Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

8823 – 7103

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

This question asks you to explore some properties of the family of curves $y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

(a) By systematically varying the value of b, or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x-axis intercepts. [2]

(b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly

(i) one x-axis intercept; [1]

(ii) three x-axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

(c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly

(i) two x-axis intercepts; [1]

(ii) one x-axis intercept; [1]

(iii) three x-axis intercepts. [1]

(This question continues on the following page)

(Question 1 continued)

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

(d) Consider the case where the curve has exactly three x-axis intercepts. State whether each point of zero gradient is located above or below the x-axis. [1]

(e) Show that the curve has a point of zero gradient at $\mathrm{P}(0, b)$ and a point of zero gradient at $\mathrm{Q}\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$. [5]

(f) Consider the points P and Q for $a > 0$ and $b > 0$.

(i) Find an expression for $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]

(ii) Determine whether each point is located above or below the x-axis. [1]

(g) Consider the points P and Q for $a < 0$ and $b > 0$.

(i) State whether P is a local maximum or a local minimum and whether it is above or below the x-axis. [1]

(ii) State the conditions on a and b that determine when Q is below the x-axis. [1]

(h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x-axis intercepts. [5]

2. [Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α.

(a) Consider a family of straight lines, L, with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C, at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

(i) Write down an expression for the gradient of L in terms of x and y. [1]

(ii) Hence show that the gradient of C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. [1]

(iii) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, show that the family of curves, C, has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

(b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x-intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

(c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $\mathrm{M}\left(a-b,\, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b,\, -2\sqrt{ab}\right)$. [6]

(d) At point M, show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

(This question continues on the following page)

(Question 2 continued)

Consider two families of curves, F and G.

The gradient of F is denoted by $f(x, y)$.

The gradient of G is denoted by $g(x, y)$.

Each member of F intersects every member of G at an acute angle, α.

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}.$$

In part (e), consider the specific case where $f(x, y) = -\frac{x}{y}$, for $x \neq 0$, $y \neq 0$ and $\alpha = \frac{\pi}{4}$.

(e) (i) Show that $g(x, y) = \frac{y - x}{y + x}$. [2]

(ii) Hence, by solving the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x}{y + x}$, find a general equation that represents this family of curves, G. Give your answer in the form $h(x, y) = d$ where d is a parameter. [9]

(f) By considering $\lim\limits_{\alpha \to \frac{\pi}{2}} \tan\alpha$, show that, for all finite $f(x, y)$,

$$\lim_{\alpha \to \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}.$$ [2]

Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

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2223 – 7113

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1. [Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a.

In this question you may either use the change of logarithm base formula $\log_a x = \dfrac{\ln x}{\ln a}$ or a graphic display calculator "logarithm to any base feature".

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+,\ a \neq 1.$$

(a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x-axis intercepts. [4]

(This question continues on the following page)

(Question 1 continued)

In parts (b) and (c), consider the case where $a = \mathrm{e}$. Note that $\ln x \equiv \log_{\mathrm{e}} x$.

(b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a minimum. [5]

(c) Hence deduce that $x > \ln x$. [1]

(d) There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a.

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a, write down the values of p, q and r. [4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

(e) Find the exact coordinates of P and the exact value of a. [8]

(f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have

(i) two intersection points; [1]

(ii) no intersection points. [1]

2. [Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where m, $c \in \mathbb{R}$ and m, $c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c, in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

(a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

(b) (i) Show that $r = -\dfrac{c}{m}$. [1]

(ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \dfrac{m^2}{m+2}$. [4]

(iii) State any further restrictions on the value of m. [1]

There are only three **integer** sets of values of m, r and c, that form an AS-linear function. One of these is $L(x) = -x - 1$.

(c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c. [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}$, $a \neq 0$ and b, $c \in \mathbb{R}$.

Let r_1, $r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

(d) Write down an expression for

(i) the sum of roots, $r_1 + r_2$, in terms of a and b. [1]

(ii) the product of roots, $r_1 r_2$, in terms of a and c. [1]

(This question continues on the following page)

(Question 2 continued)

If a, r_1, b, r_2 and c, in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(i) write down an expression for $r_2 - r_1$ in terms of a and b; [1]

(ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \dfrac{a^2 - ab - b}{2a}$; [2]

(iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\dfrac{1}{2}$. [3]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where $a = -\dfrac{1}{2}$.

(g) (i) Find an expression for r_1 in terms of b. [2]

(ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\dfrac{-p \pm q\sqrt{s}}{2}$ where p, q, $s \in \mathbb{Z}^+$. [5]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

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2223 – 7108

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1. [Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = xe^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x-axis and the line $x = b$, where $b > 0$, is given by $\dfrac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, A_n, of the region between the graph $y = f_n(x)$ and the x-axis can be written as $A_n = \int_0^{\infty} f_n(x)\,dx$ and is given by $\lim\limits_{b\to\infty}\int_0^b f_n(x)\,dx$.

(c) (i) Use l'Hôpital's rule to find $\lim\limits_{b\to\infty}\dfrac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1. [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4; [2]

(ii) A_5. [1]

(e) Suggest an expression for A_n in terms of n, where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m, $\lim\limits_{x\to\infty} x^m e^{-x} = 0$. [8]

2. [Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers x_1, $x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of x_1 and x_2 as a function, f, of x_1 only. [2]

(b) (i) Find the value of x_1 for which the function is maximum. [1]

(ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S, where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, $x_1, x_2, \ldots, x_n$.

The geometric mean is defined as $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \le \frac{(x_1 + x_2 + \ldots + x_n)}{n}$.

(d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \ldots = x_n$. [2]

(ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]

(e) Hence determine the value of

(i) $M_3(12)$; [1]

(ii) $M_4(12)$; [1]

(iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n.

(f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]

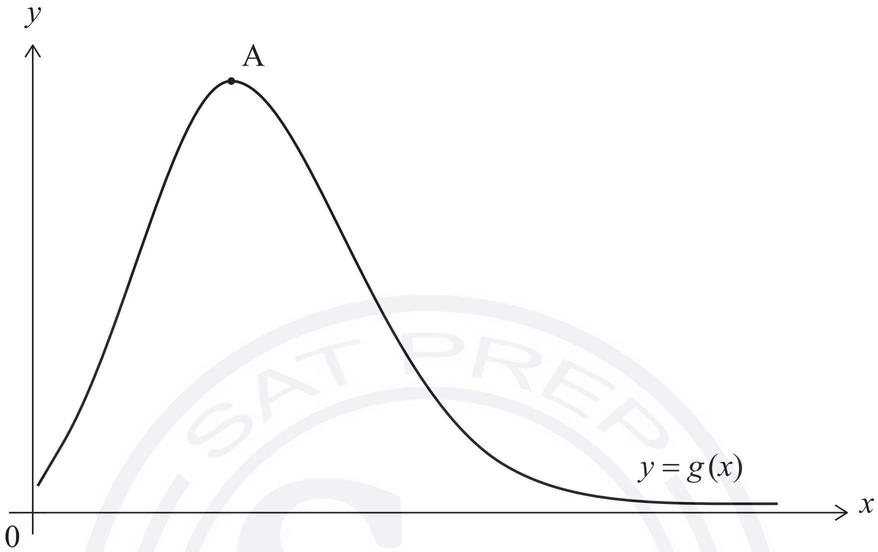
(g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the function g, defined by $\ln\left(g(x)\right) = x\ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



(h) Find, in terms of S, the x-coordinate of point A. [6]

(i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]

(j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}^+$. [3]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 8 November 2022 (afternoon)

1 hour

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8822 – 7103

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1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^{n} i^q = 1^q + 2^q + 3^q + \ldots + n^q \text{ **where** } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n, for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^{n} i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^{n} i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p. [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^{n} i^2 = a_1 n + a_2 n^2 + a_3 n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1, a_2 and a_3. [3]

(iii) Hence, find the values of a_1, a_2 and a_3. [2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q.

Consider the function $f(x) = 1 + x + x^2 + \ldots + x^n,\ n \in \mathbb{Z}^+$.

(c) Show that $x f'(x) = x + 2x^2 + 3x^3 + \ldots + nx^n$. [1]

Let $f_1(x) = x f'(x)$ and consider the following family of functions:

$$f_2(x) = x f_1'(x)$$

$$f_3(x) = x f_2'(x)$$

$$f_4(x) = x f_3'(x)$$

$$\ldots$$

$$f_q(x) = x f_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^{n} i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^{n} i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \ldots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \dfrac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

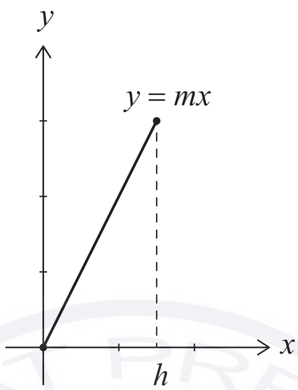
(g) (i) Show that $\lim\limits_{x \to 1} f_1(x)$ is in indeterminate form. [1]

(ii) Hence, by applying l'Hôpital's rule, show that $\lim\limits_{x \to 1} f_1(x) = \dfrac{1}{2} n(n+1)$. [5]

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \le x \le h$ and m, h are positive constants.



When this line is rotated through $360°$ about the x-axis, a cone is formed with a curved surface area A given by:

$$A = 2\pi\int_0^h y\sqrt{1+m^2}\,\mathrm{d}x\,.$$

(a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\,\pi$. [2]

(b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \le x \le h$ through $360°$ about the x-axis.

(i) Deduce an expression for the radius of this cone r in terms of h and m. [1]

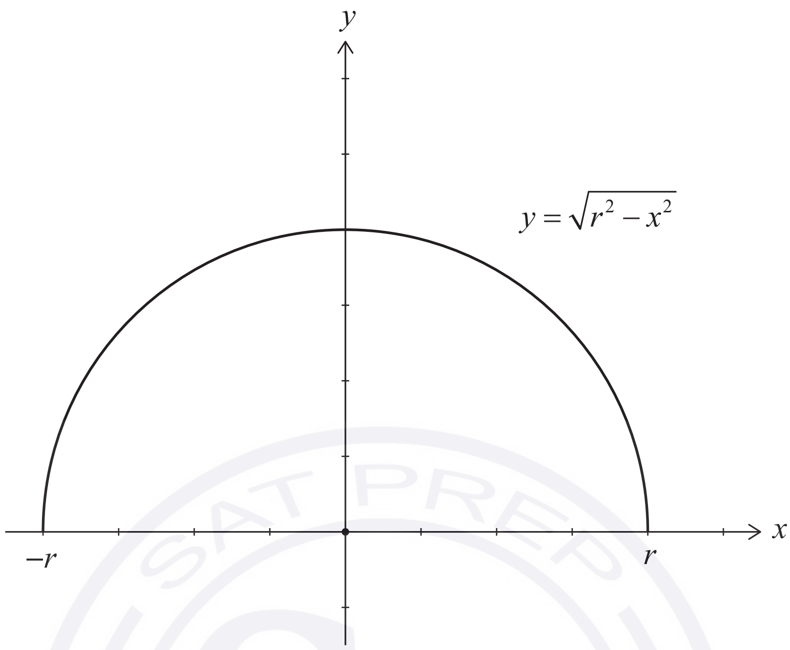
(ii) Deduce an expression for the slant height l in terms of h and m. [2]

(iii) Hence, by using the above integral, show that $A = \pi rl$. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the semi-circle, with radius r, defined by $y=\sqrt{r^2-x^2}$ where $-r \le x \le r$.



(c) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

A differentiable curve $y=f(x)$ is defined for $x_1 \le x \le x_2$ and $y \ge 0$. When any such curve is rotated through $360°$ about the x-axis, the surface formed has an area A given by:

$$A = 2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x\,.$$

(d) A sphere is formed by rotating the semi-circle $y=\sqrt{r^2-x^2}$ where $-r \le x \le r$ through $360°$ about the x-axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(This question continues on the following page)

(Question 2 continued)

(e) Let $f\left(x\right)=\sqrt{r^2-x^2}$ where $-r\le x\le r$.

The graph of $y=f(x)$ is transformed to the graph of $y=f(kx)$, $k>0$. This forms a different curve, called a semi-ellipse.

(i) Describe this geometric transformation. [2]

(ii) Write down the x-intercepts of the graph $y=f(kx)$ in terms of r and k. [1]

(iii) For $y=f(kx)$, find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, r and k. [2]

(iv) The semi-ellipse $y=f(kx)$ is rotated $360°$ about the x-axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A, of the ellipsoid.

Give your answer in the form $2\pi\int_{x_1}^{x_2}\sqrt{p\left(x\right)}\,\mathrm{d}x$, where $p(x)$ is a polynomial. [4]

(v) Planet Earth can be modelled as an ellipsoid. In this model:

- the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
- the distance from the North Pole to the South Pole is $12\ 714\,\mathrm{km}$.
- the diameter of the equator is $12\ 756\,\mathrm{km}$.

By choosing suitable values for r and k, find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a\times 10^q$ where $1\le a<10$ and $q\in\mathbb{Z}^+$. [4]

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Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

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2222 – 7113

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ **for various values of** a **and** b**, where** $a, b \in \mathbb{N}$.

(a) On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3$, $x \ge 0$ [2]

(ii) $y^2 = x^3 + 1$, $x \ge -1$ [2]

(b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

(ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \ge -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

(c) By varying the value of b, suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x$, $x \ge 0$.

(d) (i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \pm\dfrac{3x^2 + 1}{2\sqrt{x^3 + x}}$, for $x > 0$. [3]

(ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

(e) Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\dfrac{p\sqrt{3} + q}{r}}$, where p, q, $r \in \mathbb{Z}$. [7]

(This question continues on the following page)

(Question 1 continued)

$P(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $P(-1, -1)$ lies on C.

(f) (i) Find the equation of the tangent to C at P. [2]

(ii) Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. [2]

(g) The point $S(-1, 1)$ also lies on C. The line $[QS]$ intersects C at a further point. Determine the coordinates of this point. [5]

2. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q, $r \in \mathbb{R}$, has roots α, β and γ.

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma.$$

[3]

(b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

(e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

(ii) Explain why the equation will have at least one real root for all values of q. [1]

(This question continues on the following page)

(Question 2 continued)

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p,\ q,\ r,\ s \in \mathbb{R}$, has roots $\alpha,\ \beta,\ \gamma$ and δ.

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

(f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

(ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

(h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

(ii) Write down the integer root of this equation. [1]

(iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

2222 – 7108

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

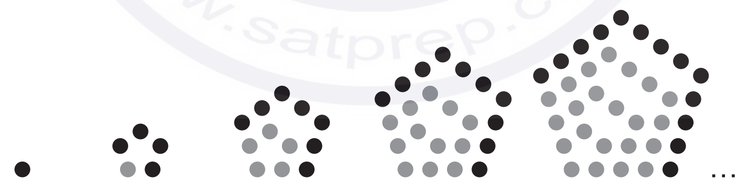
1. [Maximum marks: 27]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are $1, 3, 6, 10$ and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		$1, 3, 6, 10, 15, \ldots$
Square numbers		$1, 4, 9, 16, 25, \ldots$
Pentagonal numbers		$1, 5, 12, 22, 35, \ldots$

For an r-sided regular polygon, where $r \in \mathbb{Z}^+$, $r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

(This question continues on the following page)

(Question 1 continued)

Hence, for square numbers, $P_4(n) = \dfrac{(4-2)n^2 - (4-4)n}{2} = n^2$.

(a) (i) For triangular numbers, verify that $P_3(n) = \dfrac{n(n+1)}{2}$. [2]

(ii) The number 351 is a triangular number. Determine which one it is. [2]

(b) (i) Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. [2]

(ii) State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. [1]

(iii) For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). [1]

(c) Show that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3]

The nth pentagonal number can be represented by the arithmetic series

$$P_5(n) = 1 + 4 + 7 + \ldots + (3n-2).$$

(d) Hence show that $P_5(n) = \dfrac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. [3]

(e) By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. [5]

A polygonal number, $P_r(n)$, can be represented by the series

$$\sum_{m=1}^{n} \left(1 + (m-1)(r-2)\right) \text{ where } r \in \mathbb{Z}^+,\ r \geq 3.$$

(f) Use mathematical induction to prove that $P_r(n) = \dfrac{(r-2)n^2 - (r-4)n}{2}$ where $n \in \mathbb{Z}^+$. [8]

2. [Maximum marks: 28]

This question asks you to explore cubic polynomials of the form $(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r=1$, $a=4$ and $b=1$.

Consider the equation $(z-1)(z^2-8z+17)=0$ for $z \in \mathbb{C}$.

(a) (i) Given that 1 and $4+\mathrm{i}$ are roots of the equation, write down the third root. [1]

(ii) Verify that the mean of the two complex roots is 4. [1]

Consider the function $f(x)=(x-1)(x^2-8x+17)$ for $x \in \mathbb{R}$.

(b) Show that the line $y=x-1$ is tangent to the curve $y=f(x)$ at the point $\mathrm{A}(4, 3)$. [4]

(c) Sketch the curve $y=f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x-axis. [2]

Consider the function $g(x)=(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(d) (i) Show that $g'(x)=2(x-r)(x-a)+x^2-2ax+a^2+b^2$. [2]

(ii) Hence, or otherwise, prove that the tangent to the curve $y=g(x)$ at the point $\mathrm{A}(a, g(a))$ intersects the x-axis at the point $\mathrm{R}(r, 0)$. [6]

The equation $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$ has roots r and $a \pm b\mathrm{i}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

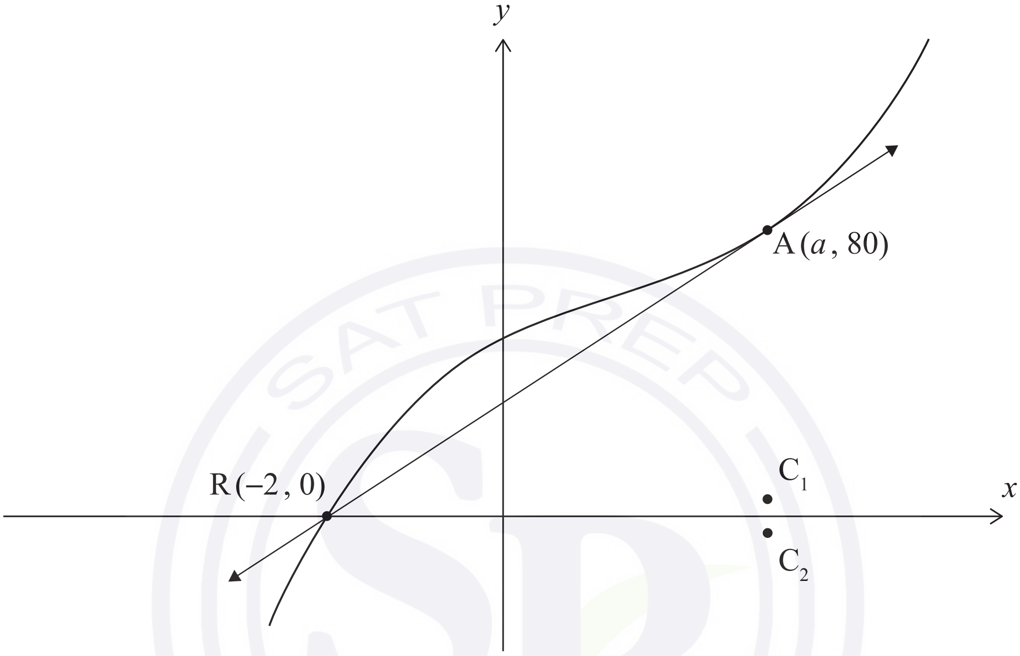
(e) Deduce from part (d)(i) that the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$ can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$. [1]

(This question continues on the following page)

(Question 2 continued)

On the Cartesian plane, the points $C_1\left(a, \sqrt{g'(a)}\right)$ and $C_2\left(a, -\sqrt{g'(a)}\right)$ represent the real and imaginary parts of the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$.

The following diagram shows a particular curve of the form $y=(x-r)(x^2-2ax+a^2+16)$ and the tangent to the curve at the point $A(a, 80)$. The curve and the tangent both intersect the x-axis at the point $R(-2, 0)$. The points C_1 and C_2 are also shown.



(f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z-r)(z^2-2az+a^2+16)=0$ for $z \in \mathbb{C}$. [4]

(ii) State the coordinates of C_2. [1]

Consider the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a \neq r$, $b>0$. The points $A\left(a, g(a)\right)$ and $R(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

(g) (i) Show that the x-coordinate of P is $\frac{1}{3}(2a+r)$.

You are **not** required to demonstrate a change in concavity. [2]

(ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A. [1]

Consider the special case where $a=r$ and $b>0$.

(h) (i) Sketch the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a=r=1$ and $b=2$. [2]

(ii) For $a=r$ and $b>0$, state in terms of r, the coordinates of points P and A. [1]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 9 November 2021 (morning)

1 hour

Instructions to candidates

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- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
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- The maximum mark for this examination paper is **[55 marks]**.

3 pages

8821 – 7103

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1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z)=\dfrac{e^{z}+e^{-z}}{2}$ and $g(z)=\dfrac{e^{z}-e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u, such that $t,\ u \in \mathbb{R}$.

(a) Verify that $u=f(t)$ satisfies the differential equation $\dfrac{d^2u}{dt^2}=u$. [2]

(b) Show that $(f(t))^2+(g(t))^2=f(2t)$. [3]

(c) Using $e^{iu}=\cos u+i\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

(i) $f(iu)$; [3]

(ii) $g(iu)$. [2]

(d) Hence find, and simplify, an expression for $(f(iu))^2+(g(iu))^2$. [2]

(e) Show that $(f(t))^2-(g(t))^2=(f(iu))^2-(g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta,\ \sin\theta)$ defines points on the unit circle with equation $x^2+y^2=1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta),\ g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2-y^2=1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2-y^2=1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2-y^2=1$ can be rotated to coincide with the curve defined by $xy=k,\ k \in \mathbb{R}$.

(g) Find the possible values of k. [5]

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{dx}{dt} = x - y \text{ and } \frac{dy}{dt} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a) (i) By solving the differential equation $\frac{dy}{dt} = y$, show that $y = Ae^t$ where A is a constant. [3]

(ii) Show that $\frac{dx}{dt} - x = -Ae^t$. [1]

(iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

(b) (i) By differentiating $\frac{dy}{dt} = -x + y$ with respect to t, show that $\frac{d^2y}{dt^2} = 2\frac{dy}{dt}$. [3]

(ii) By substituting $Y = \frac{dy}{dt}$, show that $Y = Be^{2t}$ where B is a constant. [3]

(iii) Hence find y as a function of t. [2]

(iv) Hence show that $x = -\frac{B}{2}e^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c) (i) Show that $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = Fe^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(ii) Find the two values for λ that satisfy $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2.

(iii) Verify that $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

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2221 – 7113

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1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$;
- the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

(c) Show that $f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(This question continues on the following page)

(Question 1 continued)

(f) Hence, or otherwise, show that $f_n'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a - x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a - x)^n = k$ has four solutions for x. [5]

2. [Maximum mark: 24]

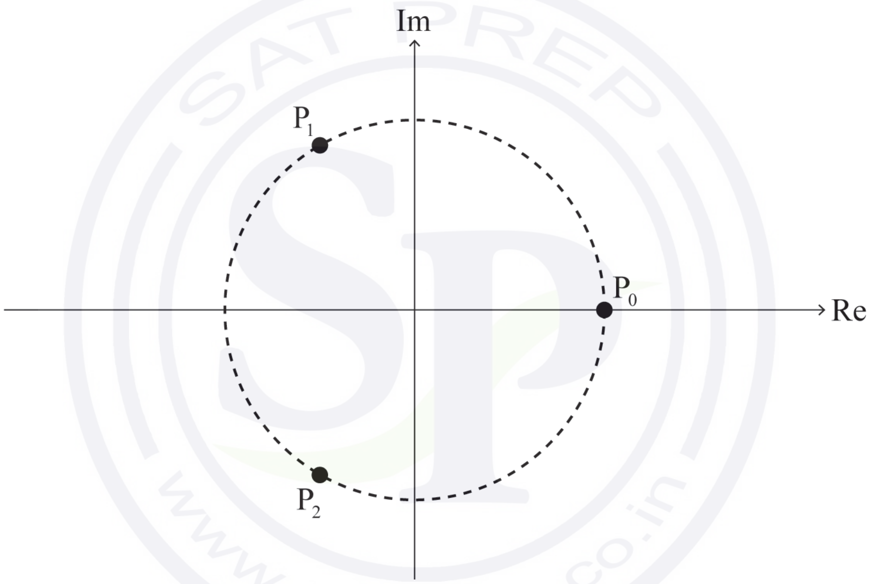
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2, ..., ω^{n-1}, where $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$. Each root can be represented by a point $\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_{n-1}$, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω. On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1.

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1, ω and ω^2. On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $\mathrm{O}(0, 0)$.



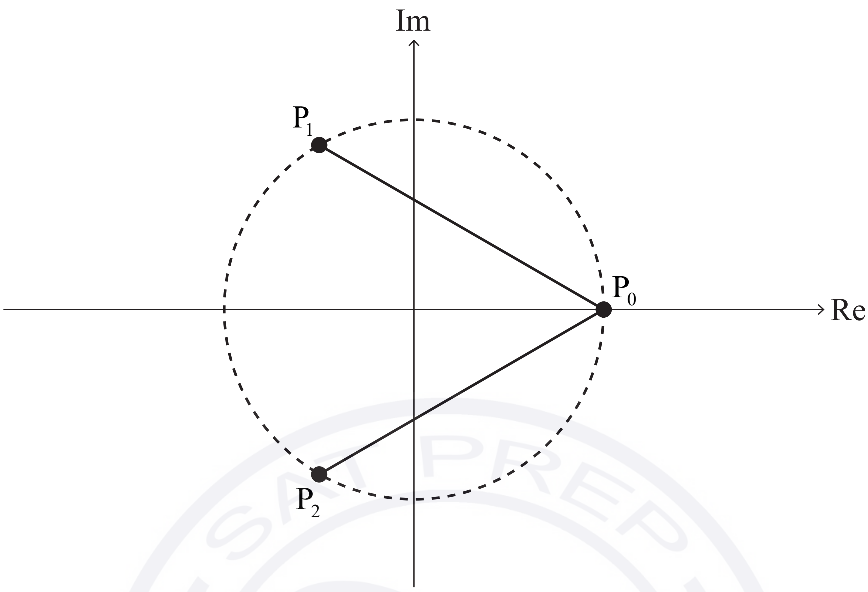
(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

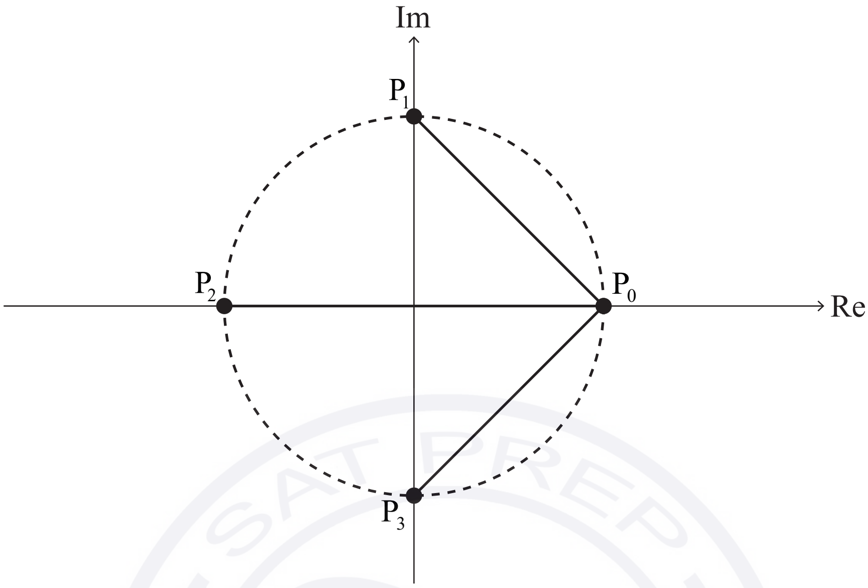
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2 and ω^3.

(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $O(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1, ω, ω^2, ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2, ..., ω^{n-1}. On an Argand diagram, these roots can be represented by the points P_0, P_1, P_2, ..., P_{n-1} respectively where $[P_0P_1]$, $[P_0P_2]$, ..., $[P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $O(0, 0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times ... \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω. [2]

(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω. [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \ldots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

References:

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

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- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
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- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

2221 – 7108

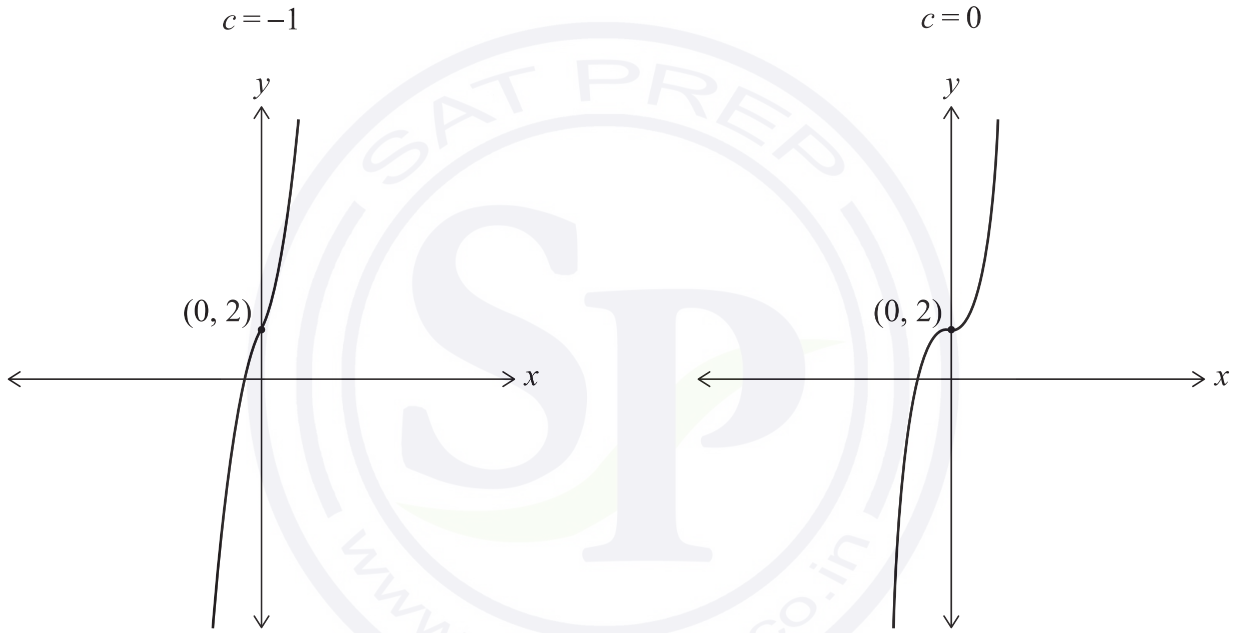
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y-intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

(c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has

(i) a point of inflexion with zero gradient; [1]

(ii) one local maximum point and one local minimum point; [2]

(iii) no points where the gradient is equal to zero. [1]

(d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that

(i) the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]

(ii) the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]

(e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has

(i) exactly one x-axis intercept; [2]

(ii) exactly two x-axis intercepts; [2]

(iii) exactly three x-axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c,\ d \in \mathbb{R}$.

(f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x-axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

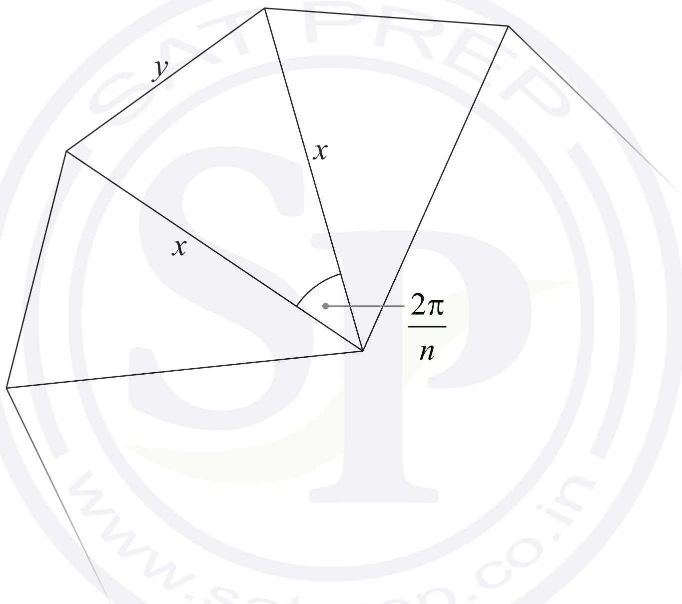
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P.

(a) Find the side length, s, where $s > 0$, of a square such that $A = P$. [3]

An n-sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n-sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n, an expression for the area, A_T, of one of these isosceles triangles. [1]

(c) Show that $y = 2x\sin\frac{\pi}{n}$. [2]

Consider a n-sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n\tan\frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n \to \infty}\left(4n \tan \frac{\pi}{n}\right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2 + b^2}$, where $a \ge b$, such that $A = P$.

(f) Show that $a = \frac{8}{b-4} + 4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a,\ b,\ A,\ P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

Specimen

1 hour

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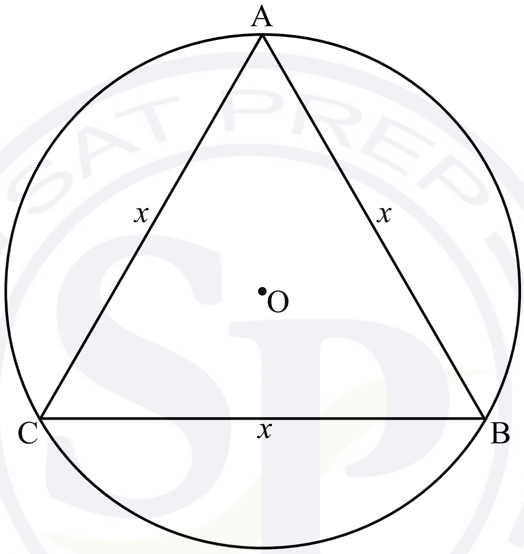
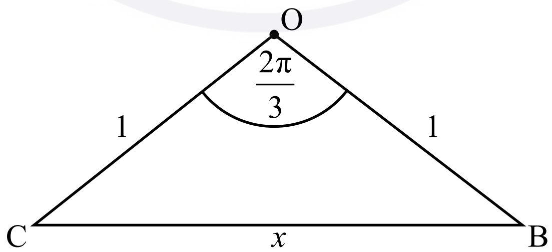
1. [Maximum mark: 30]

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

(a) Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram.



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units. [3]

(b) Consider a square of side length, x units, inscribed in a circle of radius 1 unit. By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. [3]

(This question continues on the following page)

(Question 1 continued)

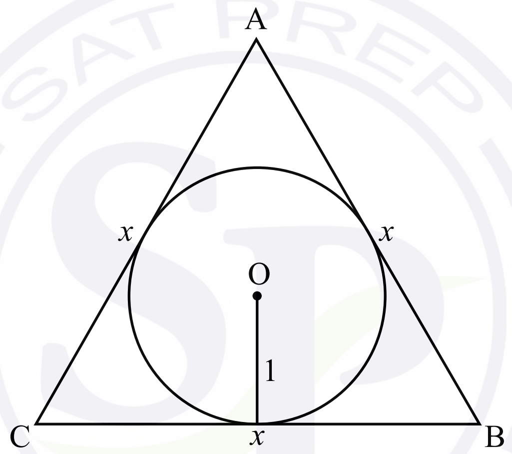
(c) Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. [2]

Let $P_i(n)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

(d) Show that $P_i(n) = 2n\sin\left(\frac{\pi}{n}\right)$. [3]

(e) Use an appropriate Maclaurin series expansion to find $\lim_{n\to\infty} P_i(n)$ and interpret this result geometrically. [5]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

(f) Show that $P_c(n) = 2n\tan\left(\frac{\pi}{n}\right)$. [4]

Consider the function $P(x) = 2x\tan\left(\frac{\pi}{x}\right)$ where $x \in \mathbb{R}, x > 2$.

(g) (i) By writing $P(x)$ in the form $\dfrac{2\tan\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$, find $\lim_{x\to\infty} P(x)$.

(ii) Hence state the value of $\lim_{n\to\infty} P_c(n)$ for integers $n > 2$. [5]

(This question continues on the following page)

(Question 1 continued)

(h) Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. [2]

The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π.

(i) Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π. [3]

2. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$. [2]

(b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$. [2]

(d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

(g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$. [2]

(h) Hence

(i) show that $f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x)$, $n \in \mathbb{Z}^+$;

(ii) express $f_3(x)$ as a cubic polynomial. [5]