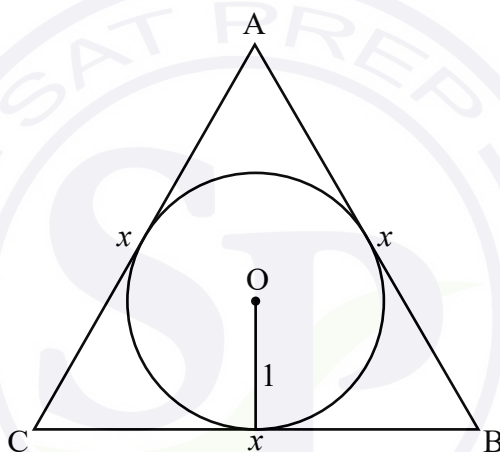
- (c) Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. [2]

Let $P_i(n)$ represent the perimeter of any n -sided regular polygon inscribed in a circle of radius 1 unit.

- (d) Show that $P_i(n) = 2n \sin\left(\frac{\pi}{n}\right)$. [3]

- (e) Use an appropriate Maclaurin series expansion to find $\lim_{n \rightarrow \infty} P_i(n)$ and interpret this result geometrically. [5]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n -sided regular polygon circumscribed about a circle of radius 1 unit.

- (f) Show that $P_c(n) = 2n \tan\left(\frac{\pi}{n}\right)$. [4]

Consider the function $P(x) = 2x \tan\left(\frac{\pi}{x}\right)$ where $x \in \mathbb{R}, x > 2$.

- (g) (i) By writing $P(x)$ in the form $\frac{2 \tan\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$, find $\lim_{x \rightarrow \infty} P(x)$.

- (ii) Hence state the value of $\lim_{n \rightarrow \infty} P_c(n)$ for integers $n > 2$. [5]

(This question continues on the following page)

(Question 1 continued)

- (h) Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n . [2]

The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π .

- (i) Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π . [3]



2. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \leq x \leq 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

- (a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \leq x \leq 1$. [2]
- (b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n . Hence suggest an expression for odd values of n describing, in terms of n , the number of
- (i) local maximum points;
- (ii) local minimum points. [4]
- (c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \leq x \leq 1$. [2]
- (d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n . Hence suggest an expression for even values of n describing, in terms of n , the number of
- (i) local maximum points;
- (ii) local minimum points. [4]
- (e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos \frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n .

- (f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1) \arccos x)$.

- (g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x) \cos(\arccos x) - \sin(n \arccos x) \sin(\arccos x)$. [2]
- (h) Hence
- (i) show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x)$, $n \in \mathbb{Z}^+$;
- (ii) express $f_3(x)$ as a cubic polynomial. [5]