8823 – 7103M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 3

23 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a) varies the value of b with $a = 3$ ***(M1)***

> **Note:** The ***(M1)*** in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii).
> Award ***(M1)*** for evidence that $b = 0$ case is considered/determined.

$b = -4,\ 0$ ***A1***

[2 marks]

(b) (i) $b < -4$ or $b > 0$ ***A1***

[1 mark]

(ii) $-4 < b < 0$ ***A1***

[1 mark]

continued…

Question 1 continued

(c) (i) $b = 0, 4$ ***A1***

[1 mark]

(ii) $b < 0$ or $b > 4$ ***A1***

[1 mark]

(iii) $0 < b < 4$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) one point of zero gradient is located on either side (of the x- axis) (or equivalent) ***A1***

[1 mark]

(e) **METHOD 1**

$\frac{dy}{dx} = 3x^2 + 2ax$ ***(A1)***

attempts to solve their $\frac{dy}{dx} = 0$ for x ***M1***

$x(3x+2a)(=0)$ OR $x = \frac{-2a \pm \sqrt{4a^2}}{6}$ OR $x + \frac{a}{3} = \pm\frac{a}{3}$

$x = -\frac{2}{3}a,\ 0$ ***A1***

when $x = 0,\ y = b$ and so $\mathrm{P}(0,\ b)$ is a point of zero gradient ***AG***

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b\ \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $\mathrm{Q}\left(-\frac{2}{3}a,\ \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

continued…

Question 1 continued

METHOD 2

$\frac{dy}{dx} = 3x^2 + 2ax$ ***(A1)***

substitutes either $x = 0$ or $x = -\frac{2}{3}a$ into their $\frac{dy}{dx}$ ***M1***

when $x = 0$, $\frac{dy}{dx} = 0$ and $y = b$ so $P(0, b)$ is a point of zero gradient ***AG***

$\frac{dy}{dx} = 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right)$

$= \frac{4}{3}a^2 - \frac{4}{3}a^2 \ (= 0) \ \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2 (= 0), = \frac{12}{9}a^2 - \frac{4}{3}a^2 (= 0)\right)$ ***A1***

and so $\frac{dy}{dx} = 0$ when $x = -\frac{2}{3}a$ ***AG***

Note: The following two marks are independent of the first three marks.

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \ \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

continued…

Question 1 continued

(f) (i) $\frac{d^2y}{dx^2} = 6x + 2a$ ***A1***

when $x = 0$, $\frac{d^2y}{dx^2} = 2a$ ($a > 0$) and so (P) is a (local) minimum (point) ***R1***

when $x = -\frac{2}{3}a$, $\frac{d^2y}{dx^2} = -2a$ ($a > 0$) and so (Q) is a (local) maximum (point) ***R1***

[3 marks]

(ii) (P and Q are) both above (the x- axis) ***A1***

Note: Award ***A1*** if it is made clear that both points are above (the x- axis).
Accept a labelled sketch that clearly shows this information.

[1 mark]

continued…

Question 1 continued

(g) (i) (P) is a (local) maximum (point) and is above (the x- axis) ***A1***

[1 mark]

(ii) (Q is below the x- axis when) $\frac{4}{27}a^3 + b < 0$ ***A1***

Note: Award ***A1*** for an equivalent correct inequality, eg. $\frac{4}{27}a^3 < -b$. Accept a labelled sketch that clearly shows this information.

[1 mark]

continued…

Question 1 continued

(h) **METHOD 1**

attempts to factorize $4a^3b + 27b^2\ (< 0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3 + b\right)(< 0)$ OR $b\left(4a^3 + 27b\right)(< 0)$ ***A1***

$b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.
Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

when b and $\frac{4}{27}a^3 + b$ have opposite sign, P and Q are located on either side (of the x- axis) (or equivalent) ***R1***

Note: Accept labelled sketches that clearly show this information.

P and Q are located on either side (of the x- axis) if (and only if) the curve has exactly three x- axis intercepts ***R1***

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x- axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

continued…

Question 1 continued

METHOD 2

attempts to factorize $4a^3b+27b^2\ (<0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3+b\right)(<0)$ OR $b\left(4a^3+27b\right)(<0)$ ***A1***

either $b>0$ and $\frac{4}{27}a^3+b<0$ or $b<0$ and $\frac{4}{27}a^3+b>0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.
Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3+b$ is less than zero (or equivalent).

$b>0$ and $\frac{4}{27}a^3+b<0$, $\left(\Rightarrow 0<b<-\frac{4}{27}a^3\right)\Rightarrow a<0$ and hence three x- axis intercepts ***R1***

$b<0$ and $\frac{4}{27}a^3+b>0$, $\left(\Rightarrow -\frac{4}{27}a^3<b<0\right)\Rightarrow a>0$ and hence three x- axis intercepts ***R1***

Note: Accept labelled sketches that clearly show this information.

if $4a^3b+27b^2<0$, then the graph of $y=x^3+ax^2+b$ has exactly three x- axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

[5 marks]

Total [24 marks]

2. (a) (i) $\frac{y}{x}$ ***A1***

[1 mark]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(\frac{y}{x}\right)} \quad \left(= -\frac{1}{m}\right)$ ***A1***

> **Note:** Award ***A1*** for responses such as 'the gradient is the negative (opposite) reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).
>
> Award ***A1*** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.
>
> Do not award ***FT*** from part (a) (i).

so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***AG***

[1 mark]

Question 2 continued

(iii) attempts to separate variables x and y ***(M1)***

$$\int y \, \mathrm{d}y = -\int x \, \mathrm{d}x$$

Note: Award ***(M1)*** for $y \, \mathrm{d}y = -x \, \mathrm{d}x$.

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$ ***A1***

Note: Award ***A1*** for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.   Award ***A0*** for $\frac{y^2}{2} = -\frac{x^2}{2}$.

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

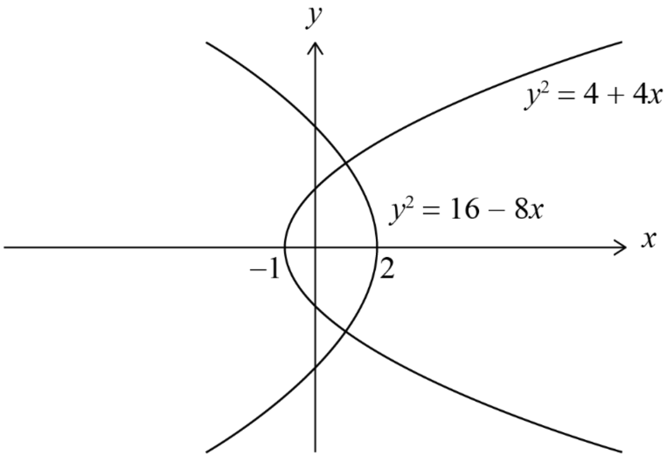
$\Rightarrow x^2 + y^2 = k$ (where $k = 2c$) ***AG***

[2 marks]

continued…

Question 2 continued

(b)



two parabolic shaped curves with approximately correct shape/position
(e.g. two intersection points, in first and fourth quadrant) ***A1A1***

x-intercepts -1 and 2 ***A1***

[3 marks]

continued…

Question 2 continued

(c) at intersection, $4a^2 - 4ax = 4b^2 + 4bx$

$4a^2 - 4b^2 = 4ax + 4bx$ ($a^2 - b^2 = ax + bx,\ 4a^2 - 4b^2 - 4ax - 4bx = 0$) ***A1***

attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of $a^2 - b^2 - ax - bx = 0$ (or equivalent) ***(M1)***

$(a+b)(a-b) = (a+b)x$

Note: Accept alternative forms such as $4(a+b)(a-b) = 4(a+b)x$ or $(a+b)((a-b)-x) = 0$.

recognition that $a+b > 0$ (or equivalent, eg. $a > 0, b > 0$) (allows division by $a+b$) ***R1***

Note: Subsequent marks are not dependent on this ***R1***.

$x = a - b$ ***A1***

Note: As $x = a - b$ is an ***AG***, only award the above ***A1*** if $a^2 - b^2 = (a+b)(a-b)$ has been used.

substitutes $x = a - b$ into either $y^2 = 4a^2 - 4ax$ or $y^2 = 4b^2 + 4bx$ and attempts to simplify ***(M1)***

$y^2 = 4a^2 - 4a(a-b)$ OR $y^2 = 4b^2 + 4b(a-b)$

$y^2 = 4a^2 - 4a^2 + 4ab \Rightarrow y = \pm 2\sqrt{ab}$ ***A1***

so $\mathrm{M}\left(a-b,\ 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b,\ -2\sqrt{ab}\right)$ ***AG***

[6 marks]

continued…

Question 2 continued

(d) **METHOD 1**

attempts implicit differentiation on either curve ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a}{2y}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4b}{2y}$ (or equivalent) ***A1***

substitutes $y = 2\sqrt{ab}$ into either $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a}{2y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4b}{2y}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\frac{a}{b}}$ $\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{b}{a}}$ $\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

OR

eg. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

continued…

Question 2 continued

METHOD 2

attempts chain rule differentiation on either $y=\sqrt{4a^2-4ax}$ or $y=\sqrt{4b^2+4bx}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ (or equivalent) ***A1***

substitutes $x=a-b$ into either $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{\frac{a}{b}}$ $\left(=-\frac{a}{\sqrt{ab}}\right)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{b}{a}}$ $\left(=\frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}}\times\sqrt{\frac{b}{a}}=-1$ OR $-\frac{a}{\sqrt{ab}}\times\frac{b}{\sqrt{ab}}=-1$ (or equivalent) ***A1***

OR

eg. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

[5 marks]

continued…

Question 2 continued

(e) (i) $g(x,\, y) = \dfrac{-\dfrac{x}{y} + \tan\dfrac{\pi}{4}}{1 - \left(-\dfrac{x}{y}\right)\tan\dfrac{\pi}{4}}$ ***(A1)***

$g(x,\, y) = \dfrac{-\dfrac{x}{y} + 1}{1 + \dfrac{x}{y}} \left(= \dfrac{\dfrac{-x+y}{y}}{\dfrac{y+x}{y}} \right)$ ***A1***

so $g(x,\, y) = \dfrac{y-x}{y+x}$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) let $y = vx$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x} =\right)\frac{vx - x}{vx + x}\ \left(= \frac{v-1}{v+1}\right)$ ***(A1)***

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single fraction in v ***(M1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{v+1}$ (or equivalent) ***(A1)***

attempts to separate variables x and v ***(M1)***

$\int \frac{v+1}{v^2+1}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$ (or equivalent)

$\frac{1}{2}\ln\left(v^2+1\right) + \arctan v = -\ln|x|\,(+d)$ (or equivalent) ***A1A1***

$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\frac{y}{x} + \ln|x| = d$ (or equivalent) ***A1***

[9 marks]

continued…

Question 2 continued

(f) **METHOD 1**

$$g(x,\ y)=\frac{\frac{1}{\tan\alpha}f\left(x,y\right)+1}{\frac{1}{\tan\alpha}-f\left(x,y\right)}$$ ***M1***

EITHER

as $\alpha\to\frac{\pi}{2}$, $\frac{1}{\tan\alpha}\to 0$, (hence $g(x,\ y)\to-\frac{1}{f(x,\ y)}$) ***R1***

OR

as $\alpha\to\frac{\pi}{2}$, $\tan\alpha\to\infty$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

THEN

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

METHOD 2

uses either $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$ or $\frac{1}{\tan\alpha}=\frac{\cos\alpha}{\sin\alpha}$ to form $g(x,\ y)=\frac{\cos\alpha f\left(x,y\right)+\sin\alpha}{\cos\alpha-\sin\alpha f\left(x,y\right)}$ ***M1***

as $\alpha\to\frac{\pi}{2}$, $\cos\alpha\to 0$ and $\sin\alpha\to 1$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

[2 marks]

Total [31 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 3

32 pages

Instructions to Examiners

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R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

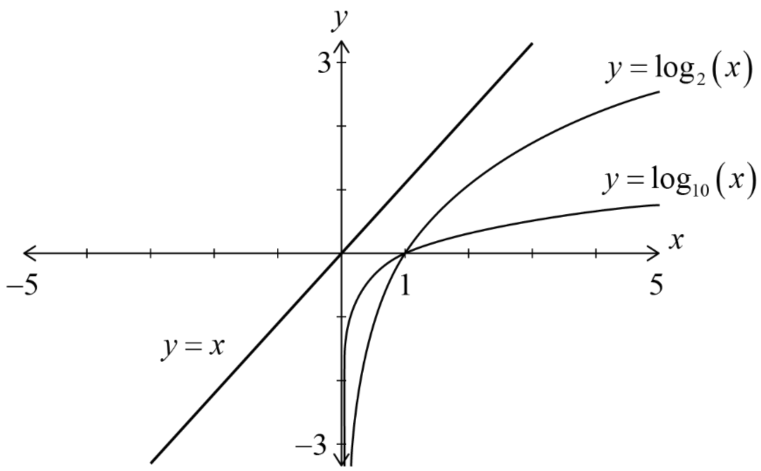
A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a)



clearly labelled graphs of $y=\log_2 x$ and $y=\log_{10} x$ with correct domain, asymptotic behaviour and concavity evident **_A1_**

correct relative positions of the two log graphs both above and below the x-axis **_A1_**

$(1,0)$ indicated (coordinates not required) **_A1_**

correct graph of $y=x$ **_A1_**

[4 marks]

continued…

Question 1 continued

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(x-\ln x)$

$=1-\frac{1}{x}$ ***A1***

attempts to solve their $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ for x ***(M1)***

$1-\frac{1}{x}=0 \Rightarrow x=1$

(when $x=1$,) $x-\ln x=1$ ***A1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-\frac{1}{x}\right)$

$=\frac{1}{x^2}$ ***A1***

$\frac{1}{x^2}>0$ (when $x=1$) ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1*** for either '$1>0$' or a graph of $y=\frac{1}{x^2}>0$ or 'the graph of $y=x-\ln x$ is concave-up'. Do not award ***R1*** if the second derivative is incorrect.

continued…

Question 1 continued

OR

for $(0<)x<1$, $1-\frac{1}{x}<0$ ***R1***

for $x>1$, $1-\frac{1}{x}>0$ ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1R1*** for either a clearly labelled sign diagram/table (accept correct numerical values) or the graph of $y=1-\frac{1}{x}$ with sign change in gradient indicated.

Note: Award a maximum of ***A0(M1)A1A0R1*** or ***A0(M1)A1R0R1*** if no symbolic derivatives are seen.

[5 marks]

(c)

EITHER

$x-\ln x\geq 1$ $\left(x\in\mathbb{R}^{+}\right)$ ***R1***

OR

$x-\ln x>0$ $\left(x\in\mathbb{R}^{+}\right)$ ***R1***

THEN

so $x>\ln x$ ***AG***

[1 mark]

continued…

Question 1 continued

(d)

Interval	Number of intersection points
$0 < a < 1$	$p = 1$
$1 < a < 1.4$	$q = 2$
$1.5 < a < 2$	$r = 0$

A1A2A1

Note: Award ***A1*** for $p = 1$, ***A2*** for $q = 2$ and ***A1*** for $r = 0$.

[4 marks]

(e) **METHOD 1**

EITHER

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

continued…

Question 1 continued

THEN

$x=\dfrac{1}{\ln a}$ OR $x\ln a=1$ OR $\ln a=\dfrac{1}{x}$ OR $\ln a^{x}=1$ OR $\dfrac{1}{a^{x}\ln a}=1$ ***A1***

at $x=\dfrac{1}{\ln a}$, $\log_{a}x=x$

attempts to solve $\dfrac{\ln x}{\ln a}=\dfrac{1}{\ln a}$ OR $\ln x=1$ OR $\left(\mathrm{e}^{\frac{1}{x}}\right)^{x}=x$ for x ***(M1)***

$x=\mathrm{e}$

coordinates of P are (e,e) (accept $x=\mathrm{e}$, $y=\mathrm{e}$) ***A1A1***

attempts to solve $\dfrac{1}{\ln a}=\mathrm{e}$ OR $\log_{a}\mathrm{e}=\mathrm{e}$ for a analytically ***(M1)***

$\ln a=\dfrac{1}{\mathrm{e}}$ OR $a^{\mathrm{e}}=\mathrm{e}$

$a=\mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

continued…

Question 1 continued

METHOD 2

EITHER

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\log_a\left(\frac{1}{\ln a}\right) = \frac{1}{\ln a}$ for a ***(M1)***

EITHER

$\frac{\ln\left(\frac{1}{\ln a}\right)}{\ln a} = \frac{1}{\ln a} \Rightarrow \ln\left(\frac{1}{\ln a}\right) = 1$

OR

for example, writes $a^{\log_a\left(\frac{1}{\ln a}\right)} = a^{\frac{1}{\ln a}}$ and then attempts to apply appropriate index/log laws to both sides: $\ln a = \frac{\log_a a}{\log_a \mathrm{e}}$ and so $\frac{1}{\ln a} = \log_a \mathrm{e}$

$a^{\frac{1}{\ln a}} = a^{\log_a \mathrm{e}} = \mathrm{e}$

continued…

Question 1 continued

THEN

attempts to solve $\frac{1}{\ln a} = \mathrm{e}$ OR $\log_a \mathrm{e} = \mathrm{e}$ for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ OR $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

$x = \frac{1}{\ln \mathrm{e}^{\frac{1}{\mathrm{e}}}} = \frac{1}{\frac{1}{\mathrm{e}}}$

coordinates of P are (e, e) (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

continued…

Question 1 continued

METHOD 3

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

(equation of the tangent at (x_1, y_1) is) $y = \frac{1}{x_1 \ln a}(x - x_1) + \frac{\ln x_1}{\ln a}$ (or equivalent) ***A1***

compares this equation with $y = x$ and attempts to form at least one of the following ***M1***

$\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$

attempts to solve $\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$ for x_1 ***(M1)***

$x_1 = \mathrm{e}$

coordinates of P are (e, e) (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

attempts to solve $\frac{1}{\mathrm{e} \ln a} = 1$ (or equivalent) for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ **OR** $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

[8 marks]

continued…

Question 1 continued

(f) (i) $1 < a < e^{\frac{1}{e}}$ ***A1***

Note: Award ***A0*** for $a < e^{\frac{1}{e}}$.

[1 mark]

(ii) $a > e^{\frac{1}{e}}$ ***A1***

Note: Only award ***FT*** for $1.4 < a < 1.5$. If the value of a is not exact, e.g. 1.44, award at most ***A0A1*** in part (f) for a consistent approximate endpoint value.
If a value of a is not found in part (e), award at most ***A0A1*** in part (f) for a consistent approximate endpoint value provided that $1.4 < a < 1.5$.

[1 mark]

Total [24 marks]

2. (a) $m = 2$, $c = -1$

$r = \frac{1}{2}$ ***(A1)***

$2, \frac{1}{2}, -1$

EITHER

$d \left(= \frac{1}{2} - 2 = -1 - \frac{1}{2} \right) = -\frac{3}{2}$ ***A1***

OR

this sequence has a common difference of $-\frac{3}{2}$ ***A1***

OR

the (arithmetic) mean of 2 and -1 is $\frac{1}{2}$ ***A1***

THEN

hence $L(x) = 2x - 1$ is an AS-linear function ***AG***

[2 marks]

(b) (i) $\left(L(r) = 0 \Rightarrow \right) mr + c = 0$ ***A1***

$r = -\frac{c}{m}$ ***AG***

Note: Award ***A0*** for numerical verification from $L(x) = 2x - 1$ in part (a).

[1 mark]

continued…

Question 2 continued

(ii) **METHOD 1**

EITHER

attempts to use $(d =)\, r - m = c - r$ ***(M1)***

Note: Award ***(M1)*** for attempting to use $(d =)\, m - r = r - c$.

$(d =) -\dfrac{c}{m} - m = c - \left(-\dfrac{c}{m}\right)$ ***A1***

Note: Award ***A1*** for $(d =) -\dfrac{c}{m} - m = \dfrac{c-m}{2}$.

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $\dfrac{m+c}{2} = r$ ***(M1)***

$m + c = -\dfrac{2c}{m}$ ***A1***

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $c = m + 2d$ ***(M1)***

$c = m + 2\left(-\dfrac{c}{m} - m\right)$ ***A1***

Note: Award ***A1*** for $c = m + 2\left(c - \left(-\dfrac{c}{m}\right)\right)$.

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

continued…

Question 2 continued

OR

attempts to use $r = m + d$ and $c = m + 2d$ $\left(c = m + 2(r - m)\right)$ ***(M1)***

$m^2 + dm + m + 2d = 0$ (or equivalent) ***A1***

substitutes $d = \frac{c - m}{2}$ into their expression involving m and d ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

THEN

$c(m + 2) = -m^2 \Rightarrow c = -\frac{m^2}{m + 2}$ ***A1***

Note: Award ***A1*** for a convincing demonstration that $c = -\frac{m^2}{m + 2}$.

so $L(x) = mx - \frac{m^2}{m + 2}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

METHOD 2

considers $L(x) = mx - mr$

attempts to use $(d =)\, r - m = c - r$ ***(M1)***

Note: Award ***(M1)*** for attempting to use $(d =)\, m - r = r - c$.

$(d =)\, r - m = -mr - r$ ***A1***

attempts to express r in terms of m ***(M1)***

$2r + mr = m \Rightarrow r = \frac{m}{m + 2}$ ***A1***

so $L(x) = mx - \frac{m^2}{m + 2}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

[4 marks]

continued…

Question 2 continued

(iii) $m \neq -2 \ (m \neq 0)$ **A1**

[1 mark]

(c)

attempts to find an integer value of m **(M1)**

e.g. uses the result that $m+2$ exactly divides 2 OR uses a table OR uses a graph and slider OR uses systematic trial and error

Note: Award ***(M1)*** for solving $m^2 = k(m+2)$ for m or solving $mr - \frac{m^2}{m+2} = 0$ for m or solving $m^2 + cm + 2c = 0$ for m.

$m = -4$ OR $m = -3$ **(A1)**

$-4, 2, 8$ OR $-3, 3, 9$

$L(x) = -4x + 8, \ L(x) = -3x + 9$ **A1**

Note: Award ***(M1)(A1)A0*** for $-4x + 8$ and $-3x + 9$.

[3 marks]

(d) (i) $-\frac{b}{a}$ **A1**

[1 mark]

(ii) $\frac{c}{a}$ **A1**

[1 mark]

continued…

Question 2 continued

(e) (i) $b-a$ ***A1***

Note: Award marks as appropriate in parts (e) (ii) and (iii) for use of $r_1 - r_2 = a - b$.

[1 mark]

(ii) attempts to eliminate r_2 ***M1***

$$2r_1 = -\frac{b}{a} - (b-a) \Rightarrow 2r_1 = \frac{a^2 - ab - b}{a} \text{ (or equivalent)}$$ ***A1***

Note: Award ***A1*** for a correct alternative form of $\pm r_1$ or $\pm 2r_1$.

so $r_1 = \dfrac{a^2 - ab - b}{2a}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

[2 marks]

continued…

Question 2 continued

(iii) **METHOD 1**

EITHER

$(r_1 =)\ \dfrac{a+b}{2}$ ***(A1)***

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and b ***M1***

$\dfrac{a+b}{2} = \dfrac{a^2-ab-b}{2a}$ OR $a+b = \dfrac{a^2-ab-b}{a}$

OR

$b - r_1 = r_1 - a$ ***(A1)***

attempts to use $b - r_1 = r_1 - a$ with $r_1 = \dfrac{a^2-ab-b}{2a}$ ***M1***

$b - \left(\dfrac{a^2-ab-b}{2a}\right) = \dfrac{a^2-ab-b}{2a} - a$

OR

$(r_1 =)\, a + d$ ***(A1)***

attempts to use $r_1 = a + d$ with $r_1 = \dfrac{a^2-ab-b}{2a}$ and $d = \dfrac{b-a}{2}$ ***M1***

$\dfrac{a^2-ab-b}{2a} = a + \dfrac{b-a}{2}$

THEN

$2a^2 + 2ab = 2a^2 - 2ab - 2b$ OR $a^2 + ab = a^2 - ab - b$

$4ab + 2b = 0$ OR $2ab + b = 0$

$2b(2a+1) = 0$ OR $b(2a+1) = 0$ ***A1***

Note: Award ***(A1)M1*** for any valid approach that correctly leads to $2ab + b = 0$ (or equivalent).
Do not accept numerical verification from the ***AG***.

so $b = 0$ or $a = -\dfrac{1}{2}$ ***AG***

continued…

Question 2 continued

METHOD 2

$(b=)a+2d$ OR $(r_1=)a+d$ ***(A1)***

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and d ***M1***

$a+d=\frac{a^2-a(a+2d)-(a+2d)}{2a}$ **OR** $2(a+d)=\frac{a^2-a(a+2d)-(a+2d)}{a}$

$2a^2+4ad+a+2d=0$

$(2a+1)(a+2d)=0$ ***A1***

Note: Do not accept numerical verification from the ***AG***.

so $b=0$ or $a=-\frac{1}{2}$ ***AG***

[3 marks]

continued…

Question 2 continued

(f) **METHOD 1**

$r_1 = \frac{a}{2}$ OR $r_2 = -\frac{a}{2}$ OR $d = -\frac{a}{2}$ ***(A1)***

$c = -a$ ***(A1)***

attempts to find the values of a ***(M1)***

EITHER

the roots of $ax^2 - a = 0$ are ± 1 and $\frac{a}{2} = \pm 1$

OR

substitutes $x = \pm\frac{a}{2}$ into $ax^2 - a = 0$ giving $\frac{a^3}{4} - a = 0$

OR

$(r_1 r_2 =)\frac{c}{a} = -\frac{a^2}{4} \Rightarrow c = -\frac{a^3}{4}$ and so $-a = -\frac{a^3}{4} \Rightarrow \frac{a^3}{4} - a = 0$

Note: Award ***(M1)*** for attempting to find the values of a from their arithmetic sequence expressed in terms of a.

OR

$c - r_2 = r_2 - b \Rightarrow -\frac{a^3}{4} - \left(-\frac{a}{2}\right) = -\frac{a}{2} \Rightarrow \frac{a^3}{4} - a = 0$

THEN

$a = \pm 2$ ***(A1)***

$(r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2$, $Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $a = 2$ or $a = -2$.

continued…

Question 2 continued

METHOD 2

$r_1 = -d$ OR $r_2 = d$ OR $a = -2d$ ***(A1)***

$c = 2d$ ***(A1)***

attempts to find the values of d ***(M1)***

EITHER

the roots of $-2dx^2 + 2d = 0$ are ± 1

OR

substitutes $x = \pm d$ into $-2dx^2 + 2d = 0$ giving $-2d^3 + 2d = 0$

OR

attempts to use $r_1 r_2 = \frac{c}{a}$ to form $-d^2 = \frac{2d}{-2d}$

Note: Award ***(M1)*** for attempting to find the values of d from their arithmetic sequence expressed in terms of d.

THEN

$d = \pm 1$ ***(A1)***

$\left(a = \pm 2,\ r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2\right)$

so $Q(x) = 2x^2 - 2$, $Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $d = 1$ or $d = -1$.

continued…

Question 2 continued

METHOD 3

$a = 2r_1$ OR $r_2 = -r_1$ OR $d = -r_1$ ***(A1)***

$c = -2r_1$ ***(A1)***

attempts to find the values of r_1 ***(M1)***

EITHER

the roots of $2r_1x^2 - 2r_1 = 0$ are ± 1

OR

substitutes $x = \pm r_1$ into $2r_1x^2 - 2r_1 = 0$ giving $2r_1^{\,3} - 2r_1 = 0$

OR

attempts to use $r_1r_2 = \frac{c}{a}$ to form $-r_1^{\,2} = \frac{-2r_1}{2r_1}$

Note: Award ***(M1)*** for attempting to find the values of r_1 from their arithmetic sequence expressed in terms of r_1.

THEN

$r_1 = \pm 1$ ***(A1)***

$(a = \pm 2,\ r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2$, $Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $r_1 = 1$ or $r_1 = -1$.

[5 marks]

continued…

Question 2 continued

(g) (i) attempts to express r_1 in terms of b with $a=-\frac{1}{2}$ ***(M1)***

Note: Do not award ***(M1)*** if $a=\frac{1}{2}$ is used.

EITHER

uses $r_1=\frac{a+b}{2}$

OR

uses $r_1=\frac{a^2-ab-b}{2a}$

OR

uses $r_1-a=b-r_1$

THEN

$r_1=\frac{2b-1}{4}\left(=\frac{b}{2}-\frac{1}{4},=\frac{b-\frac{1}{2}}{2}\right)$ ***A1***

[2 marks]

continued…

Question 2 continued

(ii) **METHOD 1**

EITHER

substitutes their expression for r_1 with $a=-\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$$Q\left(\frac{2b-1}{4}\right)(=0)\Rightarrow -\frac{1}{2}\left(\frac{2b-1}{4}\right)^2+b\left(\frac{2b-1}{4}\right)+c(=0)$$

OR

$$r_2=\frac{6b+1}{4}\left(=\frac{3b}{2}+\frac{1}{4}\right)$$

substitutes their expression for r_2 with $a=-\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$$Q\left(\frac{6b+1}{4}\right)(=0)\Rightarrow -\frac{1}{2}\left(\frac{6b+1}{4}\right)^2+b\left(\frac{6b+1}{4}\right)+c(=0)$$

THEN

$c=\frac{4b+1}{2}\left(=2b+\frac{1}{2}\right)$ (seen anywhere) ***A1***

$$4b^2+20b+5=0$$

attempts to solve their quadratic in b ***(M1)***

$b=\frac{-5\pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c=\frac{4b+1}{2}$

$c=\frac{-9\pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 2

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $Q(x)$ ***(M1)***

$$-\frac{1}{2}\left(x-\left(\frac{2b-1}{4}\right)\right)\left(x-\left(\frac{6b+1}{4}\right)\right)$$

$$-\frac{1}{2}x^2+bx-\frac{3}{8}b^2+\frac{1}{8}b+\frac{1}{32}$$

$c=\frac{4b+1}{2}\left(=2b+\frac{1}{2}\right)$ (seen anywhere) ***A1***

$$2b+\frac{1}{2}=-\frac{3}{8}b^2+\frac{1}{8}b+\frac{1}{32}$$

$$4b^2+20b+5=0$$

attempts to solve their quadratic in b ***(M1)***

$b=\frac{-5\pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c=\frac{4b+1}{2}$

$c=\frac{-9\pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 3

$r_2 = \frac{6b+1}{4}\left(= \frac{3b}{2} + \frac{1}{4}\right)$

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ ***(M1)***

$\left(\frac{2b-1}{4}\right)\left(\frac{6b+1}{4}\right) = \frac{c}{-\frac{1}{2}}$ (or equivalent)

EITHER

$c = \frac{4b+1}{2}\left(= 2b + \frac{1}{2}\right)$ (seen anywhere) ***A1***

OR

$-\frac{1}{2}\left(\frac{2b-1}{4}\right)\left(\frac{6b+1}{4}\right) - \left(\frac{6b+1}{4}\right) = \frac{2b-1}{4} - \left(-\frac{1}{2}\right)$ ***A1***

THEN

$4b^2 + 20b + 5 = 0$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 4

attempts to equate two expressions for r_1 with $a = -\frac{1}{2}$ ***(M1)***

$$\frac{-b \pm \sqrt{b^2 + 2c}}{-1} = \frac{2b-1}{4} \left(\pm\sqrt{b^2 + 2c} = \frac{2b+1}{4} \right)$$

$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right)$ (seen anywhere) ***A1***

$$12b^2 - 4b - 1 + 32\left(2b + \frac{1}{2}\right) = 0 \ \left(4b^2 + 20b + 5 = 0\right)$$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 5

EITHER

$r_1 = d - \frac{1}{2}$

substitutes their expression for r_1 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$Q\left(d - \frac{1}{2}\right)(=0) \Rightarrow -\frac{1}{2}\left(d - \frac{1}{2}\right)^2 + b\left(d - \frac{1}{2}\right) + c(=0)$

OR

$r_2 = 3d - \frac{1}{2}$

substitutes their expression for r_2 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$Q\left(3d - \frac{1}{2}\right)(=0) \Rightarrow -\frac{1}{2}\left(3d - \frac{1}{2}\right)^2 + b\left(3d - \frac{1}{2}\right) + c(=0)$

THEN

$b = 2d - \frac{1}{2}$ and $c = 4d - \frac{1}{2}$ (seen anywhere) ***A1***

$4d^2 + 8d - 1 = 0$

attempts to solve their quadratic in d ***(M1)***

$d = \frac{-2 \pm \sqrt{5}}{2}$

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 6

$r_1 = d - \frac{1}{2}$ and $r_2 = 3d - \frac{1}{2}$

substitutes their expressions for r_1 and r_2 in terms of d with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ ***(M1)***

$\left(d - \frac{1}{2}\right)\left(3d - \frac{1}{2}\right) = \frac{c}{-\frac{1}{2}}$ (or equivalent)

$c = 4d - \frac{1}{2}$ (seen anywhere) ***A1***

$4d^2 + 8d - 1 = 0$

attempts to solve their quadratic in d ***(M1)***

$d = \frac{-2 \pm \sqrt{5}}{2}$

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

[5 marks]

Total [31 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

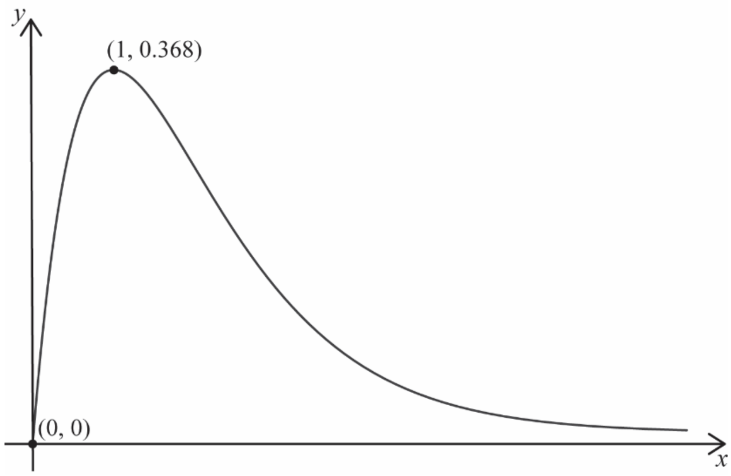
A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a)



A1 for $(1,\ 0.368)$ or $\left(1,\ \frac{1}{e}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through,the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

continued…

Question 1 continued

(b) $\int_0^b x\,\mathrm{e}^{-x}\,\mathrm{d}x$ ***(A1)***

Note: Award (***A1)*** for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$=\left[-x\,\mathrm{e}^{-x}\right]_0^b+\int_0^b \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for each part (including the correct sign with each)

$=\left[-x\,\mathrm{e}^{-x}\right]_0^b-\left[\mathrm{e}^{-x}\right]_0^b$ ***A1***

Note: Award ***A1*** for correct second term. Condone absence of limits to this point

attempt to substitute limits ***M1***

$=-b\,\mathrm{e}^{-b}-\mathrm{e}^{-b}+1$ ***A1***

$=\dfrac{\mathrm{e}^{b}-b-1}{\mathrm{e}^{b}}$ ***AG***

[6 marks]

(c) (i) $\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^{b}-b-1}{\mathrm{e}^{b}}=\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^{b}-1}{\mathrm{e}^{b}}$ ***A1***

Note: Award ***A1*** for correct quotient. Condone absence of limit.

$\left(=\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^{b}}{\mathrm{e}^{b}}\right)=1$ ***A1***

[2 marks]

(ii) $\left(\int_0^\infty x\,\mathrm{e}^{-x}\,\mathrm{d}x=\right)1$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) (i) correct integral ***(M1)***

Note: Award ***M1*** for correct integrand with limits from 0 to a larger number.

24 ***A1***

[2 marks]

(ii) 120 ***A1***

Note: The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) $A_n = n!$ ***A1***

[1 mark]

continued…

Question 1 continued

(f)

Note: Accept starting at $n=0$, throughout this proof.

$n=1$

$A_1=1$ and $1!=1$ ***M1A1***

Note: Award ***M1*** for considering the case where $n=1$, and ***A1*** if it is clear that both $A_1=1$ and $1!=1$ have been considered.

so true for $n=1$

assume true for $n=k$, ($A_k=\int_0^{\infty} x^k e^{-x}\,dx=k!$) ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".   **Note:** Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

when $n=k+1$

attempt to integrate by parts ***M1***

Note: To obtain the ***M1***, a minimum of an expression +/- an integral must be seen.

$\int_0^{\infty} x^{k+1} e^{-x}\,dx=\left[-x^{k+1} e^{-x}\right]_0^{\infty}+(k+1)\int_0^{\infty} x^k e^{-x}\,dx$ ***A1***

$(k+1)\int_0^{\infty} x^k e^{-x}\,dx$ simplified to $(k+1)k!$ seen ***A1***

$=0+(k+1)k!$

Note: Condone omission of the zero.

$=(k+1)!$ ***A1***

Hence if true for $n=k$ then also true for $n=k+1$. As true for $n=1$ so true for $n\in\mathbb{Z}^+$. ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[8 marks]

Total [25 marks]

2. (a) $x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b) (i) $(x_1 =)\ 6$ ***A1***

Note: Award the ***A1*** if 6 seen in part (ii).

[1 mark]

(ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at $(6, 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

(c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

continued…

Question 2 continued

(d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(ii) $x_1 + x_2 + ... + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times ... \times x_n)^{\frac{1}{n}} \leq \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times ... \times x_n \leq \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n(S) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) (i) $M_3(12) = 4^3 = 64$ ***A1***

[1 mark]

(ii) $M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(iii) $M_5(12) = 2.4^5 = 79.6\ (79.6262...)$ ***A1***

[1 mark]

continued…

Question 2 continued

(f) considering $M_n(12)$ for higher values of n

$P(12) = 81$ ***A1***

$n = 4$
A1

Note: Award ***A0A0*** for $P(12) = 82.6$ and $n = 4.41$.

[2 marks]

(g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n = 7$ ***A1***

$P(20) = \left(\frac{20}{7}\right)^7 = 1550 \ (1554.260...)$ ***A1***

Note: Award ***(M1)A0A1*** for $n = 7.36$ and $P(20) = 1570$.

[3 marks]

continued…

Question 2 continued

(h) **EITHER**

$\ln(g(x)) = x(\ln(S) - \ln x)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g'(x)}{g(x)} = \ln S - \ln x - x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g'(x)}{g(x)} = \ln\frac{S}{x} + \left(x\frac{x}{S} \times \frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g(x) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g'(x) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x \times \frac{x}{S} \times \left(-Sx^{-2}\right) + \ln\left(\frac{S}{x}\right)\right]$

$= \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right) - 1\right]$ ***A1***

THEN

$g'(x) = \left(\ln\frac{S}{x} - 1\right)g(x)$

$g(x) \neq 0$

$g'(x) = 0 \Rightarrow \ln\frac{S}{x} - 1 = 0$ ***M1***

$x = \frac{S}{\mathrm{e}}$ $(0.368S,\ 0.36789...S)$ ***A1***

[6 marks]

continued…

Question 2 continued

(i) $\ln\left(g(x)\right) = x\ln\left(\frac{S}{x}\right) \Rightarrow \ln\left(g(x)\right) = \ln\left(\frac{S}{x}\right)^x$ ***M1***

$g(x) = \left(\frac{S}{x}\right)^x$ ***A1***

$= M_x(S)$ for $x \in \mathbb{Z}^+$ ***AG***

[2 marks]

(j) $\frac{100}{\mathrm{e}} = 36.8$ ***M1***

$\left(\frac{100}{36}\right)^{36} = 9.3996... \times 10^{15}$ **AND** $\left(\frac{100}{37}\right)^{37} = 9.47406... \times 10^{15}$ ***R1***

largest possible product is 9.47×10^{15} $(9.47406... \times 10^{15})$ ***A1***

Note: Award ***A1*** independently of the ***R1*** (but not independently of the ***M1***).

[3 marks]

Total [30 marks]

Markscheme

November 2022

Mathematics: analysis and approaches

Higher level

Paper 3

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*

2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a) **EITHER**

$S_n = \frac{n}{2}\left(2\times 1+\left(n-1\right)\times 1\right)$ ***A1***

OR

$u_1 = 1$ and either $u_n = n$ or $d = 1$ stated explicitly ***A1***

OR

$1+2+...+n$ (or equivalent) stated explicitly ***A1***

THEN

$S_n = \frac{n}{2}\left(1+n\right)$ ***AG***

Note: Award ***A0*** for a numerical verification.

[1 mark]

(b) (i) 14 ***A1***

[1 mark]

(ii) $a_1 + a_2 + a_3 = 1$ ***A1***

$2a_1 + 4a_2 + 8a_3 = 5$ ***A1***

$3a_1 + 9a_2 + 27a_3 = 14$ ***A1***

Note: For the third ***A*** mark, award ***A1FT*** for $3a_1 + 9a_2 + 27a_3 = p$ where p is their answer to part (b) (i).

[3 marks]

continued...

Question 1 continued

(iii) $a_1 = \frac{1}{6}$ $(= 0.166666... \approx 0.167)$, $a_2 = \frac{1}{2}$ $(= 0.5)$,

$a_3 = \frac{1}{3}$ $(= 0.333333... \approx 0.333)$ ***A2***

Note: Award ***A1*** if only two of a_1, a_2, a_3 are correct. Only award ***FT*** if three linear equations, each in a_1, a_2 and a_3 are stated in part (b) (ii) or (iii). Award ***A2FT*** for their a_1, a_2 and a_3. Award ***A1FT*** for their a_1, a_2 and $a_3 = 0$.

[2 marks]

(c) $f'(x) = 1 + 2x + 3x^2 + ... + nx^{n-1}$ ***A1***

Note: Award ***A1*** for $f'(x) = \sum_{i=1}^{n} ix^{i-1}$.

$\Rightarrow x\,f'(x) = x + 2x^2 + 3x^3 + ... + nx^n$ ***AG***

[1 mark]

continued...

Question 1 continued

(d) (i) **METHOD 1**

$f_2(x) = xf_1'(x)$

$f_1'(x) = 1^2 + 2^2 x + (3^2 x^2) + \ldots + n^2 x^{n-1} \quad (= 1 + 4x + (9x^2) + \ldots + n^2 x^{n-1})$ ***A1***

> **Note:** Award ***A1*** for
> $xf_1'(x) = x(1^2 + 2^2 x + (3^2 x^2) + \ldots + n^2 x^{n-1}) \quad (= x(1 + 4x + (9x^2) + \ldots + n^2 x^{n-1}))$
> .

$xf_1'(x) = 1^2 x + 2^2 x^2 + (3^2 x^3) + \ldots + n^2 x^n \quad (= x + 4x^2 + (9x^3) + \ldots + n^2 x^n)$ ***A1***

> **Note:** Award ***A1*** for $f_1'(x) = \sum_{i=1}^{n} i^2 x^{i-1}$ and ***A1*** for $xf_1'(x) = x\sum_{i=1}^{n} i^2 x^{i-1}$.
> The second ***A1*** is dependent on the first ***A1***.
> Award a maximum of ***A0A1*** if a general term is not considered.

$= \sum_{i=1}^{n} i^2 x^i$ ***AG***

METHOD 2

$f_2(x) = x\frac{\mathrm{d}}{\mathrm{d}x}(xf'(x))$

$= x(f'(x) + xf''(x)) \quad (= xf'(x) + x^2 f''(x))$ ***A1***

$= x\sum_{i=1}^{n} ix^{i-1} + x^2 \sum_{i=1}^{n} i(i-1)x^{i-2}$

$= \sum_{i=1}^{n} ix^i + \sum_{i=1}^{n} i(i-1)x^i \quad \left(= \sum_{i=1}^{n} (i + i^2 - i)x^i\right)$ ***A1***

$= \sum_{i=1}^{n} i^2 x^i$ ***AG***

[2 marks]

continued...

Question 1 continued

(ii) consider $q = 1$

$f_1(x) = x + 2x^2 + ... + nx^n$ (reference to part (c)) and $f_1(x) = \sum_{i=1}^{n} ix^i$ ***R1***

assume true for $q = k$, ($f_k(x) = \sum_{i=1}^{n} i^k x^i$) ***M1***

Note: Do not award ***M1*** for statements such as "let $q = k$" or "$q = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $q = k + 1$

$f_{k+1}(x) = xf_k'(x)$ ***M1***

$= x\sum_{i=1}^{n} i^{k+1} x^{i-1}$ **OR** $x(1 + 2^{k+1}x + 3^{k+1}x^2 + ... + n^{k+1}x^{n-1})$ ***A1***

Note: Award the above ***M1*** if $f_{k+1}(x) = x\sum_{i=1}^{n} i^{k+1} x^{i-1}$ or $xf_k'(x) = x\sum_{i=1}^{n} i^{k+1} x^{i-1}$ (or equivalent) is stated.

$= \sum_{i=1}^{n} i^{k+1} x^i$ **OR** $x + 2^{k+1}x^2 + 3^{k+1}x^3 + ... + n^{k+1}x^n$ ***A1***

since true for $q = 1$ and true for $q = k + 1$ if true for $q = k$, hence true for all q $(\in \mathbb{Z}^+)$ ***R1***

Note: To obtain the final ***R1***, three of the previous five marks must have been awarded.

[6 marks]

(iii) $f_q(1) = 1^q + 2^q + 3^q + ... + n^q$

$= \sum_{i=1}^{n} i^q \left(= \sum_{i=1}^{n} 1^i i^q \right)$ ***A1***

[1 mark]

continued...

Question 1 continued

(e) uses $S_n = \dfrac{u_1\left(r^n - 1\right)}{r-1}$ with $r = x$ and $u_1 = 1$ ***M1***

clear indication there are $(n+1)$ terms ***R1***

$f(x) = \dfrac{x^{n+1} - 1}{x - 1}$ ***AG***

[2 marks]

(f) **METHOD 1**

$f_1(x) = xf'(x)$

$= x\dfrac{(x-1)(n+1)x^n - 1\times\left(x^{n+1} - 1\right)}{(x-1)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to use the quotient or the product rule to find $f'(x)$.

$= x\dfrac{(nx + x - n - 1)x^n - \left(x^{n+1} - 1\right)}{(x-1)^2}\left(= x\dfrac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2}\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$= \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$ ***AG***

METHOD 2

attempts to form $(x-1)f_1(x)$ ***M1***

$(x-1)f_1(x) = nx^{n+1} - \left(x + x^2 + x^3 + \ldots + x^n\right)$

$f_1(x) = \dfrac{1}{x-1}\left(nx^{n+1} - \left(\dfrac{x^{n+1} - 1}{x - 1} - 1\right)\right)$ ***A1***

$f_1(x) = \dfrac{1}{x-1}\left(\dfrac{nx^{n+1}(x-1) - x^{n+1} + x}{x-1}\right)\ \left(= \dfrac{1}{x-1}\left(\dfrac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{x-1}\right)\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$$f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$ ***AG***

[3 marks]

(g) (i) $\lim_{x \to 1} f_1(x) = \frac{n-(n+1)+1}{0} \left(= \frac{0}{0}\right)$ ***R1***

Note: Only award ***R1*** for sufficient simplification of the numerator, for example, as shown above.
Do not award ***R1*** if $\lim_{x \to 1}$ is not referred to or stated.

[1 mark]

continued...

Question 1 continued

(ii) attempts to differentiate both the numerator and the denominator **M1**

$$\lim_{x\to1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}$$ **A1**

Note: Award ***A1*** for $\left(\left(\lim_{x\to1}\right)\frac{n(n+2)x^{n+1}-n(n+1)x^n-(n+1)x^n+1}{2(x-1)}\right)$. This form can be used in subsequent work.

(l'Hôpital's rule applies again since)

$$\lim_{x\to1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}=\frac{0}{0}$$ **R1**

Note: Do not award ***R1*** if $\lim_{x\to1}$ is not referred to or stated.
Subsequent marks are independent of this ***R*** mark.

attempts to differentiate both the numerator and the denominator **M1**

$$\lim_{x\to1}\frac{n(n+2)(n+1)x^n-n(n+1)^2x^{n-1}}{2}$$

$$=\frac{n(n+2)(n+1)-n(n+1)^2}{2}\quad\left(=\frac{n^3+3n^2+2n-(n^3+2n^2+n)}{2}\right)$$

$$=\frac{n(n+1)((n+2)-(n+1))}{2}\quad\left(=\frac{n^2+n}{2}\right)$$ **A1**

$$=\frac{1}{2}n(n+1)$$ **AG**

[5 marks]

Total [28 marks]

2. (a) **EITHER**

$A = 2\pi\int_0^3 2x\sqrt{1+2^2}\ \mathrm{d}x\ \left(= 4\sqrt{5}\pi\int_0^3 x\,\mathrm{d}x\right)$ ***(A1)***

$= 2\pi\sqrt{5}\ \left[x^2\right]_0^3\ \left(= 2\pi\sqrt{5}\left(3^2 - 0^2\right)\right)$ ***A1***

OR

$A = 2\pi m\sqrt{1+m^2}\ \left[\frac{x^2}{2}\right]_0^h\ \left(= 2\pi m\sqrt{1+m^2}\left(\frac{h^2}{2}\right)\right)$ ***(A1)***

$= 2\pi(2)\sqrt{5}\ \left[\frac{x^2}{2}\right]_0^3\ \left(= 2\pi(2)\sqrt{5}\left(\frac{3^2}{2}\right)\right)$ ***A1***

THEN

$= 18\sqrt{5}\,\pi$ ***AG***

[2 marks]

(b) (i) $r = mh$ ***A1***

[1 mark]

(ii) $l = \sqrt{h^2 + r^2}$ ***(M1)***

$l = \sqrt{h^2 + h^2m^2}\ \left(= h\sqrt{1+m^2}\right)$ ***A1***

[2 marks]

continued...

Question 2 continued

(iii) $A = 2\pi\int_0^h mx\sqrt{1+m^2}\ \mathrm{d}x$ ***(A1)***

$= 2\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$ ***(M1)***

Note: Award ***(M1)*** for $(c)\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$.
At least one of the above two lines needs to be seen.

$= \pi h^2 m\sqrt{1+m^2}\ \left(= \pi hm\times\sqrt{\left(h^2+h^2m^2\right)}\right)$ ***A1***

$= \pi rl$ ***AG***

Note: Award as above if $\frac{l}{h} = \sqrt{1+m^2}$ is used, for example.

[3 marks]

(c) **METHOD 1**

attempts to use the chain rule ***(M1)***

Note: Award ***(M1)*** for $\frac{\mathrm{d}y}{\mathrm{d}x} = (c)x\left(r^2-x^2\right)^{-\frac{1}{2}}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2-x^2\right)^{-\frac{1}{2}}\ (-2x)\ \left(= -x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***A1***

[2 marks]

continued...

Question 2 continued

(d) **EITHER**

attempts to substitute $y=\sqrt{r^2-x^2}$ and their $\frac{dy}{dx}$ into A ***(M1)***

$$A=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(-x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)^2}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$ ***A1***

OR

attempts to substitute y and their $\frac{dy}{dx}$ in terms of x and y into A ***(M1)***

$$A=2\pi\int_{-r}^{r}y\sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx$$

$$=2\pi\int_{-r}^{r}y\sqrt{1+\frac{x^2}{y^2}}\,dx\ \left(=2\pi\int_{-r}^{r}\sqrt{x^2+y^2}\,dx\right)$$ ***A1***

THEN

attempts to perform valid algebraic simplification to form a definite integral in terms of r only ***M1***

$$=2\pi\int_{-r}^{r}r\,dx$$

$$=2\pi r\left[x\right]_{-r}^{r}\ \left(=2\pi r\left(r-(-r)\right)\right)$$ ***A1***

$$=4\pi r^2$$ ***AG***

Note: Award marks as above for $A=4\pi\int_{0}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(-x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)^2}\,dx$.

[4 marks]

continued...

Question 2 continued

(e) (i) **EITHER**

horizontal stretch ***A1***

factor $\frac{1}{k}$ ***A1***

OR

horizontal compression ***A1***

factor k (invariant line y- axis) ***A1***

Note: Award ***A1A1*** as above for correct alternative descriptions.
For example, dilation by a factor of $\frac{1}{k}$ from the y- axis.

[2 marks]

(ii) $\pm\frac{r}{k}$ ***A1***

Note: Award ***A0*** for $\frac{r}{k}$ only and ***A0*** for $-\frac{r}{k}$ only.

[1 mark]

continued...

Question 2 continued

(iii) **METHOD 1**

attempts to use the chain rule ***(M1)***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2 - (kx)^2\right)^{-\frac{1}{2}} \times \left(-k^2 2x\right) \quad \left(= -k^2 x\left(r^2 - (kx)^2\right)^{-\frac{1}{2}} \quad \left(= \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}\right)\right)$$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - k^2 x^2$ (or equivalent) ***(M1)***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{k^2 x}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}$$ ***A1***

[2 marks]

continued...

Question 2 continued

(iv) **EITHER**

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2}\sqrt{1 + \frac{k^4x^2}{r^2 - k^2x^2}}\ \mathrm{d}x$$ ***(A1)***

Note: Award ***(A1)*** for the correct substitution of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of $r^2 - k^2x^2$ and then cancelling $r^2 - k^2x^2$ ***(M1)***

OR

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \left(-\frac{k^2x}{y}\right)^2}\ \mathrm{d}x \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \frac{k^4x^2}{y^2}}\ \mathrm{d}x \right)$$ ***(A1)***

Note: Award ***(A1)*** for the correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of y^2 and cancelling y^2 ***(M1)***

THEN

$$= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2 + k^4x^2}\ \mathrm{d}x \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 + \left(k^4 - k^2\right)x^2}\,\mathrm{d}x \right)$$ ***A1A1***

Note: Award ***A1*** for correct limits (seen anywhere) and ***A1*** for $p(x)$ correct.
The above ***A1*** for correct limits is independent of the ***(M1)***.

[4 marks]

continued...

Question 2 continued

(v) $r = 6378\ (\text{km})$ ***(A1)***

$k = 1.00330...\left(=\dfrac{6378}{6357}=\dfrac{2126}{2119}\right)$ ***(A1)***

attempts to form a definite integral for surface area ***(M1)***

$$= 2\pi \int_{-6357}^{6357} \sqrt{6378^2 - \left(\frac{6378}{6357}\right)^2 x^2 + \left(\frac{6378}{6357}\right)^4 x^2}\ \mathrm{d}x$$

$= 510064226.3...$

$= 5.101\times 10^8\ \left(\text{km}^2\right)$ ***A1***

> **Note:** Award ***A0A1FT(M1)A0*** for using $r = 6357\ (\text{km})$ and $k = 0.996707...$ leading to an answer of $5.089\times 10^8\ \left(\text{km}^2\right)$.

[4 marks]

Total [27 marks]

M22/5/MATHX/HP3/ENG/TZ2/XX/M

Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 3

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

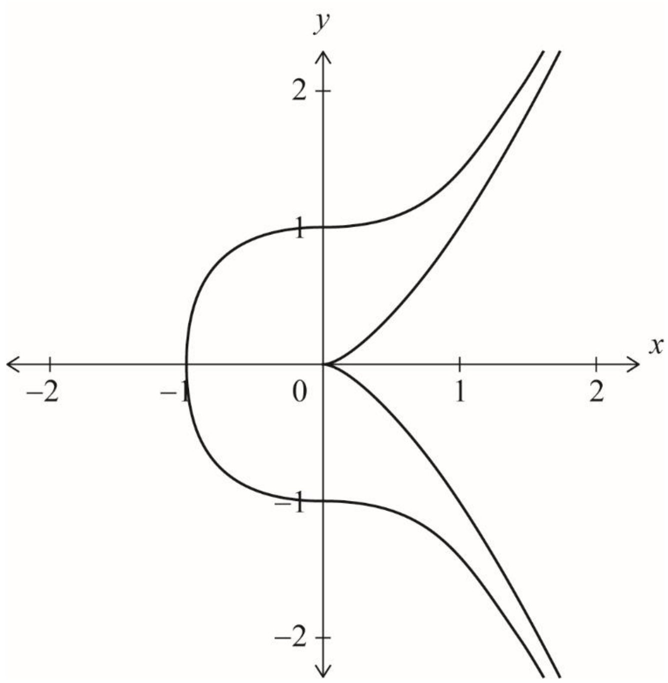
A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) (i)



approximately symmetric about the x-axis graph of $y^2 = x^3$ ***A1***

including cusp/sharp point at (0, 0) ***A1***

[2 marks]

(ii) approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts ***A1***

some indication of position of intersections at $x = -1,\ y = \pm 1$ ***A1***

[2 marks]

Note: Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct. Award ***A1A0A0A0*** if only one correct branch of both curves are seen.

Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0.***

continued…

Question 1 continued

(b) (i) $(0, 1)$ and $(0, -1)$ ***A1***

[1 mark]

(ii) Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different x-axis intercepts (one goes through the origin, and the other does not)

graphs have different y-axis intercepts ***A1***

Note: Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent.

[1 mark]

continued…

Question 1 continued

(c) Any **two** from:

as $x \to \infty$, $y \to \pm\infty$

as $x \to \infty$, $y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have x intercepts at $x = -\sqrt[3]{b}$

they have y intercepts at $y = (\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or x-axis) is a line of symmetry

they all have the same line of symmetry $(y = 0)$

they have one x-axis intercept

they have two y-axis intercepts

they have two points of inflexion

at x-axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at x-axis intercepts ***A1A1***

Note: The last example is the only valid answer for things "not" present. Do not credit an answer of "they are all symmetrical" without some reference to the line of symmetry.

Note: Do not allow same/ similar shape or equivalent.

Note: In accordance with marking rules, mark their first two responses and ignore any subsequent.

[2 marks]

continued…

Question 1 continued

(d) (i) **METHOD 1**

attempt to differentiate implicitly **_M1_**

$2y\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+1$ **_A1_**

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2+1}{2y}$ OR $(\pm)2\sqrt{x^3+x}\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+1$ **_A1_**

$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ **_AG_**

METHOD 2

attempt to use chain rule $y=(\pm)\sqrt{x^3+x}$ **_M1_**

$\frac{\mathrm{d}y}{\mathrm{d}x}=(\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+1)$ **_A1A1_**

Note: Award ***A1*** for $(\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}$, ***A1*** for $(3x^2+1)$.

$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ **_AG_**

[3 marks]

(ii) **EITHER**

local minima/maxima occur when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$1+3x^2=0$ has no (real) solutions (or equivalent) **_R1_**

OR

$\left(x^2\ge 0\Rightarrow\right)$ $3x^2+1>0$, so $\frac{\mathrm{d}y}{\mathrm{d}x}\neq 0$ **_R1_**

THEN

so, no local minima/maxima exist **_AG_**

[1 mark]

continued…

Question 1 continued

(e) **EITHER**

attempt to use quotient rule to find $\frac{d^2y}{dx^2}$ ***M1***

$$\frac{d^2y}{dx^2} = (\pm)\frac{12x\sqrt{x+x^3} - (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)}{4(x+x^3)}$$ ***A1A1***

Note: Award ***A1*** for correct $12x\sqrt{x+x^3}$ and correct denominator, ***A1*** for correct $-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$.

Note: Future ***A*** marks may be awarded if the denominator is missing or incorrect.

stating or using $\frac{d^2y}{dx^2} = 0$ (may be seen anywhere) ***(M1)***

$$12x\sqrt{x+x^3} = (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$$

OR

attempt to use product rule to find $\frac{d^2y}{dx^2}$ ***M1***

$$\frac{d^2y}{dx^2} = \frac{1}{2}(3x^2+1)\left(-\frac{1}{2}\right)(3x^2+1)(x^3+x)^{-\frac{3}{2}} + 3x(x^3+x)^{-\frac{1}{2}}$$ ***A1A1***

Note: Award ***A1*** for correct first term, ***A1*** for correct second term.

setting $\frac{d^2y}{dx^2} = 0$ ***(M1)***

continued…

Question 1 continued

OR

attempts implicit differentiation on $2y\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+1$ ***M1***

$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x$ ***A1***

recognizes that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{3x}$

$(\pm)\frac{3x^2+1}{2\sqrt{x^3+x}}=(\pm)\sqrt{3x}$ ***(A1)***

THEN

$12x(x+x^3)=(1+3x^2)^2$

$12x^2+12x^4=9x^4+6x^2+1$

$3x^4+6x^2-1=0$ ***A1***

attempt to use quadratic formula or equivalent ***(M1)***

$x^2=\frac{-6\pm\sqrt{48}}{6}$

$(x>0\Rightarrow)\,x=\sqrt{\frac{2\sqrt{3}-3}{3}}\quad(p=2,\ q=-3,\ r=3)$ ***A1***

Note: Accept any integer multiple of p, q and r (e.g. $4, -6$ and 6).

[7 marks]

continued…

Question 1 continued

(f) (i) attempt to find tangent line through $(-1,-1)$ ***(M1)***

$y+1=-\frac{3}{2}(x+1)$ OR $y=-1.5x-2.5$ ***A1***

[2 marks]

(ii) attempt to solve simultaneously with $y^2=x^3+2$ ***(M1)***

Note: The ***M1*** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. (4.25, -8.875)).

obtain $\left(\frac{17}{4},-\frac{71}{8}\right)$ ***A1***

[2 marks]

continued…

Question 1 continued

(g) attempt to find equation of $[\mathrm{QS}]$ ***(M1)***

$\frac{y-1}{x+1}=-\frac{79}{42}(=-1.88095\ldots)$ ***(A1)***

solve simultaneously with $y^2=x^3+2$ ***(M1)***

$x=0.28798\ldots\left(=\frac{127}{441}\right)$ ***A1***

$y=-1.4226\ldots\left(=\frac{13175}{9261}\right)$ ***A1***

$(0.288,-1.42)$

OR

attempt to find vector equation of $[\mathrm{QS}]$ ***(M1)***

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}+\lambda\begin{pmatrix}\frac{21}{4}\\-\frac{79}{8}\end{pmatrix}$ ***(A1)***

$x=-1+\frac{21}{4}\lambda$

$y=1-\frac{79}{8}\lambda$

attempt to solve $\left(1-\frac{79}{8}\lambda\right)^2=\left(-1+\frac{21}{4}\lambda\right)^3+2$ ***(M1)***

$\lambda=0.2453\ldots$

$x=0.28798\ldots\left(=\frac{127}{441}\right)$ ***A1***

$y=-1.4226\ldots\left(=\frac{13175}{9261}\right)$ ***A1***

$(0.288,-1.42)$

[5 marks]

[Total 28 marks]

2. (a) attempt to expand $(x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$=\left(x^2-(\alpha+\beta)x+\alpha\beta\right)(x-\gamma)$ **OR** $=(x-\alpha)\left(x^2-(\beta+\gamma)x+\beta\gamma\right)$ ***A1***

$\left(x^3+px^2+qx+r\right)=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$ ***A1***

comparing coefficients:

$p=-(\alpha+\beta+\gamma)$ ***AG***

$q=(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$r=-\alpha\beta\gamma$ ***AG***

Note: For candidates who do not include the ***AG*** lines award full marks.

[3 marks]

(b) (i) $p^2-2q=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***(A1)***

attempt to expand $(\alpha+\beta+\gamma)^2$ ***(M1)***

$=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ or equivalent ***A1***

$=\alpha^2+\beta^2+\gamma^2$ ***AG***

Note: Accept equivalent working from RHS to LHS.

[3 marks]

continued…

Question 2 continued

(ii) **EITHER**

attempt to expand $(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***(M1)***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=2\left(p^2-2q\right)-2q$ or equivalent ***A1***

$=2p^2-6q$ ***AG***

OR

attempt to write $2p^2-6q$ in terms of α,β,γ ***(M1)***

$=2\left(p^2-2q\right)-2q$

$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***A1***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***AG***

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

continued…

Question 2 continued

(c) $p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$\Rightarrow (\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 < 0$ ***A1***

if all roots were real $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 \geq 0$ ***R1***

Note: Condone strict inequality in the ***R1*** line.

Note: Do not award ***A0R1.***

$\Rightarrow$ roots cannot all be real ***AG***

[2 marks]

(d) $p^2 = (-7)^2 = 49$ and $3q = 51$ ***A1***

so $p^2 < 3q \Rightarrow$ the equation has at least one complex root ***R1***

Note: Allow equivalent comparisons; e.g. checking $2p^2 < 6q$

[2 marks]

(e) (i) use of GDC (*eg* graphs or tables) ***(M1)***

$q = 12$ ***A1***

[2 marks]

(ii) complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).
OR
a cubic curve always crosses the x-axis at at least one point. ***R1***

[1 mark]

continued…

Question 2 continued

(f) (i) attempt to expand $(\alpha+\beta+\gamma+\delta)^2$ ***(M1)***

$(\alpha+\beta+\gamma+\delta)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

(A1)

$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$\left(\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2 =\right) p^2-2q$ ***A1***

[3 marks]

(ii) $p^2 < 2q$ **OR** $p^2-2q<0$ ***A1***

Note: Allow ***FT*** on their result from part (f)(i).

[1 mark]

(g) $4<6$ OR $2^2-2\times3<0$ ***R1***

hence there is at least one complex root. ***AG***

Note: Allow ***FT*** from part (f)(ii) for the ***R*** mark provided numerical reasoning is seen.

[1 mark]

continued…

Question 2 continued

(h) (i) $\left(p^2 > 2q\right)$ $\left(81 > 2\times 24\right)$ (so) nothing can be deduced ***R1***

Note: Do not allow ***FT*** for the ***R*** mark.

[1 mark]

(ii) -1 ***A1***

[1 mark]

(iii) attempt to express as a product of a linear and cubic factor ***M1***

$(x+1)(x^3-10x^2+34x-12)$ ***A1A1***

Note: Award ***A1*** for each factor. Award at most ***A1A0*** if not written as a product.

since for the cubic, $p^2 < 3q$ $(100 < 102)$ ***R1***

there is at least one complex root ***AG***

[4 marks]

[Total: 27 marks]

Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 3

27 pages

Instructions to Examiners

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R Marks awarded for clear **Reasoning**.

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- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.

- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a) (i) $P_3(n) = \frac{(3-2)n^2 - (3-4)n}{2}$ OR $P_3(n) = \frac{n^2 - (-n)}{2}$ ***A1***

$P_3(n) = \frac{n^2 + n}{2}$ ***A1***

Note: Award ***A0A1*** if $P_3(n) = \frac{n^2+n}{2}$ only is seen. Do not award any marks for numerical verification.

so for triangular numbers, $P_3(n) = \frac{n(n+1)}{2}$ ***AG***

[2 marks]

(ii) **METHOD 1**

uses a table of values to find a positive integer that satisfies $P_3(n) = 351$ ***(M1)***

for example, a list showing at least 3 consecutive terms $(\ldots\ 325,\ 351,\ 378\ldots)$

Note: Award ***(M1)*** for use of a GDC's numerical solve or graph feature.

$n = 26$ (26th triangular number) ***A1***

Note: Award ***A0*** for $n = -27, 26$. Award ***A0*** if additional solutions besides $n = 26$ are given.

METHOD 2

attempts to solve $\frac{n(n+1)}{2} = 351$ $(n^2 + n - 702 = 0)$ for n ***(M1)***

$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-702)}}{2}$ OR $(n-26)(n+27) = 0$

$n = 26$ (26th triangular number) ***A1***

Note: Award ***A0*** for $n = -27, 26$. Award ***A0*** if additional solutions besides $n = 26$ are given.

[2 marks]

continued…

Question 1 continued

(b) (i) attempts to form an expression for $P_3(n)+P_3(n+1)$ in terms of n ***M1***

EITHER

$$P_3(n)+P_3(n+1)\equiv\frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}$$

$$\equiv\frac{(n+1)(2n+2)}{2}\quad\left(\equiv\frac{2(n+1)(n+1)}{2}\right)$$ ***A1***

OR

$$P_3(n)+P_3(n+1)\equiv\left(\frac{n^2}{2}+\frac{n}{2}\right)+\left(\frac{(n+1)^2}{2}+\frac{n+1}{2}\right)$$

$$\equiv\left(\frac{n^2+n}{2}\right)+\left(\frac{n^2+2n+1+n+1}{2}\right)\ \left(\equiv n^2+2n+1\right)$$ ***A1***

THEN

$$\equiv(n+1)^2$$ ***AG***

[2 marks]

(ii) the sum of the nth and $(n+1)$th triangular numbers is the $(n+1)$th square number ***A1***

[1 mark]

continued…

Question 1 continued

(iii)

X	X	X	X	X
O	X	X	X	X
O	O	X	X	X
O	O	O	X	X
O	O	O	O	X

A1

Note: Accept equivalent single diagrams, such as the one above, where the 4th and 5th triangular numbers and the 5th square number are clearly shown. Award ***A1*** for a diagram that show $P_3(4)$ (a triangle with 10 dots) and $P_3(5)$ (a triangle with 15 dots) and $P_4(5)$ (a square with 25 dots).

[1 mark]

continued…

Question 1 continued

(c) **METHOD 1**

$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1\ \left(=4n(n+1)+1\right)$ ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$=4n^2+4n+1$

$=(2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

METHOD 2

$8P_3(n)+1=8\left((n+1)^2-P_3(n+1)\right)+1\ \left(=8\left((n+1)^2-\frac{(n+1)(n+2)}{2}\right)+1\right)$ ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$8\left(n^2+2n+1\right)-4\left(n^2+3n+2\right)+1\ \left(=4n^2+4n+1\right)$

$=(2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

continued…

Question 1 continued

METHOD 3

$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1 \; \left(=(An+B)^2\right)$ (where $A,B\in\mathbb{Z}^+$) ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$4n^2+4n+1 \; \left(=A^2n^2+2ABn+B^2\right)$

now equates coefficients and obtains $B=1$ and $A=2$

$=(2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

[3 marks]

continued…

Question 1 continued

(d) **EITHER**

$u_1 = 1$ and $d = 3$ ***(A1)***

substitutes their u_1 and their d into $P_5(n) = \frac{n}{2}\left(2u_1 + (n-1)d\right)$ ***M1***

$P_5(n) = \frac{n}{2}\left(2 + 3(n-1)\right) \left(= \frac{n}{2}(2 + 3n - 3)\right)$ ***A1***

OR

$u_1 = 1$ and $u_n = 3n - 2$ ***(A1)***

substitutes their u_1 and their u_n into $P_5(n) = \frac{n}{2}(u_1 + u_n)$ ***M1***

$P_5(n) = \frac{n}{2}(1 + 3n - 2)$ ***A1***

OR

$P_5(n) = \left(3(1) - 2\right) + \left(3(2) - 2\right) + \left(3(3) - 2\right) + ... + 3n - 2$

$P_5(n) = \left(3(1) + 3(2) + 3(3) + ... + 3n\right) - 2n \left(= 3\left(1 + 2 + 3 + ... + n\right) - 2n\right)$ ***(A1)***

substitutes $\frac{n(n+1)}{2}$ into their expression for $P_5(n)$ ***M1***

$P_5(n) = 3\left(\frac{n(n+1)}{2}\right) - 2n$

$P_5(n) = \frac{n}{2}\left(3(n+1) - 4\right)$ ***A1***

OR

attempts to find the arithmetic mean of n terms ***(M1)***

$= \frac{1 + (3n - 2)}{2}$ ***A1***

multiplies the above expression by the number of terms n

$P_5(n) = \frac{n}{2}(1 + 3n - 2)$ ***A1***

THEN

so $P_5(n) = \frac{n(3n-1)}{2}$ ***AG***

[3 marks]

continued…

Question 1 continued

(e) **METHOD 1**

forms a table of $P_3(n)$ values that includes some values for $n > 5$ ***(M1)***

forms a table of $P_5(m)$ values that includes some values for $m > 5$ ***(M1)***

Note: Award ***(M1)*** if at least one $P_3(n)$ value is correct. Award ***(M1)*** if at least one $P_5(m)$ value is correct. Accept as above for $(n^2 + n)$ values and $(3m^2 - m)$ values.

$n = 20$ for triangular numbers ***(A1)***

$m = 12$ for pentagonal numbers ***(A1)***

Note: Award ***(A1)*** if $n = 20$ is seen in or out of a table. Award ***(A1)*** if $m = 12$ is seen in or out of a table. Condone the use of the same parameter for triangular numbers and pentagonal numbers, for example, $n = 20$ for triangular numbers and $n = 12$ for pentagonal numbers.

210 (is a triangular number and a pentagonal number) ***A1***

Note: Award all five marks for 210 seen anywhere with or without working shown.

continued…

Question 1 continued

METHOD 2

EITHER

attempts to express $P_3(n) = P_5(m)$ as a quadratic in n ***(M1)***

$n^2 + n + \left(m - 3m^2\right)(= 0)$ (or equivalent)

attempts to solve their quadratic in n ***(M1)***

$$n = \frac{-1 \pm \sqrt{12m^2 - 4m + 1}}{2} \quad \left(= \frac{-1 \pm \sqrt{1^2 - 4\left(m - 3m^2\right)}}{2}\right)$$

OR

attempts to express $P_3(n) = P_5(m)$ as a quadratic in m ***(M1)***

$3m^2 - m - \left(n^2 + n\right)(= 0)$ (or equivalent)

attempts to solve their quadratic in m ***(M1)***

$$m = \frac{1 \pm \sqrt{12n^2 + 12n + 1}}{6} \quad \left(= \frac{1 \pm \sqrt{(-1)^2 + 12\left(n^2 + n\right)}}{6}\right)$$

THEN

$n = 20$ for triangular numbers ***(A1)***

$m = 12$ for pentagonal numbers ***(A1)***

210 (is a triangular number and a pentagonal number) ***A1***

continued…

Question 1 continued

METHOD 3

$$\frac{n(n+1)}{2}=\frac{m(3m-1)}{2}$$

let $n=m+k$ ($n>m$) and so $3m^2-m=(m+k)(m+k+1)$ ***M1***

$2m^2-2(k+1)m-\left(k^2+k\right)=0$ ***A1***

attempts to find the discriminant of their quadratic
and recognises that this must be a perfect square ***M1***

$\Delta=4(k+1)^2+8(k^2+k)$

$N^2=4(k+1)^2+8(k^2+k)$ $\left(=4(k+1)(3k+1)\right)$

determines that $k=8$ leading to $2m^2-18m-72=0\Rightarrow m=-3,12$ and so $m=12$ ***A1***

210 (is a triangular number and a pentagonal number) ***A1***

METHOD 4

$$\frac{n(n+1)}{2}=\frac{m(3m-1)}{2}$$

let $m=n-k$ ($m<n$) and so $n^2+n=(n-k)\left(3\left(n-k\right)-1\right)$ ***M1***

$2n^2-2(3k+1)n+\left(3k^2+k\right)=0$ ***A1***

attempts to find the discriminant of their quadratic
and recognises that this must be a perfect square ***M1***

$\Delta=4(3k+1)^2-8(3k^2+k)$

$N^2=4(3k+1)^2-8(3k^2+k)$ $\left(=4(k+1)(3k+1)\right)$

determines that $k=8$ leading to $2n^2-50n+200=0\Rightarrow n=5,20$ and so $n=20$ ***A1***

210 (is a triangular number and a pentagonal number) ***A1***

[5 marks]

continued…

Question 1 continued

(f)

Note: Award a maximum of ***R1M0M0A1M1A1A1R0*** for a 'correct' proof using n and $n+1$.

consider $n=1$: $P_r(1)=1+(1-1)(r-2)=1$ and $P_r(1)=\dfrac{(r-2)(1^2)-(r-4)(1)}{2}=1$

so true for $n=1$ ***R1***

Note: Accept $P_r(1)=1$ and $P_r(1)=\dfrac{(r-2)(1^2)-(r-4)(1)}{2}=1$.
Do not accept one-sided considerations such as '$P_r(1)=1$ and so true for $n=1$'.
Subsequent marks after this ***R1*** are independent of this mark can be awarded.

Assume true for $n=k$, *ie*. $P_r(k)=\dfrac{(r-2)k^2-(r-4)k}{2}$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$". The assumption of truth must be clear.
Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Consider $n = k + 1$:

($P_r(k+1)$ can be represented by the sum

$$\sum_{m=1}^{k+1}\left(1+(m-1)(r-2)\right) = \sum_{m=1}^{k}\left(1+(m-1)(r-2)\right) + \left(1+k(r-2)\right)$$ and so

$$P_r(k+1) = \frac{(r-2)k^2-(r-4)k}{2} + \left(1+k(r-2)\right) \quad \left(P_r(k+1) = P_r(k) + \left(1+k(r-2)\right)\right)$$ ***M1***

$$= \frac{(r-2)k^2-(r-4)k+2+2k(r-2)}{2}$$ ***A1***

$$= \frac{(r-2)(k^2+2k)-(r-4)k+2}{2}$$

$$= \frac{(r-2)(k^2+2k+1)-(r-2)-(r-4)k+2}{2}$$ ***M1***

$$= \frac{(r-2)(k+1)^2-(r-4)k-(r-4)}{2}$$ ***(A1)***

$$= \frac{(r-2)(k+1)^2-(r-4)(k+1)}{2}$$ ***A1***

hence true for $n = 1$ and $n = k$ true $\Rightarrow n = k + 1$ true ***R1***

therefore true for all $n \in \mathbb{Z}^+$

Note: Only award the final ***R1*** if the first five marks have been awarded. Award marks as appropriate for solutions that expand both the LHS and (given) RHS of the equation.

[8 marks]

Total [27 marks]

2. (a) (i) $4-\mathrm{i}$ ***A1***

[1 mark]

(ii) mean $=\frac{1}{2}(4+\mathrm{i}+4-\mathrm{i})$ ***A1***

$=4$ ***AG***

[1 mark]

(b) **METHOD 1**

attempts product rule differentiation ***(M1)***

Note: Award ***(M1)*** for attempting to express $f(x)$ as $f(x)=x^3-9x^2+25x-17$.

$f'(x)=(x-1)(2x-8)+x^2-8x+17\ \left(f'(x)=3x^2-18x+25\right)$ ***A1***

$f'(4)=1$ ***A1***

Note: Where $f'(x)$ is correct, award ***A1*** for solving $f'(x)=1$ and obtaining $x=4$.

EITHER

$y-3=1(x-4)$ ***A1***

OR

$y=x+c$

$3=4+c\Rightarrow c=-1$ ***A1***

OR

states the gradient of $y=x-1$ is also 1 and verifies that $(4,\ 3)$ lies on the line $y=x-1$ ***A1***

THEN

so $y=x-1$ is the tangent to the curve at $\mathrm{A}(4,\ 3)$ ***AG***

Note: Award a maximum of ***(M0)A0A1A1*** to a candidate who does not attempt to find $f'(x)$.

continued…

Question 2 continued

METHOD 2

sets $f(x) = x - 1$ to form $x - 1 = (x-1)(x^2 - 8x + 17)$ ***(M1)***

EITHER

$(x-1)(x^2 - 8x + 16) = 0$ $(x^3 - 9x^2 + 24x - 16 = 0)$ ***A1***

attempts to solve a correct cubic equation ***(M1)***

$(x-1)(x-4)^2 = 0 \Rightarrow x = 1, 4$

OR

recognises that $x \neq 1$ and forms $x^2 - 8x + 17 = 1$ $(x^2 - 8x + 16 = 0)$ ***A1***

attempts to solve a correct quadratic equation ***(M1)***

$(x-4)^2 = 0 \Rightarrow x = 4$

THEN

$x = 4$ is a double root ***R1***

so $y = x - 1$ is the tangent to the curve at $\mathrm{A}(4, 3)$ ***AG***

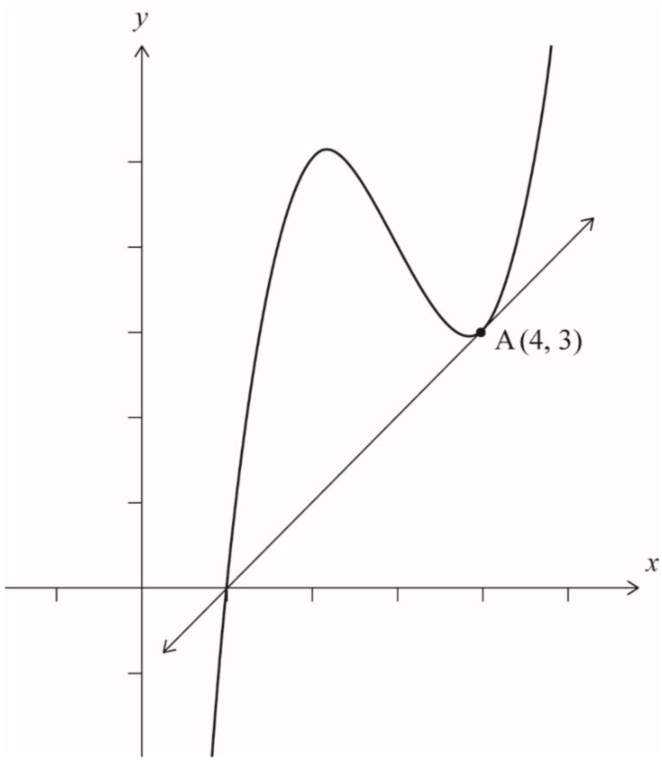
Note: Candidates using this method are not required to verify that $y = 3$.

[4 marks]

continued…

Question 2 continued

(c)



a positive cubic with an x- intercept $(x = 1)$, and a local maximum and local minimum in the first quadrant both positioned to the left of A ***A1***

Note: As the local minimum and point A are very close to each other, condone graphs that seem to show these points coinciding. For the point of tangency, accept labels such as A, $(4,3)$ or the point labelled from both axes. Coordinates are not required.

a correct sketch of the tangent passing through A and crossing the x-axis at the same point ($x = 1$) as the curve ***A1***

Note: Award ***A1A0*** if both graphs cross the x- axis at distinctly different points.

[2 marks]

continued…

Question 2 continued

(d) (i) **EITHER**

$g'(x) = (x-r)(2x-2a) + x^2 - 2ax + a^2 + b^2$ ***(M1)A1***

OR

$g(x) = x^3 - (2a+r)x^2 + (a^2 + b^2 + 2ar)x - (a^2 + b^2)r$

attempts to find $g'(x)$ ***M1***

$g'(x) = 3x^2 - 2(2a+r)x + a^2 + b^2 + 2ar$

$= 2x^2 - 2(a+r)x + 2ar + x^2 - 2ax + a^2 + b^2$ ***A1***

$\left(= 2(x^2 - ax - rx + ar) + x^2 - 2ax + a^2 + b^2\right)$

THEN

$g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) **METHOD 1**

$g(a) = b^2(a-r)$ ***(A1)***

$g'(a) = b^2$ ***(A1)***

attempts to substitute their $g(a)$ and $g'(a)$ into $y - g(a) = g'(a)(x-a)$ ***M1***

$y - b^2(a-r) = b^2(x-a)$

EITHER

$y = b^2(x-r)$ $\left(y = b^2x - b^2r\right)$ ***A1***

sets $y = 0$ so $b^2(x-r) = 0$ ***M1***

$b > 0 \Rightarrow x = r$ OR $b \neq 0 \Rightarrow x = r$ ***R1***

OR

sets $y = 0$ so $-b^2(a-r) = b^2(x-a)$ ***M1***

$b > 0$ OR $b \neq 0$ $\Rightarrow -(a-r) = x - a$ ***R1***

$x = r$ ***A1***

THEN

so the tangent intersects the x-axis at the point $\mathrm{R}(r, 0)$ ***AG***

continued…

Question 2 continued

METHOD 2

$g'(a) = b^2$ ***(A1)***

$g(a) = b^2(a - r)$ ***(A1)***

attempts to substitute their $g(a)$ and $g'(a)$ into $y = g'(a)x + c$ and attempts to find c ***M1***

$c = -b^2 r$

EITHER

$y = b^2(x - r)$ $\left(y = b^2 x - b^2 r\right)$ ***A1***

sets $y = 0$ so $b^2(x - r) = 0$ ***M1***

$b > 0 \Rightarrow x = r$ OR $b \neq 0 \Rightarrow x = r$ ***R1***

OR

sets $y = 0$ so $b^2(x - r) = 0$ ***M1***

$b > 0$ OR $b \neq 0$ $\Rightarrow x - r = 0$ ***R1***

$x = r$ ***A1***

METHOD 3

$g'(a) = b^2$ ***(A1)***

the line through $\mathrm{R}(r,\ 0)$ parallel to the tangent at A has equation

$y = b^2(x - r)$ ***A1***

sets $g(x) = b^2(x - r)$ to form $b^2(x - r) = (x - r)\left(x^2 - 2ax + a^2 + b^2\right)$ ***M1***

$b^2 = x^2 - 2ax + a^2 + b^2$, $(x \neq r)$ ***A1***

$(x - a)^2 = 0$ ***A1***

since there is a double root ($x = a$), this parallel line through

$\mathrm{R}(r,\ 0)$ is the required tangent at A ***R1***

[6 marks]

continued…

Question 2 continued

(e) **EITHER**

$g'(a) = b^2 \Rightarrow b = \sqrt{g'(a)}$ (since $b > 0$) ***R1***

Note: Accept $b = \pm\sqrt{g'(a)}$.

OR

$(a \pm b\mathrm{i} =)\, a \pm \mathrm{i}\sqrt{b^2}$ and $g'(a) = b^2$ ***R1***

THEN

hence the complex roots can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$ ***AG***

[1 mark]

(f) (i) $b = 4$ (seen anywhere) ***A1***

EITHER

attempts to find the gradient of the tangent in terms of a and equates to 16 ***(M1)***

OR

substitutes $r = -2, x = a$ and $y = 80$ to form $80 = \left(a - (-2)\right)(a^2 - 2a^2 + a^2 + 16)$ ***(M1)***

OR

substitutes $r = -2, x = a$ and $y = 80$ into $y = 16(x - r)$ ***(M1)***

THEN

$\frac{80}{a+2} = 16 \Rightarrow a = 3$

roots are -2 (seen anywhere) and $3 \pm 4\mathrm{i}$ ***A1A1***

Note: Award ***A1*** for -2 and ***A1*** for $3 \pm 4\mathrm{i}$. Do not accept coordinates.

[4 marks]

(ii) $(3, -4)$ ***A1***

Note: Accept "$x = 3$ and $y = -4$". Do not award ***A1FT*** for $(a, -4)$.

[1 mark]

continued…

Question 2 continued

(g) (i) $g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$

attempts to find $g''(x)$ ***M1***

$g''(x) = 2(x-a) + 2(x-r) + 2x - 2a \ (= 6x - 2r - 4a)$

sets $g''(x) = 0$ and correctly solves for x ***A1***

for example, obtaining $x - r + 2(x-a) = 0$ leading to $3x = 2a + r$

so $x = \frac{1}{3}(2a + r)$ ***AG***

Note: Do not award ***A1*** if the answer does not lead to the ***AG***.

[2 marks]

(ii) point P is $\frac{2}{3}$ of the horizontal distance (way) from point R to point A ***A1***

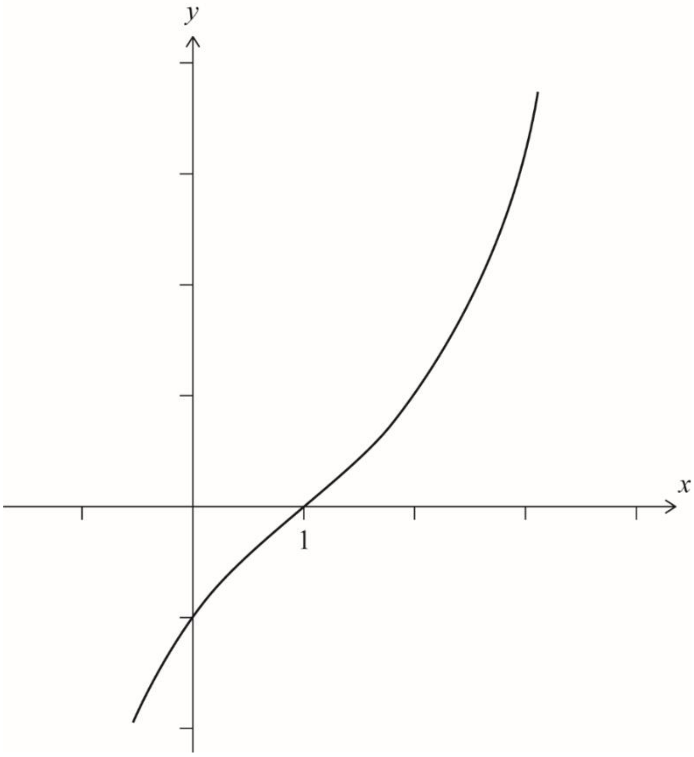
Note: Accept equivalent numerical statements or a clearly labelled diagram displaying the numerical relationship. Award ***A0*** for non-numerical statements such as "P is between R and A, closer to A".

[1 mark]

continued…

Question 2 continued

(h) (i) $y=(x-1)(x^2-2x+5)$ ***(A1)***



a positive cubic with no stationary points and a non-stationary point of inflexion at $x=1$ ***A1***

Note: Graphs may appear approximately linear. Award this ***A1*** if a change of concavity either side of $x=1$ is apparent. Coordinates are not required and the y- intercept need not be indicated.

[2 marks]

(ii) $(r\,,\,0)$ ***A1***

[1 mark]

Total [28 marks]

Markscheme

November 2021

Mathematics: analysis and approaches

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
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Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

**3	Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4	Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

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includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

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5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) $f'(t) = \dfrac{e^t - e^{-t}}{2}$ ***A1***

$f''(t) = \dfrac{e^t + e^{-t}}{2}$ ***A1***

$= f(t)$ ***AG***

[2 marks]

(b) **METHOD 1**

$\left(f(t)\right)^2 + \left(g(t)\right)^2$

substituting f and g ***M1***

$= \dfrac{(e^t + e^{-t})^2 + (e^t - e^{-t})^2}{4}$

$= \dfrac{(e^t)^2 + 2 + (e^{-t})^2 + (e^t)^2 - 2 + (e^{-t})^2}{4}$ ***(M1)***

$= \dfrac{(e^t)^2 + (e^{-t})^2}{2} \quad \left(= \dfrac{e^{2t} + e^{-2t}}{2}\right)$ ***A1***

$= f(2t)$ ***AG***

METHOD 2

$f(2t) = \dfrac{e^{2t} + e^{-2t}}{2}$

$= \dfrac{(e^t)^2 + (e^{-t})^2}{2}$ ***M1***

$= \dfrac{(e^t + e^{-t})^2 + (e^t - e^{-t})^2}{4}$ ***M1A1***

$= \left(f(t)\right)^2 + \left(g(t)\right)^2$ ***AG***

[3 marks]

Note: Accept combinations of METHODS 1 & 2 that meet at equivalent expressions.

(c) (i) substituting $e^{iu} = \cos u + i\sin u$ into the expression for f ***(M1)***

obtaining $e^{-iu} = \cos u - i\sin u$ ***(A1)***

$f(iu) = \dfrac{\cos u + i\sin u + \cos u - i\sin u}{2}$

Note: The ***M1*** can be awarded for the use of sine and cosine being odd and even respectively.

$= \dfrac{2\cos u}{2}$

$= \cos u$ ***A1***

[3 marks]

(ii) $g(iu) = \dfrac{\cos u + i\sin u - \cos u + i\sin u}{2}$

substituting and attempt to simplify ***(M1)***

$= \dfrac{2i\sin u}{2}$

$= i\sin u$ ***A1***

[2 marks]

(d) **METHOD 1**

$(f(iu))^2 + (g(iu))^2$

substituting expressions found in part (c) ***(M1)***

$= \cos^2 u - \sin^2 u \ (= \cos 2u)$ ***A1***

METHOD 2

$f(2iu) = \dfrac{e^{2iu} + e^{-2iu}}{2}$

$= \dfrac{\cos 2u + i\sin 2u + \cos 2u - i\sin 2u}{2}$ ***M1***

$= \cos 2u$ ***A1***

Note: Accept equivalent final answers that have been simplified removing all imaginary parts eg $2\cos^2 u - 1$ etc

[2 marks]

(e) $\left(f(t)\right)^2-\left(g(t)\right)^2=\dfrac{(\mathrm{e}^t+\mathrm{e}^{-t})^2-(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}$ ***M1***

$=\dfrac{(\mathrm{e}^{2t}+\mathrm{e}^{-2t}+2)-(\mathrm{e}^{2t}+\mathrm{e}^{-2t}-2)}{4}$ ***A1***

$=\dfrac{4}{4}=1$ ***A1***

> **Note:** Award ***A1*** for a value of 1 obtained from either LHS or RHS of given expression.

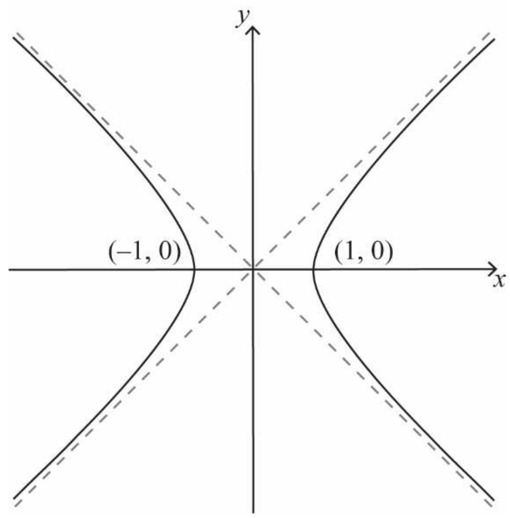
$\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2=\cos^2 u+\sin^2 u$ ***M1***

$=1$ $\left(\text{hence}\left(f(t)\right)^2-\left(g(t)\right)^2=\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2\right)$ ***AG***

> **Note:** Award full marks for showing that $\left(f(z)\right)^2-\left(g(z)\right)^2=1, \forall z\in\mathbb{C}$.

[4 marks]

(f)



A1A1A1A1

> **Note:** Award ***A1*** for correct curves in the upper quadrants, ***A1*** for correct curves in the lower quadrants ***, A1*** for correct x-intercepts of $(-1, 0)$ and $(1, 0)$ (condone $x=-1$ and 1), ***A1*** for $y=x$ and $y=-x$.

[4 marks]

(g) attempt to rotate by 45° in either direction ***(M1)***

Note: Evidence of an attempt to relate to a sketch of $xy = k$ would be sufficient for this ***(M1).***

attempting to rotate a particular point, *eg* $(1, 0)$ ***(M1)***

$(1, 0)$ rotates to $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ (or similar) ***(A1)***

hence $k = \pm\frac{1}{2}$ ***A1A1***

[5 marks]
Total [25 marks]

2. (a) (i) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}t} = y$

$\int \frac{\mathrm{d}y}{y} = \int \mathrm{d}t$ ***(M1)***

$\ln y = t + c$ **OR** $\ln|y| = t + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ and ***A1*** for t and c.

$y = Ae^{t}$ ***AG***

METHOD 2

rearranging to $\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$ AND multiplying by integrating factor e^{-t} ***M1***

$ye^{-t} = A$ ***A1A1***

$y = Ae^{t}$ ***AG***

[3 marks]

(ii) substituting $y = Ae^{t}$ into differential equation in x ***M1***

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - Ae^{t}$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -Ae^{t}$ ***AG***

[1 mark]

(iii) integrating factor (IF) is $e^{\int -1 \mathrm{d}t}$ ***(M1)***

$= e^{-t}$ ***(A1)***

$e^{-t}\frac{\mathrm{d}x}{\mathrm{d}t} - xe^{-t} = -A$

$xe^{-t} = -At + D$ ***(A1)***

$x = (-At + D)e^{t}$ ***A1***

Note: The first constant must be A**,** and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[4 marks]

(b) (i) $\frac{d^2y}{dt^2} = -\frac{dx}{dt} + \frac{dy}{dt}$ ***A1***

EITHER

$= -x + y + \frac{dy}{dt}$ ***(M1)***

$= \frac{dy}{dt} + \frac{dy}{dt}$ ***A1***

OR

$= -x + y + (-x + y)$ ***(M1)***

$= 2(-x + y)$ ***A1***

THEN

$= 2\frac{dy}{dt}$ ***AG***

[3 marks]

(ii) $\frac{dY}{dt} = 2Y$ ***A1***

$\int \frac{dY}{Y} = \int 2dt$ ***M1***

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ ***A1***

$Y = Be^{2t}$ ***AG***

[3 marks]

(iii) $\frac{dy}{dt} = Be^{2t}$

$y = \int Be^{2t}\,dt$ ***M1***

$y = \frac{B}{2}e^{2t} + C$ ***A1***

Note: The first constant must be B, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[2 marks]

(iv) **METHOD 1**

substituting $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$ and their (iii) into $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ ***M1(M1)***

$B\mathrm{e}^{2t} = -x + \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

Note: Follow through from incorrect part (iii) cannot be awarded if it does not lead to the ***AG.***

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -\frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}(x\mathrm{e}^{-t})}{\mathrm{d}t} = -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}$ ***M1***

$x\mathrm{e}^{-t} = \int -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}\ \mathrm{d}t$

$x\mathrm{e}^{-t} = -\frac{B}{2}\mathrm{e}^{t} + C\mathrm{e}^{-t} + D$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C + D\mathrm{e}^{t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y \Rightarrow B\mathrm{e}^{2t} = \frac{B}{2}\mathrm{e}^{2t} - C - D\mathrm{e}^{t} + \frac{B}{2}\mathrm{e}^{2t} + C \Rightarrow D = 0$ ***M1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

[3 marks]

(c) (i) $\frac{\mathrm{d}y}{\mathrm{d}t} = -4x + y$

$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ seen anywhere ***M1***

METHOD 1

$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -4(x - y) + \frac{\mathrm{d}y}{\mathrm{d}t}$

attempt to eliminate x ***M1***

$= -4\left(\frac{1}{4}\left(y - \frac{\mathrm{d}y}{\mathrm{d}t}\right) - y\right) + \frac{\mathrm{d}y}{\mathrm{d}t}$

$= 2\frac{\mathrm{d}y}{\mathrm{d}t} + 3y$ ***A1***

$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$ ***AG***

METHOD 2

rewriting LHS in terms of x and y ***M1***

$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ ***A1***

$= 0$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda \mathrm{e}^{\lambda t}$, $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F\lambda^2 \mathrm{e}^{\lambda t}$ ***(A1)***

$F\lambda^2 \mathrm{e}^{\lambda t} - 2F\lambda \mathrm{e}^{\lambda t} - 3F\mathrm{e}^{\lambda t} = 0$ ***(M1)***

$\lambda^2 - 2\lambda - 3 = 0$ (since $\mathrm{e}^{\lambda t} \neq 0$) ***A1***

λ_1 and λ_2 are 3 and -1 (either order) ***A1***

[4 marks]

(iii) **METHOD 1**

$y = Fe^{3t} + Ge^{-t}$

$\frac{dy}{dt} = 3Fe^{3t} - Ge^{-t}$, $\frac{d^2y}{dt^2} = 9Fe^{3t} + Ge^{-t}$ ***(A1)(A1)***

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 9Fe^{3t} + Ge^{-t} - 2\left(3Fe^{3t} - Ge^{-t}\right) - 3\left(Fe^{3t} + Ge^{-t}\right)$ ***M1***

$= 9Fe^{3t} + Ge^{-t} - 6Fe^{3t} + 2Ge^{-t} - 3Fe^{3t} - 3Ge^{-t}$ ***A1***

$= 0$ ***AG***

METHOD 2

$y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$

$\frac{dy}{dt} = F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}$, $\frac{d^2y}{dt^2} = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t}$ ***(A1)(A1)***

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t} - 2\left(F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}\right) - 3\left(Fe^{\lambda_1 t} + Ge^{\lambda_2 t}\right)$ ***M1***

$= Fe^{\lambda_1 t}\left(\lambda^2 - 2\lambda - 3\right) + Ge^{\lambda_2 t}\left(\lambda^2 - 2\lambda - 3\right)$ ***A1***

$= 0$ ***AG***

[4 marks]

Total [30 marks]

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 3

Instructions to Examiners

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	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
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If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

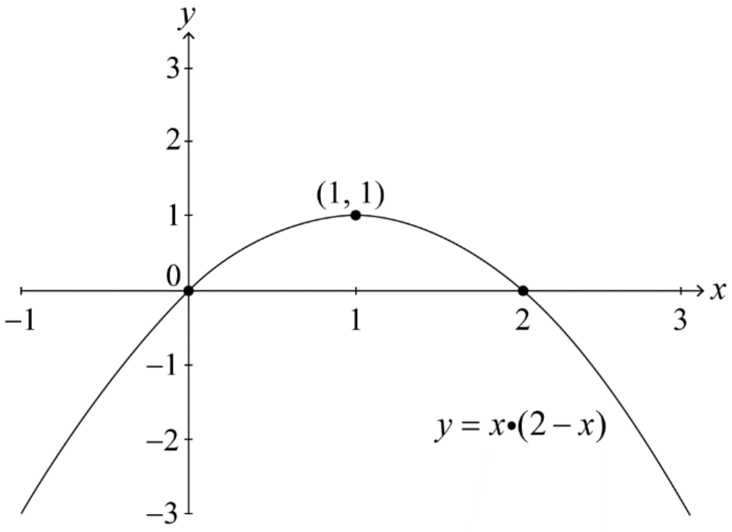
A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a)



inverted parabola extended below the x-axis ***A1***

x-axis intercept values $x = 0, 2$ ***A1***

Note: Accept a graph passing through the origin as an indication of $x = 0$.

local maximum at $(1,1)$ ***A1***

Note: Coordinates must be stated to gain the final ***A1***. Do not accept decimal approximations.

[3 marks]

(b)

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n=3$ and $n=5$	1	0	2
$n=2$ and $n=4$	1	2	0

A1A1A1A1A1A1

Note: Award ***A1*** for each correct value.

For a table not sufficiently or clearly labelled, assume that their values are in the same order as the table in the question paper and award marks accordingly.

[6 marks]

(c) **METHOD 1**

attempts to use the product rule ***(M1)***

$f_n'(x) = -nx^n(a-x)^{n-1} + nx^{n-1}(a-x)^n$ ***A1A1***

Note: Award ***A1*** for a correct $u\frac{\mathrm{d}v}{\mathrm{d}x}$ and ***A1*** for a correct $v\frac{\mathrm{d}u}{\mathrm{d}x}$.

EITHER

attempts to factorise $f_n'(x)$ (involving at least one of nx^{n-1} or $(a-x)^{n-1}$) ***(M1)***

$= nx^{n-1}(a-x)^{n-1}((a-x)-x)$ ***A1***

OR

attempts to express $f_n'(x)$ as the difference of two products with each product containing at least one of nx^{n-1} or $(a-x)^{n-1}$ ***(M1)***

$= (-x)(nx^{n-1})(a-x)^{n-1} + (a-x)(nx^{n-1})(a-x)^{n-1}$ ***A1***

THEN

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

Note: Award the final ***(M1)A1*** for obtaining any of the following forms:

$$f_n'(x) = nx^n(a-x)^n\left(\frac{a-x-x}{x(a-x)}\right); \quad f_n'(x) = \frac{nx^n(a-x)^n}{x(a-x)}(a-x-x);$$

$$f_n'(x) = nx^{n-1}\left((a-x)^n - x(a-x)^{n-1}\right);$$

$$f_n'(x) = (a-x)^{n-1}\left(nx^{n-1}(a-x)^n - nx^n\right)$$

METHOD 2

$f_n(x) = (x(a-x))^n$ ***(M1)***

$= (ax-x^2)^n$ ***A1***

attempts to use the chain rule ***(M1)***

$f_n'(x) = n(a-2x)(ax-x^2)^{n-1}$ ***A1A1***

Note: Award ***A1*** for $n(a-2x)$ and ***A1*** for $(ax-x^2)^{n-1}$.

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

[5 marks]

(d) $x=0,\ x=\frac{a}{2},\ x=a$ ***A2***

Note: Award ***A1*** for either two correct solutions or for obtaining $x=0,\ x=-a,\ x=-\frac{a}{2}$. Award ***A0*** otherwise.

[2 marks]

(e) attempts to find an expression for $f_n\left(\frac{a}{2}\right)$ ***(M1)***

$f_n\left(\frac{a}{2}\right) = \left(\frac{a}{2}\right)^n\left(a-\frac{a}{2}\right)^n$

$= \left(\frac{a}{2}\right)^n\left(\frac{a}{2}\right)^n \ \left(=\left(\frac{a}{2}\right)^{2n}\right), \left(=\left(\left(\frac{a}{2}\right)^n\right)^2\right)$ ***A1***

EITHER

since $a \in \mathbb{R}^+$, $\left(\frac{a}{2}\right)^{2n} > 0$ (for $n \in \mathbb{Z}^+, n>1$ and so $f_n\left(\frac{a}{2}\right) > 0$) ***R1***

Note: Accept any logically equivalent conditions/statements on a and n. Award ***R0*** if any conditions/statements specified involving a, n or both are incorrect.

OR

(since $a \in \mathbb{R}^+$), $\frac{a}{2}$ raised to an even power ($2n$) (or equivalent reasoning) is always positive (and so $f_n\left(\frac{a}{2}\right) > 0$) ***R1***

Note: The condition $a \in \mathbb{R}^+$ is given in the question. Hence some candidates will assume $a \in \mathbb{R}^+$ and not state it. In these instances, award ***R1*** for a convincing argument. Accept any logically equivalent conditions/statements on a and n. Award ***R0*** if any conditions/statements specified involving a, n or both are incorrect.

THEN

so $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ is always above the horizontal axis ***AG***

Note: Do not award ***(M1)A0R1***.

[3 marks]

(f) **METHOD 1**

$f_n'\left(\frac{a}{4}\right) = n\left(\frac{a}{4}\right)^{n-1}\left(a-\frac{a}{2}\right)\left(a-\frac{a}{4}\right)^{n-1}$ $\left(= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1}\right)$ ***A1***

EITHER

$n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1} > 0$ as $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ ***R1***

OR

$n\left(\frac{a}{4}\right)^{n-1}, \left(a-\frac{a}{2}\right)$ and $\left(a-\frac{a}{4}\right)^{n-1}$ are all > 0 ***R1***

Note: Do not award ***A0R1***. Accept equivalent reasoning on correct alternative expressions for $f_n'\left(\frac{a}{4}\right)$ and accept any logically equivalent conditions/statements on a and n. Exceptions to the above are condone $n > 1$ and condone $n > 0$. An alternative form for $f_n'\left(\frac{a}{4}\right)$ is $(2n)(3)^{n-1}\left(\frac{a}{4}\right)^{2n-1}$.

THEN

hence $f_n'\left(\frac{a}{4}\right) > 0$ ***AG***

[2 marks]

METHOD 2

$f_n(0) = 0$ and $f_n\left(\frac{a}{2}\right) > 0$ ***A1*** (since f_n is continuous and there are no stationary points between $x=0$ and $x=\frac{a}{2}$)

the gradient (of the curve) must be positive between $x=0$ and $x=\frac{a}{2}$ ***R1***

Note: Do not award ***A0R1***.

hence $f_n'\left(\frac{a}{4}\right)>0$ ***AG***

[2 marks]

(g) (i) $f_n'(-1)=n(-1)^{n-1}(a+2)(a+1)^{n-1}$

for n even:

$n(-1)^{n-1}(=-n)<0$ (and $(a+2),(a+1)^{n-1}$ are both >0) ***R1***

$f_n'(-1)<0$ ***A1***

$f_n'(0)=0$ and $f_n'\left(\frac{a}{4}\right)>0$ (seen anywhere) ***A1***

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If n is even, then $n-1$ is odd and hence $(-1)^{n-1}<0$ $(=-1)$.

Do not award ***R0A1***.

The second ***A1*** is independent of the other two marks.

The ***A*** marks can be awarded for correct descriptions expressed in words.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0)=0$. The last ***A*** mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last ***A1*** can be awarded for use of a specific case (*e.g.* $n=2$).

hence $(0,0)$ is a local minimum point ***AG***

[3 marks]

(ii) for n odd:

$n(-1)^{n-1}(=n)>0$, (and $(a+2)$, $(a+1)^{n-1}$ are both >0) so $f_n{}'(-1)>0$ ***R1***

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.
For example, award ***R1*** for the following:
If n is odd, then $n-1$ is even and hence $(-1)^{n-1}>0$ $(=1)$.

$f_n{}'(0)=0$ and $f_n{}'\left(\frac{a}{4}\right)>0$ (seen anywhere) ***A1***

Note: The ***A1*** is independent of the ***R1***.
Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n{}'(0)=0$. The last *A* mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.
The last *A1* can be awarded for use of a specific case (*e.g.* $n=3$).

hence $(0,0)$ is a point of inflexion with zero gradient ***AG***

[2 marks]

(h) considers the parity of n ***(M1)***

Note: Award ***M1*** for stating at least one specific even value of n.

n must be even (for four solutions) ***A1***

Note: The above 2 marks are independent of the 3 marks below.

$$0 < k < \left(\frac{a}{2}\right)^{2n}$$ ***A1A1A1***

Note: Award ***A1*** for the correct lower endpoint, ***A1*** for the correct upper endpoint and ***A1*** for strict inequality signs.

The third ***A1*** (strict inequality signs) can only be awarded if ***A1A1*** has been awarded.

For example, award ***A1A1A0*** for $0 \le k \le \left(\frac{a}{2}\right)^{2n}$. Award ***A1A0A0*** for $k > 0$.

Award ***A1A0A0*** for $0 < k < f_n\left(\frac{a}{2}\right)$.

[5 marks]

Total [31 marks]

2. (a) (i) **METHOD 1**

attempts to expand $(\omega - 1)(\omega^2 + \omega + 1)$ ***(M1)***

$= \omega^3 + \omega^2 + \omega - \omega^2 - \omega - 1$ ***A1***

$= \omega^3 - 1$ ***AG***

[2 marks]

METHOD 2

attempts polynomial division on $\dfrac{\omega^3 - 1}{\omega - 1}$ ***M1***

$= \omega^2 + \omega + 1$ ***A1***

so $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$ ***AG***

[2 marks]

(ii) (since ω is a root of $z^3 = 1) \Rightarrow \omega^3 - 1 = 0$ ***R1***

and $\omega \neq 1$ ***R1***

$\Rightarrow \omega^2 + \omega + 1 = 0$ ***AG***

Note: In part (a), award marks as appropriate where ω has been converted into Cartesian, modulus-argument (polar) or Euler form.

[2 marks]

(b) **METHOD 1**

attempts to find either P_0P_1 or P_0P_2 ***(M1)***

accept any valid method

e.g. $2\sin\frac{\pi}{3}$, $1^2+1^2-2\cos\frac{2\pi}{3}$, $\frac{1}{\sin\frac{\pi}{6}}=\frac{\text{P}_0\text{P}_1}{\sin\frac{2\pi}{3}}$ from either $\Delta\text{OP}_0\text{P}_1$ or $\Delta\text{OP}_0\text{P}_2$

e.g. use of Pythagoras' theorem

e.g. $\left|1-e^{i\frac{2\pi}{3}}\right|$, $\left|1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right|$ by calculating the distance between 2 points

$\text{P}_0\text{P}_1=\sqrt{3}$ ***A1***

$\text{P}_0\text{P}_2=\sqrt{3}$ ***A1***

Note: Award a maximum of ***M1A1A0*** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_2 or both.

so $\text{P}_0\text{P}_1\times\text{P}_0\text{P}_2=3$ ***AG***

METHOD 2

attempts to find $\text{P}_0\text{P}_1\times\text{P}_0\text{P}_2=|1-\omega||1-\omega^2|$ ***(M1)***

$\text{P}_0\text{P}_1\times\text{P}_0\text{P}_2=\left|\omega^3-\omega^2-\omega+1\right|$ ***A1***

$=\left|1-\left(\omega^2+\omega+1\right)+2\right|$ and since $\omega^2+\omega+1=0$ ***R1***

so $\text{P}_0\text{P}_1\times\text{P}_0\text{P}_2=3$ ***AG***

[3 marks]

(c) **METHOD 1**

$z^4 - 1 = (z-1)(z^3 + z^2 + z + 1)$ ***A1***

(ω is a root hence) $\omega^4 - 1 = 0$ and $\omega \neq 1$ ***R1***

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

Note: Condone the use of ω throughout.

[2 marks]

METHOD 2

considers the sum of roots of $z^4 - 1 = 0$ ***(M1)***

the sum of roots is zero (there is no z^3 term) ***A1***

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

[2 marks]

METHOD 3

substitutes for ω ***(M1)***

e.g. LHS $= e^{i\frac{3\pi}{2}} + e^{\pi i} + e^{i\frac{\pi}{2}} + 1$

$= -i - 1 + i + 1$ ***A1***

Note: This can be demonstrated geometrically or by using vectors. Accept Cartesian or modulus-argument (polar) form.

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

[2 marks]

METHOD 4

$\omega^3 + \omega^2 + \omega + 1 = \dfrac{\omega^4 - 1}{\omega - 1}$ ***A1***

$= \dfrac{0}{\omega - 1} = 0$ as $\omega \neq 1$ ***R1***

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

[2 marks]

(d) **METHOD 1**

$P_0P_2 = 2$ ***A1***

attempts to find either P_0P_1 or P_0P_3 ***(M1)***

> **Note:** For example, $P_0P_1 = |1-i|$ and $P_0P_3 = |1+i|$.
> Various geometric and trigonometric approaches can be used by candidates.

$P_0P_1 = \sqrt{2}$, $P_0P_3 = \sqrt{2}$ ***A1A1***

> **Note:** Award a maximum of ***A1M1A1A0*** if labels such as P_0P_1 are not clearly shown.
> Award full marks if the lengths are shown on a clearly labelled diagram.
> Award a maximum of ***A1M1A1A0*** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_3 or both.

$P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ ***AG***

[4 marks]

METHOD 2

attempts to find $P_0P_1 \times P_0P_2 \times P_0P_3 = |1-\omega||1-\omega^2||1-\omega^3|$ ***M1***

$P_0P_1 \times P_0P_2 \times P_0P_3 = |-\omega^6 + \omega^5 + \omega^4 - \omega^2 - \omega + 1|$ ***A1***

$= |-(-1) + \omega^5 + 1 - (-1) - \omega + 1|$ since $\omega^6 = \omega^2 = -1$ and $\omega^4 = 1$ ***A1***

$= |\omega^5 - \omega + 4|$ and since $\omega^5 = \omega$ ***R1***

so $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ ***AG***

[4 marks]

METHOD 3

$P_0P_2 = 2$ ***A1***

attempts to find $P_0P_1 \times P_0P_3 = |1-\omega||1-\omega^3|$ ***M1***

$P_0P_1 \times P_0P_3 = |\omega^4 - \omega^3 - \omega + 1|$ ***A1***

$= |2-(-\omega)-\omega|$ since $\omega^4 = 1$ and $\omega^3 = -\omega$ ***R1***

so $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ ***AG***

[4 marks]

(e) $\left(\mathrm{P_0P_1} \times \mathrm{P_0P_2} \times \ldots \times \mathrm{P_0P_{n\text{-}1}}\right) = n$ ***A1***

[1 mark]

(f) (i) $\mathrm{P_0P_2} = \left|1-\omega^2\right|,\ \mathrm{P_0P_3} = \left|1-\omega^3\right|$ ***A1A1***

[2 marks]

(ii) $\mathrm{P_0P}_{n-1} = \left|1-\omega^{n-1}\right|$ ***A1***

Note: Accept $\left|1-\omega\right|$ from symmetry.

[1 mark]

(g) (i) $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + ... + z + 1)$

considers the equation $z^{n-1} + z^{n-2} + ... + z + 1 = 0$ ***(M1)***

the roots are $\omega, \omega^2, ..., \omega^{n-1}$ ***(A1)***

so $(z-\omega)(z-\omega^2)...(z-\omega^{n-1})$ ***A1***

[3 marks]

(ii) **METHOD 1**

substitutes $z = 1$ into $(z-\omega)(z-\omega^2)...(z-\omega^{n-1}) \equiv z^{n-1} + z^{n-2} + ... + z + 1$

M1

$(1-\omega)(1-\omega^2)...(1-\omega^{n-1}) = n$ ***(A1)***

takes modulus of both sides ***M1***

$\left|(1-\omega)(1-\omega^2)...(1-\omega^{n-1})\right| = |n|$

$|1-\omega||1-\omega^2|...|1-\omega^{n-1}| = n$ ***A1***

so $P_0P_1 \times P_0P_2 \times ... \times P_0P_{n-1} = n$ ***AG***

Note: Award a maximum of ***M1A1FTM1A0*** from part (e).

[4 marks]

METHOD 2

$(1-\omega),(1-\omega^2),\ldots,(1-\omega^{n-1})$ are the roots of $(1-v)^{n-1}+(1-v)^{n-2}+\ldots+(1-v)+1=0$

M1

coefficient of v^{n-1} is $(-1)^{n-1}$ and the coefficient of 1 is n ***A1***

product of the roots is $\frac{(-1)^{n-1}n}{(-1)^{n-1}}=n$ ***A1***

$|1-\omega||1-\omega^2|\ldots|1-\omega^{n-1}|=n$ ***A1***

so $P_0P_1\times P_0P_2\times\ldots\times P_0P_{n-1}=n$ ***AG***

[4 marks]

Total[24 marks]

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

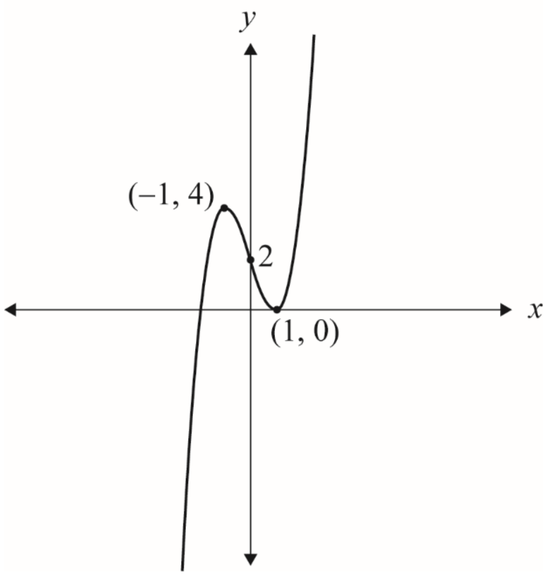
A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) (i)



$c=1$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

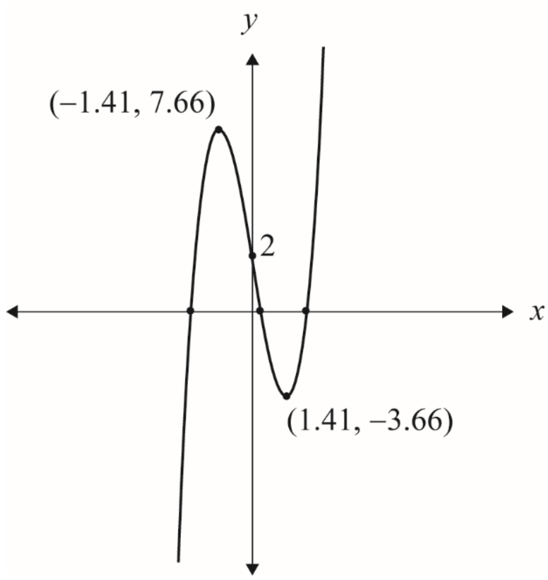
local minimum point correctly labelled ***A1***

[3 marks]

continued…

Question 1 continued

(ii)



$c = 2$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

local minimum point correctly labelled ***A1***

Note: Accept the following exact answers:

Local maximum point coordinates $\left(-\sqrt{2}, 2+4\sqrt{2}\right)$.

Local minimum point coordinates $\left(\sqrt{2}, 2-4\sqrt{2}\right)$.

[3 marks]

continued…

Question 1 continued

(b) $f'(x) = 3x^2 - 3c$ ***A1***

Note: Accept $3x^2 - 3c$ (an expression).

[1 mark]

(c) (i) $c = 0$ ***A1***

[1 mark]

(ii) considers the number of solutions to their $f'(x) = 0$ ***(M1)***

$3x^2 - 3c = 0$

$c > 0$ ***A1***

[2 marks]

(iii) $c < 0$ ***A1***

Note: The ***(M1)*** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

[1 mark]

(d) attempts to solve their $f'(x)=0$ for x ***(M1)***

$x=\pm\sqrt{c}$ ***(A1)***

Note: Award ***(A1)*** if either $x=-\sqrt{c}$ or $x=\sqrt{c}$ is subsequently considered.
Award the above ***(M1)(A1)*** if this work is seen in part (c).

(i) correctly evaluates $f\left(-\sqrt{c}\right)$ ***A1***

$f\left(-\sqrt{c}\right)=-c^{\frac{3}{2}}+3c^{\frac{3}{2}}+2\ \left(=-c\sqrt{c}+3c\sqrt{c}+2\right)$

the y-coordinate of the local maximum point is $2c^{\frac{3}{2}}+2$ ***AG***

[3 marks]

(ii) correctly evaluates $f\left(\sqrt{c}\right)$ ***A1***

$f\left(\sqrt{c}\right)=c^{\frac{3}{2}}-3c^{\frac{3}{2}}+2\ \left(=c\sqrt{c}-3c\sqrt{c}+2\right)$

the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}}+2$ ***AG***

[1 mark]

continued…

Question 1 continued

(e) (i) the graph of $y = f(x)$ will have one $x-$ axis intercept if

EITHER

$-2c^{\frac{3}{2}} + 2 > 0$ (or equivalent reasoning) ***R1***

OR

the minimum point is above the $x-$ axis ***R1***

Note: Award ***R1*** for a rigorous approach that does not (only) refer to sketched graphs.

THEN

$0 < c < 1$ ***A1***

Note: Condone $c < 1$. The ***A1*** is independent of the ***R1***.

[2 marks]

(ii) the graph of $y = f(x)$ will have two $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 = 0$ (or equivalent reasoning) ***(M1)***

OR

evidence from the graph in part(a)(i) ***(M1)***

THEN

$c = 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(iii) the graph of $y = f(x)$ will have three $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 < 0$ (or equivalent reasoning) ***(M1)***

OR

reasoning from the results in both parts (e)(i) and (e)(ii) ***(M1)***

THEN

$c > 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(f) case 1:

$c \leq 0$ (independent of the value of d) ***A1***

EITHER

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution) ***R1***

OR

$\Rightarrow g'(x) \geq 0$ for $x \in$ ~ ***R1***

OR

the graph of $y = f(x)$ has no local maximum or local minimum points,
hence any vertical translation of this graph ($y = g(x)$) will also have
no local maximum or local minimum points ***R1***

THEN

therefore there is only one x- axis intercept ***AG***

Note: Award at most ***A0R1*** if only $c < 0$ is considered.

continued…

Question 1 continued

case 2

$c > 0$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a local minimum point ***(A1)***

Note: Award ***(A1)*** for a correct y-coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point ***(M1)***

EITHER

considers both points above the x-axis or both points below the x-axis

OR

considers either the local minimum point only above the x-axis OR the local maximum point only below the x-axis

THEN

$d > 2c^{\frac{3}{2}}$ (both points above the x-axis) ***A1***

$d < -2c^{\frac{3}{2}}$ (both points below the x-axis) ***A1***

Note: Award at most ***(A1)(M1)A0A0*** for case 2 if $c > 0$ is not clearly stated.

[6 marks]

Total [27 marks]

2. (a) $A = s^2$ and $P = 4s$ ***(A1)***

$A = P \Rightarrow s^2 = 4s$ ***(M1)***

$s(s-4) = 0$

$\Rightarrow s = 4\,(s > 0)$ ***A1***

> **Note:** Award ***A1M1A0*** if both $s = 4$ and $s = 0$ are stated as final answers.

[3 marks]

(b) $A_T = \frac{1}{2}x^2 \sin\frac{2\pi}{n}$ ***A1***

> **Note:** Award ***A1*** for a correct alternative form expressed in terms of x and n only.
>
> For example, using Pythagoras' theorem, $A_T = x\sin\frac{\pi}{n}\sqrt{x^2 - x^2\sin^2\frac{\pi}{n}}$ or $A_T = 2\left(\frac{1}{2}\left(x\sin\frac{\pi}{n}\right)\left(x\cos\frac{\pi}{n}\right)\right)$ or $A_T = x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$.

[1 mark]

continued…

Question 2 continued

(c) **METHOD 1**

uses $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ ***(M1)***

$\frac{\frac{y}{2}}{x} = \sin\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

METHOD 2

uses Pythagoras' theorem $\left(\frac{y}{2}\right)^2 + h^2 = x^2$ and $h = x\cos\frac{\pi}{n}$ ***(M1)***

$\left(\frac{y}{2}\right)^2 + \left(x\cos\frac{\pi}{n}\right)^2 = x^2 \quad \left(y^2 = 4x^2\left(1-\cos^2\frac{\pi}{n}\right)\right)$

$= 4x^2\sin^2\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

continued…

Question 2 continued

METHOD 3

uses the cosine rule ***(M1)***

$$y^2 = 2x^2 - 2x^2\cos\frac{2\pi}{n} \quad \left(= 2x^2\left(1-\cos\frac{2\pi}{n}\right)\right)$$

$$= 4x^2\sin^2\frac{\pi}{n}$$ ***A1***

$$y = 2x\sin\frac{\pi}{n}$$ ***AG***

[2 marks]

METHOD 4

uses the sine rule ***(M1)***

$$\frac{y}{\sin\frac{2\pi}{n}} = \frac{x}{\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)}$$

$$y\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\cos\frac{\pi}{n}$$ ***A1***

$$y = 2x\sin\frac{\pi}{n}$$ ***AG***

[2 marks]

continued…

Question 2 continued

(d) $A = P \Rightarrow nA_T = ny$ ***(M1)***

Note: Award ***M1*** for equating correct expressions for A and P.

$\frac{1}{2}nx^2 \sin\frac{2\pi}{n} = 2nx\sin\frac{\pi}{n}$ $\left(nx^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2nx\sin\frac{\pi}{n}\right)$

$\frac{1}{2}x^2 \sin\frac{2\pi}{n} = 2x\sin\frac{\pi}{n}$ $\left(x^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\right)$ ***A1***

uses $\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ (seen anywhere in part (d) or in part (b)) ***(M1)***

$x^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}$

attempts to either factorise or divide their expression ***(M1)***

$x\sin\frac{\pi}{n}\left(x\cos\frac{\pi}{n} - 2\right) = 0$

$x = \frac{2}{\cos\frac{\pi}{n}}, \left(x\sin\frac{\pi}{n} \neq 0\right)$(or equivalent) ***A1***

continued…

Question 2 continued

EITHER

substitutes $x = \dfrac{2}{\cos\frac{\pi}{n}}$ (or equivalent) into $P = ny$ ***(M1)***

$P = 2n\left(\dfrac{2}{\cos\frac{\pi}{n}}\right)\left(\sin\dfrac{\pi}{n}\right)$ ***A1***

> **Note:** Other approaches are possible. For example, award ***A1*** for $P = 2nx\cos\dfrac{\pi}{n}\tan\dfrac{\pi}{n}$ and ***M1*** for substituting $x = \dfrac{2}{\cos\frac{\pi}{n}}$ into P.

OR

substitutes $x = \dfrac{2}{\cos\frac{\pi}{n}}$ (or equivalent) into $A = nA_T$ ***(M1)***

$A = \dfrac{1}{2}n\left(\dfrac{2}{\cos\frac{\pi}{n}}\right)^2\left(\sin\dfrac{2\pi}{n}\right)$

$A = \dfrac{1}{2}n\left(\dfrac{2}{\cos\frac{\pi}{n}}\right)^2\left(2\sin\dfrac{\pi}{n}\cos\dfrac{\pi}{n}\right)$ ***A1***

THEN

$A = P = 4n\tan\dfrac{\pi}{n}$ ***AG***

[7 marks]

continued…

Question 2 continued

(e) (i) attempts to use the Maclaurin series for $\tan x$ with $x = \frac{\pi}{n}$ ***(M1)***

$$\tan\frac{\pi}{n} = \frac{\pi}{n} + \frac{\left(\frac{\pi}{n}\right)^3}{3} + \frac{2\left(\frac{\pi}{n}\right)^5}{15}(+...)$$

$$4n\tan\frac{\pi}{n} = 4n\left(\frac{\pi}{n} + \frac{\pi^3}{3n^3} + \frac{2\pi^5}{15n^5}(+...)\right)$$ (or equivalent) ***A1***

$$= 4\left(\pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + ...\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right) = 4\pi$$ ***A1***

Note: Award a maximum of ***M1A1A0*** if $\lim_{n\to\infty}$ is not stated anywhere.

[3 marks]

(ii) (as $n \to \infty$, $P \to 4\pi$ and $A \to 4\pi$)

the polygon becomes a circle of radius 2 ***R1***

Note: Award ***R1*** for alternative responses such as: the polygon becomes a circle of area 4π OR the polygon becomes a circle of perimeter 4π OR the polygon becomes a circle with $A = P = 4\pi$. Award ***R0*** for polygon becomes a circle.

[1 mark]

(f) $A=\frac{1}{2}ab$ and $P=a+b+\sqrt{a^2+b^2}$ ***(A1)(A1)***

equates their expressions for A and P ***M1***

$A=P\Rightarrow a+b+\sqrt{a^2+b^2}=\frac{1}{2}ab$

$\sqrt{a^2+b^2}=\frac{1}{2}ab-(a+b)$ ***M1***

Note: Award ***M1*** for isolating $\sqrt{a^2+b^2}$ or $\pm 2\sqrt{a^2+b^2}$. This step may be seen later.

$a^2+b^2=\left(\frac{1}{2}ab-(a+b)\right)^2$

$a^2+b^2=\frac{1}{4}a^2b^2-2\left(\frac{1}{2}ab\right)(a+b)+(a+b)^2$ ***M1***

$\left(=\frac{1}{4}a^2b^2-a^2b-ab^2+a^2+2ab+b^2\right)$

Note: Award ***M1*** for attempting to expand their RHS of either $a^2+b^2=...$ or $4(a^2+b^2)=....$

EITHER

$ab\left(\frac{1}{4}ab-a-b+2\right)=0\ (ab\neq 0)$ ***A1***

$\frac{1}{4}ab-a-b+2=0$

$ab-4a=4b-8$

OR

$\frac{1}{4}a^2b^2-a^2b-ab^2+2ab=0$

$a\left(\frac{1}{4}b^2-b\right)+(2b-b^2)=0\ \left(a(b^2-4b)+(8b-4b^2)=0\right)$ ***A1***

$a=\frac{4b^2-8b}{b^2-4b}$

continued…

Question 2 continued

THEN

$\Rightarrow a = \frac{4b - 8}{b - 4}$ **A1**

$a = \frac{4b - 16 + 8}{b - 4}$

$a = \frac{8}{b - 4} + 4$ **AG**

Note: Award a maximum of ***A1A1M1M1M0A0A0*** for attempting to verify. For example, verifying that $A = P = \frac{16}{b - 4} + 2b + 4$ gains 4 of the 7 marks.

[7 marks]

continued…

Question 2 continued

(g) (i) using an appropriate method ***(M1)***

eg substituting values for b or using divisibility properties

$(5,12,13)$ and $(6,8,10)$ ***A1A1***

> **Note:** Award ***A1A0*** for either one set of three correct side lengths or two sets of two correct side lengths.

[3 marks]

(ii) $A = P = 30$ and $A = P = 24$ ***A1***

> **Note:** Do not award ***A1FT***.

[1 mark]

Total [28 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 3

9 pages

3 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s) or subpart(s). Usually, to award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then* ***FT*** *marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, use of $r>1$ for the sum of an infinite GP, $\sin\theta=1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta=1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features/ CAS functionality are not allowed.

Calculator notation
The subject guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

1. (a) ***METHOD 1***

consider right-angled triangle OCX where $\mathrm{CX} = \frac{x}{2}$

$\sin\frac{\pi}{3} = \frac{\frac{x}{2}}{1}$ ***M1A1***

$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2(1)(1)\cos\frac{2\pi}{3}$ ***M1A1***

$x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

Note: Accept use of sine rule.

[3 marks]

(b) $\sin\frac{\pi}{4} = \frac{1}{x}$ where x = side of square ***M1***

$x = \sqrt{2}$ ***A1***

$P_i = 4\sqrt{2}$ ***A1***

[3 marks]

(c) 6 equilateral triangles $\Rightarrow x = 1$ ***A1***

$P_i = 6$ ***A1***

[2 marks]

(d) in right-angled triangle $\sin\left(\frac{\pi}{n}\right) = \frac{\frac{x}{2}}{1}$ ***M1***

$\Rightarrow x = 2\sin\left(\frac{\pi}{n}\right)$ ***A1***

$P_i = n \times x$

$P_i = n \times 2\sin\left(\frac{\pi}{n}\right)$ ***M1***

$P_i = 2n\sin\left(\frac{\pi}{n}\right)$ ***AG***

[3 marks]

continued…

Question 1 continued

(e) consider $\lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ ***M1***

$2n\sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \ldots\right)$ ***(A1)***

$= 2\left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \ldots\right)$ ***A1***

$\Rightarrow \lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\pi$ ***A1***

as $n \to \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$ ***R1***

[5 marks]

(f) consider an n-sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre ***M1A1***

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2\tan\left(\frac{\pi}{n}\right)$ ***M1A1***

Perimeter $P_c = 2n\tan\left(\frac{\pi}{n}\right)$ ***AG***

[4 marks]

(g) (i) $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{2\tan\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$ ***R1***

attempt to use L'Hôpital's rule ***M1***

$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{-\frac{2\pi}{x^2}\sec^2\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$ ***A1***

$\lim_{x\to\infty} P(x) = 2\pi$ ***A1***

(ii) $\lim_{n\to\infty} P_c(n) = 2\pi$ ***A1***

[5 marks]

continued…

Question 1 continued

(h) $P_i < 2\pi < P_c$

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$ ***M1***

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$ ***A1***

[2 marks]

(i) attempt to find the lower bound and upper bound approximations within 0.005 of π ***(M1)***

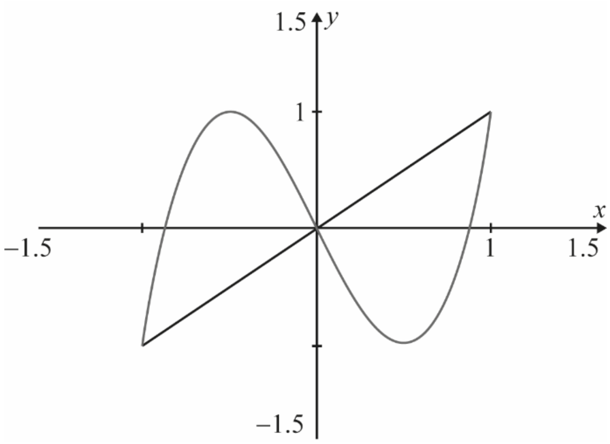
$n = 46$ ***A2***

[3 marks]

Total [30 marks]

2. (a) correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

(b) (i) graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

(ii) $\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

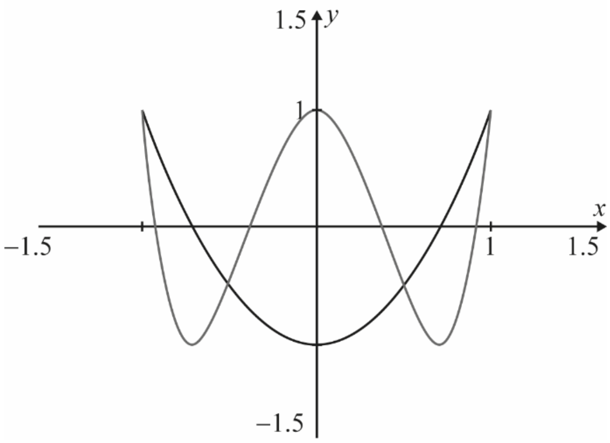
[4 marks]

continued…

Question 2 continued

(c) correct graph of $y = f_2(x)$ **A1**

correct graph of $y = f_4(x)$ **A1**



[2 marks]

(d) (i) graphical or tabular evidence that n has been systematically varied **M1**

eg $n = 2$, 0 local maximum point and 1 local minimum point
$n = 4$, 1 local maximum points and 2 local minimum points
$n = 6$, 2 local maximum points and 3 local minimum points **(A1)**

$\frac{n-2}{2}$ local maximum points **A1**

(ii) $\frac{n}{2}$ local minimum points **A1**

[4 marks]

(e) $f_n(x) = \cos(n\arccos(x))$

$$f_n'(x) = \frac{n\sin(n\arccos(x))}{\sqrt{1-x^2}}$$ **M1A1**

Note: Award ***M1*** for attempting to use the chain rule.

$f_n'(x) = 0 \Rightarrow n\sin(n\arccos(x)) = 0$ **M1**

$n\arccos(x) = k\pi$ ($k \in \mathbb{Z}^+$) **A1**

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and $0 < k < n$) **AG**

[4 marks]

continued…

Question 2 continued

(f) $f_2(x) = \cos(2\arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

(g) $f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A+B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h) (i) $f_{n-1}(x) = \cos((n-1)\arccos x)$ ***A1***

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ ***A1***

$= 2xf_n(x)$ ***AG***

(ii) $f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$ ***A1***

[5 marks]

Total [25 marks]