

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and any

values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$.

However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

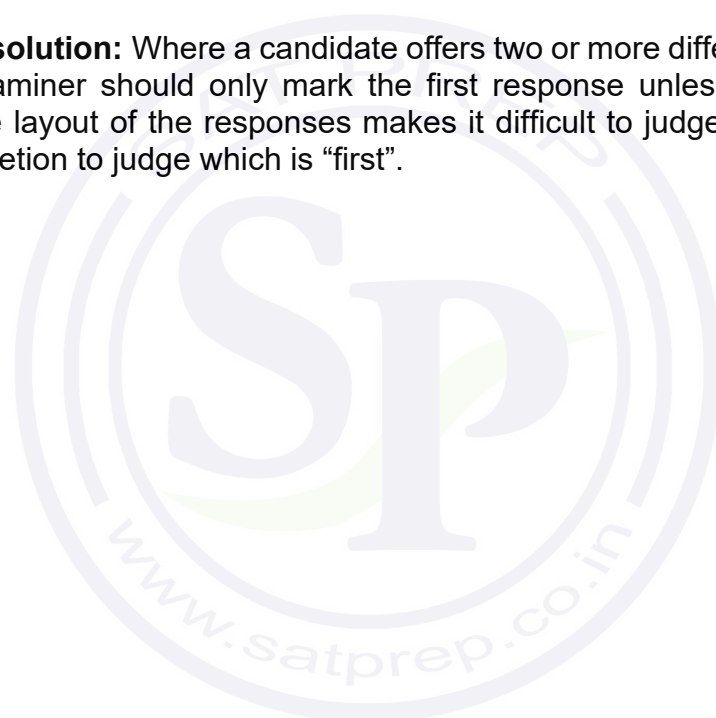
9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.



1. (a) attempts to use $E(M) = \sum mP(M = m)$ (M1)

$$= 1 \times \frac{1}{16} + 2 \times \frac{3}{16} + 3 \times \frac{5}{16} + 4 \times \frac{7}{16} \text{ (or equivalent)}$$

$$= \frac{25}{8} \left(= 3.125, = \frac{50}{16}, = 3.13 \right) \quad \text{A1}$$

[2 marks]

- (b) $G(1) = 1$ A1

[1 mark]

- (c) (i) $G'(t) = \frac{1}{16} + \frac{3}{8}t + \frac{15}{16}t^2 + \frac{7}{4}t^3 \left(= \frac{1}{16} + \frac{6}{16}t + \frac{15}{16}t^2 + \frac{28}{16}t^3 \right)$ A2

Note: Award **A1A0** for three correct terms.

[2 marks]

- (ii) substitutes $t = 1$ into their $G'(t)$ (M1)

$$G'(1) = \frac{1}{16} + \frac{3}{8} + \frac{15}{16} + \frac{7}{4} \left(= \frac{1}{16} + \frac{6}{16} + \frac{15}{16} + \frac{28}{16} \right) \quad \text{A1}$$

$$= \frac{25}{8} \left(= 3.125, = \frac{50}{16} \right) \quad \text{A1}$$

$$= E(M) \quad \text{AG}$$

Note: Award a maximum of **M1A1A0FT** for an incorrect $E(M)$ obtained with an incorrect $G'(t)$.

Award a maximum of **M1A1A0FT** for a correct $E(M)$ obtained with an incorrect $G'(t)$.

[3 marks]

continued...

Question 1 continued.

(d) attempts to find at least one required probability (M1)

$$P(X=0) = \frac{3}{5} \times \frac{2}{4} \text{ OR } P(X=0) = \frac{{}^2C_0 \times {}^3C_2}{{}^5C_2} \left(= \frac{{}^3C_2}{{}^5C_2} \right) \quad \text{A1}$$

$$= \frac{3}{10}$$

$$P(X=1) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} \left(= 2 \times \frac{6}{20} \right) \quad \text{A1A1}$$

Note: Award **A1** for $\frac{2}{5} \times \frac{3}{4}$ OR $\frac{6}{20}$ and **A1** for $\frac{3}{5} \times \frac{2}{4}$ OR $\frac{6}{20}$.

$$\text{Award } \mathbf{A2} \text{ for } P(X=1) = \frac{{}^2C_1 \times {}^3C_1}{{}^5C_2} \left(= \frac{2 \times {}^3C_1}{{}^5C_2} \right).$$

$$= \frac{3}{5}$$

EITHER

$$P(X=2) = \frac{2}{5} \times \frac{1}{4} \text{ OR } P(X=2) = \frac{{}^2C_2 \times {}^3C_0}{{}^5C_2} \left(= \frac{{}^2C_2}{{}^5C_2}, = \frac{{}^3C_0}{{}^5C_2} \right) \quad \text{A1}$$

OR

$$P(X=2) = 1 - \left(\frac{3}{10} + \frac{3}{5} \right) \quad \text{A1}$$

THEN

$$= \frac{1}{10}$$

$$\text{leading to } G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2 \quad \text{AG}$$

Note: Award **A** marks as appropriate for clearly labelled correct tree diagrams.

Do not award **A** marks for $P(X=0) = \frac{3}{10}$, $P(X=1) = \frac{3}{5}$ and

$P(X=2) = \frac{1}{10}$ stated without any working shown.

[5 marks]

continued...

Question 1 continued.

(e) (i) **EITHER**

$$P(Y = 2) = \frac{1}{2}p \quad (\text{A1})$$

OR

$$\frac{1}{2}p = \frac{1}{3} \quad (\text{A1})$$

THEN

$$p = \frac{2}{3} (= 0.667) \quad \text{A1}$$

[2 marks]

(ii) uses their $G_Y(t) = P(Y = 0)t^0 + P(Y = 1)t^1 + \frac{1}{3}t^2$ with their value of p (M1)

$$P(Y = 0) = \frac{1}{6} \quad (\text{A1})$$

$$P(Y = 1) = \frac{1}{2} \quad (\text{A1})$$

$$G_Y(t) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2 \quad \text{A1}$$

Note: Award a maximum (M1)(A1)(A1)A0FT from an incorrect value of p .

Award (M1)(A1)(A1)A0FT from $p = \frac{1}{3}$ leading to $G_Y(t) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$.

[4 marks]

continued...

Question 1 continued.

(f) **METHOD 1**

$$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2 \right) \left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2 \right) \left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4 \right) \quad (\text{A1})$$

attempts to evaluate their $G_Z'(1)$ (M1)

$$G_Z'(1) = 1.96666\dots$$

$$E(Z) = 1.97 \left(= \frac{59}{30} \right) \quad \text{A2}$$

Note: Award a maximum of **(A1)FT(M1)(M1)A0** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).

Do not award **(A1)FT** if $G_Y(t)$ is not a quadratic of the required form.

METHOD 2

$$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2 \right) \left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2 \right) \left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4 \right) \quad (\text{A1})$$

attempts to find $G_Z'(t)$ by differentiation M1

$$G_Z'(t) = \frac{1}{4} + \frac{5}{6}t + \frac{3}{4}t^2 + \frac{2}{15}t^3 \text{ OR}$$

$$G_Z'(t) = \left(\frac{3}{5} + \frac{1}{5}t \right) \left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2 \right) + \left(\frac{1}{2} + \frac{2}{3}t \right) \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2 \right)$$

substitutes $t = 1$ into their $G_Z'(t)$ (M1)

$$G_Z'(1) = 1.96666\dots$$

$$E(Z) = 1.97 \left(= \frac{59}{30} \right) \quad \text{A1}$$

Note: Award a maximum of **(A1)FT(M1)(M1)A0** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).

Do not award **(A1)FT** if $G_Y(t)$ is not a quadratic of the required form.

continued...

Question 1 continued.

METHOD 3

$$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2 \right) \left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2 \right) \left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4 \right) \quad (\text{A1})$$

$$\text{recognizes that } G_Z'(1) = G_X'(1) + G_Y'(1) \Rightarrow E(Z) = E(X) + E(Y) \quad (\text{M1})$$

$$\text{attempts to evaluate their } G_X'(1) \text{ and } G_Y'(1) \text{ (or their } E(X) \text{ and } E(Y)) \quad (\text{M1})$$

$$G_X'(1) = \frac{3}{5} + \frac{1}{5} \left(= \frac{4}{5} \right) \text{ and } G_Y'(1) = \frac{1}{2} + \frac{2}{3} \left(= \frac{7}{6} \right)$$

$$E(Z) = 1.97 \left(= \frac{59}{30} \right) \quad \text{A1}$$

Note: Award a maximum of **(A1)FT(M1)(M1)A0** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).

Do not award **(A1)FT** if $G_Y(t)$ is not a quadratic of the required form.

[4 marks]

Total [23 marks]

2. (a) $g'(x) = m$ OR $g''(x) = 0$ (seen anywhere) (A1)

Note: Award **A0** for $g''(x) = 0$ obtained from an incorrect $g'(x)$, for example, $g'(x) = 1$ OR $g'(x) = 0$.

EITHER

$$k(x) = \frac{|0|}{(1+m^2)^{\frac{3}{2}}} \text{ OR } k(x) = \frac{0}{(1+m^2)^{\frac{3}{2}}} \text{ OR } k(x) = \frac{0}{\sqrt{1+3m^2+3m^4+m^6}} \quad \text{A1}$$

OR

the numerator of $k(x)$ is zero and the denominator is non-zero A1

THEN

$\Rightarrow k(x) = 0$ AG
[2 marks]

- (b) (i) $x = -\frac{b}{2a}$ A1

Note: Do not award **A1** if $a = 0$ is included as a solution.

[1 mark]

- (ii) substitutes their value of x into $k(x)$ (M1)

$$k\left(-\frac{b}{2a}\right) = \frac{2|a|}{\left(1 + \left(2a\left(-\frac{b}{2a}\right) + b\right)^2\right)^{\frac{3}{2}}} \left(= \frac{2|a|}{\left(1 + (-b+b)^2\right)^{\frac{3}{2}}} \right)$$

$$= 2|a| \quad \text{A1}$$

Note: Award **(M1)A0** if the modulus sign is not considered.

[2 marks]

continued...

Question 2 continued.

(iii) **EITHER**

$$\lim_{x \rightarrow \infty} k(x) = 0$$

A1

OR

the curvature (of a quadratic) approaches zero as $x \rightarrow \infty$

A1

THEN

for large positive values of x , quadratic functions are close to being straight

R1

Note: Award **R1** for similar statements such as ‘for large positive values of x , a quadratic function behaves like a linear function.’

[2 marks]

(iv) **C** ($k_{\max}$ of $p = k_{\max}$ of q)

A1

$k_{\max}$ of p is $(2 \times |-2| =) 4$ and $k_{\max}$ of q is $(2 \times |2| =) 4$

R1

Note: Accept arguments such as ‘ $k_{\max} = 2|a|$ ’ and both quadratics have $|a| = 2$.
Do not award **A1R0**.
If modulus signs were omitted in part (b)(ii), award **FT** in part (b)(iv) for an answer of **B** with appropriate values shown.

[2 marks]

(c) (i) sets $k'(x) = 0$ and attempts to solve for x

(M1)

$$x^2 = \frac{1}{2}$$

$$x = \frac{1}{\sqrt{2}}$$

A1

Note: Award **A0** for $x = \pm \frac{1}{\sqrt{2}}$ ($= \pm 0.707$). Award **A0** for $x = 0.707$.

[2 marks]

continued...

Question 2 continued.

(ii) substitutes their value of x into $k(x)$

(M1)

Note: Award a maximum of **(M1)(A0)A0A0** if their value of x is a decimal.
Award **(M1)(A0)A0A0** for $(0.707, 0.385)$.

$$k\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(1 + \left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above denominator.

$$\left(1 + \left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{3}{2}} \quad \textbf{(A1)}$$

$$= \frac{3\sqrt{3}}{2\sqrt{2}} \text{ (or equivalent unsimplified fractional surd e.g. } \sqrt{\frac{27}{8}} \text{)} \quad \textbf{A1}$$

$$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}} \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}} \right)$$

$$= \frac{2}{3\sqrt{3}} \text{ (or equivalent surd e.g. } \frac{\sqrt{12}}{9} \text{)} \quad \textbf{A1}$$

$$k_{\max} = \frac{2\sqrt{3}}{9} \quad \textbf{AG}$$

[4 marks]

continued...

Question 2 continued.

(d) (i) **EITHER**

sets $k'(x) = 0$ and attempts to solve for e^{2x} (M1)

$$e^{2x} = \frac{1}{2} \quad \text{A1}$$

$$e^x = \frac{1}{\sqrt{2}}$$

substitutes their values for e^{2x} and e^x into $k(x)$ (M1)

OR

attempts to solve $k'(x) = 0$ for x (M1)

$$x = \frac{1}{2} \ln \frac{1}{2} \left(= \ln \frac{1}{\sqrt{2}} = -\frac{1}{2} \ln 2 \right) \quad \text{A1}$$

Note: Award a maximum of (M1)(A0)(M0)A0A0 if their value of x is a decimal.
Award (M1)(A0)(M0)A0A0 for $(-0.347, 0.385)$.

substitutes their value of x into $k(x)$ (M1)

THEN

for example, $\left(k \left(\ln \frac{1}{\sqrt{2}} \right) = \right) \frac{\frac{1}{\sqrt{2}}}{\left(1 + \left(\frac{1}{\sqrt{2}} \right)^2 \right)^{\frac{3}{2}}} \left(= \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2} \right)^{\frac{3}{2}}} \right)$ A1

$$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}} \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}} \right) \quad \text{A1}$$

$$= \frac{2}{3\sqrt{3}}$$

$$k_{\max} = \frac{2\sqrt{3}}{9} \quad \text{AG}$$

Note: After obtaining the penultimate **A1** above, candidates may state $k_{\max} = \frac{2\sqrt{3}}{9}$
as per correct work seen in part (c) (ii). In such cases, award the final **A1**.

[5 marks]

continued...

Question 2 continued.

(ii) **EITHER**

w and v are inverse functions of each other (and as such are equally 'curved') (or equivalent)

R1

OR

the graphs of w and v are reflections of each other in the line $y = x$

R1

OR

w and v have the same shape (or equivalent)

R1

[1 mark]

(e) (i) **METHOD 1**

attempts to use the chain rule to find $\frac{dy}{dx}$

(M1)

$$\frac{dy}{dx} = \frac{1}{2}(-2x)(r^2 - x^2)^{-\frac{1}{2}} \left(= -\frac{x}{\sqrt{r^2 - x^2}}, = -\frac{x}{y} \right)$$

A1

attempts to use the quotient or product rule to find $\frac{d^2y}{dx^2}$

(M1)

$$\frac{d^2y}{dx^2} = \frac{-y + x \frac{dy}{dx}}{y^2} \left(= \frac{x}{y^2} \frac{dy}{dx} - \frac{1}{y} \right)$$

A1

$$\text{correct substitution of } \frac{dy}{dx} = -\frac{x}{y} \text{ to form } = \frac{-y + x \left(-\frac{x}{y} \right)}{y^2} \left(= \frac{x}{y^2} \left(-\frac{x}{y} \right) - \frac{1}{y} \right)$$

A1

$$= \frac{-y^2 - x^2}{y^3} \left(= -\frac{y^2 + x^2}{y^3} \right)$$

A1

$$\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$$

AG

continued...

Question 2 continued.

METHOD 2

attempts to use the chain rule to find $\frac{dy}{dx}$ **(M1)**

$$\frac{dy}{dx} = \frac{1}{2}(-2x)(r^2 - x^2)^{-\frac{1}{2}} \left(= -\frac{x}{\sqrt{r^2 - x^2}} \right) \quad \textbf{A1}$$

attempts to use the quotient or product rule to find $\frac{d^2y}{dx^2}$ **(M1)**

$$= \frac{-\left(r^2 - x^2\right)^{\frac{1}{2}} - x^2 \left(r^2 - x^2\right)^{-\frac{1}{2}}}{r^2 - x^2} \left(= -x^2 \left(r^2 - x^2\right)^{-\frac{3}{2}} - \left(r^2 - x^2\right)^{-\frac{1}{2}} \right) \quad \textbf{A1}$$

$$= -\frac{r^2}{\left(r^2 - x^2\right)^{\frac{3}{2}}} \left(= \frac{-(r^2 - x^2) - x^2}{\left(r^2 - x^2\right)^{\frac{3}{2}}}, = -\frac{r^2}{\left(y^2\right)^{\frac{3}{2}}} \right) \quad \textbf{A1A1}$$

Note: Award **A1** for a correct numerator and **A1** for a correct denominator.

Candidates may express their $\frac{d^2y}{dx^2}$ in terms of r and y instead of r and x .

$$\frac{d^2y}{dx^2} = -\frac{r^2}{y^3} \quad \textbf{AG}$$

continued...

Question 2 continued.

METHOD 3

attempts to use implicit differentiation on $y^2 = r^2 - x^2$ **(M1)**

$$2y \frac{dy}{dx} = -2x \quad \left(y \frac{dy}{dx} = -x \right) \quad \text{A1}$$

attempts to use the product rule and implicit differentiation **(M1)**

$$\left(\frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} = -1 \text{ (or equivalent)} \quad \text{A1}$$

correct substitution of $\frac{dy}{dx} = -\frac{x}{y}$ to form $\left(-\frac{x}{y} \right)^2 + y \frac{d^2y}{dx^2} = -1$ **A1**

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{y} \left(-1 - \frac{x^2}{y^2} \right) \\ &= \frac{-y^2 - x^2}{y^3} \quad \left(= -\frac{y^2 + x^2}{y^3} \right) \end{aligned} \quad \text{A1}$$

$$\frac{d^2y}{dx^2} = -\frac{r^2}{y^3} \quad \text{AG}$$

[6 marks]

continued...

Question 2 continued.

(ii) **METHOD 1**

substitutes their $\frac{dy}{dx}$ and the given $\frac{d^2y}{dx^2}$ into $k(x)$ **(M1)**

$$= \frac{\left| -\frac{r^2}{y^3} \right|}{\left(1 + \left(-\frac{x}{y} \right)^2 \right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left| -\frac{r^2}{y^3} \right| = \frac{r^2}{y^3}$ (seen anywhere) **A1**

denominator: attempts to express $\left(1 + \left(-\frac{x}{y} \right)^2 \right)$ as $\frac{y^2 + x^2}{y^2}$ **M1**

denominator is $\left(\frac{r^2}{y^2} \right)^{\frac{3}{2}} \left(= \frac{r^3}{y^3} \right)$ **A1**

Note: Award the following **A** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{r^2}{y^3}}{\frac{r^3}{y^3}} \left(k = \frac{r^2}{y^3} \times \frac{y^3}{r^3} \right) \Rightarrow k = \frac{1}{r} \quad \text{A1}$$

which is a constant **AG**

Note: Award a maximum of **(M1)A0M1A1A1** if $\left| -\frac{r^2}{y^3} \right| = \frac{r^2}{y^3}$ is not considered.

continued...

Question 2 continued.

METHOD 2

substitutes their $\frac{dy}{dx}$ and the given $\frac{d^2y}{dx^2}$ into $k(x)$ **(M1)**

$$= \frac{\left| -\frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}} \right|}{\left(1 + \left(-\frac{x}{\sqrt{r^2 - x^2}} \right)^2 \right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left| -\frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}} \right| = \frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}}$ (seen anywhere) **A1**

denominator: attempts to express $\left(1 + \left(-\frac{x}{\sqrt{r^2 - x^2}} \right)^2 \right)$ as $\frac{r^2}{r^2 - x^2}$ **M1**

denominator is $\left(\frac{r^2}{r^2 - x^2} \right)^{\frac{3}{2}}$ **A1**

Note: Award the following **A** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}}}{\frac{r^3}{(r^2 - x^2)^{\frac{3}{2}}}} \left(k = \frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}} \times \frac{(r^2 - x^2)^{\frac{3}{2}}}{r^3} \right) \Rightarrow k = \frac{1}{r}$$

which is a constant **AG**

Note: Award a maximum of **(M1)A0M1A1A1** if $\left| -\frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}} \right| = \frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}}$ is not considered.

continued...

Question 2 continued.

METHOD 3

substitutes their $\frac{dy}{dx}$ and the given $\frac{d^2y}{dx^2}$ into $k(x)$ **(M1)**

$$= \frac{\left| -\frac{x^2 + y^2}{y^3} \right|}{\left(1 + \left(-\frac{x}{y} \right)^2 \right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left| -\frac{x^2 + y^2}{y^3} \right| = \frac{x^2 + y^2}{y^3}$ (seen anywhere) **A1**

denominator: attempts to express $\left(1 + \left(-\frac{x}{y} \right)^2 \right)^{\frac{3}{2}}$ as $\left(\frac{y^2 + x^2}{y^2} \right)^{\frac{3}{2}}$ **M1**

Note: Award the following **A1A1** for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{x^2 + y^2}{y^3}}{\left(\frac{x^2 + y^2}{y^2} \right)^{\frac{3}{2}}} = \frac{\frac{x^2 + y^2}{y^3}}{\frac{(x^2 + y^2)^{\frac{3}{2}}}{y^3}}$$

$$= \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \quad \text{A1}$$

$$k = \frac{1}{(r^2)^{\frac{1}{2}}} \Rightarrow k = \frac{1}{r} \quad \text{A1}$$

which is a constant **AG**

Note: Award a maximum of **(M1)A0M1A1A1** if $\left| -\frac{x^2 + y^2}{y^3} \right| = \frac{x^2 + y^2}{y^3}$ is not considered.

[5 marks]

Total [32 marks]

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg M1, A2*.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g. M1A1*, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** *etc.*, do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and any

values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An

exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$.

However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

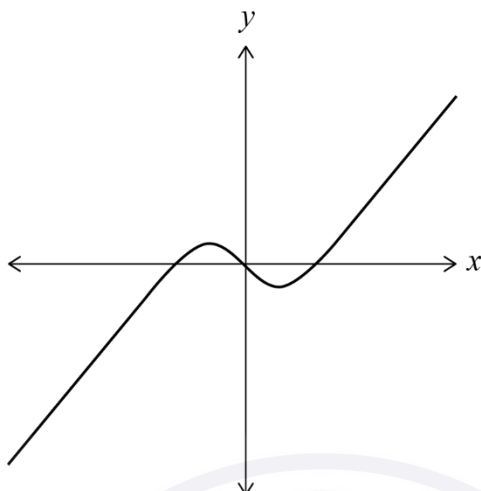
A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) (i)



general shape of curve passing through $(0,0)$

A1

[1 mark]

(ii) $(-4,0), (0,0), (4,0)$

A1

Note: Award **A0** if additional points are given.

[1 mark]

(iii) Max $(-1.94, 1.20)$ $(-1.94347..., 1.20113...)$

A1

Min $(1.94, -1.20)$ $(1.94347..., -1.20113...)$

A1

[2 marks]

continued

Question 1 continued.

(b) **EITHER**

odd

A1

$$\text{since } f(-x) = \frac{(-x)((-x)^2 - 16)}{(-x)^2 + 16} = \frac{-x(x^2 - 16)}{x^2 + 16} = -f(x) \text{ (or equivalent)}$$

R1

OR

odd

A1

since graph of f has rotational symmetry order 2 (of 180 degrees) about $(0, 0)$

R1

OR

odd

A1

since graph has reflection symmetry over the x-axis followed by reflection symmetry over the y-axis (or vice versa)

R1

Note: Do not award **A1R0**.

[2 marks]

(c) attempt to use quotient rule or product rule

(M1)

$$f'(x) = \frac{(x^2 + A)(3x^2 - A) - 2x(x^3 - Ax)}{(x^2 + A)^2} \left(f'(x) = \frac{2x^2 + (x^2 - A)}{x^2 + A} - \frac{2x^2(x^2 - A)}{(x^2 + A)^2} \right)$$

(or equivalent)

A1A1

Note: Award **A1** for numerator, **A1** for denominator.

Note: If product rule used, award **A1** for each correct term.

Note: Award **(M1)** for an attempt to use quotient rule or product rule with $A = 16$ or any other constant. Award no further marks.

$$f'(\sqrt{A}) = \frac{(2A)(2A) - 0}{(2A)^2} = 1$$

A1

hence independent of A

AG

[4 marks]

continued

Question 1 continued.

(d) (i) **METHOD 1**

$$x - \frac{2Ax}{x^2 + A} \equiv \frac{x(x^2 + A) - 2Ax}{x^2 + A} \left(\equiv \frac{x^3 + Ax - 2Ax}{x^2 + A} \right) \quad \text{M1}$$

$$\equiv \frac{x^3 - Ax}{x^2 + A} \left(\equiv \frac{x(x^2 + A - 2A)}{x^2 + A} \right) \quad \text{A1}$$

$$x - \frac{2Ax}{x^2 + A} \equiv \frac{x(x^2 - A)}{x^2 + A} \quad \text{AG}$$

Note: Award no marks for numerical approaches.

METHOD 2

EITHER

attempt to do long division on $\frac{x(x^2 - A)}{x^2 + A}$ or $\frac{x^3 - Ax}{x^2 + A}$ M1

quotient of x and remainder of $-2Ax$ A1

$$x - \frac{2Ax}{x^2 + A} \equiv \frac{x(x^2 - A)}{x^2 + A} \quad \text{AG}$$

OR

$$\frac{x^3}{x^2 + A} - \frac{Ax}{x^2 + A} \equiv \frac{x^3 + Ax}{x^2 + A} - \frac{Ax + Ax}{x^2 + A} \quad \text{M1}$$

$$\equiv \frac{x(x^2 + A)}{(x^2 + A)} - \frac{2Ax}{(x^2 + A)} \quad \text{A1}$$

$$\equiv x - \frac{2Ax}{x^2 + A} \equiv \frac{x(x^2 - A)}{x^2 + A} \quad \text{AG}$$

Note: Award no marks for numerical approaches.

[2 marks]

continued

Question 1 continued.

(ii) hence oblique asymptote is $y = x$ (since $\frac{2Ax}{x^2 + A} \rightarrow 0$ as $x \rightarrow \pm\infty$)

A1

Note: Award **A0** if not given as an equation.

[1 mark]

(iii) $(\sqrt{A}, 0)$ OR $(-\sqrt{A}, 0)$

A1

[1 mark]

continued



Question 1 continued.

(e)

Note: Award no marks for attempts at integration that do not involve partial fractions.

attempt to write $\frac{1}{1-v^2}$ in the form $\frac{B}{1+v} + \frac{C}{1-v}$ **(M1)**

$$B = \frac{1}{2} \text{ and } C = \frac{1}{2} \quad \text{A1}$$

$$\int \frac{1}{1-v^2} dv = \int \frac{\frac{1}{2}}{1+v} + \frac{\frac{1}{2}}{1-v} dv$$

attempt to integrate their partial fractions **(M1)**

EITHER

$$\frac{1}{2} \ln|1+v| - \frac{1}{2} \ln|1-v| + \frac{1}{2} \ln A \quad \text{A1A1}$$

Note: Award **A1** for first two terms. The moduli must be present for this mark. Award **A1** for $\frac{1}{2} \ln A$.

OR

$$\frac{1}{2} \ln|1+v| - \frac{1}{2} \ln|1-v| (+c) \quad \text{A1}$$

$$= \frac{1}{2} \ln \left| \frac{e^{2c}(1+v)}{1-v} \right| \text{ where } e^{2c} = A \quad \text{A1}$$

Note: The moduli must be present for each of these marks.

THEN

$$= \frac{1}{2} \ln \left| \frac{A(1+v)}{1-v} \right| \quad \text{AG}$$

[5 marks]

continued

Question 1 continued.

- (f) attempt to separate variables and integrate (M1)

$$\int \frac{dx}{x} = \int \frac{dv}{1-v^2}$$

$$\ln|x| = \frac{1}{2} \ln \left| \frac{A(1+v)}{1-v} \right| \quad \text{A1}$$

attempt to use power rule of logs (M1)

$$\ln|x| = \ln \left| \frac{A(1+v)}{1-v} \right|^{\frac{1}{2}} \quad \text{OR} \quad \ln|x|^2 = \ln \left| \frac{A(1+v)}{1-v} \right|$$

$$|x| = \left| \frac{A(1+v)}{1-v} \right|^{\frac{1}{2}} \quad \text{OR} \quad x^2 = \left| \frac{A(1+v)}{1-v} \right| \quad \text{A1}$$

$$x^2 = \left| \frac{A \left(1 + \frac{y}{x} \right)}{1 - \frac{y}{x}} \right| \quad \text{A1}$$

$$x^2 = \left| \frac{A(x+y)}{x-y} \right| \quad \text{AG}$$

Note: Award at most **(M1)A1(M1)A1A0** for incorrect use or omission of moduli.

Note: The substitution of $\frac{y}{x}$ may be seen earlier.

[5 marks]

continued

Question 1 continued.

$$(g) \quad x^2 = \frac{A(x+y)}{x-y} \quad (A1)$$

attempt to clear denominator M1

$$x^2(x-y) = A(x+y)$$

$$x^3 - x^2y = Ax + Ay$$

attempt to isolate y M1

$$x^3 - Ax = x^2y + Ay$$

$$x(x^2 - A) = y(x^2 + A) \quad (x^3 - Ax = y(x^2 + A)) \quad (A1)$$

$$y = \frac{x(x^2 - A)}{x^2 + A} \quad (AG)$$

Note: The y must be factored out for the final A1.

[4 marks]

Total [28 marks]

2.

- (a) (i) attempt to use Pythagoras' theorem (M1)

$$\sqrt{12500^2 + 6000^2}$$

$$= 13900 \text{ (km) (13865.4...)}$$

A1

[2 marks]

- (ii) attempt to use an appropriate inverse trig ratio (M1)

$$\text{e.g. } \arctan\left(\frac{12500}{6000}\right) \text{ or } \arctan\left(\frac{6000}{12500}\right)$$

identifying the correct angle (M1)

$$= (0)64.4^\circ \text{ OR } 064^\circ (64.3589...)$$

A1

Note: The two method marks can be awarded independently.

The first **(M1)** can come from any geometry leading to an inverse trigonometric ratio.

The second **(M1)** requires recognition of which angle is the bearing, possibly on a diagram. 64.4 appearing in intermediate working is not sufficient.

[3 marks]

- (b) (i) $p \cdot n = 0$ M1

$$\text{so the angle is } 90^\circ \text{ OR } \frac{\pi}{2} \text{ rads}$$

A1

Note: Award **MOA0** for an answer of 90 degrees without any mention that the scalar product is zero.

[2 marks]

- (ii) attempt to use arc length formula $s = r\theta$ OR $\frac{90^\circ}{360}$ of circumference (M1)

$$= 6000\left(\frac{\pi}{2}\right) \text{ OR } \frac{1}{4} \times 2\pi \times 6000$$

A1

$$= 3\pi \text{ thousand km OR } 3000\pi \text{ km}$$

AG

[2 marks]

continued

Question 2 continued.

- (c) (i) attempt to find cross product (e.g. correct except for sign error)

(M1)

$$\mathbf{a} \times \mathbf{p} = \begin{pmatrix} 0 \\ -36 \\ 0 \end{pmatrix}$$

A1

[2 marks]

(ii) **METHOD 1**

$$\mathbf{a} \times \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 36 \end{pmatrix}$$

A1

choosing to use scalar product for their $\mathbf{a} \times \mathbf{p}$ and $\mathbf{a} \times \mathbf{n}$

M1

$$(\mathbf{a} \times \mathbf{p}) \cdot (\mathbf{a} \times \mathbf{n}) = 0$$

A1

Note: The final **A1** can be awarded for seeing the dot product equals zero without incorrect working. Follow through from (i) is permitted and if they explicitly state that they are using $\mathbf{a} \times \mathbf{n}$ then allow internal follow through.

so angle is 90°

AG

METHOD 2

$$\mathbf{a} \times \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 36 \end{pmatrix}$$

A1

attempt to use cross product for their $\mathbf{a} \times \mathbf{p}$ and $\mathbf{a} \times \mathbf{n}$. This must include both an

attempt to find the cross product and knowledge that $\sin \theta = \frac{|\mathbf{x} \times \mathbf{y}|}{|\mathbf{x}||\mathbf{y}|}$

M1

$$(\mathbf{a} \times \mathbf{n}) \times (\mathbf{a} \times \mathbf{p}) = \begin{pmatrix} 1296 \\ 0 \\ 0 \end{pmatrix} \text{ OR } (\mathbf{a} \times \mathbf{p}) \times (\mathbf{a} \times \mathbf{n}) = \begin{pmatrix} -1296 \\ 0 \\ 0 \end{pmatrix}$$

$$\sin \theta = \frac{1296}{36 \times 36} = 1$$

A1

so angle is 90°

AG

[3 marks]

continued

Question 2 continued.

(d) attempt to use $\frac{\theta}{360} \times 2\pi r = 6$ OR $s = r\theta$ **(M1)**

$$\frac{\theta}{360} \times 2\pi \times 6 = 6 \quad \text{OR} \quad 6 = 6\theta \Rightarrow \theta = 1 \text{ radian}$$

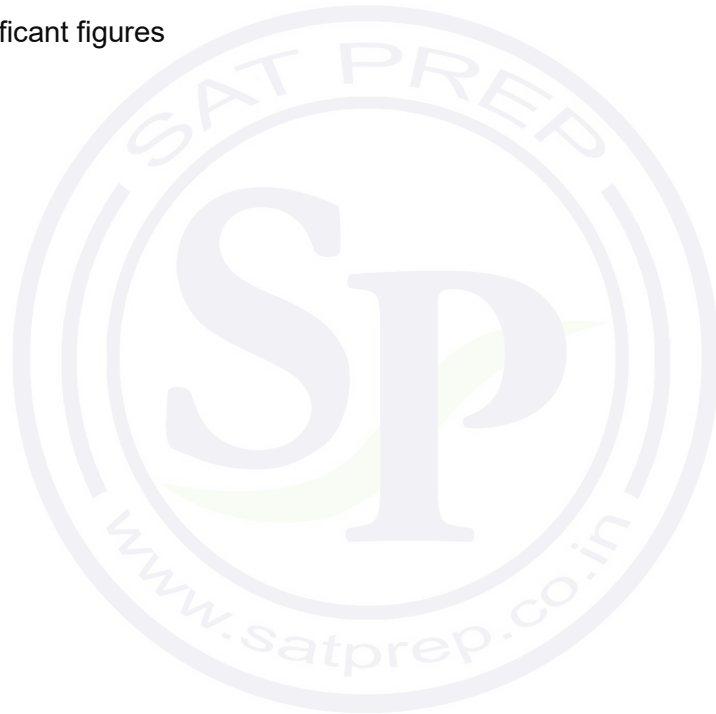
$$\theta = 57.2957\dots^\circ$$
 A1

Note: The fourth significant figure must be seen somewhere to award the **A1**.

$$\theta = 57.3^\circ \text{ to 3 significant figures}$$

AG
[2 marks]

continued



Question 2 continued.

(e) **METHOD 1**

EITHER (Scalar product)

attempt to use scalar product to find $\hat{BOM} = \arccos\left(\frac{\mathbf{b} \cdot \mathbf{m}}{|\mathbf{b}||\mathbf{m}|}\right)$ (M1)

Note: This may be written as $\cos(\hat{BOM}) = \frac{\mathbf{b} \cdot \mathbf{m}}{|\mathbf{b}||\mathbf{m}|}$.

$$= \arccos\left(\frac{(6\cos 120^\circ)(6\cos 57.3^\circ)}{6^2}\right)$$
 (A1)

Note: This (A1) can be awarded for $\cos\theta = \frac{6\cos 120^\circ \times 6\cos 57.3^\circ}{6^2}$.

$$= 106^\circ (= 105.673\dots^\circ) \text{ OR } 1.84 \text{ radians } (1.84434\dots)$$
 (A1)

OR (Cosine Rule)

$$|BM| = \sqrt{(6\sin 120 - 0)^2 + (6\cos 120 - 6\sin 57.3)^2 + (0 - 6\sin 57.3)^2}$$

$$= 9.56(9.56299\dots) \text{ OR } 9560\text{km}$$
 (A1)

attempt to use the cosine rule to find angle BOM (M1)

$$|BM|^2 = 6^2 + 6^2 - 2 \times 6 \times 6 \times \cos BOM$$

$$BOM = 106^\circ (= 105.673\dots^\circ) \text{ OR } 1.84 \text{ radians } (1.84434\dots)$$
 (A1)

THEN

attempt to use $\frac{\theta}{360} \times 2\pi r$ OR $s = r\theta$ (M1)

$$= \frac{105.671^\circ}{360} \times 2\pi \times 6$$

$$= 11.1 \text{ (11.0660... thousand km)}$$
 A1

Note: Accept an answer of 11100 OR 11066.0... without units (i.e. “km” is implicit).

continued

Question 2 continued.

METHOD 2 (Cross product)

attempt to use cross product to find $\hat{BOM} = \arcsin \frac{|b \times m|}{|b||m|}$ **(M1)**

Note: This may be written as $\sin \hat{BOM} = \frac{|b \times m|}{|b||m|}$.

$$= \frac{\begin{vmatrix} 36 \cos 120 \sin \theta \\ -36 \sin 120 \sin \theta \\ 36 \sin 120 \cos \theta \end{vmatrix}}{6 \times 6}$$

$$= \frac{\sqrt{\sin^2 \theta + 3}}{2} = 0.962818...$$

(A1)

$$\hat{BOM} = 180 - \arcsin 0.962818...$$

$$= 106^\circ (105.673..., 1.844346 \text{ rads})$$
 (A1)

attempt to use $\frac{\theta}{360} \times 2\pi r$ OR $s = r\theta$ **(M1)**

$$= \frac{105.671}{360} \times 2\pi \times 6$$

$$= 11.1 \text{ (11.0660 thousand km)}$$

A1

Note: Award at most **(M1)(A1)(A0)(M1)A0** for an answer of 7.78 thousand km from finding the principal root of the arcsin.

[5 marks]

continued

Question 2 continued.

(f) attempt to calculate $\mathbf{b} \times \mathbf{m}$ (M1)

$$\mathbf{b} \times \mathbf{m} = \begin{pmatrix} 36 \cos 120^\circ \sin 57.3^\circ \\ -36 \sin 120^\circ \sin 57.3^\circ \\ 36 \sin 120^\circ \cos 57.3^\circ \end{pmatrix} = \begin{pmatrix} -15.1464... \\ -26.2344... \\ 16.8449... \end{pmatrix} \quad (\text{A1})$$

attempt to use scalar product to find angle between vectors (M1)

$$(\cos \theta =) \left(\frac{(\mathbf{b} \times \mathbf{m}) \cdot (\mathbf{b} \times \mathbf{p})}{|\mathbf{b} \times \mathbf{m}| |\mathbf{b} \times \mathbf{p}|} \right)$$

$$(\cos \theta =) \left(\frac{1090.54...}{34.6614... \times 36} \right) \quad (\text{A1})(\text{A1})$$

Note: Award (A1) for numerator, (A1) for denominator.

bearing is $(0)29.1^\circ (29.0770...^\circ)$ A1

Note: Do not penalise absence of degree symbol.

Award full marks for answer in radians = 0.507.

Award full marks for 029° .

Special Case: If a candidate calculates $\mathbf{m} \times \mathbf{b}$ instead of $\mathbf{b} \times \mathbf{m}$ then you can award at most (M1)(A0)(M1)(A1)(A1)A0.

[6 marks]

Total[27 marks]

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and any

values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An

exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$.

However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10 Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) (i) attempts to find an expression for $f^2(x)$ (M1)

Note: For example, award **(M1)** for $f(mx + c)$ seen.

$$= m(mx + c) + c$$

A1

$$= m^2x + cm + c$$

A1

$$f^2(x) = m^2x + c(1 + m)$$

AG

[3 marks]

- (ii) attempts to find an expression for $f^3(x)$ (M1)

Note: For example, award **(M1)** for $f^2(mx + c)$ or $f(m^2x + c(1 + m))$ seen.

EITHER

$$f^3(x) = m^2(mx + c) + c(1 + m) \quad (= m^3x + m^2c + c + cm)$$

A1

OR

$$f^3(x) = m(m^2x + c(1 + m)) + c \quad (= m^3x + mc + m^2c + c)$$

A1

THEN

$$f^3(x) = m^3x + c(1 + m + m^2)$$

AG

[2 marks]

continued...

Question 1 continued

(b) (i) $f^4(x) = m^4x + c(1 + m + m^2 + m^3)$ **A1**

Note: Award **A1** for equivalent forms, for example:

$$f^4(x) = m^4x + c \sum_{i=0}^3 m^i$$

$$f^4(x) = m^3(mx + c) + c(1 + m + m^2)$$

$$f^4(x) = m(m^3x + m^2c + mc + c) + c$$

[1 mark]

(ii) $f^n(x) = m^n x + c(1 + m + m^2 + \dots + m^{n-1})$ **A1A1**

Note: Award **A1** for $m^n x$ and **A1** for $c(1 + m + m^2 + \dots + m^{n-1})$ or equivalent,

for example: $c \sum_{i=0}^{n-1} m^i$ or $c \left(\frac{1 - m^n}{1 - m} \right)$

For alternative forms, award **A1** for the correct term in x and **A1** for the correct constant term.

[2 marks]

(iii) **METHOD 1**

attempts to substitute $m = 1$ into their $f^n(x)$ **(M1)**

$$f^n(x) = x + c(1 + 1 + 1 + \dots + 1) \quad \text{A1}$$

$$f^n(x) = x + nc \quad \text{A1}$$

METHOD 2

$$f(x) = x + c \quad \text{A1}$$

attempts to find $f^2(x)$ with their $f(x)$ and $m = 1$ **(M1)**

$$f^2(x) = f(x + c) = x + 2c$$

$$\Rightarrow f^n(x) = x + nc \quad \text{A1}$$

[3marks]

continued...

Question 1 continued

(c) consider $n = 1$

$$f^1(x) = m^1x + c \left(\frac{1-m^1}{1-m} \right) \Rightarrow f(x) = mx + c,$$

R1

hence true for $n = 1$

Note: The substitution $n = 1$ must be explicit for the **R1** to be awarded.

Award **R0** for only considering $n = 2$.

Subsequent marks after this **R1** mark are independent of this mark and can be awarded.

assume true for $n = k$ ($p(k)$ is true)

M1

$$f^k(x) = m^kx + c \left(\frac{1-m^k}{1-m} \right) \text{ (for some } k \in \mathbb{Z}^+ \text{)}$$

Note: The assumption of truth must be apparent. Do not award **M1** for statements such as “let $n = k$ ” or “ $n = k$ is true” or “assume $n = k$ is true”. Subsequent marks after this **M1** are independent of this mark and can be awarded.

EITHER

Attempting to find $f(f^k(x))$

(M1)

$$f^{k+1}(x) = f(f^k(x))$$

$$f \left(m^kx + c \left(\frac{1-m^k}{1-m} \right) \right) = m \left(m^kx + c \left(\frac{1-m^k}{1-m} \right) \right) + c$$

A1

$$= \left(m^{k+1}x + cm \left(\frac{1-m^k}{1-m} \right) \right) + c$$

OR

Attempting to find $f^k(f(x))$

(M1)

$$f^{k+1}(x) = f^k(f(x))$$

$$f^k(mx + c) = m^k(mx + c) + c \left(\frac{1-m^k}{1-m} \right)$$

A1

$$= \left(m^{k+1}x + cm^k + c \left(\frac{1-m^k}{1-m} \right) \right)$$

continued...

Question 1 continued

THEN

EITHER

attempts to put c over a common denominator

(M1)

$$m^{k+1}x + cm \left(\frac{1-m^k}{1-m} \right) + \frac{c(1-m)}{1-m}$$

$$= m^{k+1}x + c \left(\frac{m(1-m^k) + (1-m)}{1-m} \right) \text{ (or equivalent)}$$

A1

OR

attempts to put cm^k over a common denominator

(M1)

$$m^{k+1}x + \frac{cm^k(1-m)}{1-m} + c \left(\frac{1-m^k}{1-m} \right)$$

$$= m^{k+1}x + c \left(\frac{m^k(1-m) + (1-m^k)}{1-m} \right) \text{ (or equivalent)}$$

A1

OR

recognizes that $\frac{1-m^k}{1-m}$ can be expressed as a geometric series

(M1)

$$m^{k+1}x + cm(1 + m + m^2 + \dots + m^{k-1}) + c$$

$$= m^{k+1}x + c(1 + m + m^2 + m^3 + \dots + m^k)$$

A1

THEN

$$= m^{k+1}x + c \left(\frac{1-m^{k+1}}{1-m} \right)$$

A1

since true for $n = 1$ and true for $n = k + 1$ if true for $n = k$, hence true for all $n \in \mathbb{Z}^+$

R1

Note: To obtain the final **R1**, four of the previous seven marks must have been awarded.

The statement ‘true for $n = 1$ ’ may be seen anywhere in the proof.

[8 marks]

continued...

Question 1 continued

- (d) attempts to apply $m^n \rightarrow 0$ (as $n \rightarrow \infty$) to at least one term in expression (M1)

Note: Award (M1) for an equivalent statement/conclusion in words.

$$m^n x \rightarrow 0 \qquad \left(c \left(\frac{1-m^n}{1-m} \right) \rightarrow \right) \frac{c}{1-m} \text{ (as } n \rightarrow \infty \text{)}$$
(A1)A1

Note: Award (A1) for the correct term in x and A1 for the correct constant term.

$$y = \frac{c}{1-m}$$
A1

[4 marks]

(e) (i) $f^n(x) = (-1)^n x + c \left(\frac{1-(-1)^n}{1-(-1)} \right)$ A1

Note: Award A1 here if the above is seen in part (e)(ii).

Award A0 for a missing bracket, for example: $f^n(x) = -1^n x + c \left(\frac{1-(-1)^n}{1-(-1)} \right)$.

$$(-1)^n = -1 \text{ (when } n \text{ is odd)}$$
R1

$$\left(f^n(x) = -x + c \left(\frac{1-(-1)}{1-(-1)} \right) \right)$$

Note: The A1 and R1 mark can be awarded in either order. A0R1 is possible.

Award R0 for arguments based on specific numerical examples.

$$f^n(x) = -x + c$$
AG

[2 marks]

- (ii) evidence that an even n has been considered (M1)

for example, $(-1)^2 = 1$

$$f^n(x) = x$$
A1

Note: Award (M1)A1 for answer of x (need not be an equation).

[2 marks]

Total [27 marks]

2. (a) (i) 0

A1
[1 mark]

(ii) 0

A1
[1 mark]

(b) attempts to expand a product of two of the side lengths

(M1)

EITHER

$$V = x(4x^2 - 2(a+b)x + ab) \quad (V = x(4x^2 - 2ax - 2bx + ab))$$

A1

OR

$$V = (ax - 2x^2)(b - 2x)$$

$$= 4x^3 - 2ax^2 - 2bx^2 + abx$$

A1

OR

$$V = (a - 2x)(bx - 2x^2)$$

$$= 4x^3 - 2ax^2 - 2bx^2 + abx$$

A1

THEN

$$V = 4x^3 - 2(a+b)x^2 + abx$$

AG
[2 marks]

continued...

Question 2 continued

(c) attempts to find $\frac{dV}{dx}$ (M1)

$$\frac{dV}{dx} = 12x^2 - 4(a+b)x + ab \quad \text{A1}$$

Note: Award (M1) for obtaining the equivalent of two correct terms.

EITHER

attempts to solve $\frac{dV}{dx} = 0$ for x using the quadratic formula (M1)

$$x = \frac{-(-4(a+b)) \pm \sqrt{(-4(a+b))^2 - 4(12)(ab)}}{24} \quad \text{(A1)}$$

Note: Award A0 for incorrect expressions, for example:

$$x = \frac{-(-4(a+b)) \pm \sqrt{-4(a+b)^2 - 4(12)(ab)}}{24}$$

attempts to expand brackets within the square root sign and/or to take a common factor of 4 (M1)

$$x = \frac{-(-4(a+b)) \pm \sqrt{16a^2 + 16b^2 + 32ab - 48ab}}{24} \quad \text{A1}$$

or $x = \frac{4(a+b) \pm 4\sqrt{a^2 + 2ab + b^2 - 3ab}}{24}$

OR

attempt to substitute either of the given x values into their $\frac{dV}{dx}$ M1

correct manipulation with each solution to reduce each expression to zero A1A1

$\frac{dV}{dx} = 0$ is a quadratic equation and hence these are the only two solutions R1

THEN

$$x = \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6} \quad \text{AG}$$

[6 marks]

continued...

Question 2 continued

- (d) attempts to find their $\frac{d^2V}{dx^2}$ (M1)

$$\frac{d^2V}{dx^2} = 24x - 4(a + b) \quad \text{A1}$$

$$\frac{d^2V}{dx^2} < 0 \quad \text{R1}$$

Note: Award **R1** for equivalent statement in words.

substitutes $x = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6}$ into their $\frac{d^2V}{dx^2}$ (M1)

Note: Award (M1) for substituting $x = \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6}$

$$\frac{d^2V}{dx^2} = 24 \left(\frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6} \right) - 4(a+b)$$

$$= -4\sqrt{a^2 - ab + b^2} (< 0) \quad \text{A1}$$

so $x_m = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6}$ gives a local maximum of V AG

[5 marks]

- (e) the maximum volume will occur at either the endpoints or at the local maximum. R1

Note: Award **R1** for an answer which refers to the volume being 0 at both endpoints ($x = 0$ and $x = \frac{a}{2}$) and the volume of the box at $x_m > 0$

hence the maximum volume of the box occurs at x_m / the local maximum is a global maximum.

AG
[1 mark]

continued...

Question 2 continued

(f) $2st - s^2 < t^2 - s^2$ A1

$t^2 > 2st$ and since $t \in \mathbb{Z}^+$ (dividing both sides by t) R1

Note: Award **R1** for equivalent reasoning, $t > 0$ or t is positive

$\Rightarrow t > 2s$ AG

[2 marks]

(g) attempts to substitute for each of a, b and $\sqrt{a^2 - ab + b^2}$ in terms of s and t (M1)

$(x_m =) \frac{(2st - s^2) + (t^2 - s^2) - (s^2 - st + t^2)}{6}$ A1

$= \frac{-3s^2 + 3st}{6}$ A1

$x_m = \frac{s(t-s)}{2}$ AG

[3 marks]

(h) (i) **METHOD 1**

when s is even, x_m is an integer R1

hence V_m is a product of integers ($a, b \in \mathbb{Z}^+$) R1

hence V_m is an integer AG

METHOD 2

$V_m = \frac{s(t-s)}{2} \left(2st - s^2 - 2 \left(\frac{s(t-s)}{2} \right) \right) \left(t^2 - s^2 - 2 \left(\frac{s(t-s)}{2} \right) \right)$ (or equivalent) A1

Note: Award **A1** for $V_m = \frac{s(t-s)}{2} \left(a - 2 \left(\frac{s(t-s)}{2} \right) \right) \left(b - 2 \left(\frac{s(t-s)}{2} \right) \right)$

when s is even, $\frac{s(t-s)}{2}$ is an integer **AND**

hence V_m is a product of integers ($a, b \in \mathbb{Z}^+$) R1

hence V_m is an integer AG

[2 marks]

continued...

Question 2 continued

(ii) when s is odd **AND** t is odd ($s(t-s)$ is even)

A1

x_m and V_m are both integers

AG

[1 mark]

(iii) the following table shows some possible sets of values for s, t, a, b, x_m and V_m

s	t	a	b	x_m	V_m
1	3	5	8	1	18
1	5	9	24	2	200
2	5	16	21	3	450
2	6	20	32	4	1152
2	7	24	45	5	2450
2	8	28	60	6	4608
3	7	33	40	6	3528
4	10	64	84	12	28800
...	...	...	...	...	...

attempts to use values of s and t to find

EITHER x_m **OR** a and b **OR** V_m

(M1)

a corresponding correct x_m value

A1

corresponding correct a and b values

(A1)

a corresponding correct V_m value

A1

Note: Award at most **(M1)A0A0A0** if the values of s and t do not satisfy the conditions: $s, t \in \mathbb{Z}^+$, $t > 2s$, and s is even / s and t are odd.

The three **A** marks can be awarded in any order.

The **A1** for x_m and the **A1** for a and b can be awarded independently.

Some candidates may determine in part (h) that $V_m = \frac{s^2 t^2 (t-s)^2}{2}$.

[4 marks]

Total [28 marks]

Markscheme

November 2024

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and any

values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$.

However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) $T_2 = 1.1(6000) - 500$ OR $T_2 = 6600 - 500$ **A2**
 $= 6100$ **AG**
[2 marks]

(b) (i) $T_3 = 1.1(6100) - 500$ ($T_3 = 1.1(1.1(6000) - 500) - 500$) OR $T_3 = 6710 - 500$ **A1**
 $= 6210$ **AG**
[1 mark]

(ii) $T_4 = 1.1(6210) - 500$ OR $T_4 = 6831 - 500$ **(A1)**
 $= 6331$ **A1**
[2 marks]

continued...



Question 1 continued.

(c) (i) **EITHER**

attempts to eliminate the denominator

(M1)

$$T_n = 6000(1.1)^{n-1} - 5000((1.1)^{n-1} - 1) \quad (= (6000 - 5000)(1.1)^{n-1} + 5000)$$

A1

Note: Accept correct equivalent forms, eg. $T_n = 6000(1.1)^{n-1} - 5000(1.1)^{n-1} + 5000$.

OR

attempts to express T_n with a common denominator of 0.1

(M1)

$$T_n = \frac{600(1.1)^{n-1} - 500((1.1)^{n-1} - 1)}{0.1} \quad \left(= \frac{600(1.1)^{n-1} - 500(1.1)^{n-1} + 500}{0.1} \right)$$

A1

Note: Accept correct equivalent forms.

THEN

$$T_n = 1000(1.1)^{n-1} + 5000$$

AG

[2 marks]

(c) (ii) **EITHER**

attempts to find T_6 using the explicit formula for T_n

(M1)

$$T_6 = 1000(1.1)^{6-1} + 5000 \quad \text{OR} \quad T_6 = 6610.51$$

OR

attempts to find T_6 using $T_{n+1} = 1.1T_n - 500$

(M1)

$$T_4 = 6331, T_5 = 6464.1, T_6 = 6610.51$$

OR

attempts to find T_6 using a finance solver application

(M1)

$$N = 5, I = 10\%, PV = -6000 \text{ and } PMT = 500$$

THEN

$$6611$$

A1

Note: Accept 6610.

[2 marks]

continued...

Question 1 continued.

(d) (i) **METHOD 1**

attempts to use the explicit formula for D_n to find an expression for $D_{n+1} - D_n$ **(M1)**

$$= 1.1D_n - 750 - (-1500(1.1)^{n-1} + 7500)$$

$$= 1.1(-1500(1.1)^{n-1} + 7500) - 750 - (-1500(1.1)^{n-1} + 7500) \quad \mathbf{A1}$$

EITHER

$$= -1650(1.1)^{n-1} + 8250 - 750 + 1500(1.1)^{n-1} - 7500 \quad \mathbf{A1}$$

OR

$$= 1500(1.1)^{n-1} - 1500(1.1)^n$$

$$= 1500(1.1)^{n-1}(1 - 1.1) \quad (= 1500(1.1)^{n-1}(-0.1)) \quad \mathbf{A1}$$

THEN

$$D_{n+1} - D_n = -150(1.1)^{n-1} \quad \mathbf{AG}$$

METHOD 2

attempts to express $D_{n+1} - D_n$ in terms of D_n **(M1)**

$$D_{n+1} - D_n = 1.1D_n - 750 - D_n$$

$$= 0.1D_n - 750 \quad \mathbf{(A1)}$$

$$= 0.1(-1500(1.1)^{n-1} + 7500) - 750 \quad (= -150(1.1)^{n-1} + 750 - 750) \quad \mathbf{A1}$$

$$D_{n+1} - D_n = -150(1.1)^{n-1} \quad \mathbf{AG}$$

[3 marks]

continued...

Question 1 continued.

(ii) $D_{n+1} - D_n < 0$ (because $(1.1)^{n-1} > 0$ and so $-150(1.1)^{n-1} < 0$)

R1

Note: Award **R1** for responses such as:

‘The difference is negative’.

$$-150(1.1)^{n-1} < 0$$

$$D_n = D_{n+1} + 150(1.1)^{n-1}$$

Accept responses such as:

$D_n - D_{n+1} > 0$ or ‘the difference is positive’ as long as $D_n - D_{n+1}$ is referred to.

Do not accept arguments based on specific numerical examples.

therefore $D_{n+1} < D_n \Rightarrow D_n > D_{n+1}$

AG

(and hence the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year)

[1 mark]

continued...

Question 1 continued.

(e) **METHOD 1**

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $D_n \leq 0$ ($D_n = 0$ OR $D_n < 0$)

(M1)

Note: Award **(M1)** for attempting to obtain $-1500(1.1)^{n-1} \leq -7500$ or

$-1500(1.1)^{n-1} = -7500$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award **(M1)**, for considering values of n either side of $D_n = 0$.

EITHER

$$17.8863... \left(= \log_{1.1} 5 + 1, = \frac{\ln 5}{\ln 1.1} + 1 \right) \text{ (critical value of } n \text{ seen anywhere)}$$

(A2)

OR

$$D_{17} = 607.540... \text{ (} 607.540... < 750 \text{)}$$

(A2)

OR

$$D_{18} = -81.7054... (< 0)$$

(A2)

THEN

trout disappears from the lake during (at the end of) the 17th year

A1

Note: Award **A1** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

Note: Award as above for $D_{n+1} \leq 0$ ($D_{n+1} = 0$ OR $D_{n+1} < 0$).

continued...

Question 1 continued.

METHOD 2

$S_n = \sum (D_{n+1} - D_n)$ is the total decrease in the number of trout by the end of year n or start of year $n + 1$

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $S_n \leq -6000$ ($S_n = -6000$ OR $S_n < -6000$)

(M1)

$$\frac{-150((1.1)^n - 1)}{1.1 - 1} (= -1500((1.1)^n - 1)) \leq -6000$$

Note: Award **(M1)** for attempting to obtain $-1500((1.1)^n - 1) \leq -6000$ or

$-1500((1.1)^n - 1) = -6000$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award **(M1)**, for considering values of n either side of $S_n = -6000$.

$$n = 16.8863...$$

(A2)

trout disappears from the lake during (at the end of) the 17th year

A1

Note: Award **A1** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

[4 marks]

(f) **METHOD 1**

$$C_n = (6000 - 10d)(1.1)^{n-1} + 10d$$

(A1)

EITHER

$$6000 - 10d = 0 \text{ (recognizes that } C_n \text{ is independent of } n \text{)}$$

(M1)

OR

$$10d = 6000 \text{ (sets } C_n = 6000 \text{ and compares coefficients)}$$

(M1)

THEN

$$d = 600$$

A1

continued...

Question 1 continued.

METHOD 2

EITHER

$$C_{n+1} - C_n (= 0)$$

$$6000(1.1)^n - 10d((1.1)^n - 1) - (6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1)) (= 0) \quad (\text{A1})$$

OR

$$0.1C_n - d (= 0) \text{ (similar to part (d) (i) method 2)}$$

$$0.1(6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1)) - d (= 0) \quad (\text{A1})$$

THEN

attempts to expand to eliminate the linear terms in d and then attempts to factorize (M1)

$$600(1.1)^{n-1} - d(1.1)^{n-1} (= 0) \quad ((600 - d)(1.1)^{n-1} (= 0))$$

$$d = 600 \quad \text{A1}$$

METHOD 3

considers $C_{n+1} = 1.1C_n - d$ and $C_1 = 6000$ (A1)

EITHER

$$6000 = 1.1 \times 6000 - d \quad (C_1 = C_2 = 6000) \quad (\text{A1})$$

OR

systematically varies the value of d (M1)

THEN

$$d = 600 \quad \text{A1}$$

continued...

Question 1 continued.

METHOD 4

EITHER

considers a specific $C_n = C_{n+1}$, for example, $C_2 = C_3$ (A1)

$$6000(1.1) - 10d(1.1 - 1) = 6000(1.1)^2 - 10d((1.1)^2 - 1)$$

attempts to solve their $C_n = C_{n+1}$ for d (M1)

for example, $1.1d = 660$

OR

considers a specific $C_n = 6000$, for example, $C_2 = 6000$ (A1)

$$6000(1.1)^{2-1} - 10d((1.1)^{2-1} - 1) = 6000$$

attempts to solve their $C_n = 6000$ for d (M1)

$$6600 - d = 6000$$

THEN

$$d = 600$$

A1

[3 marks]

continued...

Question 1 continued.

$$(g) \quad n = 1: \text{LHS} = u_1 \text{ and } \text{RHS} = \left(u_1 r^{1-1} - \frac{d(r^{1-1} - 1)}{r-1} \right) = u_1 r^0 - \frac{d(r^0 - 1)}{r-1} = u_1 \quad \mathbf{R1}$$

LHS = RHS and so true for $n = 1$

Note: Award **R0** for considering $n = 2$.

Subsequent marks after this **R1** are independent of this mark and can be awarded.

$$\text{assume true for } n = k \quad \left(u_k = u_1 r^{k-1} - \frac{d(r^{k-1} - 1)}{r-1} \right) \quad \mathbf{M1}$$

Note: The assumption of truth must be apparent. Do not award **M1** for statements such as “let $n = k$ ” or “assume that $n = k$ is true”. Subsequent marks after this **M1** are independent of this mark and can be awarded.

attempts to substitute for u_k in $u_{k+1} = ru_k - d$ **M1**

$$\begin{aligned} &= r \left(u_1 r^{k-1} - \frac{d(r^{k-1} - 1)}{r-1} \right) - d \\ &= u_1 r^k - \frac{dr(r^{k-1} - 1)}{r-1} - \frac{d(r-1)}{r-1} \left(= u_1 r^k - \frac{dr^k - dr}{r-1} - d \right) \quad \mathbf{A1} \end{aligned}$$

$$= u_1 r^k - \frac{dr^k + dr - dr - d}{r-1} \left(= u_1 r^k - \frac{dr^k - d}{r-1} \right) \quad \mathbf{A1}$$

$$= u_1 r^k - \frac{d(r^k - 1)}{r-1} \quad \mathbf{A1}$$

since true for $n = 1$ and true for $n = k + 1$ if true for $n = k$, hence true for all $n \in \mathbb{Z}^+$ **R1**

Note: To obtain the final **R1**, four of the previous six marks must have been awarded.

[7 marks]

Total [27 marks]

2. (a) (i) attempts to solve

(M1)

EITHER

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

OR

$$(x+2)^2 - 3 (= 0)$$

OR

$$(x+2+\sqrt{3})(x+2-\sqrt{3})(= 0)$$

THEN

$$x = -2 \pm \sqrt{3}$$

A1A1

Note: Award **A1** for $-2 + \sqrt{3}$ and **A1** for $-2 - \sqrt{3}$.

[3 marks]

continued...

Question 2 continued.

(ii) **METHOD 1**

attempts to find their $\frac{1}{-2-\sqrt{3}} \times \frac{-2+\sqrt{3}}{-2+\sqrt{3}}$ (M1)

Note: Only award (M1) if the roots are in the form $s \pm \sqrt{t}$.

$$= \frac{-2+\sqrt{3}}{(-2)^2 - (\sqrt{3})^2} \text{ and } (-2)^2 - (\sqrt{3})^2 = 1$$
A1

hence the roots form a reciprocal pair AG

Note: Award as above for attempting to show that $\frac{1}{-2+\sqrt{3}} = -2-\sqrt{3}$.

METHOD 2

considers the product of their roots (M1)

Note: Only award (M1) if the roots are in the form $s \pm \sqrt{t}$.

$$(-2+\sqrt{3})(-2-\sqrt{3}) = 1$$
A1

hence the roots form a reciprocal pair AG

METHOD 3

considers $\frac{c}{a} \left(= \frac{1}{1} \right)$ (M1)

the product of roots is 1 A1

hence the roots form a reciprocal pair AG

[2 marks]

continued...

Question 2 continued.

(b) roots are $\pm i \left(= i, \frac{1}{i}, = \text{cis} \left(\pm \frac{\pi}{2} \right), = e^{\pm i \frac{\pi}{2}} \right)$ (A1)

EITHER

$\frac{1}{-i} \times \frac{i}{i} = i$ OR $\frac{1}{-i} = i$ A1

Note: Award as above for showing that $\frac{1}{i} = -i$ OR stating that $\frac{1}{i} = -i$.

OR

$i \times -i = 1$ OR $i \times \frac{1}{i} = 1$ A1

OR

the product of roots is 1 A1

THEN

hence the roots form a reciprocal pair AG
[2 marks]

(c) $p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$ (A1)

$x^2 p\left(\frac{1}{x}\right) = x^2 \left(\frac{a}{x^2} + \frac{b}{x} + a \right)$ A1

$= ax^2 + bx + a$

$= p(x)$ AG

Note: Award as above for showing that $\frac{p(x)}{x^2} = p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$.

[2 marks]

continued...

Question 2 continued.

(d) **METHOD 1**

$$p(\alpha) = 0 \Rightarrow \alpha^n p\left(\frac{1}{\alpha}\right) = 0$$

A1

EITHER

$$\alpha^n \neq 0 \text{ (as } \alpha \neq 0) \Rightarrow p\left(\frac{1}{\alpha}\right) = 0$$

R1

OR

$$\alpha \neq 0 \text{ (hence } \alpha^n \neq 0) \Rightarrow p\left(\frac{1}{\alpha}\right) = 0$$

R1

THEN

so $\frac{1}{\alpha}$ is also a root

AG

Note: The **R1** is dependent on the **A1**.

[2 marks]

METHOD 2

$$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p\left(\frac{1}{\frac{1}{\alpha}}\right)$$

$$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p(\alpha)$$

A1

for $\alpha \neq 0$ (seen anywhere), $p(\alpha) = 0 \Rightarrow \left(\frac{1}{\alpha}\right)^n p(\alpha) = 0$ and so $p\left(\frac{1}{\alpha}\right) = 0$

R1

so $\frac{1}{\alpha}$ is also a root

AG

Note: The **R1** is dependent on the **A1**.

[2 marks]

continued...

Question 2 continued.

(e) **METHOD 1**

$$p(x) = x^n p\left(\frac{1}{x}\right)$$

$$q(x) = x^m q\left(\frac{1}{x}\right) \quad (\text{A1})$$

attempts to substitute for $p(x)$ and $q(x)$ into $f(x) = p(x)q(x)$ (M1)

$$f(x) = x^n p\left(\frac{1}{x}\right) x^m q\left(\frac{1}{x}\right) \left(= x^{n+m} p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right) \right) \quad \text{A1}$$

$$= x^{n+m} f\left(\frac{1}{x}\right) \text{ (and } n+m \text{ is the degree of } f \text{)} \quad \text{A1}$$

hence f is a palindromic polynomial. AG

METHOD 2

$$p\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}$$

$$q\left(\frac{1}{x}\right) = \frac{q(x)}{x^m} \quad (\text{A1})$$

attempts to substitute for $p\left(\frac{1}{x}\right)$ and $q\left(\frac{1}{x}\right)$ into $f\left(\frac{1}{x}\right) = p\left(\frac{1}{x}\right)q\left(\frac{1}{x}\right)$ (M1)

$$f\left(\frac{1}{x}\right) = \frac{p(x)}{x^n} \frac{q(x)}{x^m} \left(= \frac{p(x)q(x)}{x^{n+m}} \right) \quad \text{A1}$$

$$x^{n+m} f\left(\frac{1}{x}\right) = p(x)q(x) \text{ and } f(x) = p(x)q(x) \text{ (and } n+m \text{ is the degree of } f \text{)} \quad \text{A1}$$

hence f is a palindromic polynomial. AG

[4 marks]

continued...

Question 2 continued.

METHOD 3

for example, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_{n-1} x + a_n$ AND

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_{m-1} x + b_m$$

A1

attempts to multiply out their $p(x)q(x)$

(M1)

Note: Only award **(M1)** for polynomials of degree n and m respectively with different corresponding coefficients.

$$p(x)q(x) = a_n b_m x^{m+n} + (a_{n-1} b_m + a_n b_{m-1}) x^{m+n-1} + \dots + (a_{n-1} b_m + a_n b_{m-1}) x + a_n b_m$$

A1A1

Note: Award **A1** for the first and second terms and **A1** for the penultimate and last terms.

hence f is a palindromic polynomial

AG

[4 marks]

(f) **METHOD 1**

attempts to expand the product of two quadratic factors

(M1)

EITHER

$$x^4 + ux^3 + vx^3 + uvx^2 + 2x^2 + ux + vx + 1$$

(A1)

OR

$$x^4 + (u+v)x^3 + (uv+2)x^2 + (u+v)x + 1$$

(A1)

THEN

equates coefficients and forms a system of two equations in u and v

(M1)

$$u+v=2 \text{ and } uv+2=-1 \text{ (} uv=-3 \text{)}$$

A1

attempts to solve their system of two equations

(M1)

$$u=-1, v=3 \text{ OR } (f(x)=(x^2-x+1)(x^2+3x+1))$$

A1

Note: Candidates must attempt to form and solve a system of equations in u and v . Hence, award no marks for working backwards from either decimal or exact values of roots given by a GDC.

Award **A0** for $u=-1, v=3$ and $u=3, v=-1$ both specified.

continued...

Question 2 continued.

METHOD 2

attempts to divide one of $(x^2 + ux + 1)$ or $(x^2 + vx + 1)$ into $f(x)$ (M1)

EITHER

for example, the quotient is $x^2 + (2 - u)x + u^2 - 2u - 2$ A1

sets their quotient equal to $x^2 + vx + 1$ (M1)

$$x^2 + (2 - u)x + u^2 - 2u - 2 = x^2 + vx + 1$$

$$u^2 - 2u - 2 = 1 \text{ and } 2 - u = v \text{ A1}$$

attempts to solve their equations for u and then v (or vice versa) (M1)

OR

for example, the remainder is $u(-u^2 + 2u + 3)x + (-u^2 + 2u + 3)$ A1

sets their remainder equal to zero (M1)

attempts to solve for u (or v) (M1)

$$u = -1, 3 \text{ and } u + v = 2 \text{ A1}$$

THEN

$$u = -1, v = 3 \text{ OR } (f(x) =)(x^2 - x + 1)(x^2 + 3x + 1) \text{ A1}$$

Note: Award **A0** for $u = -1, v = 3$ and $u = 3, v = -1$ both specified.

continued...

Question 2 continued.

METHOD 3

considers $f(1)$, for example (M1)

$$(u+2)(v+2)=5 \quad \text{A1}$$

considers $f(2)$, for example (M1)

$$(2u+5)(2v+5)=33 \quad \text{A1}$$

attempts to solve their system of two equations (M1)

$$u=-1, v=3 \text{ OR } (f(x)=(x^2-x+1)(x^2+3x+1)) \quad \text{A1}$$

Note: Award **A0** for $u=-1, v=3$ and $u=3, v=-1$ both specified.

[6 marks]

(g) attempts to solve $x^4 + 2x^3 - x^2 + 2x + 1 = 0$ (M1)

Note: Award **(M1)** for attempting to solve one of their quadratic equations.

$$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{3}{2} \pm \frac{\sqrt{5}}{2} \quad \text{A1A1}$$

Note: Award **A1** for $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and **A1** for $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$.

[3 marks]

continued...

Question 2 continued.

(h) **METHOD 1**

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + a_0 (= 0)$

attempts to find $p(-1)$ (M1)

$$p(-1) = (-1)^n + a_1(-1)^{n-1} + a_2(-1)^{n-2} + \dots + a_2(-1)^2 + a_1(-1) + 1$$
 (A1)

$$= (-1) + a_1 - a_2 + \dots + a_2 - a_1 + 1$$
 A1

(since n is odd), $p(x)$ has

EITHER

$n+1$ terms so all terms cancel giving $p(-1) = 0$ R1

OR

an even number of terms so all terms cancel giving $p(-1) = 0$ R1

THEN

hence -1 is always a root AG

METHOD 2

considers the product of roots of a polynomial equation of degree n (M1)

$$\frac{(-1)^n a_0}{a_n}$$

product of roots is $(-1)^n$ (seen anywhere) A1

(for n odd) there are $\frac{n-1}{2}$ reciprocal pairs of roots A1

product of each reciprocal pair is 1 and $(-1)^n = -1$ (since n is odd) R1

hence -1 is always a root AG

continued...

Question 2 continued.

METHOD 3

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + a_0 (=0)$

considers $p(x) = x^n p\left(\frac{1}{x}\right)$ with $x = -1$ **(M1)**

$p(-1) = (-1)^n p(-1)$ **A1**

$p(-1) = -p(-1)$ ($2p(-1) = 0$) (since n is odd) **A1**

hence $p(-1) = 0$ **R1**

hence -1 is always a root **AG**

[4 marks]

Total [28 marks]



Markscheme

May 2024

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) (i) $0.9 \times 0.9 \times 0.1$ (M1)

Note: Award **M1** for 0.9 seen as part of a multiplication.

$$= 0.081$$

A1

[2 marks]

- (ii) let A = number of boosts in the first 6 actions

METHOD 1

Recognition of binomial distribution

(M1)

with parameters $n = 6$ and $p = 0.1$

(A1)

$$(P(A \geq 1) =) 0.469 \quad (0.468558...)$$

A1

Note: Award **(M1)(A1)A0** for an answer of 0.114 (from finding $P(A > 1)$).

METHOD 2

$$(P(A \geq 1) =) 1 - P(\text{no boost in first 6})$$

(M1)

$$= 1 - 0.9^6$$

(A1)

$$= 0.469 \quad (0.468558...)$$

A1

METHOD 3

$$0.1 + 0.9 \times 0.1 + 0.9^2 \times 0.1 + 0.9^3 \times 0.1 + 0.9^4 \times 0.1 + 0.9^5 \times 0.1$$

(M1)(A1)

Note: Award **(M1)** for at least 3 correct terms in a sum.

$$= 0.469 \quad (0.468558...)$$

A1

[3 marks]

continued...

Question 1 continued

(b) (i) **EITHER**

there will be $x-1$ actions not boosted, followed by one boosted action

A1

OR

The x^{th} action is boosted and all previous actions are not boosted

A1

OR

$$\underbrace{(1-p) \times (1-p) \times \dots \times (1-p)}_{(x-1) \text{ times}} \times p$$

A1

Note: For the **OR** method, the order of the operations must be clear (accept also a tree diagram etc.).

THEN

$$p(1-p)^{x-1}$$

AG

[1 mark]

(ii) $E(X) = \sum_{x=1}^{\infty} xp(1-p)^{x-1}$

A1

[1 mark]

continued...

Question 1 continued

(c) (i) LHS:

$$a + 2ar + 3ar^2 + \dots$$

A1

Note: Award A1 for writing LHS = $\sum ar^n$ and differentiating to get = $\sum anr^{n-1}$.

RHS:

attempt to differentiate $a(1-r)^{-1}$ or use quotient rule

(M1)

$$\frac{a}{(1-r)^2}$$

A1

$$\sum_{n=1}^{\infty} nar^{n-1} = \frac{a}{(1-r)^2}$$

A1

Note: To award the final **A1** " $\sum_{n=1}^{\infty} nar^{n-1}$ " must be seen either as part of the LHS working or as part of the final answer and all previous marks must be awarded.

[4 marks]

(ii) recognition that $a = p$ and $r = 1 - p$ (and $x = n$)

(M1)

$$E(X) = \frac{p}{(1-(1-p))^2}$$

A1

Note: Award **A0 FT** if their answer does not lead to the **AG**.

$$= \frac{1}{p}$$

AG

[2 marks]

(d) $E(X) = 10$, $\text{Var}(X) = 90$

A1A1

[2 marks]

continued...

Question 1 continued

(e) $0.8 \times 0.6 \times 0.6$

(M1)A1

Note: Award **M1** for evidence of P(“not boosted”) changing, e.g. a labelled tree diagram and/or 0.8 AND 0.6 seen.

$= 0.288$

AG

[2 marks]

- (f) after four actions that are not boosted, the probability that the next action is boosted is 1 (and $p = 1$ is a certainty)

R1

Note: Accept “when $Y = 5$, the probability of a boost is 1”.

Do not accept “on the 5th action the probability of a boost is 1”, unless there is reference to the previous actions being not boosted.

[1 mark]

(g) (i) $m = 0.32, n = 0.1536$ (0.154)

A1A1

[2 marks]

- (ii) attempt to multiply outcomes by probabilities and find the sum
 $0.2 + 2 \times 0.32 + 3 \times 0.288 + 4 \times 0.1536 + 5 \times 0.0384$

(M1)

A1

Note: The **A1** can only be awarded if m and n are correct and the exact value of n is used.

$= 2.5104$

AG

[2 marks]

(iii) $0.2 + 2^2 m + 3^2 \times 0.288 + 4^2 \times n + 5^2 \times 0.0384 - 2.5104^2$
 $= 1.19$ (1.18749...)

(A1)

A1

Note: Award **A1A1** for a correct answer found using the GDC.

[2 marks]

continued...

Question 1 continued

(h) (i) $\left(p = \frac{1}{2.5104}\right) = 0.398 \text{ (0.398342...)}$

A1

[1 mark]

(ii) $\text{Var}(X) = 3.79 \text{ (3.79170...)}$

A1

Note: Award **A1** for an answer of 3.80 (3.80040...) (from using their 3sf answer from part (i).)

Condone 3.8.

[1 mark]

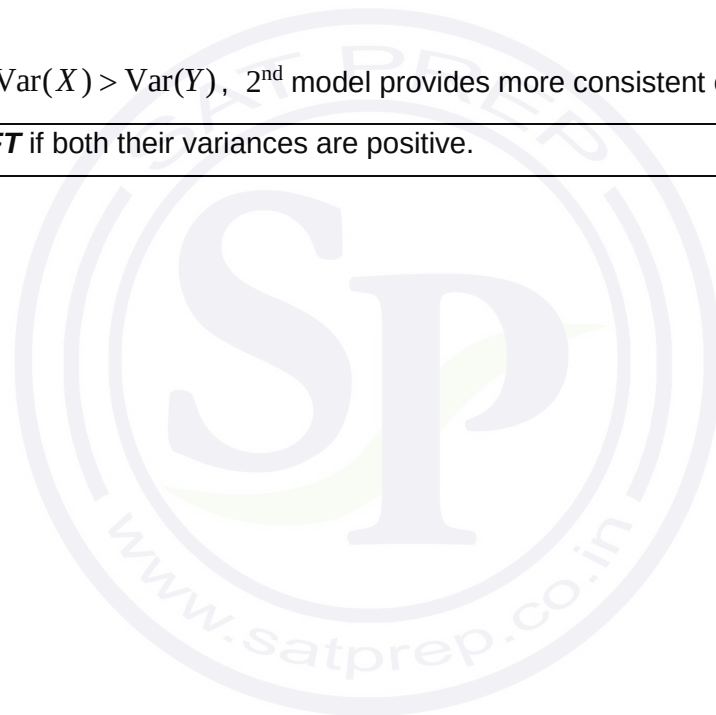
(iii) since $\text{Var}(X) > \text{Var}(Y)$, 2nd model provides more consistent experience

R1

Note: Only award **R1 FT** if both their variances are positive.

[1 mark]

Total [27 marks]



2. (a) $\frac{dy}{dx} = 4x^3 - 9x^2 + 3$ (M1)(A1)

Note: Award **M1** for at least two correct terms.

$$\frac{d^2y}{dx^2} = 12x^2 - 18x \quad \text{A1}$$

[3 marks]

(b) valid attempt to find x -coordinates (e.g. solving $12x^2 - 18x = 0$ or graphing $\frac{dy}{dx} = 4x^3 - 9x^2 + 3$) (M1)

$x = 0, 1.5 \left(\frac{3}{2} \right)$ (A1)

point B is (0, 0) A1

point C is (1.5, -0.5625) $\left((1.5, -0.563), \left(\frac{3}{2}, -\frac{9}{16} \right) \right)$ A1

Note: Award **M0A0A1A0** for an unsupported answer of point B is (0, 0).

[4 marks]

(c) y -intercept = 0 (as line passes through (0, 0)) R1

Note: Award **R1** for correctly substituting point B or point C to show that $y = 0$.

Award **R1** for $y = -0.375x + 0$.

$$\text{gradient} = \frac{-0.5625}{1.5} \left(\begin{array}{c} -\frac{9}{16} \\ \frac{3}{2} \end{array} \right) \quad \text{A1}$$

Note: Award **A0FT** if their answer to (b) doesn't lead to the given answer, but condone $\frac{-0.563}{1.5}$.

$= -0.375$

so equation is $y = -0.375x$ AG

Note: Award at most **A1R0** for working backwards to verify points B and C lie on the given line.

[2 marks]

continued...

Question 2 continued

(d) $x^4 - 3x^3 + 3x = -0.375x$

(M1)

2.427

A1

Note: Correct answer must be given to three decimal places for the **A1** to be awarded.

Award **(M1)A1** for $(2.427, -0.910)$.

Award **(M1)A0** for an unsupported answer of 2.43.

[2 marks]

(e) $\frac{dy}{dx} = 4x^3 - 3mx^2 + n$

(A1)

$\frac{d^2y}{dx^2} = 12x^2 - 6mx$

(A1)

$6x(2x - m) = 0$

$x = 0, \frac{m}{2}$

A1

Note: Accept $0m$ in place of 0.

[3 marks]

(f) (i) point B is $(0, 0)$

A1

[1 mark]

(ii) $y_c = \left(\frac{m}{2}\right)^4 - m\left(\frac{m}{2}\right)^3 + n\left(\frac{m}{2}\right) \quad \left(= \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}\right)$

(A1)

$= -\frac{m^4}{16} + \frac{nm}{2} \quad \text{OR} \quad \frac{8nm - m^4}{16} \quad \text{OR} \quad \frac{m}{2}\left(n - \frac{m^3}{8}\right) \quad \text{OR} \quad \text{equivalent simplification} \quad \text{A1}$

Point C is $\left(\frac{m}{2}, -\frac{m^4}{16} + \frac{nm}{2}\right)$

Note: Award the second **A1** if the simplified y-coordinate is seen here or in part (g).

[2 marks]

continued...

Question 2 continued

(g) attempt to divide their y_c by their x_c OR substitute into $\frac{y_2 - y_1}{x_2 - x_1}$ (M1)

e.g. $\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \div \frac{m}{2}$

$\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \times \frac{2}{m}$ OR equivalent manipulation leading to given answer A1

Note: Award **A0 FT** if their answer does not lead to the **AG**.

$= -\frac{m^3}{8} + n$ and y -intercept = 0,

so equation is $y = \left(-\frac{m^3}{8} + n\right)x$ AG

Note: Award at most **M1A0** for working backwards to verify points B and C lie on the given line.

[2 marks]

(h) **METHOD 1**

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero (M1)

$x^4 - mx^3 + \frac{m^3}{8}x = 0$

recognizing their $(x - x_c)$ is a factor of this expression (M1)

$x\left(x - \frac{m}{2}\right)\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right) (= 0)$ OR equivalent A1M1A1

Note: Award **A1** for $x\left(x - \frac{m}{2}\right)$, **M1A1** for $\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)$.

If a candidate divides by x without justification, do not award the first **A1**, but all subsequent marks can still be awarded.

continued...

Question 2 continued

use of quadratic formula to find roots, x_A (and x_D), of the quadratic

M1

$$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2} \quad \text{OR equivalent}$$

A1

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$$

AG

Note: Award **(M1)(M0)A0M0A0M0A0** for attempting to verify the given answer satisfies

$$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx.$$

METHOD 2

$$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$$

attempt to rearrange this equation to equal zero

(M1)

$$x^4 - mx^3 + \frac{m^3}{8}x = 0$$

$$x \left(x^3 - mx^2 + \frac{m^3}{8} \right) = 0$$

Recognise that their x_C is a root of this equation

(M1)

Attempt to find sum and product of roots

(M1)

$$x_A + x_D + \frac{m}{2} = m \quad \text{AND} \quad x_A \times x_D \times \frac{m}{2} = -\frac{m^3}{8}$$

A1

$$x_A \times \left(\frac{m}{2} - x_A \right) \times \frac{m}{2} = -\frac{m^3}{8}$$

$$x_A^2 - \frac{m}{2}x_A - \frac{m^2}{4} = 0$$

A1

use of quadratic formula to find roots, x_A (and x_D), of the quadratic

M1

continued...

Question 2 continued

$$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2} \quad \text{OR equivalent}$$

A1

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$$

AG

[7 marks]

$$\begin{aligned} \text{(i)} \quad & \frac{0 - \left(\frac{m}{4} - \frac{m}{4}\sqrt{5} \right)}{\frac{m}{2} - 0} \\ &= \frac{\sqrt{5} - 1}{2} \end{aligned}$$

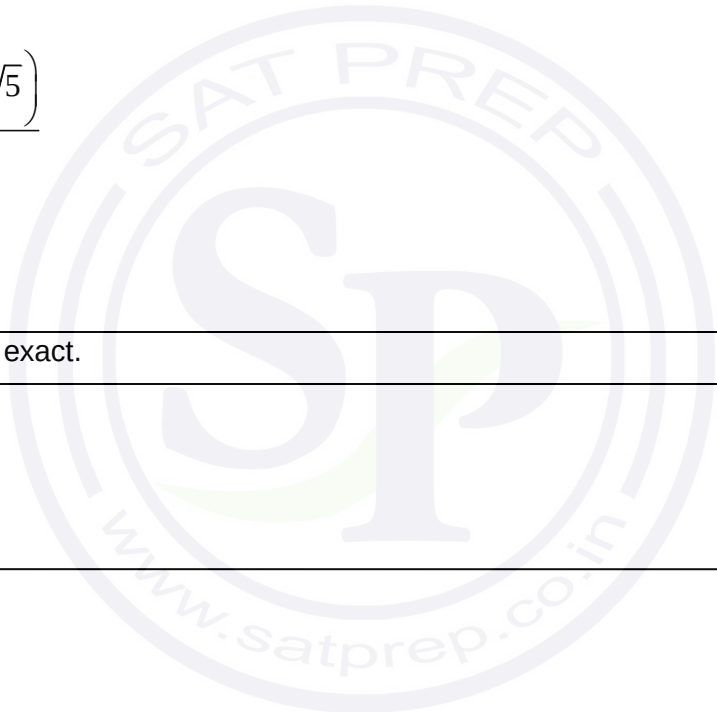
(A1)

A1

Note: Answer must be exact.

[2 marks]

Total [28 marks]



Markscheme

May 2024

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg M1, A2*.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g. M1A1*, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** *etc.*, do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)

2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or

written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) (i) attempts chain rule differentiation to find $f'(x)$ (M1)

$$f'(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3} \right) \quad \text{A1}$$

Note: Award (M1) for attempting chain rule differentiation on $(4 - 4x + x^2)^{-1}$ or

attempting quotient rule differentiation on $\frac{1}{(4 - 4x + x^2)} \left(= \frac{1}{(2-x)^2} \right)$.

Award A1 for $f'(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3} \right)$.

[2 marks]

- (ii) $g'(x) = 2x$ (A1)

$$f'(x)g'(x) = (2(2-x)^{-3})(2x) \left(= \frac{2(2x)}{(2-x)^3} \right) \text{ (or equivalent)} \quad \text{A1}$$

$$= \frac{4x}{(2-x)^3} \quad \text{AG}$$

[2 marks]

continued...

Question 1 continued.

(iii)

Note: Award a maximum of **(M1)A1(M1)A0FT** from parts (a) (i) and (ii).

substitutes $f(x), g(x)$ and their $g'(x), f'(x)$ into the given expression **(M1)**

EITHER

$$f(x)g'(x) + g(x)f'(x) = 2x(2-x)^{-2} + 2x^2(2-x)^{-3} \quad \text{A1}$$

Note: Award **A1** if $f(x)g'(x) = 2x(2-x)^{-2}$ and $g(x)f'(x) = 2x^2(2-x)^{-3}$ are stated separately.

attempts to factorise their expression **(M1)**

$$= 2x(2-x)^{-3}((2-x) + x) \quad \text{A1}$$

OR

$$f(x)g'(x) + g(x)f'(x) = \frac{2x}{(2-x)^2} + \frac{2x^2}{(2-x)^3} \quad \text{A1}$$

Note: Award **A1** if $f(x)g'(x) = \frac{2x}{(2-x)^2}$ and $g(x)f'(x) = \frac{2x^2}{(2-x)^3}$ are stated separately.

attempts to form an expression with a common denominator **(M1)**

Note: Award **(M1)** for $(2-x)^2(2-x)^3$ as a common denominator.

$$= \frac{2x(2-x)}{(2-x)^3} + \frac{2x^2}{(2-x)^3} \left(= \frac{4x - 2x^2 + 2x^2}{(2-x)^3} \right) \quad \text{A1}$$

THEN

$$= \frac{4x}{(2-x)^3} \quad \text{AG}$$

Note: Award marks as appropriate for attempting to find the derivative of

$$f(x)g(x) = \frac{x^2}{(2-x)^2} \text{ (or equivalent).}$$

[4 marks]

continued...

Question 1 continued.

(b) **METHOD 1**

$$f'(x)g'(x) - g(x)f'(x) = f(x)g'(x) \quad (\text{A1})$$

$$(f'(x)g'(x) - g(x)f'(x) - f(x)g'(x) = 0)$$

$$f'(x)(g'(x) - g(x)) = f(x)g'(x) \quad (f'(x)(g(x) - g'(x)) = -f(x)g'(x)) \quad \text{A1}$$

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)} \quad \text{AG}$$

Note: Award **(A0)A0** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 2

$$g'(x) = \frac{f'(x)g'(x)}{f(x)} - \frac{g(x)f'(x)}{f(x)} \quad (\text{A1})$$

$$g'(x) = \frac{f'(x)}{f(x)}(g'(x) - g(x)) \quad \text{A1}$$

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)} \quad \text{AG}$$

Note: Candidates may not show the steps exactly as shown above.
Award **(A0)A0** for use of $f(x)$ and $g(x)$ from part (a).

continued...

Question 1 continued.

METHOD 3

$$g'(x) = \frac{f(x)g'(x)}{f'(x)} + g(x) \quad (\text{A1})$$

$$\frac{f(x)}{f'(x)} = \frac{g'(x) - g(x)}{g'(x)} \quad \text{A1}$$

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)} \quad \text{AG}$$

Note: Candidates may not show the steps exactly as shown above.

Award **(A0)A0** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 4

$$\frac{f(x)}{f'(x)} + \frac{g(x)}{g'(x)} = 1 \quad (\text{A1})$$

$$\frac{f(x)}{f'(x)} = \frac{g'(x) - g(x)}{g'(x)} \quad \text{A1}$$

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)} \quad \text{AG}$$

Note: Candidates may not show the steps exactly as shown above.

Award **(A0)A0** for use of $f(x)$ and $g(x)$ from part (a).

[2 marks]

continued...

Question 1 continued.

(c) **METHOD 1**

Note: Condone the absence of 'dx' and the modulus sign throughout.
Only award the second **A** mark if the constant of integration has been dealt with correctly.

EITHER

$$\ln f(x) = \int \frac{g'(x)}{g'(x) - g(x)} dx (+C) \quad \mathbf{A1}$$

Note: Condone the absence of '+ C' when awarding the first **A** mark.

$$f(x) = e^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} e^C \left(f(x) = e^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx + C\right)} \right) \quad \mathbf{A1}$$

Note: Award **A1** for $f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$, where $A = e^C$.

OR

$$\ln f(x) (+C) = \int \frac{g'(x)}{g'(x) - g(x)} dx \quad \mathbf{A1}$$

Note: Condone the absence of '+ C' when awarding the first **A** mark.

$$f(x) = e^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} e^{-C} \left(f(x) e^C = e^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}, f(x) = e^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx - C\right)} \right) \quad \mathbf{A1}$$

Note: Award **A1** for $f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$, where $A = e^{-C}$.

THEN

$$f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} \quad \mathbf{AG}$$

continued...

Question 1 continued.

METHOD 2

Note: Condone the absence of 'dx' throughout.

$$f'(x) - \frac{g'(x)}{g'(x) - g(x)} f(x) = 0$$

Integrating factor: $e^{\left(-\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$

A1

Note: Award **A1** for $f(x)e^{\left(-\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} = C$.

$$\frac{d}{dx} \left[f(x)e^{\left(-\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} \right] = 0 \Rightarrow f(x)e^{\left(-\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} = A$$

A1

Note: Award **A1** for $f(x)e^{\left(-\int \frac{g'(x)}{g'(x) - g(x)} dx\right)} = C$.

$$f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$$

AG

[2 marks]

continued...

Question 1 continued.

(d) $g'(x) = xe^x + e^x$ (seen anywhere) (A1)

attempts to find an expression for $\frac{g'(x)}{g'(x) - g(x)}$ (M1)

$$= \frac{xe^x + e^x}{e^x} \left(= \frac{e^x(x+1)}{e^x} \right)$$

$= x+1$ (as $e^x \neq 0$) (A1)

attempts to integrate their $\frac{g'(x)}{g'(x) - g(x)}$ (M1)

$$\int (x+1) dx = \frac{1}{2}x^2 + x (+C)$$

$$f(x) = e^{\left(\frac{1}{2}x^2 + x\right)}$$

A1

Note: Award **A0** for $f(x) = e^{\left(\frac{1}{2}x^2 + x + C\right)}$ (or equivalent expressed with an arbitrary constant).

[5 marks]

continued...

Question 1 continued.

(e) $g'(x) = \cos x - \sin x$ (seen anywhere) (A1)

attempts to find an expression for $\frac{g'(x)}{g'(x) - g(x)}$ (M1)

$$= \frac{\cos x - \sin x}{\cos x - \sin x - \sin x - \cos x} \left(= \frac{\sin x - \cos x}{2 \sin x} \right)$$

$$= \frac{1}{2} - \frac{1}{2} \cot x \text{ (as } \sin x \neq 0 \text{)} \text{ OR } = \frac{1}{2} - \frac{1}{2} \frac{\cos x}{\sin x} \text{ (as } \sin x \neq 0 \text{)} \quad \text{A1}$$

$$f(x) = e^{\int \left(\frac{1}{2} - \frac{1}{2} \cot x \right) dx}$$

attempts to find the indefinite integral of $(\pm k) \cot x$ OR $(\pm k) \frac{\cos x}{\sin x}$ (M1)

Note: As $|\sin x| = \sin x$ for $0 < x < \pi$, condone the presence or omission of the modulus sign throughout a candidate's solution.
Condone the presence of an arbitrary constant except when awarding the final **A** mark.

$$\int \left(\frac{1}{2} - \frac{1}{2} \cot x \right) dx = \frac{x}{2} - \frac{1}{2} \ln |\sin x| (+C) \left(= \frac{1}{2} (x - \ln |\sin x| (+C)) \right) \quad \text{A1}$$

$$f(x) = e^{\frac{x}{2}} e^{-\frac{1}{2} \ln |\sin x|} (e^C)$$

$$= e^{\frac{x}{2}} e^{\ln \sqrt{\frac{1}{\sin x}}} (e^C) \left(= e^{\frac{x}{2}} e^{\frac{1}{2} \ln \left(\frac{1}{\sin x} \right)} (e^C), = \sqrt{e^{x - \ln(\sin x)}} (e^C) \right) \quad \text{A1}$$

$$= e^{\frac{x}{2}} \sqrt{\frac{1}{\sin x}}$$

$$= \sqrt{e^x \operatorname{cosec} x} \left(= \sqrt{\frac{e^x}{\sin x}} \right) \text{ (where } h(x) = \frac{1}{\sin x} \text{)} \quad \text{A1}$$

[7 marks]

Total [24 marks]

2. (a) 6^3 OR $6 \times 6 \times 6$

A1

Note: Accept a labelled diagram that clearly illustrates correct application of the multiplication principle leading to 216.

$= 216$

AG

[1 mark]

(b) **EITHER**

attempts to find Δ

(M1)

$$\Delta (= (4^2 - 4(1)(4))) = 0$$

A1

OR

attempts to solve $x^2 + 4x + 4 = 0$

(M1)

$$((x + 2)^2 = 0 \Rightarrow) x = -2$$

A1

OR

attempts to express $x^2 + 4x + 4 (= 0)$ as a perfect square

(M1)

$(x + 2)^2 (= 0)$ is a perfect square

A1

OR

a graph of $y = x^2 + 4x + 4$ with the vertex touching the x -axis at $x = -2$

A2

THEN

graph of f has only one x -intercept

AG

[2 marks]

continued...

Question 2 continued.

Note: In parts (c) – (f), $(a, b, c) = (1, 2, 1)$, for example, represents an ordered 3-tuple $a = 1, b = 2$ and $c = 1$.

(c) recognizes that $b^2 - 4ac = 0$ (or equivalent) (M1)

EITHER

attempts to use $\frac{b^2}{ac} = 4 \left(\frac{b^2}{4} = ac \right)$ (M1)

determines one value of b from $b = 2, 4$ or 6 only (seen anywhere) OR one value of ac from $ac = 1, 4$ or 9 only (seen anywhere) (A1)

OR

attempts to find a possible value of b (M1)

determines one value of b from $b = 2, 4$ or 6 only (seen anywhere) (A1)

OR

recognizes that b^2 must be a multiple of 4 OR b must be a multiple of 2 (M1)

determines one value of b from $b = 2, 4$ or 6 only (seen anywhere) (A1)

OR

attempts to find a possible value of ac (M1)

determines one value of ac from $ac = 1, 4$ or 9 only (seen anywhere) (A1)

THEN

$b = 2$ and $ac = 1$:

$(a, b, c) = (1, 2, 1)$ OR 1 possible way OR $\frac{1}{216}$ A1

$b = 4$ and $ac = 4$:

$(a, b, c) = (1, 4, 4), (4, 4, 1), (2, 4, 2)$ OR 3 possible ways OR $\frac{3}{216}$ A1

$b = 6$ and $ac = 9$:

$(a, b, c) = (3, 6, 3)$ OR 1 possible way OR $\frac{1}{216}$ A1

therefore the required probability is $\frac{1}{216} + \frac{3}{216} + \frac{1}{216}$

$= \frac{5}{216}$ AG

[6 marks]

continued...

Question 2 continued.

(d) recognizes that $b^2 - 4ac > 0$ (or equivalent eg. $\frac{b^2}{4} > ac$) (M1)

maximum value of b^2 is 36 OR maximum value of ac is 8 (A1)

Note: The above (A1) is independent of the (M1).

$$ac = 1, 2, 3, 4, 5, 6, 8$$

A1

[3 marks]

(e) (i) $ac = 1$ ($b^2 > 4$)

$b = 3, 4, 5, 6$ OR 1×4 (quadratics) OR $6 - 2$ (quadratics)

A1

there are four quadratic functions

AG

[1 mark]

(ii)

$$ac = 2$$
 ($b^2 > 8$)

$$b = 3, 4, 5, 6$$

(A1)

Note: Award (A1) for referencing their result shown in part (e) (i).

EITHER

$$(a, b, c) = (1, 3, 2), (1, 4, 2), (1, 5, 2), (1, 6, 2), (2, 3, 1), (2, 4, 1), (2, 5, 1), (2, 6, 1)$$

A1

Note: Award A1 for listing the eight quadratic expressions.

OR

2×4 (quadratics)

A1

THEN

there are eight quadratics functions

AG

[2 marks]

continued...

Question 2 continued.

(f) **METHOD 1**

varies ac ($ac \neq 1, 2$) and determines possible values of b such that $\Delta > 0$ **(M1)**

correctly determines one of the following five cases **(A1)**

correctly determines a further two of the following five cases **(A1)**

correctly determines the remaining two cases **(A1)**

case 1: $ac = 3$ ($b^2 > 12 \Rightarrow b = 4, 5, 6$)

$(a, b, c) = (1, 4, 3), (1, 5, 3), (1, 6, 3), (3, 4, 1), (3, 5, 1), (3, 6, 1)$ OR

6 possible ways OR $\frac{6}{216}$

case 2: $ac = 4$ ($b^2 > 16 \Rightarrow b = 5, 6$)

$(a, b, c) = (1, 5, 4), (1, 6, 4), (2, 5, 2), (2, 6, 2), (4, 5, 1), (4, 6, 1)$ OR

6 possible ways OR $\frac{6}{216}$

case 3: $ac = 5$ ($b^2 > 20 \Rightarrow b = 5, 6$)

$(a, b, c) = (1, 5, 5), (1, 6, 5), (5, 5, 1), (5, 6, 1)$ OR 4 possible ways OR $\frac{4}{216}$

case 4: $ac = 6$ ($b^2 > 24 \Rightarrow b = 5, 6$)

$(a, b, c) = (1, 5, 6), (2, 5, 3), (3, 5, 2), (6, 5, 1), (1, 6, 6), (2, 6, 3), (3, 6, 2), (6, 6, 1)$

OR 8 possible ways OR $\frac{8}{216}$

case 5: $ac = 8$ ($b^2 > 32 \Rightarrow b = 6$)

$(a, b, c) = (2, 6, 4), (4, 6, 2)$ OR 2 possible ways OR $\frac{2}{216}$

adds their probabilities **(M1)**

Note: Award **(M1)** for adding at least 3 of their probabilities (denominator 216).

$$\begin{aligned} (p) &= \frac{4}{216} + \frac{8}{216} + \frac{6}{216} + \frac{6}{216} + \frac{4}{216} + \frac{8}{216} + \frac{2}{216} \\ &= 0.0185... + 0.0370... + 0.0277... + 0.0277... + 0.0185... + 0.0370... + 0.0092... \\ &= \frac{38}{216} \quad (= \frac{19}{108}, = 0.176) \end{aligned}$$

A1

continued...

Question 2 continued.

METHOD 2

varies $b^2 (\neq 1, 4)$ OR $b (\neq 1, 2)$ and determines possible values of ac such that $\Delta > 0$

(M1)

correctly determines one of the following four cases

(A1)

correctly determines another case from the following four cases

(A1)

correctly determines the remaining two cases

(A1)

case 1: $b^2 = 9$ ($b = 3$) ($ac = 1, 2$)

$(a, b, c) = (1, 3, 1), (1, 3, 2), (2, 3, 1)$ OR 3 possible ways OR $\frac{3}{216}$

case 2: $b^2 = 16$ ($b = 4$) ($ac = 1, 2, 3$)

$(a, b, c) = (1, 4, 1), (1, 4, 2), (2, 4, 1), (1, 4, 3), (3, 4, 1)$ OR 5 possible ways OR $\frac{5}{216}$

case 3: $b^2 = 25$ ($b = 5$) ($ac = 1, 2, 3, 4, 5, 6$)

$(a, b, c) = (1, 5, 1), (1, 5, 2), (2, 5, 1), (1, 5, 3), (3, 5, 1), (1, 5, 4), (2, 5, 2)$
 $(4, 5, 1), (1, 5, 5), (5, 5, 1), (1, 5, 6), (2, 5, 3), (3, 5, 2), (6, 5, 1)$

OR 14 possible ways OR $\frac{14}{216}$

case 4: $b^2 = 36$ ($b = 6$) ($ac = 1, 2, 3, 4, 5, 6, 8$)

$(a, b, c) = (1, 6, 1), (1, 6, 2), (2, 6, 1), (1, 6, 3), (3, 6, 1), (1, 6, 4), (2, 6, 2), (4, 6, 1)$
 $(1, 6, 5), (5, 6, 1), (1, 6, 6), (2, 6, 3), (3, 6, 2), (6, 6, 1), (2, 6, 4), (4, 6, 2)$

OR 16 possible ways OR $\frac{16}{216}$

adds their probabilities

(M1)

Note: Award (M1) for adding at least 3 of their probabilities (denominator 216).

$$(p =) \frac{3}{216} + \frac{5}{216} + \frac{14}{216} + \frac{16}{216}$$

$$(= 0.013889... + 0.023148... + 0.064815... + 0.074074...)$$

$$= \frac{38}{216} \quad (= \frac{19}{108}, = 0.176)$$

A1

continued...

Question 2 continued.

METHOD 3

varies b^2 OR b and determines possible values of ac such that $\Delta < 0$ (M1)

correctly determines two of the following six cases (A1)

correctly determines a further two of the following six cases (A1)

correctly determines the remaining two cases (A1)

case 1: $b^2 = 1$ ($b = 1$) 36 possible ways OR $\frac{36}{216}$

case 2: $b^2 = 4$ ($b = 2$) 35 possible ways OR $\frac{35}{216}$

case 3: $b^2 = 9$ ($b = 3$) 33 possible ways OR $\frac{33}{216}$

case 4: $b^2 = 16$ ($b = 4$) 28 possible ways OR $\frac{28}{216}$

case 5: $b^2 = 25$ ($b = 5$) 22 possible ways OR $\frac{22}{216}$

case 6: $b^2 = 36$ ($b = 6$) 19 possible ways OR $\frac{19}{216}$

$(p =) 1 - \left(\frac{36}{216} + \frac{35}{216} + \frac{33}{216} + \frac{28}{216} + \frac{22}{216} + \frac{19}{216} + \frac{5}{216} \right)$ (M1)

$\left(= 1 - \left(0.16666... + 0.16203... + 0.15277... + 0.12962... + 0.10185... + 0.087962... + 0.023148... \right) \right)$

Note: Award (M1) for adding at least 3 of their probabilities inside the above bracket (denominator 216).

$= \frac{38}{216} \left(= \frac{19}{108}, = 0.176 \right)$ A1

[6 marks]

continued...

Question 2 continued.

(g) recognizes that $4Z^2 - 4 > 0$ ($Z^2 > 1$) (M1)

probability of two x -intercepts is

EITHER

$P(|Z| > 1)$ (A1)

OR

$P(Z < -1)$ or $P(Z > 1)$ (can be shown on a labelled diagram) (A1)

$= 0.158655... + 0.158655...$

OR

$1 - P(-1 \leq Z \leq 1)$ (can be shown on a labelled diagram) (A1)

$= 1 - 0.682689...$

THEN

$= 0.317310...$

$= 0.317$

A1

[3 marks]

(h) attempts to solve $X_1 > 0.5$ for Z (M1)

$-1.25 < Z \leq -1$ (A1)(A1)

Note: Award (M1)(A1) for obtaining $Z = -1.25$ from solving $X_1 = 0.5$ and award (A1) for stating the correct inequality.

Award (M1)(A1)(A1) for $-1.25 < Z < -1$.

Award (M1)(A1)(A0) for $-1.25 < Z$.

Award (M1) for rearranging to form $-\sqrt{Z^2 - 1} = Z + 0.5$ and then attempting to square both sides $Z^2 - 1 = (Z + 0.5)^2$ ($= Z^2 + Z + 0.25$).

attempts to calculate their $P(X_1, X_2 \text{ both } > 0.5)$ (M1)

$P(-1.25 < Z \leq -1) = 0.053005...$ (A1)

attempts to calculate their $P(X_1, X_2 \text{ both } > 0.5 | x\text{-intercepts})$ (M1)

$= \frac{0.053005...}{0.317310...}$

$= 0.167$

A1

[7 marks]

Total [31 marks]

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg M1, A2*.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g. M1A1*, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** *etc.*, do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) varies the value of b with $a = 3$ (M1)

Note: The (M1) in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii).

Award (M1) for evidence that $b = 0$ case is considered/determined.

$$b = -4, 0$$

A1

[2 marks]

- (b) (i) $b < -4$ or $b > 0$

A1

[1 mark]

- (ii) $-4 < b < 0$

A1

[1 mark]

continued...



Question 1 continued

(c) (i) $b = 0, 4$

A1

[1 mark]

(ii) $b < 0$ or $b > 4$

A1

[1 mark]

(iii) $0 < b < 4$

A1

[1 mark]

continued...



Question 1 continued

- (d) one point of zero gradient is located on either side (of the x -axis) (or equivalent) **A1**

[1 mark]

- (e) **METHOD 1**

$$\frac{dy}{dx} = 3x^2 + 2ax \quad \textbf{(A1)}$$

attempts to solve their $\frac{dy}{dx} = 0$ for x **M1**

$$x(3x + 2a) = 0 \text{ OR } x = \frac{-2a \pm \sqrt{4a^2}}{6} \text{ OR } x + \frac{a}{3} = \pm \frac{a}{3}$$

$$x = -\frac{2}{3}a, 0 \quad \textbf{A1}$$

when $x = 0$, $y = b$ and so $P(0, b)$ is a point of zero gradient **AG**

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ **(M1)**

$$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$$

$$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b \right) \quad \textbf{A1}$$

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient **AG**

[5 marks]

continued...

Question 1 continued

METHOD 2

$$\frac{dy}{dx} = 3x^2 + 2ax \quad (\text{A1})$$

substitutes either $x = 0$ or $x = -\frac{2}{3}a$ into their $\frac{dy}{dx}$ **M1**

when $x = 0$, $\frac{dy}{dx} = 0$ and $y = b$ so $P(0, b)$ is a point of zero gradient **AG**

$$\begin{aligned} \frac{dy}{dx} &= 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right) \\ &= \frac{4}{3}a^2 - \frac{4}{3}a^2 (=0) \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2 (=0), = \frac{12}{9}a^2 - \frac{4}{3}a^2 (=0)\right) \end{aligned} \quad \text{A1}$$

and so $\frac{dy}{dx} = 0$ when $x = -\frac{2}{3}a$ **AG**

Note: The following two marks are independent of the first three marks.

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ **(M1)**

$$\begin{aligned} y &= \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b \\ y &= -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right) \end{aligned} \quad \text{A1}$$

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient **AG**

[5 marks]

continued...

Question 1 continued

(f) (i) $\frac{d^2y}{dx^2} = 6x + 2a$ **A1**

when $x = 0$, $\frac{d^2y}{dx^2} = 2a$ ($a > 0$) and so (P) is a (local) minimum (point) **R1**

when $x = -\frac{2}{3}a$, $\frac{d^2y}{dx^2} = -2a$ ($a > 0$) and so (Q) is a (local) maximum (point) **R1**

[3 marks]

(ii) (P and Q are) both above (the x -axis) **A1**

Note: Award **A1** if it is made clear that both points are above (the x -axis).
Accept a labelled sketch that clearly shows this information.

[1 mark]

continued...

Question 1 continued

(g) (i) (P) is a (local) maximum (point) and is above (the x - axis)

A1

[1 mark]

(ii) (Q is below the x - axis when) $\frac{4}{27}a^3 + b < 0$

A1

Note: Award **A1** for an equivalent correct inequality, eg. $\frac{4}{27}a^3 < -b$.

Accept a labelled sketch that clearly shows this information.

[1 mark]

continued...



Question 1 continued

(h) **METHOD 1**

attempts to factorize $4a^3b + 27b^2 (< 0)$ **(M1)**

$27b\left(\frac{4}{27}a^3 + b\right)(< 0)$ OR $b(4a^3 + 27b)(< 0)$ **A1**

$b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ **A1**

Note: Only award this **A1** if both cases are stated.

Award **A1** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

when b and $\frac{4}{27}a^3 + b$ have opposite sign, P and Q are located on either side (of the x -axis) (or equivalent) **R1**

Note: Accept labelled sketches that clearly show this information.

P and Q are located on either side (of the x -axis) if (and only if) the curve has exactly three x -axis intercepts **R1**

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x -axis intercepts **AG**

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd **A1** line not necessary in reverse method).

continued...

Question 1 continued

METHOD 2

attempts to factorize $4a^3b + 27b^2 (< 0)$ **(M1)**

$27b\left(\frac{4}{27}a^3 + b\right)(< 0)$ OR $b(4a^3 + 27b)(< 0)$ **A1**

either $b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ **A1**

Note: Only award this **A1** if both cases are stated.

Award **A1** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

$b > 0$ and $\frac{4}{27}a^3 + b < 0$, $\left(\Rightarrow 0 < b < -\frac{4}{27}a^3\right) \Rightarrow a < 0$ and hence three x -axis intercepts **R1**

$b < 0$ and $\frac{4}{27}a^3 + b > 0$, $\left(\Rightarrow -\frac{4}{27}a^3 < b < 0\right) \Rightarrow a > 0$ and hence three x -axis intercepts **R1**

Note: Accept labelled sketches that clearly show this information.

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x -axis intercepts **AG**

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd **A1** line not necessary in reverse method).

[5 marks]

Total [24 marks]

2. (a) (i) $\frac{y}{x}$

A1

[1 mark]

(ii) $\frac{dy}{dx} = -\frac{1}{\left(\frac{y}{x}\right)} \left(= -\frac{1}{m} \right)$

A1

Note: Award **A1** for responses such as 'the gradient is the negative (opposite)

reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).

Award **A1** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.

Do not award **FT** from part (a) (i).

so $\frac{dy}{dx} = -\frac{x}{y}$

AG

[1 mark]

Question 2 continued

(iii) attempts to separate variables x and y

(M1)

$$\int y \, dy = -\int x \, dx$$

Note: Award (M1) for $y \, dy = -x \, dx$.

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

A1

Note: Award A1 for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.

Award A0 for $\frac{y^2}{2} = -\frac{x^2}{2}$.

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$\Rightarrow x^2 + y^2 = k \quad (\text{where } k = 2c)$$

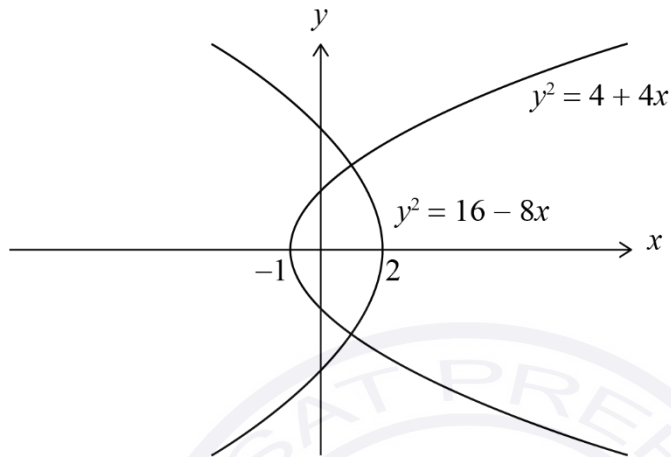
AG

[2 marks]

continued...

Question 2 continued

(b)



two parabolic shaped curves with approximately correct shape/position
(e.g. two intersection points, in first and fourth quadrant)

A1A1

x-intercepts -1 and 2

A1

[3 marks]

continued...

Question 2 continued

(c) at intersection, $4a^2 - 4ax = 4b^2 + 4bx$

$$4a^2 - 4b^2 = 4ax + 4bx \quad (a^2 - b^2 = ax + bx, 4a^2 - 4b^2 - 4ax - 4bx = 0)$$

A1

attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of

$$a^2 - b^2 - ax - bx = 0 \quad (\text{or equivalent})$$

(M1)

$$(a+b)(a-b) = (a+b)x$$

Note: Accept alternative forms such as $4(a+b)(a-b) = 4(a+b)x$ or

$$(a+b)((a-b)-x) = 0.$$

recognition that $a+b > 0$ (or equivalent, eg. $a > 0, b > 0$) (allows division by $a+b$)

R1

Note: Subsequent marks are not dependent on this **R1**.

$$x = a - b$$

A1

Note: As $x = a - b$ is an **AG**, only award the above **A1** if $a^2 - b^2 = (a+b)(a-b)$ has been used.

substitutes $x = a - b$ into either $y^2 = 4a^2 - 4ax$ or $y^2 = 4b^2 + 4bx$ and attempts to simplify

(M1)

$$y^2 = 4a^2 - 4a(a-b) \quad \text{OR} \quad y^2 = 4b^2 + 4b(a-b)$$

$$y^2 = 4a^2 - 4a^2 + 4ab \Rightarrow y = \pm 2\sqrt{ab}$$

A1

$$\text{so } M(a-b, 2\sqrt{ab}) \text{ and } N(a-b, -2\sqrt{ab})$$

AG

[6 marks]

continued...

Question 2 continued

(d) **METHOD 1**

attempts implicit differentiation on either curve **(M1)**

$$\frac{dy}{dx} = -\frac{4a}{2y} \text{ (or equivalent) and } \frac{dy}{dx} = \frac{4b}{2y} \text{ (or equivalent)} \quad \textbf{A1}$$

substitutes $y = 2\sqrt{ab}$ into either $\frac{dy}{dx} = -\frac{4a}{2y}$ or $\frac{dy}{dx} = \frac{4b}{2y}$ **(M1)**

$$\frac{dy}{dx} = -\sqrt{\frac{a}{b}} \text{ } (= -\frac{a}{\sqrt{ab}}) \text{ and } \frac{dy}{dx} = \sqrt{\frac{b}{a}} \text{ } (= \frac{b}{\sqrt{ab}}) \text{ (or equivalent)} \quad \textbf{A1}$$

EITHER

$$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1 \text{ OR } -\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1 \text{ (or equivalent)} \quad \textbf{A1}$$

OR

eg. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) **A1**

OR

the product of the two gradients is -1 **A1**

THEN

so at point M, the curves intersect at right angles **AG**

continued...

Question 2 continued

METHOD 2

attempts chain rule differentiation on either $y = \sqrt{4a^2 - 4ax}$ or $y = \sqrt{4b^2 + 4bx}$ **(M1)**

$$\frac{dy}{dx} = -\frac{2a}{\sqrt{4a^2 - 4ax}} \text{ (or equivalent) and } \frac{dy}{dx} = \frac{2b}{\sqrt{4b^2 + 4bx}} \text{ (or equivalent)} \quad \mathbf{A1}$$

substitutes $x = a - b$ into either $\frac{dy}{dx} = -\frac{2a}{\sqrt{4a^2 - 4ax}}$ or $\frac{dy}{dx} = \frac{2b}{\sqrt{4b^2 + 4bx}}$ **(M1)**

$$\frac{dy}{dx} = -\sqrt{\frac{a}{b}} \left(= -\frac{a}{\sqrt{ab}} \right) \text{ and } \frac{dy}{dx} = \sqrt{\frac{b}{a}} \left(= \frac{b}{\sqrt{ab}} \right) \text{ (or equivalent)} \quad \mathbf{A1}$$

EITHER

$$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1 \text{ OR } -\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1 \text{ (or equivalent)} \quad \mathbf{A1}$$

OR

eg. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) **A1**

OR

the product of the two gradients is -1 **A1**

THEN

so at point M, the curves intersect at right angles **AG**

[5 marks]

continued...

Question 2 continued

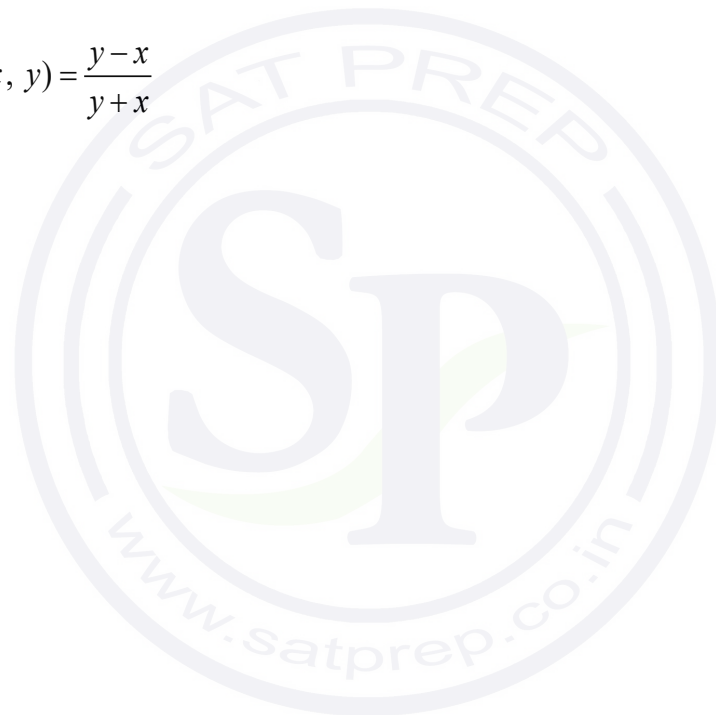
$$(e) \quad (i) \quad g(x, y) = \frac{-\frac{x}{y} + \tan \frac{\pi}{4}}{1 - \left(-\frac{x}{y}\right) \tan \frac{\pi}{4}} \quad (A1)$$

$$g(x, y) = \frac{-\frac{x}{y} + 1}{1 + \frac{x}{y}} \left(= \frac{\frac{-x+y}{y}}{\frac{y+x}{y}} \right) \quad A1$$

$$\text{so } g(x, y) = \frac{y-x}{y+x} \quad AG$$

[2 marks]

continued...



Question 2 continued

(ii) let $y = vx$ (M1)

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (A1)$$

$$\left(v + x \frac{dv}{dx} \right) \frac{vx - x}{vx + x} \left(= \frac{v-1}{v+1} \right) \quad (A1)$$

attempts to express $x \frac{dv}{dx}$ as a single fraction in v (M1)

$$x \frac{dv}{dx} = -\frac{v^2 + 1}{v + 1} \text{ (or equivalent)} \quad (A1)$$

attempts to separate variables x and v (M1)

$$\int \frac{v+1}{v^2+1} dv = -\int \frac{1}{x} dx \text{ (or equivalent)}$$

$$\frac{1}{2} \ln(v^2 + 1) + \arctan v = -\ln|x| (+d) \text{ (or equivalent)} \quad A1A1$$

$$\frac{1}{2} \ln\left(\frac{y^2}{x^2} + 1\right) + \arctan \frac{y}{x} + \ln|x| = d \text{ (or equivalent)} \quad A1$$

[9 marks]

continued...

Question 2 continued

(f) **METHOD 1**

$$g(x, y) = \frac{\frac{1}{\tan \alpha} f(x, y) + 1}{\frac{1}{\tan \alpha} - f(x, y)} \quad \text{M1}$$

EITHER

$$\text{as } \alpha \rightarrow \frac{\pi}{2}, \frac{1}{\tan \alpha} \rightarrow 0, \text{ (hence } g(x, y) \rightarrow -\frac{1}{f(x, y)}) \quad \text{R1}$$

OR

$$\text{as } \alpha \rightarrow \frac{\pi}{2}, \tan \alpha \rightarrow \infty \text{ and so } g(x, y) \rightarrow \frac{0 \times f(x, y) + 1}{0 - f(x, y)} \quad \text{R1}$$

THEN

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)} \quad \text{AG}$$

Note: The **R1** is dependent on the **M1**.

METHOD 2

$$\text{uses either } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} \text{ or } \frac{1}{\tan \alpha} = \frac{\cos \alpha}{\sin \alpha} \text{ to form } g(x, y) = \frac{\cos \alpha f(x, y) + \sin \alpha}{\cos \alpha - \sin \alpha f(x, y)} \quad \text{M1}$$

$$\text{as } \alpha \rightarrow \frac{\pi}{2}, \cos \alpha \rightarrow 0 \text{ and } \sin \alpha \rightarrow 1 \text{ and so } g(x, y) \rightarrow \frac{0 \times f(x, y) + 1}{0 - f(x, y)} \quad \text{R1}$$

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)} \quad \text{AG}$$

Note: The **R1** is dependent on the **M1**.

[2 marks]

Total [31 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg M1, A2*.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g.* **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** *etc.*, do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or

written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

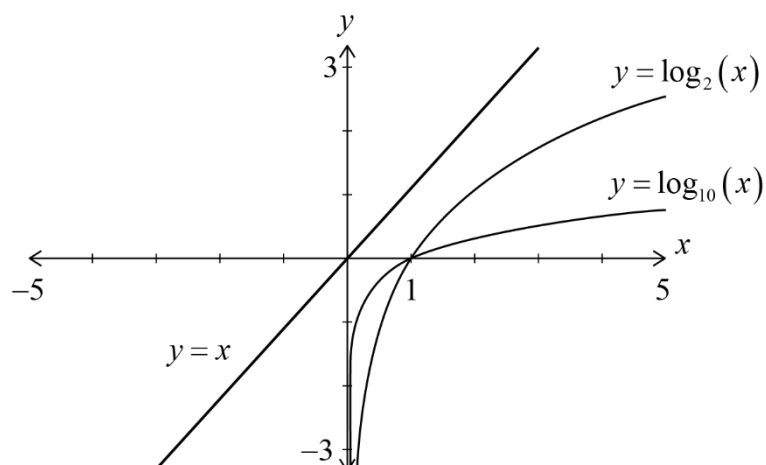
A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a)



clearly labelled graphs of $y = \log_2 x$ and $y = \log_{10} x$ with correct domain,
asymptotic behaviour and concavity evident

A1

correct relative positions of the two log graphs both above and below the x -axis

A1

$(1,0)$ indicated (coordinates not required)

A1

correct graph of $y = x$

A1

[4 marks]

continued...

Question 1 continued

(b) $\frac{d}{dx}(x - \ln x)$

$$= 1 - \frac{1}{x}$$

A1

attempts to solve their $\frac{dy}{dx} = 0$ for x

(M1)

$$1 - \frac{1}{x} = 0 \Rightarrow x = 1$$

(when $x = 1$,) $x - \ln x = 1$

A1

EITHER

$$\frac{d}{dx}\left(1 - \frac{1}{x}\right)$$

$$= \frac{1}{x^2}$$

A1

$$\frac{1}{x^2} > 0 \text{ (when } x = 1 \text{)}$$

R1

hence $x - \ln x$ has a minimum value of 1

Note: Award **R1** for either ' $1 > 0$ ' or a graph of $y = \frac{1}{x^2} > 0$ or 'the graph of $y = x - \ln x$ is concave-up'. Do not award **R1** if the second derivative is incorrect.

continued...

Question 1 continued

OR

$$\text{for } (0 <) x < 1, 1 - \frac{1}{x} < 0$$

R1

$$\text{for } x > 1, 1 - \frac{1}{x} > 0$$

R1

hence $x - \ln x$ has a minimum value of 1

Note: Award **R1R1** for either a clearly labelled sign diagram/table (accept correct numerical values) or the graph of $y = 1 - \frac{1}{x}$ with sign change in gradient indicated.

Note: Award a maximum of **A0(M1)A1A0R1** or **A0(M1)A1R0R1** if no symbolic derivatives are seen.

[5 marks]

(c)

EITHER

$$x - \ln x \geq 1 \quad (x \in \mathbb{R}^+)$$

R1

OR

$$x - \ln x > 0 \quad (x \in \mathbb{R}^+)$$

R1

THEN

$$\text{so } x > \ln x$$

AG

[1 mark]

continued...

Question 1 continued

(d)

Interval	Number of intersection points
$0 < a < 1$	$p = 1$
$1 < a < 1.4$	$q = 2$
$1.5 < a < 2$	$r = 0$

A1A2A1

Note: Award **A1** for $p = 1$, **A2** for $q = 2$ and **A1** for $r = 0$.

[4 marks]

(e) **METHOD 1**

EITHER

$$y = \log_a x$$

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

(A1)

attempts to solve $\frac{1}{x \ln a} = 1$ for x

(M1)

OR

$$y = x - \log_a x$$

$$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$$

(A1)

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x

(M1)

continued...

Question 1 continued

THEN

$$x = \frac{1}{\ln a} \text{ OR } x \ln a = 1 \text{ OR } \ln a = \frac{1}{x} \text{ OR } \ln a^x = 1 \text{ OR } \frac{1}{a^x \ln a} = 1 \quad \mathbf{A1}$$

$$\text{at } x = \frac{1}{\ln a}, \log_a x = x$$

$$\text{attempts to solve } \frac{\ln x}{\ln a} = \frac{1}{\ln a} \text{ OR } \ln x = 1 \text{ OR } \left(e^{\frac{1}{x}} \right)^x = x \text{ for } x \quad \mathbf{(M1)}$$

$$x = e$$

$$\text{coordinates of P are } (e, e) \text{ (accept } x = e, y = e) \quad \mathbf{A1A1}$$

$$\text{attempts to solve } \frac{1}{\ln a} = e \text{ OR } \log_a e = e \text{ for } a \text{ analytically} \quad \mathbf{(M1)}$$

$$\ln a = \frac{1}{e} \text{ OR } a^e = e$$

$$a = e^{\frac{1}{e}} \quad \mathbf{A1}$$

continued...

Question 1 continued

METHOD 2

EITHER

$$y = \log_a x$$

$$\frac{dy}{dx} = \frac{1}{x \ln a} \quad (\text{A1})$$

$$\text{attempts to solve } \frac{1}{x \ln a} = 1 \text{ for } x \quad (\text{M1})$$

OR

$$y = x - \log_a x$$

$$\frac{dy}{dx} = 1 - \frac{1}{x \ln a} \quad (\text{A1})$$

$$\text{attempts to solve } 1 - \frac{1}{x \ln a} = 0 \text{ for } x \quad (\text{M1})$$

THEN

$$x = \frac{1}{\ln a} \text{ OR } x \ln a = 1 \text{ OR } \ln a = \frac{1}{x} \text{ OR } \ln a^x = 1 \text{ OR } \frac{1}{a^x \ln a} = 1 \quad \text{A1}$$

$$\text{at } x = \frac{1}{\ln a}, \log_a x = x$$

$$\text{attempts to solve } \log_a \left(\frac{1}{\ln a} \right) = \frac{1}{\ln a} \text{ for } a \quad (\text{M1})$$

EITHER

$$\frac{\ln \left(\frac{1}{\ln a} \right)}{\ln a} = \frac{1}{\ln a} \Rightarrow \ln \left(\frac{1}{\ln a} \right) = 1$$

OR

for example, writes $a^{\log_a \left(\frac{1}{\ln a} \right)} = a^{\frac{1}{\ln a}}$ and then attempts to apply appropriate

index/log laws to both sides: $\ln a = \frac{\log_a a}{\log_a e}$ and so $\frac{1}{\ln a} = \log_a e$

$$a^{\frac{1}{\ln a}} = a^{\log_a e} = e$$

continued...

Question 1 continued

THEN

attempts to solve $\frac{1}{\ln a} = e$ OR $\log_a e = e$ for a analytically **(M1)**

$$\ln a = \frac{1}{e} \text{ OR } a^e = e$$

$$a = e^{\frac{1}{e}} \span style="float: right;">**A1**$$

$$x = \frac{1}{\ln e^{\frac{1}{e}}} = \frac{1}{\frac{1}{e}}$$

coordinates of P are (e, e) (accept $x = e, y = e$) **A1A1**

continued...



Question 1 continued

METHOD 3

$$y = \log_a x$$

$$\frac{dy}{dx} = \frac{1}{x \ln a} \quad (\mathbf{A1})$$

(equation of the tangent at (x_1, y_1) is) $y = \frac{1}{x_1 \ln a}(x - x_1) + \frac{\ln x_1}{\ln a}$ (or equivalent) **A1**

compares this equation with $y = x$ and attempts to form at least one of the following **M1**

$$\frac{1}{x_1 \ln a} = 1 \text{ OR } \frac{\ln x_1 - 1}{\ln a} = 0$$

attempts to solve $\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$ for x_1 **(M1)**

$$x_1 = e$$

coordinates of P are (e, e) (accept $x = e$, $y = e$) **A1A1**

attempts to solve $\frac{1}{e \ln a} = 1$ (or equivalent) for a analytically **(M1)**

$$\ln a = \frac{1}{e} \text{ OR } a^e = e$$

$$a = e^{\frac{1}{e}}$$

A1

[8 marks]

continued...

Question 1 continued

(f) (i) $1 < a < e^{\frac{1}{e}}$

A1

Note: Award **A0** for $a < e^{\frac{1}{e}}$.

[1 mark]

(ii) $a > e^{\frac{1}{e}}$

A1

Note: Only award **FT** for $1.4 < a < 1.5$. If the value of a is not exact, e.g. 1.44, award at most **A0A1** in part (f) for a consistent approximate endpoint value. If a value of a is not found in part (e), award at most **A0A1** in part (f) for a consistent approximate endpoint value provided that $1.4 < a < 1.5$.

[1 mark]

Total [24 marks]



2. (a) $m = 2, c = -1$

$$r = \frac{1}{2}$$

(A1)

$$2, \frac{1}{2}, -1$$

EITHER

$$d \left(= \frac{1}{2} - 2 = -1 - \frac{1}{2} \right) = -\frac{3}{2}$$

A1

OR

this sequence has a common difference of $-\frac{3}{2}$

A1

OR

the (arithmetic) mean of 2 and -1 is $\frac{1}{2}$

A1

THEN

hence $L(x) = 2x - 1$ is an AS-linear function

AG

[2 marks]

- (b) (i) $(L(r) = 0 \Rightarrow) mr + c = 0$

A1

$$r = -\frac{c}{m}$$

AG

Note: Award **A0** for numerical verification from $L(x) = 2x - 1$ in part (a).

[1 mark]

continued...

Question 2 continued

(ii) **METHOD 1**

EITHER

attempts to use $(d =) r - m = c - r$

(M1)

Note: Award **(M1)** for attempting to use $(d =) m - r = r - c$.

$$(d =) -\frac{c}{m} - m = c - \left(-\frac{c}{m}\right)$$

A1

Note: Award **A1** for $(d =) -\frac{c}{m} - m = \frac{c - m}{2}$.

removes the denominator m from their expression involving m and c

(M1)

$$m^2 + cm + 2c = 0 \text{ (or equivalent)}$$

OR

attempts to use $\frac{m + c}{2} = r$

(M1)

$$m + c = -\frac{2c}{m}$$

A1

removes the denominator m from their expression involving m and c

(M1)

$$m^2 + cm + 2c = 0 \text{ (or equivalent)}$$

OR

attempts to use $c = m + 2d$

(M1)

$$c = m + 2\left(-\frac{c}{m} - m\right)$$

A1

Note: Award **A1** for $c = m + 2\left(c - \left(-\frac{c}{m}\right)\right)$.

removes the denominator m from their expression involving m and c

(M1)

$$m^2 + cm + 2c = 0 \text{ (or equivalent)}$$

continued...

Question 2 continued

OR

attempts to use $r = m + d$ and $c = m + 2d$ ($c = m + 2(r - m)$) **(M1)**

$m^2 + dm + m + 2d = 0$ (or equivalent) **A1**

substitutes $d = \frac{c-m}{2}$ into their expression involving m and d **(M1)**

$m^2 + cm + 2c = 0$ (or equivalent)

THEN

$c(m+2) = -m^2 \Rightarrow c = -\frac{m^2}{m+2}$ **A1**

Note: Award **A1** for a convincing demonstration that $c = -\frac{m^2}{m+2}$.

so $L(x) = mx - \frac{m^2}{m+2}$ **AG**

Note: Do not accept working backwards from the **AG**.

METHOD 2

considers $L(x) = mx - mr$

attempts to use $(d =) r - m = c - r$ **(M1)**

Note: Award **(M1)** for attempting to use $(d =) m - r = r - c$.

$(d =) r - m = -mr - r$ **A1**

attempts to express r in terms of m **(M1)**

$2r + mr = m \Rightarrow r = \frac{m}{m+2}$ **A1**

so $L(x) = mx - \frac{m^2}{m+2}$ **AG**

Note: Do not accept working backwards from the **AG**.

[4 marks]

continued...

Question 2 continued

(iii) $m \neq -2$ ($m \neq 0$)

A1

[1 mark]

(c)

attempts to find an integer value of m

(M1)

e.g. uses the result that $m + 2$ exactly divides 2 OR uses a table OR uses a graph and slider OR uses systematic trial and error

Note: Award (M1) for solving $m^2 = k(m + 2)$ for m or solving $mr - \frac{m^2}{m + 2} = 0$ for m or solving $m^2 + cm + 2c = 0$ for m .

$m = -4$ OR $m = -3$

(A1)

$-4, 2, 8$ OR $-3, 3, 9$

$L(x) = -4x + 8$, $L(x) = -3x + 9$

A1

Note: Award (M1)(A1)A0 for $-4x + 8$ and $-3x + 9$.

[3 marks]

(d) (i) $-\frac{b}{a}$

A1

[1 mark]

(ii) $\frac{c}{a}$

A1

[1 mark]

continued...

Question 2 continued

(e) (i) $b - a$

A1

Note: Award marks as appropriate in parts (e) (ii) and (iii) for use of $r_1 - r_2 = a - b$.

[1 mark]

(ii) attempts to eliminate r_2

M1

$$2r_1 = -\frac{b}{a} - (b - a) \Rightarrow 2r_1 = \frac{a^2 - ab - b}{a} \text{ (or equivalent)}$$

A1

Note: Award **A1** for a correct alternative form of $\pm r_1$ or $\pm 2r_1$.

$$\text{so } r_1 = \frac{a^2 - ab - b}{2a}$$

AG

Note: Do not accept working backwards from the **AG**.

[2 marks]

continued...

Question 2 continued

(iii) **METHOD 1**

EITHER

$$(r_1 =) \frac{a+b}{2} \quad (\text{A1})$$

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and b **M1**

$$\frac{a+b}{2} = \frac{a^2-ab-b}{2a} \text{ OR } a+b = \frac{a^2-ab-b}{a}$$

OR

$$b-r_1 = r_1 - a \quad (\text{A1})$$

attempts to use $b-r_1 = r_1 - a$ with $r_1 = \frac{a^2-ab-b}{2a}$ **M1**

$$b - \left(\frac{a^2-ab-b}{2a} \right) = \frac{a^2-ab-b}{2a} - a$$

OR

$$(r_1 =) a + d \quad (\text{A1})$$

attempts to use $r_1 = a + d$ with $r_1 = \frac{a^2-ab-b}{2a}$ and $d = \frac{b-a}{2}$ **M1**

$$\frac{a^2-ab-b}{2a} = a + \frac{b-a}{2}$$

THEN

$$2a^2 + 2ab = 2a^2 - 2ab - 2b \text{ OR } a^2 + ab = a^2 - ab - b$$

$$4ab + 2b = 0 \text{ OR } 2ab + b = 0$$

$$2b(2a+1) = 0 \text{ OR } b(2a+1) = 0 \quad \text{A1}$$

Note: Award **(A1)M1** for any valid approach that correctly leads to $2ab + b = 0$ (or equivalent).
Do not accept numerical verification from the **AG**.

$$\text{so } b = 0 \text{ or } a = -\frac{1}{2} \quad \text{AG}$$

continued...

Question 2 continued

METHOD 2

$$(b =) a + 2d \text{ OR } (r_1 =) a + d \quad \textbf{(A1)}$$

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and d **M1**

$$a + d = \frac{a^2 - a(a + 2d) - (a + 2d)}{2a} \text{ OR } 2(a + d) = \frac{a^2 - a(a + 2d) - (a + 2d)}{a}$$

$$2a^2 + 4ad + a + 2d = 0$$

$$(2a + 1)(a + 2d) = 0 \quad \textbf{A1}$$

Note: Do not accept numerical verification from the **AG**.

$$\text{so } b = 0 \text{ or } a = -\frac{1}{2} \quad \textbf{AG}$$

[3 marks]

continued...

Question 2 continued

(f) **METHOD 1**

$$r_1 = \frac{a}{2} \text{ OR } r_2 = -\frac{a}{2} \text{ OR } d = -\frac{a}{2} \quad (\text{A1})$$

$$c = -a \quad (\text{A1})$$

attempts to find the values of a (M1)

EITHER

the roots of $ax^2 - a = 0$ are ± 1 and $\frac{a}{2} = \pm 1$

OR

substitutes $x = \pm \frac{a}{2}$ into $ax^2 - a = 0$ giving $\frac{a^3}{4} - a = 0$

OR

$$(r_1 r_2) \frac{c}{a} = -\frac{a^2}{4} \Rightarrow c = -\frac{a^3}{4} \text{ and so } -a = -\frac{a^3}{4} \Rightarrow \frac{a^3}{4} - a = 0$$

Note: Award **(M1)** for attempting to find the values of a from their arithmetic sequence expressed in terms of a .

OR

$$c - r_2 = r_2 - b \Rightarrow -\frac{a^3}{4} - \left(-\frac{a}{2}\right) = -\frac{a}{2} \Rightarrow \frac{a^3}{4} - a = 0$$

THEN

$$a = \pm 2 \quad (\text{A1})$$

$$(r_1 = \pm 1, b = 0, r_2 = \mp 1, c = \mp 2)$$

$$\text{so } Q(x) = 2x^2 - 2, Q(x) = -2x^2 + 2 \quad \text{A1}$$

Note: Award **A0** for $2x^2 - 2$ and $-2x^2 + 2$.
Award **(A1)(A1)(M1)(A0)A0** for obtaining either $a = 2$ or $a = -2$.

continued...

Question 2 continued

METHOD 2

$$r_1 = -d \text{ OR } r_2 = d \text{ OR } a = -2d$$

(A1)

$$c = 2d$$

(A1)

attempts to find the values of d

(M1)

EITHER

the roots of $-2dx^2 + 2d = 0$ are ± 1

OR

substitutes $x = \pm d$ into $-2dx^2 + 2d = 0$ giving $-2d^3 + 2d = 0$

OR

attempts to use $r_1 r_2 = \frac{c}{a}$ to form $-d^2 = \frac{2d}{-2d}$

Note: Award **(M1)** for attempting to find the values of d from their arithmetic sequence expressed in terms of d .

THEN

$$d = \pm 1$$

(A1)

$$(a = \pm 2, r_1 = \pm 1, b = 0, r_2 = \mp 1, c = \mp 2)$$

$$\text{so } Q(x) = 2x^2 - 2, Q(x) = -2x^2 + 2$$

A1

Note: Award **A0** for $2x^2 - 2$ and $-2x^2 + 2$.
Award **(A1)(A1)(M1)(A0)A0** for obtaining either $d = 1$ or $d = -1$.

continued...

Question 2 continued

METHOD 3

$$a = 2r_1 \text{ OR } r_2 = -r_1 \text{ OR } d = -r_1 \quad (\text{A1})$$

$$c = -2r_1 \quad (\text{A1})$$

attempts to find the values of r_1 (M1)

EITHER

the roots of $2r_1x^2 - 2r_1 = 0$ are ± 1

OR

substitutes $x = \pm r_1$ into $2r_1x^2 - 2r_1 = 0$ giving $2r_1^3 - 2r_1 = 0$

OR

attempts to use $r_1r_2 = \frac{c}{a}$ to form $-r_1^2 = \frac{-2r_1}{2r_1}$

Note: Award **(M1)** for attempting to find the values of r_1 from their arithmetic sequence expressed in terms of r_1 .

THEN

$$r_1 = \pm 1 \quad (\text{A1})$$

$$(a = \pm 2, r_1 = \pm 1, b = 0, r_2 = \mp 1, c = \mp 2)$$

$$\text{so } Q(x) = 2x^2 - 2, Q(x) = -2x^2 + 2 \quad \text{A1}$$

Note: Award **A0** for $2x^2 - 2$ and $-2x^2 + 2$.
Award **(A1)(A1)(M1)(A0)A0** for obtaining either $r_1 = 1$ or $r_1 = -1$.

[5 marks]

continued...

Question 2 continued

- (g) (i) attempts to express r_1 in terms of b with $a = -\frac{1}{2}$ **(M1)**

Note: Do not award **(M1)** if $a = \frac{1}{2}$ is used.

EITHER

uses $r_1 = \frac{a+b}{2}$

OR

uses $r_1 = \frac{a^2 - ab - b}{2a}$

OR

uses $r_1 - a = b - r_1$

THEN

$$r_1 = \frac{2b-1}{4} \left(= \frac{b}{2} - \frac{1}{4}, = \frac{b - \frac{1}{2}}{2} \right)$$

A1

[2 marks]

continued...

Question 2 continued

(ii) **METHOD 1**

EITHER

substitutes their expression for r_1 with $a = -\frac{1}{2}$ into $Q(x)(=0)$ **(M1)**

$$Q\left(\frac{2b-1}{4}\right)(=0) \Rightarrow -\frac{1}{2}\left(\frac{2b-1}{4}\right)^2 + b\left(\frac{2b-1}{4}\right) + c(=0)$$

OR

$$r_2 = \frac{6b+1}{4} \left(= \frac{3b}{2} + \frac{1}{4} \right)$$

substitutes their expression for r_2 with $a = -\frac{1}{2}$ into $Q(x)(=0)$ **(M1)**

$$Q\left(\frac{6b+1}{4}\right)(=0) \Rightarrow -\frac{1}{2}\left(\frac{6b+1}{4}\right)^2 + b\left(\frac{6b+1}{4}\right) + c(=0)$$

THEN

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \mathbf{A1}$$

$$4b^2 + 20b + 5 = 0$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \mathbf{A1}$$

$$\text{substitutes into } c = \frac{4b+1}{2}$$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \mathbf{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

continued...

Question 2 continued

METHOD 2

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $Q(x)$ **(M1)**

$$-\frac{1}{2}\left(x - \left(\frac{2b-1}{4}\right)\right)\left(x - \left(\frac{6b+1}{4}\right)\right)$$

$$-\frac{1}{2}x^2 + bx - \frac{3}{8}b^2 + \frac{1}{8}b + \frac{1}{32}$$

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \text{A1}$$

$$2b + \frac{1}{2} = -\frac{3}{8}b^2 + \frac{1}{8}b + \frac{1}{32}$$

$$4b^2 + 20b + 5 = 0$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \text{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \text{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

continued...

Question 2 continued

METHOD 3

$$r_2 = \frac{6b+1}{4} \left(= \frac{3b}{2} + \frac{1}{4} \right)$$

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ **(M1)**

$$\left(\frac{2b-1}{4} \right) \left(\frac{6b+1}{4} \right) = \frac{c}{-\frac{1}{2}} \text{ (or equivalent)}$$

EITHER

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \mathbf{A1}$$

OR

$$-\frac{1}{2} \left(\frac{2b-1}{4} \right) \left(\frac{6b+1}{4} \right) - \left(\frac{6b+1}{4} \right) = \frac{2b-1}{4} - \left(-\frac{1}{2} \right) \quad \mathbf{A1}$$

THEN

$$4b^2 + 20b + 5 = 0$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \mathbf{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \mathbf{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

continued...

Question 2 continued

METHOD 4

attempts to equate two expressions for r_1 with $a = -\frac{1}{2}$ **(M1)**

$$\frac{-b \pm \sqrt{b^2 + 2c}}{-1} = \frac{2b-1}{4} \left(\pm \sqrt{b^2 + 2c} = \frac{2b+1}{4} \right)$$

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \textbf{A1}$$

$$12b^2 - 4b - 1 + 32 \left(2b + \frac{1}{2} \right) = 0 \quad (4b^2 + 20b + 5 = 0)$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \textbf{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \textbf{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

continued...

Question 2 continued

METHOD 5

EITHER

$$r_1 = d - \frac{1}{2}$$

substitutes their expression for r_1 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ **(M1)**

$$Q\left(d - \frac{1}{2}\right)(=0) \Rightarrow -\frac{1}{2}\left(d - \frac{1}{2}\right)^2 + b\left(d - \frac{1}{2}\right) + c(=0)$$

OR

$$r_2 = 3d - \frac{1}{2}$$

substitutes their expression for r_2 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ **(M1)**

$$Q\left(3d - \frac{1}{2}\right)(=0) \Rightarrow -\frac{1}{2}\left(3d - \frac{1}{2}\right)^2 + b\left(3d - \frac{1}{2}\right) + c(=0)$$

THEN

$$b = 2d - \frac{1}{2} \text{ and } c = 4d - \frac{1}{2} \text{ (seen anywhere)}$$

A1

$$4d^2 + 8d - 1 = 0$$

attempts to solve their quadratic in d

(M1)

$$d = \frac{-2 \pm \sqrt{5}}{2}$$

$$b = \frac{-5 \pm 2\sqrt{5}}{2}$$

A1

$$\text{substitutes into } c = \frac{4b+1}{2}$$

$$c = \frac{-9 \pm 4\sqrt{5}}{2}$$

A1

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

continued...

Question 2 continued

METHOD 6

$$r_1 = d - \frac{1}{2} \text{ and } r_2 = 3d - \frac{1}{2}$$

substitutes their expressions for r_1 and r_2 in terms of d with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ **(M1)**

$$\left(d - \frac{1}{2}\right)\left(3d - \frac{1}{2}\right) = \frac{c}{-\frac{1}{2}} \text{ (or equivalent)}$$

$$c = 4d - \frac{1}{2} \text{ (seen anywhere)}$$

A1

$$4d^2 + 8d - 1 = 0$$

attempts to solve their quadratic in d

(M1)

$$d = \frac{-2 \pm \sqrt{5}}{2}$$

$$b = \frac{-5 \pm 2\sqrt{5}}{2}$$

A1

$$\text{substitutes into } c = \frac{4b+1}{2}$$

$$c = \frac{-9 \pm 4\sqrt{5}}{2}$$

A1

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

[5 marks]

Total [31 marks]

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg M1, A2*.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g. M1A1*, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** *etc.*, do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

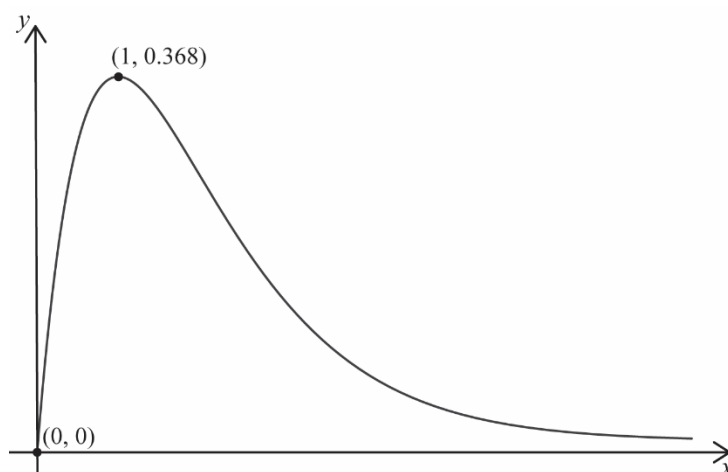
A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a)



A1 for $(1, 0.368)$ or $\left(1, \frac{1}{e}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through, the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

continued...

Question 1 continued

(b) $\int_0^b x e^{-x} dx$ (A1)

Note: Award (A1) for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$= \left[-x e^{-x} \right]_0^b + \int_0^b e^{-x} dx$ A1A1

Note: Award A1 for each part (including the correct sign with each)

$= \left[-x e^{-x} \right]_0^b - \left[e^{-x} \right]_0^b$ A1

Note: Award A1 for correct second term.
Condone absence of limits to this point

attempt to substitute limits

$= -b e^{-b} - e^{-b} + 1$ M1

$= \frac{e^b - b - 1}{e^b}$ A1

[6 marks]

(c) (i) $\lim_{b \rightarrow \infty} \frac{e^b - b - 1}{e^b} = \lim_{b \rightarrow \infty} \frac{e^b - 1}{e^b}$ A1

Note: Award A1 for correct quotient. Condone absence of limit.

$\left(= \lim_{b \rightarrow \infty} \frac{e^b}{e^b} \right) = 1$ A1

[2 marks]

(ii) $\left(\int_0^\infty x e^{-x} dx = \right) 1$ A1

[1 mark]

continued...

Question 1 continued

(d) (i) correct integral

(M1)

Note: Award **M1** for correct integrand with limits from 0 to a larger number.

24

A1

[2 marks]

(ii) 120

A1

Note: The **M1** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) $A_n = n!$

A1

[1 mark]

continued...



Question 1 continued

(f)

Note: Accept starting at $n = 0$, throughout this proof.

$$n = 1$$

$$A_1 = 1 \text{ and } 1! = 1$$

M1A1

Note: Award **M1** for considering the case where $n = 1$, and **A1** if it is clear that both $A_1 = 1$ and $1! = 1$ have been considered.

so true for $n = 1$

assume true for $n = k$, ($A_k = \int_0^\infty x^k e^{-x} dx = k!$)

M1

Note: Award **M0** for statements such as “let $n = k$ ”.

Note: Subsequent marks after this **M1** are independent of this mark and can be awarded.

when $n = k + 1$

attempt to integrate by parts

M1

Note: To obtain the **M1**, a minimum of an expression +/- an integral must be seen.

$$\int_0^\infty x^{k+1} e^{-x} dx = \left[-x^{k+1} e^{-x} \right]_0^\infty + (k+1) \int_0^\infty x^k e^{-x} dx$$

A1

$$(k+1) \int_0^\infty x^k e^{-x} dx \text{ simplified to } (k+1)k! \text{ seen}$$

A1

$$= 0 + (k+1)k!$$

Note: Condone omission of the zero.

$$= (k+1)!$$

A1

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$.

R1

Note: Award the final **R1** mark provided at least four of the previous marks are gained.

[8 marks]

Total [25 marks]

2. (a) $x_2 = 12 - x_1$ **(M1)**
 $f(x) = x_1(12 - x_1)$ **A1**
[2 marks]

- (b) (i) $(x_1 =) 6$ **A1**

Note: Award the **A1** if 6 seen in part (ii).

[1 mark]

- (ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at (6, 36) **M1**
 $= 36$ **AG**
[1 mark]

- (c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) **A1**

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word “maximum” must be seen to award the **A1**.

[1 mark]

continued...

Question 2 continued

- (d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) (M1)

Note: Do not accept use of a specific number for x or n .

$$(x^n)^{\frac{1}{n}} = x \text{ and } \frac{nx}{n} = x$$

A1

[2 marks]

- (ii) $x_1 + x_2 + \dots + x_n = S$ (A1)

$$(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \leq \frac{S}{n}$$

M1

Note: Award **M1** for use of the inequality, which may be seen as an equality.

$$x_1 \times x_2 \times \dots \times x_n \leq \left(\frac{S}{n}\right)^n \text{ (as both sides are positive)}$$

M1

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$)

R1

$$M_n(S) = \left(\frac{S}{n}\right)^n$$

AG

[4 marks]

- (e) (i) $M_3(12) = 4^3 = 64$ A1

[1 mark]

- (ii) $M_4(12) = 3^4 = 81$ A1

[1 mark]

- (iii) $M_5(12) = 2.4^5 = 79.6 \text{ (79.6262...)}$ A1

[1 mark]

continued...

Question 2 continued

- (f) considering $M_n(12)$ for higher values of n

$$P(12) = 81$$

A1

$$n = 4$$

A1

Note: Award **A0A0** for $P(12) = 82.6$ and $n = 4.41$.

[2 marks]

- (g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7

(M1)

Maximum occurs when $n = 7$

A1

$$P(20) = \left(\frac{20}{7}\right)^7 = 1550 \text{ (1554.260...)}$$

A1

Note: Award **(M1)A0A1** for $n = 7.36$ and $P(20) = 1570$.

[3 marks]

continued...

Question 2 continued

(h) **EITHER**

$$\ln(g(x)) = x(\ln(S) - \ln x)$$

M1

attempt to use implicit differentiation and product rule

M1M1

$$\frac{g'(x)}{g(x)} = \ln S - \ln x - x \frac{1}{x}$$

A1

OR

attempt to use implicit differentiation, product rule and chain rule

M1M1M1

$$\frac{g'(x)}{g(x)} = \ln \frac{S}{x} + \left(x \frac{x}{S} \times \frac{-S}{x^2} \right)$$

A1

OR

attempt to make equation explicit to $g(x) = e^{x \ln \left(\frac{S}{x} \right)}$

M1

attempt to use product rule and chain rule

M1M1

$$\begin{aligned} g'(x) &= e^{x \ln \left(\frac{S}{x} \right)} \left[x \times \frac{x}{S} \times (-Sx^{-2}) + \ln \left(\frac{S}{x} \right) \right] \\ &= e^{x \ln \left(\frac{S}{x} \right)} \left[\ln \left(\frac{S}{x} \right) - 1 \right] \end{aligned}$$

A1

THEN

$$g'(x) = \left(\ln \frac{S}{x} - 1 \right) g(x)$$

$$g(x) \neq 0$$

$$g'(x) = 0 \Rightarrow \ln \frac{S}{x} - 1 = 0$$

M1

$$x = \frac{S}{e} \quad (0.368S, 0.36789...S)$$

A1

[6 marks]

continued...

Question 2 continued

$$(i) \quad \ln(g(x)) = x \ln\left(\frac{S}{x}\right) \Rightarrow \ln(g(x)) = \ln\left(\frac{S}{x}\right)^x \quad \text{M1}$$

$$g(x) = \left(\frac{S}{x}\right)^x \quad \text{A1}$$

$$= M_x(S) \text{ for } x \in \mathbb{Z}^+ \quad \text{AG}$$

[2 marks]

$$(j) \quad \frac{100}{e} = 36.8 \quad \text{M1}$$

$$\left(\frac{100}{36}\right)^{36} = 9.3996... \times 10^{15} \quad \text{AND} \quad \left(\frac{100}{37}\right)^{37} = 9.47406... \times 10^{15} \quad \text{R1}$$

$$\text{largest possible product is } 9.47 \times 10^{15} \quad (9.47406... \times 10^{15}) \quad \text{A1}$$

Note: Award **A1** independently of the **R1** (but not independently of the **M1**).

[3 marks]

Total [30 marks]

Markscheme

November 2022

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg M1, A2*.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g. M1A1*, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** *etc.*, do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)

2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or

written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

1. (a) **EITHER**

$$S_n = \frac{n}{2}(2 \times 1 + (n-1) \times 1) \quad \text{A1}$$

OR

$u_1 = 1$ and either $u_n = n$ or $d = 1$ stated explicitly A1

OR

$1 + 2 + \dots + n$ (or equivalent) stated explicitly A1

THEN

$$S_n = \frac{n}{2}(1 + n) \quad \text{AG}$$

Note: Award **A0** for a numerical verification.

[1 mark]

(b) (i) 14

A1

[1 mark]

(ii) $a_1 + a_2 + a_3 = 1$

A1

$$2a_1 + 4a_2 + 8a_3 = 5$$

A1

$$3a_1 + 9a_2 + 27a_3 = 14$$

A1

Note: For the third **A** mark, award **A1FT** for $3a_1 + 9a_2 + 27a_3 = p$ where p is their answer to part (b) (i).

[3 marks]

continued...

Question 1 continued

$$(iii) \quad a_1 = \frac{1}{6} \quad (= 0.166666... \approx 0.167), \quad a_2 = \frac{1}{2} \quad (= 0.5),$$

$$a_3 = \frac{1}{3} \quad (= 0.333333... \approx 0.333)$$

A2

Note: Award **A1** if only two of a_1, a_2, a_3 are correct.

Only award **FT** if three linear equations, each in a_1, a_2 and a_3 are stated in part (b) (ii) or (iii).

Award **A2FT** for their a_1, a_2 and a_3 .

Award **A1FT** for their a_1, a_2 and $a_3 = 0$.

[2 marks]

$$(c) \quad f'(x) = 1 + 2x + 3x^2 + \dots + nx^{n-1}$$

A1

Note: Award **A1** for $f'(x) = \sum_{i=1}^n ix^{i-1}$.

$$\Rightarrow x f'(x) = x + 2x^2 + 3x^3 + \dots + nx^n$$

AG

[1 mark]

continued...

Question 1 continued

(d) (i) **METHOD 1**

$$f_2(x) = xf'_1(x)$$

$$f'_1(x) = 1^2 + 2^2x + (3^2x^2) + \dots + n^2x^{n-1} \quad (= 1 + 4x + (9x^2) + \dots + n^2x^{n-1}) \quad \mathbf{A1}$$

Note: Award **A1** for

$$xf'_1(x) = x(1^2 + 2^2x + (3^2x^2) + \dots + n^2x^{n-1}) \quad (= x(1 + 4x + (9x^2) + \dots + n^2x^{n-1}))$$

$$xf'_1(x) = 1^2x + 2^2x^2 + (3^2x^3) + \dots + n^2x^n \quad (= x + 4x^2 + (9x^3) + \dots + n^2x^n) \quad \mathbf{A1}$$

Note: Award **A1** for $f'_1(x) = \sum_{i=1}^n i^2x^{i-1}$ and **A1** for $xf'_1(x) = x \sum_{i=1}^n i^2x^{i-1}$.

The second **A1** is dependent on the first **A1**.

Award a maximum of **A0A1** if a general term is not considered.

$$= \sum_{i=1}^n i^2x^i \quad \mathbf{AG}$$

METHOD 2

$$f_2(x) = x \frac{d}{dx}(xf'_1(x))$$

$$= x(f'_1(x) + xf''_1(x)) \quad (= xf'_1(x) + x^2f''_1(x)) \quad \mathbf{A1}$$

$$= x \sum_{i=1}^n ix^{i-1} + x^2 \sum_{i=1}^n i(i-1)x^{i-2}$$

$$= \sum_{i=1}^n ix^i + \sum_{i=1}^n i(i-1)x^i \quad \left(= \sum_{i=1}^n (i + i^2 - i)x^i \right) \quad \mathbf{A1}$$

$$= \sum_{i=1}^n i^2x^i \quad \mathbf{AG}$$

[2 marks]

continued...

Question 1 continued

(ii) consider $q = 1$

$$f_1(x) = x + 2x^2 + \dots + nx^n \text{ (reference to part (c)) and } f_1(x) = \sum_{i=1}^n ix^i \quad \mathbf{R1}$$

$$\text{assume true for } q = k, (f_k(x) = \sum_{i=1}^n i^k x^i) \quad \mathbf{M1}$$

Note: Do not award **M1** for statements such as “let $q = k$ ” or “ $q = k$ is true”. Subsequent marks after this **M1** are independent of this mark and can be awarded.

consider $q = k + 1$

$$f_{k+1}(x) = xf'_k(x) \quad \mathbf{M1}$$

$$= x \sum_{i=1}^n i^{k+1} x^{i-1} \quad \mathbf{OR} \quad x(1 + 2^{k+1}x + 3^{k+1}x^2 + \dots + n^{k+1}x^{n-1}) \quad \mathbf{A1}$$

Note: Award the above **M1** if $f_{k+1}(x) = x \sum_{i=1}^n i^{k+1} x^{i-1}$ or $xf'_k(x) = x \sum_{i=1}^n i^{k+1} x^{i-1}$ (or equivalent) is stated.

$$= \sum_{i=1}^n i^{k+1} x^i \quad \mathbf{OR} \quad x + 2^{k+1}x^2 + 3^{k+1}x^3 + \dots + n^{k+1}x^n \quad \mathbf{A1}$$

since true for $q = 1$ and true for $q = k + 1$ if true for $q = k$, hence true for all $q (\in \mathbb{Z}^+)$

R1

Note: To obtain the final **R1**, three of the previous five marks must have been awarded.

[6 marks]

(iii) $f_q(1) = 1^q + 2^q + 3^q + \dots + n^q$

$$= \sum_{i=1}^n i^q \left(= \sum_{i=1}^n 1^i i^q \right) \quad \mathbf{A1}$$

[1 mark]

continued...

Question 1 continued

(e) uses $S_n = \frac{u_1(r^n - 1)}{r - 1}$ with $r = x$ and $u_1 = 1$ **M1**

clear indication there are $(n + 1)$ terms **R1**

$$f(x) = \frac{x^{n+1} - 1}{x - 1}$$
AG

[2 marks]

(f) **METHOD 1**

$$f_1(x) = xf'(x)$$

$$= x \frac{(x-1)(n+1)x^n - 1 \times (x^{n+1} - 1)}{(x-1)^2}$$
M1A1

Note: Award **M1** for attempting to use the quotient or the product rule to find $f'(x)$.

$$= x \frac{(nx + x - n - 1)x^n - (x^{n+1} - 1)}{(x-1)^2} \left(= x \frac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2} \right)$$
A1

Note: Award **A1** for any correct manipulation of the derivative that leads to the **AG**.

$$= \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$
AG

METHOD 2

attempts to form $(x-1)f_1(x)$

$$(x-1)f_1(x) = nx^{n+1} - (x + x^2 + x^3 + \dots + x^n)$$
M1

$$f_1(x) = \frac{1}{x-1} \left(nx^{n+1} - \left(\frac{x^{n+1} - 1}{x-1} - 1 \right) \right)$$
A1

$$f_1(x) = \frac{1}{x-1} \left(\frac{nx^{n+1}(x-1) - x^{n+1} + x}{x-1} \right) \left(= \frac{1}{x-1} \left(\frac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{x-1} \right) \right)$$
A1

Note: Award **A1** for any correct manipulation of the derivative that leads to the **AG**.

$$f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

AG

[3 marks]

(g) (i) $\lim_{x \rightarrow 1} f_1(x) = \frac{n - (n+1) + 1}{0} \left(= \frac{0}{0} \right)$

R1

Note: Only award **R1** for sufficient simplification of the numerator, for example, as shown above.

Do not award **R1** if $\lim_{x \rightarrow 1}$ is not referred to or stated.

[1 mark]

continued...



Question 1 continued

- (ii) attempts to differentiate both the numerator and the denominator

M1

$$\lim_{x \rightarrow 1} \frac{n(n+2)x^{n+1} - (n+1)^2 x^n + 1}{2(x-1)}$$

A1

Note: Award **A1** for $\left(\lim_{x \rightarrow 1} \frac{n(n+2)x^{n+1} - n(n+1)x^n - (n+1)x^n + 1}{2(x-1)} \right)$. This form can be used in subsequent work.

(l'Hôpital's rule applies again since)

$$\lim_{x \rightarrow 1} \frac{n(n+2)x^{n+1} - (n+1)^2 x^n + 1}{2(x-1)} = \frac{0}{0}$$

R1

Note: Do not award **R1** if $\lim_{x \rightarrow 1}$ is not referred to or stated.

Subsequent marks are independent of this **R** mark.

attempts to differentiate both the numerator and the denominator

M1

$$\begin{aligned} & \lim_{x \rightarrow 1} \frac{n(n+2)(n+1)x^n - n(n+1)^2 x^{n-1}}{2} \\ &= \frac{n(n+2)(n+1) - n(n+1)^2}{2} \left(= \frac{n^3 + 3n^2 + 2n - (n^3 + 2n^2 + n)}{2} \right) \\ &= \frac{n(n+1)((n+2) - (n+1))}{2} \left(= \frac{n^2 + n}{2} \right) \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

A1

AG

[5 marks]

Total [28 marks]

2. (a) **EITHER**

$$A = 2\pi \int_0^3 2x\sqrt{1+2^2} \, dx \left(= 4\sqrt{5}\pi \int_0^3 x \, dx \right) \quad (\text{A1})$$

$$= 2\pi\sqrt{5} \left[x^2 \right]_0^3 \left(= 2\pi\sqrt{5}(3^2 - 0^2) \right) \quad \text{A1}$$

OR

$$A = 2\pi m\sqrt{1+m^2} \left[\frac{x^2}{2} \right]_0^h \left(= 2\pi m\sqrt{1+m^2} \left(\frac{h^2}{2} \right) \right) \quad (\text{A1})$$

$$= 2\pi(2)\sqrt{5} \left[\frac{x^2}{2} \right]_0^3 \left(= 2\pi(2)\sqrt{5} \left(\frac{3^2}{2} \right) \right) \quad \text{A1}$$

THEN

$$= 18\sqrt{5}\pi \quad \text{AG}$$

[2 marks]

(b) (i) $r = mh$ A1

[1 mark]

(ii) $l = \sqrt{h^2 + r^2}$ (M1)

$$l = \sqrt{h^2 + h^2 m^2} \left(= h\sqrt{1+m^2} \right) \quad \text{A1}$$

[2 marks]

continued...

Question 2 continued

$$(iii) \quad A = 2\pi \int_0^h mx\sqrt{1+m^2} \, dx \quad (A1)$$

$$= 2\pi m\sqrt{1+m^2} \left[\frac{1}{2}x^2 \right]_0^h \quad (M1)$$

Note: Award **(M1)** for $(c) \pi m\sqrt{1+m^2} \left[\frac{1}{2}x^2 \right]_0^h$.

At least one of the above two lines needs to be seen.

$$= \pi h^2 m\sqrt{1+m^2} \left(= \pi h m \times \sqrt{(h^2 + h^2 m^2)} \right) \quad A1$$

$$= \pi r l \quad AG$$

Note: Award as above if $\frac{l}{h} = \sqrt{1+m^2}$ is used, for example.

[3 marks]

(c) **METHOD 1**

attempts to use the chain rule

(M1)

Note: Award **(M1)** for $\frac{dy}{dx} = (c)x(r^2 - x^2)^{-\frac{1}{2}}$.

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} (-2x) \left(= -x(r^2 - x^2)^{-\frac{1}{2}} \right) \quad A1$$

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - x^2$ (or equivalent)

(M1)

$$\frac{dy}{dx} = -\frac{x}{y} \quad A1$$

[2 marks]

continued...

Question 2 continued

(d) **EITHER**

attempts to substitute $y = \sqrt{r^2 - x^2}$ and their $\frac{dy}{dx}$ into A **(M1)**

$$A = 2\pi \int_{-r}^r \sqrt{r^2 - x^2} \sqrt{1 + \left(-x(r^2 - x^2)^{-\frac{1}{2}}\right)^2} dx$$

$$= 2\pi \int_{-r}^r \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx$$
 A1

OR

attempts to substitute y and their $\frac{dy}{dx}$ in terms of x and y into A **(M1)**

$$A = 2\pi \int_{-r}^r y \sqrt{1 + \left(-\frac{x}{y}\right)^2} dx$$

$$= 2\pi \int_{-r}^r y \sqrt{1 + \frac{x^2}{y^2}} dx \left(= 2\pi \int_{-r}^r \sqrt{x^2 + y^2} dx \right)$$
 A1

THEN

attempts to perform valid algebraic simplification to form a definite integral in terms of r only **M1**

$$= 2\pi \int_{-r}^r r dx$$

$$= 2\pi r [x]_{-r}^r \left(= 2\pi r (r - (-r)) \right)$$
 A1

$$= 4\pi r^2$$
 AG

Note: Award marks as above for $A = 4\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{1 + \left(-x(r^2 - x^2)^{-\frac{1}{2}}\right)^2} dx$.

[4 marks]

continued...

Question 2 continued

(e) (i) **EITHER**

horizontal stretch

A1

factor $\frac{1}{k}$

A1

OR

horizontal compression

A1

factor k (invariant line y -axis)

A1

Note: Award **A1A1** as above for correct alternative descriptions.

For example, dilation by a factor of $\frac{1}{k}$ from the y -axis.

[2 marks]

(ii) $\pm \frac{r}{k}$

A1

Note: Award **A0** for $\frac{r}{k}$ only and **A0** for $-\frac{r}{k}$ only.

[1 mark]

continued...

Question 2 continued

(iii) **METHOD 1**

attempts to use the chain rule

(M1)

$$\frac{dy}{dx} = \frac{1}{2} \left(r^2 - (kx)^2 \right)^{-\frac{1}{2}} \times (-k^2 2x) \left(= -k^2 x \left(r^2 - (kx)^2 \right)^{-\frac{1}{2}} \left(= \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}} \right) \right) \quad \mathbf{A1}$$

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - k^2 x^2$ (or equivalent)

(M1)

$$\frac{dy}{dx} = -\frac{k^2 x}{y}$$

$$\frac{dy}{dx} = \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}$$

A1

[2 marks]

continued...



Question 2 continued

(iv) **EITHER**

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2 x^2} \sqrt{1 + \frac{k^4 x^2}{r^2 - k^2 x^2}} \, dx \quad (\text{A1})$$

Note: Award **(A1)** for the correct substitution of y and $\frac{dy}{dx}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of $r^2 - k^2 x^2$ and then cancelling $r^2 - k^2 x^2$ (M1)

OR

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y \sqrt{1 + \left(-\frac{k^2 x}{y}\right)^2} \, dx \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y \sqrt{1 + \frac{k^4 x^2}{y^2}} \, dx \right) \quad (\text{A1})$$

Note: Award **(A1)** for the correct substitution of $\frac{dy}{dx}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of y^2 and cancelling y^2 (M1)

THEN

$$= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2 x^2 + k^4 x^2} \, dx \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 + (k^4 - k^2) x^2} \, dx \right) \quad \text{A1A1}$$

Note: Award **A1** for correct limits (seen anywhere) and **A1** for $p(x)$ correct.

The above **A1** for correct limits is independent of the **(M1)**.

[4 marks]

continued...

Question 2 continued

(v) $r = 6378 \text{ (km)}$ (A1)

$$k = 1.00330... \left(= \frac{6378}{6357} = \frac{2126}{2119} \right)$$
 (A1)

attempts to form a definite integral for surface area (M1)

$$= 2\pi \int_{-6357}^{6357} \sqrt{6378^2 - \left(\frac{6378}{6357} \right)^2 x^2 + \left(\frac{6378}{6357} \right)^4 x^2} dx$$

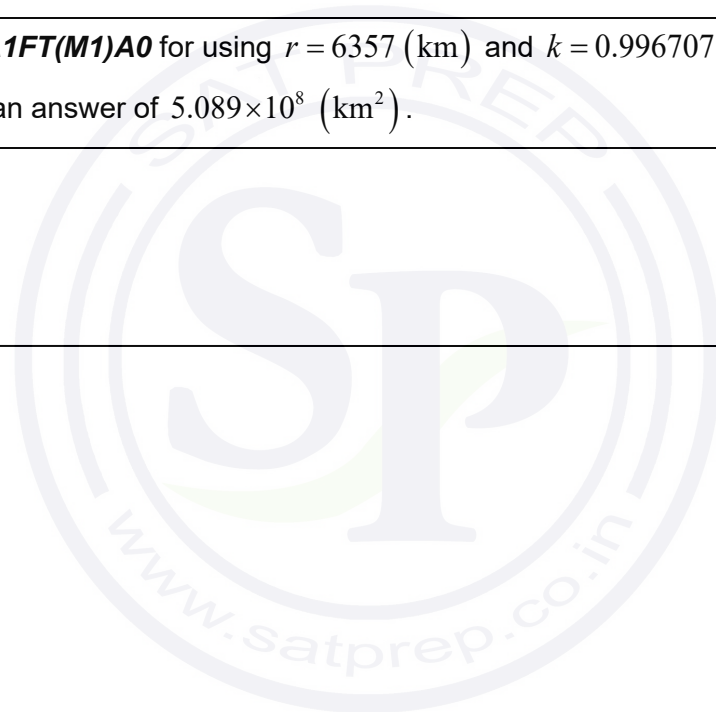
$$= 510064226.3...$$

$$= 5.101 \times 10^8 \text{ (km}^2\text{)}$$
 A1

Note: Award **A0A1FT(M1)A0** for using $r = 6357 \text{ (km)}$ and $k = 0.996707...$ leading to an answer of $5.089 \times 10^8 \text{ (km}^2\text{)}$.

[4 marks]

Total [27 marks]



Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This

includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or

written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

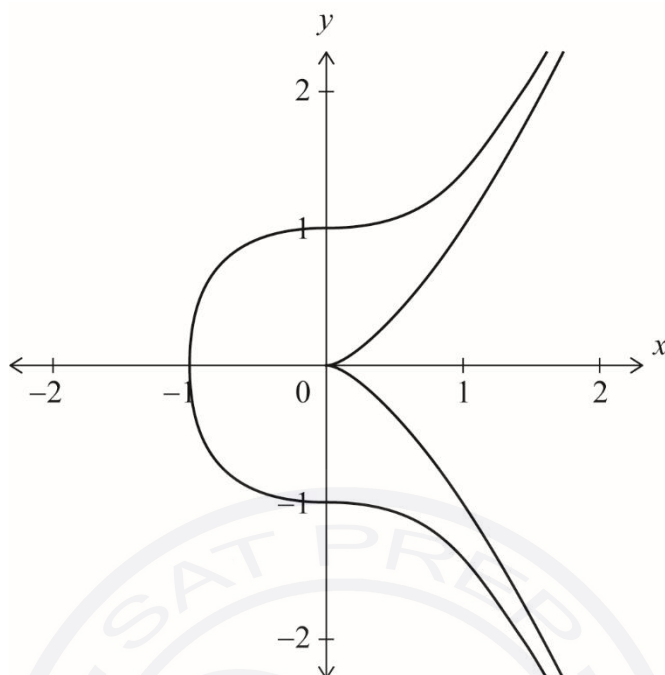
10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.



1. (a) (i)



approximately symmetric about the x-axis graph of $y^2 = x^3$

A1

including cusp/sharp point at (0, 0)

A1

[2 marks]

(ii) approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts

A1

some indication of position of intersections at $x = -1$, $y = \pm 1$

A1

[2 marks]

Note: Final **A1** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award **A1A1A1A0** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct. Award **A1A0A0A0** if only one correct branch of both curves are seen.

Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be **A1A1A0A0**.

continued...

Question 1 continued

(b) (i) $(0, 1)$ and $(0, -1)$

A1

[1 mark]

(ii) Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different x -axis intercepts (one goes through the origin, and the other does not)

graphs have different y -axis intercepts

A1

Note: Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent.

[1 mark]

continued...

Question 1 continued

(c) Any **two** from:

as $x \rightarrow \infty$, $y \rightarrow \pm\infty$

as $x \rightarrow \infty$, $y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have x intercepts at $x = -\sqrt[3]{b}$

they have y intercepts at $y = (\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or x -axis) is a line of symmetry

they all have the same line of symmetry ($y = 0$)

they have one x -axis intercept

they have two y -axis intercepts

they have two points of inflexion

at x -axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at x -axis intercepts

A1A1

Note: The last example is the only valid answer for things “not” present. Do not credit an answer of “they are all symmetrical” without some reference to the line of symmetry.

Note: Do not allow same/ similar shape or equivalent.

Note: In accordance with marking rules, mark their first two responses and ignore any subsequent.

[2 marks]

continued...

Question 1 continued

(d) (i) **METHOD 1**

attempt to differentiate implicitly

M1

$$2y \frac{dy}{dx} = 3x^2 + 1$$

A1

$$\frac{dy}{dx} = \frac{3x^2 + 1}{2y} \quad \text{OR} \quad (\pm) 2\sqrt{x^3 + x} \frac{dy}{dx} = 3x^2 + 1$$

A1

$$\frac{dy}{dx} = \pm \frac{3x^2 + 1}{2\sqrt{x^3 + x}}$$

AG

METHOD 2

attempt to use chain rule $y = (\pm)\sqrt{x^3 + x}$

M1

$$\frac{dy}{dx} = (\pm) \frac{1}{2} (x^3 + x)^{-\frac{1}{2}} (3x^2 + 1)$$

A1A1

Note: Award **A1** for $(\pm) \frac{1}{2} (x^3 + x)^{-\frac{1}{2}}$, **A1** for $(3x^2 + 1)$.

$$\frac{dy}{dx} = \pm \frac{3x^2 + 1}{2\sqrt{x^3 + x}}$$

AG

[3 marks]

(ii) **EITHER**

local minima/maxima occur when $\frac{dy}{dx} = 0$

$1 + 3x^2 = 0$ has no (real) solutions (or equivalent)

R1

OR

$(x^2 \geq 0 \Rightarrow) 3x^2 + 1 > 0$, so $\frac{dy}{dx} \neq 0$

R1

THEN

so, no local minima/maxima exist

AG

[1 mark]

continued...

Question 1 continued

(e) **EITHER**

attempt to use quotient rule to find $\frac{d^2y}{dx^2}$ **M1**

$$\frac{d^2y}{dx^2} = (\pm) \frac{12x\sqrt{x+x^3} - (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)}{4(x+x^3)}$$
A1A1

Note: Award **A1** for correct $12x\sqrt{x+x^3}$ and correct denominator, **A1** for correct $-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$.

Note: Future **A** marks may be awarded if the denominator is missing or incorrect.

stating or using $\frac{d^2y}{dx^2} = 0$ (may be seen anywhere) **(M1)**

$$12x\sqrt{x+x^3} = (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$$

OR

attempt to use product rule to find $\frac{d^2y}{dx^2}$ **M1**

$$\frac{d^2y}{dx^2} = \frac{1}{2}(3x^2+1)\left(-\frac{1}{2}\right)(3x^2+1)(x^3+x)^{-\frac{3}{2}} + 3x(x^3+x)^{-\frac{1}{2}}$$
A1A1

Note: Award **A1** for correct first term, **A1** for correct second term.

setting $\frac{d^2y}{dx^2} = 0$ **(M1)**

continued...

Question 1 continued

OR

attempts implicit differentiation on $2y \frac{dy}{dx} = 3x^2 + 1$

M1

$$2\left(\frac{dy}{dx}\right)^2 + 2y \frac{d^2y}{dx^2} = 6x$$

A1

recognizes that $\frac{d^2y}{dx^2} = 0$

(M1)

$$\frac{dy}{dx} = \pm\sqrt{3x}$$

$$(\pm) \frac{3x^2 + 1}{2\sqrt{x^3 + x}} = (\pm)\sqrt{3x}$$

(A1)

THEN

$$12x(x + x^3) = (1 + 3x^2)^2$$

$$12x^2 + 12x^4 = 9x^4 + 6x^2 + 1$$

$$3x^4 + 6x^2 - 1 = 0$$

A1

attempt to use quadratic formula or equivalent

(M1)

$$x^2 = \frac{-6 \pm \sqrt{48}}{6}$$

$$(x > 0 \Rightarrow) x = \sqrt{\frac{2\sqrt{3} - 3}{3}} \quad (p = 2, q = -3, r = 3)$$

A1

Note: Accept any integer multiple of p , q and r (e.g. 4, -6 and 6).

[7 marks]

continued...

Question 1 continued

- (f) (i) attempt to find tangent line through $(-1, -1)$ **(M1)**

$$y + 1 = -\frac{3}{2}(x + 1) \text{ OR } y = -1.5x - 2.5$$

A1

[2 marks]

- (ii) attempt to solve simultaneously with $y^2 = x^3 + 2$ **(M1)**

Note: The **M1** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. (4.25, -8.875)).

obtain $\left(\frac{17}{4}, -\frac{71}{8}\right)$

A1

[2 marks]

continued...

Question 1 continued

(g) attempt to find equation of [QS] (M1)

$$\frac{y-1}{x+1} = -\frac{79}{42} (= -1.88095\dots) \quad (A1)$$

solve simultaneously with $y^2 = x^3 + 2$ (M1)

$$x = 0.28798\dots \left(= \frac{127}{441} \right) \quad A1$$

$$y = -1.4226\dots \left(= \frac{13175}{9261} \right) \quad A1$$

$$(0.288, -1.42)$$

OR

attempt to find vector equation of [QS] (M1)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{21}{4} \\ -\frac{79}{8} \end{pmatrix} \quad (A1)$$

$$x = -1 + \frac{21}{4}\lambda$$

$$y = 1 - \frac{79}{8}\lambda$$

attempt to solve $\left(1 - \frac{79}{8}\lambda\right)^2 = \left(-1 + \frac{21}{4}\lambda\right)^3 + 2$ (M1)

$$\lambda = 0.2453\dots$$

$$x = 0.28798\dots \left(= \frac{127}{441} \right) \quad A1$$

$$y = -1.4226\dots \left(= \frac{13175}{9261} \right) \quad A1$$

$$(0.288, -1.42)$$

[5 marks]

[Total 28 marks]

2. (a) attempt to expand $(x - \alpha)(x - \beta)(x - \gamma)$ **M1**
- $= (x^2 - (\alpha + \beta)x + \alpha\beta)(x - \gamma)$ **OR** $= (x - \alpha)(x^2 - (\beta + \gamma)x + \beta\gamma)$ **A1**
- $(x^3 + px^2 + qx + r) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$ **A1**
- comparing coefficients:
- $p = -(\alpha + \beta + \gamma)$ **AG**
- $q = (\alpha\beta + \beta\gamma + \gamma\alpha)$ **AG**
- $r = -\alpha\beta\gamma$ **AG**

Note: For candidates who do not include the **AG** lines award full marks.

[3 marks]

- (b) (i) $p^2 - 2q = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ **(A1)**
- attempt to expand $(\alpha + \beta + \gamma)^2$ **(M1)**
- $= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ or equivalent **A1**
- $= \alpha^2 + \beta^2 + \gamma^2$ **AG**

Note: Accept equivalent working from RHS to LHS.

[3 marks]

continued...

Question 2 continued

(ii) **EITHER**

attempt to expand $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$ (M1)

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha) \quad \text{A1}$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 2(p^2 - 2q) - 2q \text{ or equivalent} \quad \text{A1}$$

$$= 2p^2 - 6q \quad \text{AG}$$

OR

attempt to write $2p^2 - 6q$ in terms of α, β, γ (M1)

$$= 2(p^2 - 2q) - 2q$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \quad \text{A1}$$

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha) \quad \text{A1}$$

$$= (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \quad \text{AG}$$

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

continued...

Question 2 continued

(c) $p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$\Rightarrow (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 < 0$

A1

if all roots were real $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$

R1

Note: Condone strict inequality in the **R1** line.

Note: Do not award **A0R1**.

$\Rightarrow$ roots cannot all be real

AG

[2 marks]

(d) $p^2 = (-7)^2 = 49$ and $3q = 51$

A1

so $p^2 < 3q \Rightarrow$ the equation has at least one complex root

R1

Note: Allow equivalent comparisons; e.g. checking $2p^2 < 6q$

[2 marks]

(e) (i) use of GDC (eg graphs or tables)

(M1)

$q = 12$

A1

[2 marks]

(ii) complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).

OR

a cubic curve always crosses the x -axis at at least one point.

R1

[1 mark]

continued...

Question 2 continued

- (f) (i) attempt to expand $(\alpha + \beta + \gamma + \delta)^2$ (M1)

$$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

(A1)

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$(\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 =) p^2 - 2q$$

A1

[3 marks]

- (ii) $p^2 < 2q$ OR $p^2 - 2q < 0$

A1

Note: Allow **FT** on their result from part (f)(i).

[1 mark]

- (g) $4 < 6$ OR $2^2 - 2 \times 3 < 0$

R1

hence there is at least one complex root.

AG

Note: Allow **FT** from part (f)(ii) for the **R** mark provided numerical reasoning is seen.

[1 mark]

continued...

Question 2 continued

(h) (i) $(p^2 > 2q)$ ($81 > 2 \times 24$) (so) nothing can be deduced

R1

Note: Do not allow **FT** for the **R** mark.

[1 mark]

(ii) -1

A1

[1 mark]

(iii) attempt to express as a product of a linear and cubic factor

M1

$(x+1)(x^3 - 10x^2 + 34x - 12)$

A1A1

Note: Award **A1** for each factor. Award at most **A1A0** if not written as a product.

since for the cubic, $p^2 < 3q$ ($100 < 102$)

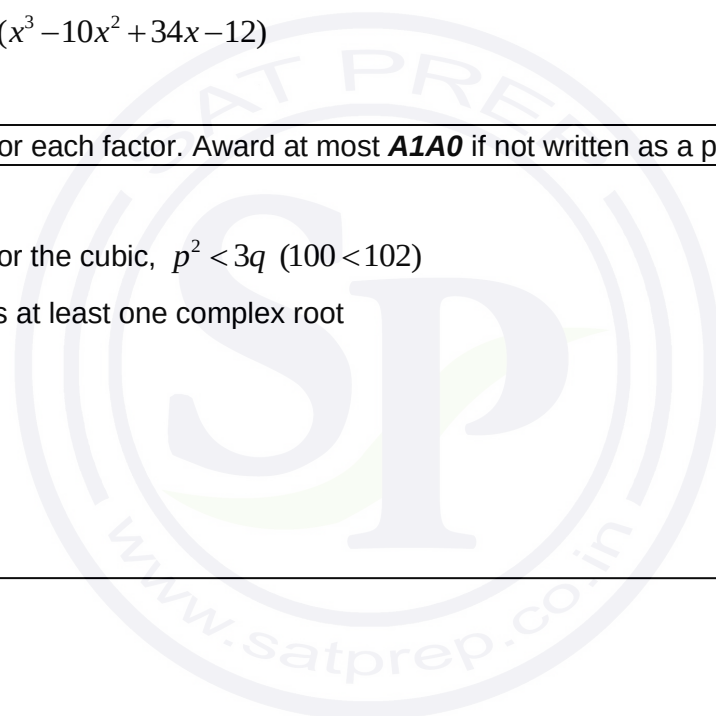
R1

there is at least one complex root

AG

[4 marks]

[Total: 27 marks]



Markscheme

May 2022

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This

includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or

written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.



1. (a) (i) $P_3(n) = \frac{(3-2)n^2 - (3-4)n}{2}$ OR $P_3(n) = \frac{n^2 - (-n)}{2}$ **A1**

$P_3(n) = \frac{n^2 + n}{2}$ **A1**

Note: Award **A0A1** if $P_3(n) = \frac{n^2 + n}{2}$ only is seen.

Do not award any marks for numerical verification.

so for triangular numbers, $P_3(n) = \frac{n(n+1)}{2}$ **AG**

[2 marks]

(ii) **METHOD 1**

uses a table of values to find a positive integer that satisfies $P_3(n) = 351$ **(M1)**

for example, a list showing at least 3 consecutive terms (... 325, 351, 378...)

Note: Award **(M1)** for use of a GDC's numerical solve or graph feature.

$n = 26$ (26th triangular number) **A1**

Note: Award **A0** for $n = -27, 26$. Award **A0** if additional solutions besides $n = 26$ are given.

METHOD 2

attempts to solve $\frac{n(n+1)}{2} = 351$ ($n^2 + n - 702 = 0$) for n **(M1)**

$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-702)}}{2}$ OR $(n-26)(n+27) = 0$

$n = 26$ (26th triangular number) **A1**

Note: Award **A0** for $n = -27, 26$. Award **A0** if additional solutions besides $n = 26$ are given.

[2 marks]

continued...

Question 1 continued

- (b) (i) attempts to form an expression for $P_3(n) + P_3(n+1)$ in terms of n

M1

EITHER

$$P_3(n) + P_3(n+1) \equiv \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}$$

$$\equiv \frac{(n+1)(2n+2)}{2} \left(\equiv \frac{2(n+1)(n+1)}{2} \right)$$

A1

OR

$$P_3(n) + P_3(n+1) \equiv \left(\frac{n^2}{2} + \frac{n}{2} \right) + \left(\frac{(n+1)^2}{2} + \frac{n+1}{2} \right)$$

$$\equiv \left(\frac{n^2 + n}{2} \right) + \left(\frac{n^2 + 2n + 1 + n + 1}{2} \right) \left(\equiv n^2 + 2n + 1 \right)$$

A1

THEN

$$\equiv (n+1)^2$$

AG

[2 marks]

- (ii) the sum of the n th and $(n+1)$ th triangular numbers
is the $(n+1)$ th square number

A1

[1 mark]

continued...

Question 1 continued

(iii)

```

X  X  X  X  X
O  X  X  X  X
O  O  X  X  X
O  O  O  X  X
O  O  O  O  X
    
```

A1

Note: Accept equivalent single diagrams, such as the one above, where the 4th and 5th triangular numbers and the 5th square number are clearly shown. Award **A1** for a diagram that show $P_3(4)$ (a triangle with 10 dots) and $P_3(5)$ (a triangle with 15 dots) and $P_4(5)$ (a square with 25 dots).

[1 mark]

continued...



Question 1 continued

(c) **METHOD 1**

$$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1 (=4n(n+1)+1) \quad \textbf{A1}$$

attempts to expand their expression for $8P_3(n)+1$ **(M1)**

$$=4n^2+4n+1$$

$$=(2n+1)^2 \quad \textbf{A1}$$

and $2n+1$ is odd **AG**

METHOD 2

$$8P_3(n)+1=8\left((n+1)^2-P_3(n+1)\right)+1 \left(=8\left((n+1)^2-\frac{(n+1)(n+2)}{2}\right)+1\right) \quad \textbf{A1}$$

attempts to expand their expression for $8P_3(n)+1$ **(M1)**

$$8(n^2+2n+1)-4(n^2+3n+2)+1 (=4n^2+4n+1)$$

$$=(2n+1)^2 \quad \textbf{A1}$$

and $2n+1$ is odd **AG**

continued...

Question 1 continued

METHOD 3

$$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1 \left(=(An+B)^2\right) \text{ (where } A,B\in\mathbb{Z}^+)$$

A1

attempts to expand their expression for $8P_3(n)+1$

(M1)

$$4n^2+4n+1 \left(=A^2n^2+2ABn+B^2\right)$$

now equates coefficients and obtains $B=1$ and $A=2$

$$=(2n+1)^2$$

A1

and $2n+1$ is odd

AG

[3 marks]

continued...



Question 1 continued

(d) **EITHER**

$$u_1 = 1 \text{ and } d = 3 \quad (\text{A1})$$

$$\text{substitutes their } u_1 \text{ and their } d \text{ into } P_5(n) = \frac{n}{2}(2u_1 + (n-1)d) \quad \text{M1}$$

$$P_5(n) = \frac{n}{2}(2 + 3(n-1)) \left(= \frac{n}{2}(2 + 3n - 3) \right) \quad \text{A1}$$

OR

$$u_1 = 1 \text{ and } u_n = 3n - 2 \quad (\text{A1})$$

$$\text{substitutes their } u_1 \text{ and their } u_n \text{ into } P_5(n) = \frac{n}{2}(u_1 + u_n) \quad \text{M1}$$

$$P_5(n) = \frac{n}{2}(1 + 3n - 2) \quad \text{A1}$$

OR

$$P_5(n) = (3(1) - 2) + (3(2) - 2) + (3(3) - 2) + \dots + 3n - 2$$

$$P_5(n) = (3(1) + 3(2) + 3(3) + \dots + 3n) - 2n \quad (= 3(1 + 2 + 3 + \dots + n) - 2n) \quad (\text{A1})$$

$$\text{substitutes } \frac{n(n+1)}{2} \text{ into their expression for } P_5(n) \quad \text{M1}$$

$$P_5(n) = 3\left(\frac{n(n+1)}{2}\right) - 2n$$

$$P_5(n) = \frac{n}{2}(3(n+1) - 4) \quad \text{A1}$$

OR

attempts to find the arithmetic mean of n terms (M1)

$$= \frac{1 + (3n - 2)}{2} \quad \text{A1}$$

multiplies the above expression by the number of terms n

$$P_5(n) = \frac{n}{2}(1 + 3n - 2) \quad \text{A1}$$

THEN

$$\text{so } P_5(n) = \frac{n(3n - 1)}{2} \quad \text{AG}$$

[3 marks]

continued...

Question 1 continued

(e) **METHOD 1**

forms a table of $P_3(n)$ values that includes some values for $n > 5$ **(M1)**

forms a table of $P_5(m)$ values that includes some values for $m > 5$ **(M1)**

Note: Award **(M1)** if at least one $P_3(n)$ value is correct. Award **(M1)** if at least one $P_5(m)$ value is correct. Accept as above for $(n^2 + n)$ values and $(3m^2 - m)$ values.

$n = 20$ for triangular numbers **(A1)**

$m = 12$ for pentagonal numbers **(A1)**

Note: Award **(A1)** if $n = 20$ is seen in or out of a table. Award **(A1)** if $m = 12$ is seen in or out of a table. Condone the use of the same parameter for triangular numbers and pentagonal numbers, for example, $n = 20$ for triangular numbers and $n = 12$ for pentagonal numbers.

210 (is a triangular number and a pentagonal number) **A1**

Note: Award all five marks for 210 seen anywhere with or without working shown.

continued...

Question 1 continued

METHOD 2

EITHER

attempts to express $P_3(n) = P_5(m)$ as a quadratic in n (M1)

$$n^2 + n + (m - 3m^2) = 0 \text{ (or equivalent)}$$

attempts to solve their quadratic in n (M1)

$$n = \frac{-1 \pm \sqrt{12m^2 - 4m + 1}}{2} \left(= \frac{-1 \pm \sqrt{1^2 - 4(m - 3m^2)}}{2} \right)$$

OR

attempts to express $P_3(n) = P_5(m)$ as a quadratic in m (M1)

$$3m^2 - m - (n^2 + n) = 0 \text{ (or equivalent)}$$

attempts to solve their quadratic in m (M1)

$$m = \frac{1 \pm \sqrt{12n^2 + 12n + 1}}{6} \left(= \frac{1 \pm \sqrt{(-1)^2 + 12(n^2 + n)}}{6} \right)$$

THEN

$n = 20$ for triangular numbers (A1)

$m = 12$ for pentagonal numbers (A1)

210 (is a triangular number and a pentagonal number) A1

continued...

Question 1 continued

METHOD 3

$$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2}$$

let $n = m + k$ ($n > m$) and so $3m^2 - m = (m+k)(m+k+1)$

M1

$$2m^2 - 2(k+1)m - (k^2 + k) = 0$$

A1

attempts to find the discriminant of their quadratic
and recognises that this must be a perfect square

M1

$$\Delta = 4(k+1)^2 + 8(k^2 + k)$$

$$N^2 = 4(k+1)^2 + 8(k^2 + k) \quad (= 4(k+1)(3k+1))$$

determines that $k=8$ leading to $2m^2 - 18m - 72 = 0 \Rightarrow m = -3, 12$ and so $m = 12$

A1

210 (is a triangular number and a pentagonal number)

A1

METHOD 4

$$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2}$$

let $m = n - k$ ($m < n$) and so $n^2 + n = (n-k)(3(n-k)-1)$

M1

$$2n^2 - 2(3k+1)n + (3k^2 + k) = 0$$

A1

attempts to find the discriminant of their quadratic
and recognises that this must be a perfect square

M1

$$\Delta = 4(3k+1)^2 - 8(3k^2 + k)$$

$$N^2 = 4(3k+1)^2 - 8(3k^2 + k) \quad (= 4(k+1)(3k+1))$$

determines that $k=8$ leading to $2n^2 - 50n + 200 = 0 \Rightarrow n = 5, 20$ and so $n = 20$

A1

210 (is a triangular number and a pentagonal number)

A1

[5 marks]

continued...

Question 1 continued

(f)

Note: Award a maximum of **R1M0M0A1M1A1A1R0** for a ‘correct’ proof using n and $n+1$.

consider $n=1$: $P_r(1) = 1 + (1-1)(r-2) = 1$ and $P_r(1) = \frac{(r-2)(1^2) - (r-4)(1)}{2} = 1$

so true for $n=1$

R1

Note: Accept $P_r(1) = 1$ and $P_r(1) = \frac{(r-2)(1^2) - (r-4)(1)}{2} = 1$.

Do not accept one-sided considerations such as ‘ $P_r(1) = 1$ and so true for $n=1$ ’.

Subsequent marks after this **R1** are independent of this mark can be awarded.

Assume true for $n=k$, ie. $P_r(k) = \frac{(r-2)k^2 - (r-4)k}{2}$

M1

Note: Award **M0** for statements such as “let $n=k$ ”. The assumption of truth must be clear.
Subsequent marks after this **M1** are independent of this mark and can be awarded.

Consider $n = k + 1$:

$(P_r(k+1))$ can be represented by the sum

$$\sum_{m=1}^{k+1} (1 + (m-1)(r-2)) = \sum_{m=1}^k (1 + (m-1)(r-2)) + (1 + k(r-2)) \text{ and so}$$

$$P_r(k+1) = \frac{(r-2)k^2 - (r-4)k}{2} + (1 + k(r-2)) \quad (P_r(k+1) = P_r(k) + (1 + k(r-2))) \quad \textbf{M1}$$

$$= \frac{(r-2)k^2 - (r-4)k + 2 + 2k(r-2)}{2} \quad \textbf{A1}$$

$$= \frac{(r-2)(k^2 + 2k) - (r-4)k + 2}{2}$$

$$= \frac{(r-2)(k^2 + 2k + 1) - (r-2) - (r-4)k + 2}{2} \quad \textbf{M1}$$

$$= \frac{(r-2)(k+1)^2 - (r-4)k - (r-4)}{2} \quad \textbf{(A1)}$$

$$= \frac{(r-2)(k+1)^2 - (r-4)(k+1)}{2} \quad \textbf{A1}$$

hence true for $n=1$ and $n=k$ true $\Rightarrow n=k+1$ true **R1**

therefore true for all $n \in \mathbb{Z}^+$

Note: Only award the final **R1** if the first five marks have been awarded. Award marks as appropriate for solutions that expand both the LHS and (given) RHS of the equation.

[8 marks]

Total [27 marks]

2. (a) (i) $4-i$

A1
[1 mark]

$$(ii) \text{ mean} = \frac{1}{2}(4+i+4-i) \\ = 4$$

A1
AG
[1 mark]

(b) **METHOD 1**

attempts product rule differentiation

(M1)

Note: Award **(M1)** for attempting to express $f(x)$ as $f(x) = x^3 - 9x^2 + 25x - 17$.

$$f'(x) = (x-1)(2x-8) + x^2 - 8x + 17 \quad (f'(x) = 3x^2 - 18x + 25)$$

A1

$$f'(4) = 1$$

A1

Note: Where $f'(x)$ is correct, award **A1** for solving $f'(x) = 1$ and obtaining $x = 4$.

EITHER

$$y - 3 = 1(x - 4)$$

A1

OR

$$y = x + c$$

$$3 = 4 + c \Rightarrow c = -1$$

A1

OR

states the gradient of $y = x - 1$ is also 1 and verifies that (4, 3) lies on the line $y = x - 1$

A1

THEN

so $y = x - 1$ is the tangent to the curve at A(4, 3)

AG

Note: Award a maximum of **(M0)A0A1A1** to a candidate who does not attempt to find $f'(x)$.

continued...

Question 2 continued

METHOD 2

sets $f(x) = x - 1$ to form $x - 1 = (x - 1)(x^2 - 8x + 17)$ **(M1)**

EITHER

$(x - 1)(x^2 - 8x + 16) = 0$ ($x^3 - 9x^2 + 24x - 16 = 0$) **A1**

attempts to solve a correct cubic equation **(M1)**

$$(x - 1)(x - 4)^2 = 0 \Rightarrow x = 1, 4$$

OR

recognises that $x \neq 1$ and forms $x^2 - 8x + 17 = 1$ ($x^2 - 8x + 16 = 0$) **A1**

attempts to solve a correct quadratic equation **(M1)**

$$(x - 4)^2 = 0 \Rightarrow x = 4$$

THEN

$x = 4$ is a double root **R1**

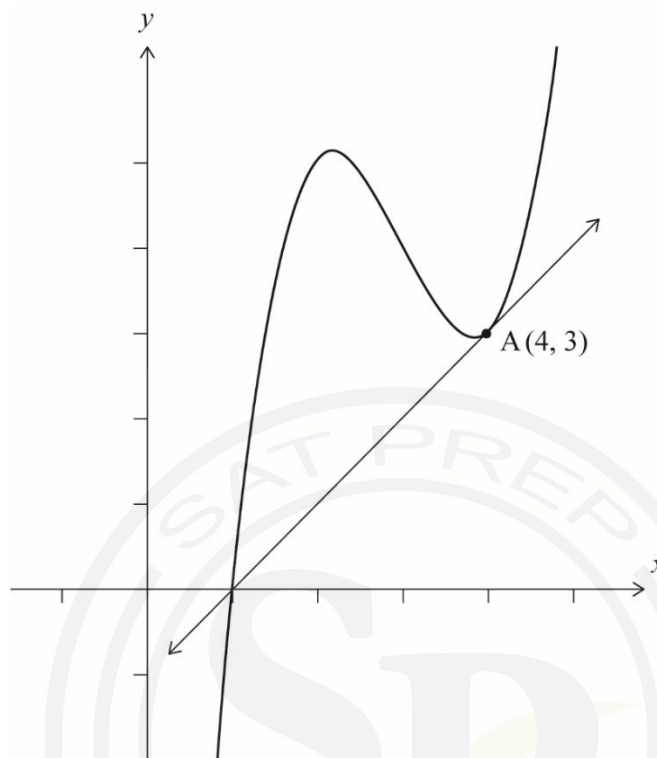
so $y = x - 1$ is the tangent to the curve at $A(4, 3)$ **AG**

Note: Candidates using this method are not required to verify that $y = 3$.

[4 marks]
continued...

Question 2 continued

(c)



a positive cubic with an x -intercept ($x=1$), and a local maximum and local minimum in the first quadrant both positioned to the left of A

A1

Note: As the local minimum and point A are very close to each other, condone graphs that seem to show these points coinciding.
For the point of tangency, accept labels such as A, (4,3) or the point labelled from both axes. Coordinates are not required.

a correct sketch of the tangent passing through A and crossing the x -axis at the same point ($x=1$) as the curve

A1

Note: Award **A1A0** if both graphs cross the x -axis at distinctly different points.

[2 marks]
continued...

Question 2 continued

(d) (i) **EITHER**

$$g'(x) = (x-r)(2x-2a) + x^2 - 2ax + a^2 + b^2$$

(M1)A1

OR

$$g(x) = x^3 - (2a+r)x^2 + (a^2 + b^2 + 2ar)x - (a^2 + b^2)r$$

attempts to find $g'(x)$

M1

$$g'(x) = 3x^2 - 2(2a+r)x + a^2 + b^2 + 2ar$$

$$= 2x^2 - 2(a+r)x + 2ar + x^2 - 2ax + a^2 + b^2$$

A1

$$\left(= 2(x^2 - ax - rx + ar) + x^2 - 2ax + a^2 + b^2 \right)$$

THEN

$$g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$$

AG

[2 marks]

continued...

Question 2 continued

(ii) **METHOD 1**

$$g(a) = b^2(a - r) \quad (A1)$$

$$g'(a) = b^2 \quad (A1)$$

attempts to substitute their $g(a)$ and $g'(a)$ into $y - g(a) = g'(a)(x - a)$ **M1**

$$y - b^2(a - r) = b^2(x - a)$$

EITHER

$$y = b^2(x - r) \quad (y = b^2x - b^2r) \quad A1$$

sets $y = 0$ so $b^2(x - r) = 0$ **M1**

$$b > 0 \Rightarrow x = r \text{ OR } b \neq 0 \Rightarrow x = r \quad R1$$

OR

sets $y = 0$ so $-b^2(a - r) = b^2(x - a)$ **M1**

$$b > 0 \text{ OR } b \neq 0 \Rightarrow -(a - r) = x - a \quad R1$$

$$x = r \quad A1$$

THEN

so the tangent intersects the x -axis at the point $R(r, 0)$ **AG**

continued...

Question 2 continued

METHOD 2

$$g'(a) = b^2 \quad (A1)$$

$$g(a) = b^2(a - r) \quad (A1)$$

attempts to substitute their $g(a)$ and $g'(a)$ into $y = g'(a)x + c$ and attempts to find c
M1

$$c = -b^2r$$

EITHER

$$y = b^2(x - r) \quad (y = b^2x - b^2r) \quad A1$$

$$\text{sets } y = 0 \text{ so } b^2(x - r) = 0 \quad M1$$

$$b > 0 \Rightarrow x = r \text{ OR } b \neq 0 \Rightarrow x = r \quad R1$$

OR

$$\text{sets } y = 0 \text{ so } b^2(x - r) = 0 \quad M1$$

$$b > 0 \text{ OR } b \neq 0 \Rightarrow x - r = 0 \quad R1$$

$$x = r \quad A1$$

METHOD 3

$$g'(a) = b^2 \quad (A1)$$

the line through $R(r, 0)$ parallel to the tangent at A has equation

$$y = b^2(x - r) \quad A1$$

$$\text{sets } g(x) = b^2(x - r) \text{ to form } b^2(x - r) = (x - r)(x^2 - 2ax + a^2 + b^2) \quad M1$$

$$b^2 = x^2 - 2ax + a^2 + b^2, (x \neq r) \quad A1$$

$$(x - a)^2 = 0 \quad A1$$

since there is a double root ($x = a$), this parallel line through

$R(r, 0)$ is the required tangent at A **R1**

[6 marks]

continued...

Question 2 continued

(e) **EITHER**

$$g'(a) = b^2 \Rightarrow b = \sqrt{g'(a)} \text{ (since } b > 0)$$

R1

Note: Accept $b = \pm\sqrt{g'(a)}$.

OR

$$(a \pm bi) = a \pm i\sqrt{b^2} \text{ and } g'(a) = b^2$$

R1

THEN

hence the complex roots can be expressed as $a \pm i\sqrt{g'(a)}$

AG

[1 mark]

(f) (i) $b = 4$ (seen anywhere)

A1

EITHER

attempts to find the gradient of the tangent in terms of a and equates to 16

(M1)

OR

substitutes $r = -2, x = a$ and $y = 80$ to form $80 = (a - (-2))(a^2 - 2a^2 + a^2 + 16)$

(M1)

OR

substitutes $r = -2, x = a$ and $y = 80$ into $y = 16(x - r)$

(M1)

THEN

$$\frac{80}{a+2} = 16 \Rightarrow a = 3$$

roots are -2 (seen anywhere) and $3 \pm 4i$

A1A1

Note: Award **A1** for -2 and **A1** for $3 \pm 4i$. Do not accept coordinates.

[4 marks]

(ii) $(3, -4)$

A1

Note: Accept “ $x = 3$ and $y = -4$ ”.

Do not award **A1FT** for $(a, -4)$.

[1 mark]

continued...

Question 2 continued

(g) (i) $g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$

attempts to find $g''(x)$

M1

$$g''(x) = 2(x-a) + 2(x-r) + 2x - 2a \quad (= 6x - 2r - 4a)$$

sets $g''(x) = 0$ and correctly solves for x

A1

for example, obtaining $x - r + 2(x - a) = 0$ leading to $3x = 2a + r$

$$\text{so } x = \frac{1}{3}(2a + r)$$

AG

Note: Do not award **A1** if the answer does not lead to the **AG**.

[2 marks]

(ii) point P is $\frac{2}{3}$ of the horizontal distance (way) from point R to point A

A1

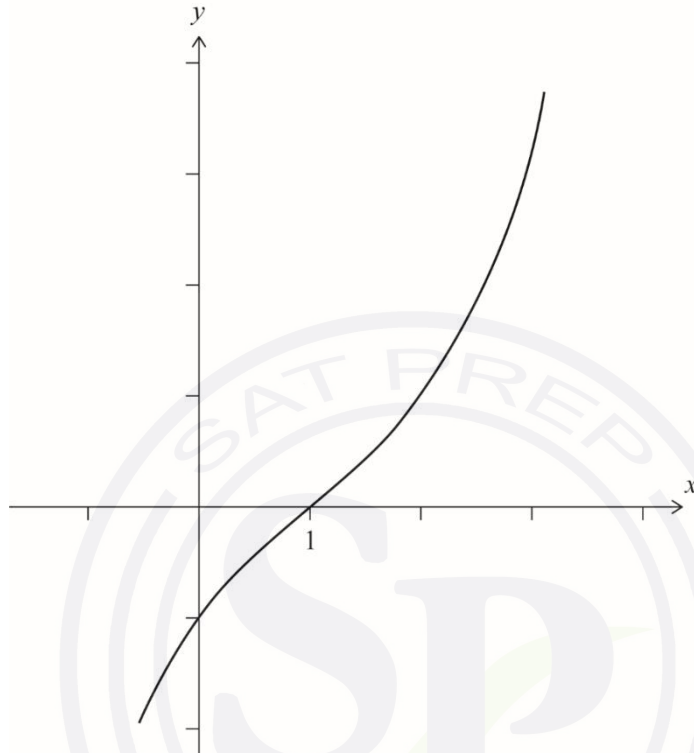
Note: Accept equivalent numerical statements or a clearly labelled diagram displaying the numerical relationship.
Award **A0** for non-numerical statements such as “P is between R and A, closer to A”.

[1 mark]

continued...

Question 2 continued

(h) (i) $y = (x-1)(x^2 - 2x + 5)$ **(A1)**



a positive cubic with no stationary points and a non-stationary point of inflexion at $x = 1$

A1

Note: Graphs may appear approximately linear. Award this **A1** if a change of concavity either side of $x = 1$ is apparent.
Coordinates are not required and the y -intercept need not be indicated.

[2 marks]

(ii) $(r, 0)$

A1

[1 mark]

Total [28 marks]

Markscheme

November 2021

Mathematics: analysis and approaches

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- R** Marks awarded for clear **Reasoning**.
- AG** Answer given in the question and so no marks are awarded.
- FT** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This

includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a “show that” question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is ‘Hence’ and not ‘Hence or otherwise’ then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed, and

any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or

written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

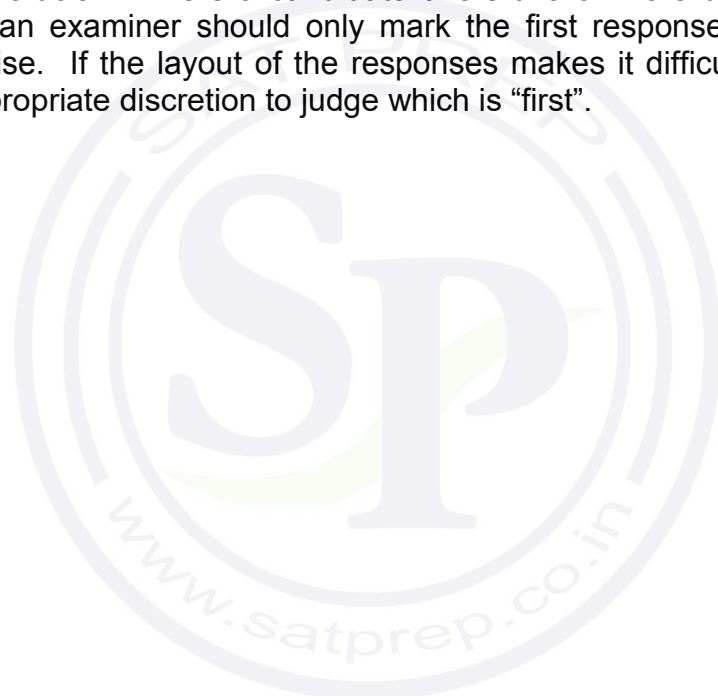
9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.



1. (a) $f'(t) = \frac{e^t - e^{-t}}{2}$ **A1**

$$f''(t) = \frac{e^t + e^{-t}}{2}$$
A1

$$= f(t)$$
AG

[2 marks]

(b) **METHOD 1**

$$(f(t))^2 + (g(t))^2$$

substituting f and g

M1

$$= \frac{(e^t + e^{-t})^2 + (e^t - e^{-t})^2}{4}$$

$$= \frac{(e^t)^2 + 2 + (e^{-t})^2 + (e^t)^2 - 2 + (e^{-t})^2}{4}$$

(M1)

$$= \frac{(e^t)^2 + (e^{-t})^2}{2} \left(= \frac{e^{2t} + e^{-2t}}{2} \right)$$

A1

$$= f(2t)$$

AG

METHOD 2

$$f(2t) = \frac{e^{2t} + e^{-2t}}{2}$$

$$= \frac{(e^t)^2 + (e^{-t})^2}{2}$$

M1

$$= \frac{(e^t + e^{-t})^2 + (e^t - e^{-t})^2}{4}$$

M1A1

$$= (f(t))^2 + (g(t))^2$$

AG

[3 marks]

Note: Accept combinations of METHODS 1 & 2 that meet at equivalent expressions.

- (c) (i) substituting $e^{iu} = \cos u + i \sin u$ into the expression for f (M1)
 obtaining $e^{-iu} = \cos u - i \sin u$ (A1)

$$f(iu) = \frac{\cos u + i \sin u + \cos u - i \sin u}{2}$$

Note: The **M1** can be awarded for the use of sine and cosine being odd and even respectively.

$$= \frac{2 \cos u}{2}$$

$$= \cos u$$

A1
[3 marks]

(ii)
$$g(iu) = \frac{\cos u + i \sin u - \cos u + i \sin u}{2}$$

substituting and attempt to simplify

(M1)

$$= \frac{2i \sin u}{2}$$

$$= i \sin u$$

A1
[2 marks]

(d) **METHOD 1**

$$(f(iu))^2 + (g(iu))^2$$

substituting expressions found in part (c)

(M1)

$$= \cos^2 u - \sin^2 u (= \cos 2u)$$

A1

METHOD 2

$$f(2iu) = \frac{e^{2iu} + e^{-2iu}}{2}$$

$$= \frac{\cos 2u + i \sin 2u + \cos 2u - i \sin 2u}{2}$$

M1

$$= \cos 2u$$

A1

Note: Accept equivalent final answers that have been simplified removing all imaginary parts eg $2 \cos^2 u - 1$ etc

[2 marks]

(e) $(f(t))^2 - (g(t))^2 = \frac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{4}$ **M1**

$= \frac{(e^{2t} + e^{-2t} + 2) - (e^{2t} + e^{-2t} - 2)}{4}$ **A1**

$= \frac{4}{4} = 1$ **A1**

Note: Award **A1** for a value of 1 obtained from either LHS or RHS of given expression.

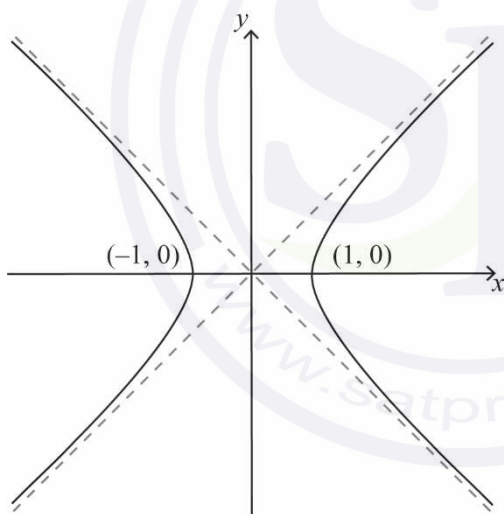
$(f(iu))^2 - (g(iu))^2 = \cos^2 u + \sin^2 u$ **M1**

$= 1$ (hence $(f(t))^2 - (g(t))^2 = (f(iu))^2 - (g(iu))^2$) **AG**

Note: Award full marks for showing that $(f(z))^2 - (g(z))^2 = 1, \forall z \in \mathbb{C}$.

[4 marks]

(f)



A1A1A1A1

Note: Award **A1** for correct curves in the upper quadrants, **A1** for correct curves in the lower quadrants, **A1** for correct x -intercepts of $(-1, 0)$ and $(1, 0)$ (condone $x = -1$ and 1), **A1** for $y = x$ and $y = -x$.

[4 marks]

- (g) attempt to rotate by 45° in either direction

(M1)

Note: Evidence of an attempt to relate to a sketch of $xy = k$ would be sufficient for this **(M1)**.

attempting to rotate a particular point, eg $(1, 0)$

(M1)

$(1, 0)$ rotates to $\left(\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$ (or similar)

(A1)

hence $k = \pm \frac{1}{2}$

A1A1

[5 marks]

Total [25 marks]



2. (a) (i) **METHOD 1**

$$\frac{dy}{dt} = y$$

$$\int \frac{dy}{y} = \int dt$$

(M1)

$$\ln y = t + c \quad \text{OR} \quad \ln|y| = t + c$$

A1A1

Note: Award **A1** for $\ln y$ and **A1** for t and c .

$$y = Ae^t$$

AG

METHOD 2

rearranging to $\frac{dy}{dt} - y = 0$ AND multiplying by integrating factor e^{-t}

M1

$$ye^{-t} = A$$

A1A1

$$y = Ae^t$$

AG

[3 marks]

(ii) substituting $y = Ae^t$ into differential equation in x

M1

$$\frac{dx}{dt} = x - Ae^t$$

$$\frac{dx}{dt} - x = -Ae^t$$

AG

[1 mark]

(iii) integrating factor (IF) is $e^{\int -1 dt}$
 $= e^{-t}$

(M1)

(A1)

$$e^{-t} \frac{dx}{dt} - xe^{-t} = -A$$

$$xe^{-t} = -At + D$$

(A1)

$$x = (-At + D)e^t$$

A1

Note: The first constant must be A , and the second can be any constant for the final **A1** to be awarded. Accept a change of constant applied at the end.

[4 marks]

(b) (i) $\frac{d^2y}{dt^2} = -\frac{dx}{dt} + \frac{dy}{dt}$ **A1**

EITHER

$= -x + y + \frac{dy}{dt}$ **(M1)**

$= \frac{dy}{dt} + \frac{dy}{dt}$ **A1**

OR

$= -x + y + (-x + y)$ **(M1)**

$= 2(-x + y)$ **A1**

THEN

$= 2\frac{dy}{dt}$ **AG**

[3 marks]

(ii) $\frac{dY}{dt} = 2Y$ **A1**

$\int \frac{dY}{Y} = \int 2dt$ **M1**

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ **A1**

$Y = Be^{2t}$ **AG**

[3 marks]

(iii) $\frac{dy}{dt} = Be^{2t}$

$y = \int Be^{2t} dt$ **M1**

$y = \frac{B}{2}e^{2t} + C$ **A1**

Note: The first constant must be B , and the second can be any constant for the final **A1** to be awarded. Accept a change of constant applied at the end.

[2 marks]

(iv) **METHOD 1**

substituting $\frac{dy}{dt} = B e^{2t}$ and their (iii) into $\frac{dy}{dt} = -x + y$ **M1(M1)**

$$B e^{2t} = -x + \frac{B}{2} e^{2t} + C$$
A1

$$x = -\frac{B}{2} e^{2t} + C$$
AG

Note: Follow through from incorrect part (iii) cannot be awarded if it does not lead to the **AG**.

METHOD 2

$$\frac{dx}{dt} = x - \frac{B}{2} e^{2t} - C$$

$$\frac{dx}{dt} - x = -\frac{B}{2} e^{2t} - C$$

$$\frac{d(xe^{-t})}{dt} = -\frac{B}{2} e^t - C e^{-t}$$
M1

$$xe^{-t} = \int -\frac{B}{2} e^t - C e^{-t} dt$$

$$xe^{-t} = -\frac{B}{2} e^t + C e^{-t} + D$$
A1

$$x = -\frac{B}{2} e^{2t} + C + D e^t$$

$$\frac{dy}{dt} = -x + y \Rightarrow B e^{2t} = \frac{B}{2} e^{2t} - C - D e^t + \frac{B}{2} e^{2t} + C \Rightarrow D = 0$$
M1

$$x = -\frac{B}{2} e^{2t} + C$$
AG

[3 marks]

(c) (i) $\frac{dy}{dt} = -4x + y$
 $\frac{d^2y}{dt^2} = -4\frac{dx}{dt} + \frac{dy}{dt}$ seen anywhere **M1**

METHOD 1

$\frac{d^2y}{dt^2} = -4(x - y) + \frac{dy}{dt}$
 attempt to eliminate x **M1**

$= -4\left(\frac{1}{4}\left(y - \frac{dy}{dt}\right) - y\right) + \frac{dy}{dt}$
 $= 2\frac{dy}{dt} + 3y$ **A1**

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$ **AG**

METHOD 2

rewriting LHS in terms of x and y **M1**

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ **A1**
 $= 0$ **AG**

[3 marks]

(ii) $\frac{dy}{dt} = F\lambda e^{\lambda t}, \frac{d^2y}{dt^2} = F\lambda^2 e^{\lambda t}$ **(A1)**

$F\lambda^2 e^{\lambda t} - 2F\lambda e^{\lambda t} - 3Fe^{\lambda t} = 0$ **(M1)**

$\lambda^2 - 2\lambda - 3 = 0$ (since $e^{\lambda t} \neq 0$) **A1**

λ_1 and λ_2 are 3 and -1 (either order) **A1**

[4 marks]

(iii)

METHOD 1

$$y = Fe^{3t} + Ge^{-t}$$

$$\frac{dy}{dt} = 3Fe^{3t} - Ge^{-t}, \quad \frac{d^2y}{dt^2} = 9Fe^{3t} + Ge^{-t} \quad (A1)(A1)$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 9Fe^{3t} + Ge^{-t} - 2(3Fe^{3t} - Ge^{-t}) - 3(Fe^{3t} + Ge^{-t}) \quad M1$$

$$= 9Fe^{3t} + Ge^{-t} - 6Fe^{3t} + 2Ge^{-t} - 3Fe^{3t} - 3Ge^{-t} \quad A1$$

$$= 0 \quad AG$$

METHOD 2

$$y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$$

$$\frac{dy}{dt} = F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}, \quad \frac{d^2y}{dt^2} = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t} \quad (A1)(A1)$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t} - 2(F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}) - 3(Fe^{\lambda_1 t} + Ge^{\lambda_2 t}) \quad M1$$

$$= Fe^{\lambda_1 t}(\lambda_1^2 - 2\lambda_1 - 3) + Ge^{\lambda_2 t}(\lambda_2^2 - 2\lambda_2 - 3) \quad A1$$

$$= 0 \quad AG$$

[4 marks]

Total [30 marks]

Markscheme

May 2021

**Mathematics:
analysis and approaches**

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part. Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed,

and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required

(although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

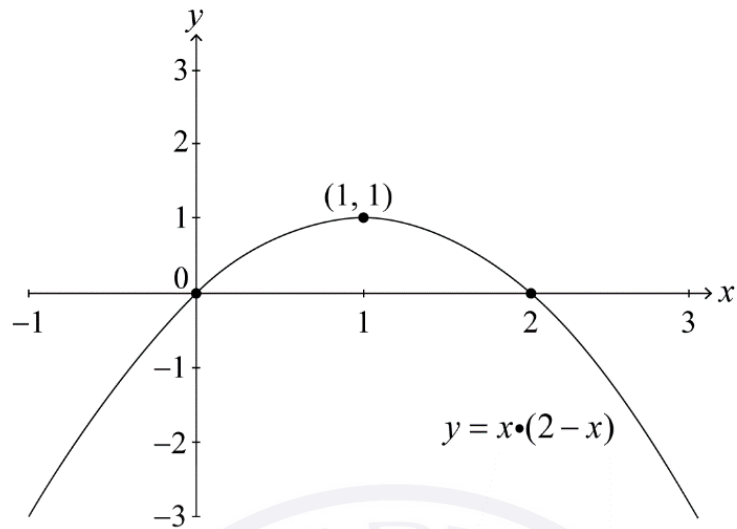
10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.



1. (a)



inverted parabola extended below the x -axis

A1

x -axis intercept values $x = 0, 2$

A1

Note: Accept a graph passing through the origin as an indication of $x = 0$.

local maximum at $(1, 1)$

A1

Note: Coordinates must be stated to gain the final **A1**.

Do not accept decimal approximations.

[3 marks]

(b)

	Number of local maximum points	Number of local minimum points	Number of points of inflexion with zero gradient
$n = 3$ and $n = 5$	1	0	2
$n = 2$ and $n = 4$	1	2	0

A1A1A1A1A1A1

Note: Award **A1** for each correct value.

For a table not sufficiently or clearly labelled, assume that their values are in the same order as the table in the question paper and award marks accordingly.

[6 marks]

(c) **METHOD 1**

attempts to use the product rule

(M1)

$$f_n'(x) = -nx^n(a-x)^{n-1} + nx^{n-1}(a-x)^n$$

A1A1

Note: Award **A1** for a correct $u \frac{dv}{dx}$ and **A1** for a correct $v \frac{du}{dx}$.

EITHER

attempts to factorise $f_n'(x)$ (involving at least one of nx^{n-1} or $(a-x)^{n-1}$)

(M1)

$$= nx^{n-1}(a-x)^{n-1}((a-x)-x)$$

A1

OR

attempts to express $f_n'(x)$ as the difference of two products with each product containing at least one of nx^{n-1} or $(a-x)^{n-1}$

(M1)

$$= (-x)(nx^{n-1})(a-x)^{n-1} + (a-x)(nx^{n-1})(a-x)^{n-1}$$

A1

THEN

$$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$$

AG

Note: Award the final **(M1)A1** for obtaining any of the following forms:

$$f_n'(x) = nx^n(a-x)^n \left(\frac{a-x-x}{x(a-x)} \right); \quad f_n'(x) = \frac{nx^n(a-x)^n}{x(a-x)}(a-x-x);$$

$$f_n'(x) = nx^{n-1}((a-x)^n - x(a-x)^{n-1});$$

$$f_n'(x) = (a-x)^{n-1}(nx^{n-1}(a-x)^n - nx^n)$$

METHOD 2

$$f_n(x) = (x(a-x))^n$$

(M1)

$$= (ax - x^2)^n$$

A1

attempts to use the chain rule

(M1)

$$f_n'(x) = n(a-2x)(ax-x^2)^{n-1}$$

A1A1

Note: Award **A1** for $n(a-2x)$ and **A1** for $(ax-x^2)^{n-1}$.

$$f'_n(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$$

AG

[5 marks]

(d) $x=0, x=\frac{a}{2}, x=a$

A2

Note: Award **A1** for either two correct solutions or for obtaining

$$x=0, x=-a, x=-\frac{a}{2}.$$

Award **A0** otherwise.

[2 marks]

(e) attempts to find an expression for $f_n\left(\frac{a}{2}\right)$

(M1)

$$f_n\left(\frac{a}{2}\right) = \left(\frac{a}{2}\right)^n \left(a - \frac{a}{2}\right)^n$$

$$= \left(\frac{a}{2}\right)^n \left(\frac{a}{2}\right)^n \left(= \left(\frac{a}{2}\right)^{2n} \right), \left(= \left(\left(\frac{a}{2}\right)^n\right)^2 \right)$$

A1

EITHER

since $a \in \mathbb{R}^+, \left(\frac{a}{2}\right)^{2n} > 0$ (for $n \in \mathbb{Z}^+, n > 1$ and so $f_n\left(\frac{a}{2}\right) > 0$)

R1

Note: Accept any logically equivalent conditions/statements on a and n .

Award **R0** if any conditions/statements specified involving a, n or both are incorrect.

OR

(since $a \in \mathbb{R}^+$), $\frac{a}{2}$ raised to an even power ($2n$) (or equivalent reasoning) is always

positive (and so $f_n\left(\frac{a}{2}\right) > 0$)

R1

Note: The condition $a \in \mathbb{R}^+$ is given in the question. Hence some candidates will assume $a \in \mathbb{R}^+$ and not state it. In these instances, award **R1** for a convincing argument.

Accept any logically equivalent conditions/statements on a and n .

Award **R0** if any conditions/statements specified involving a, n or both are incorrect.

THEN

so $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ is always above the horizontal axis

AG

Note: Do not award **(M1)A0R1**.

[3 marks]

(f) **METHOD 1**

$$f'_n\left(\frac{a}{4}\right) = n\left(\frac{a}{4}\right)^{n-1}\left(a - \frac{a}{2}\right)\left(a - \frac{a}{4}\right)^{n-1} \left(= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1} \right)$$

A1

EITHER

$$n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1} > 0 \text{ as } a \in \mathbb{R}^+ \text{ and } n \in \mathbb{Z}^+$$

R1

OR

$$n\left(\frac{a}{4}\right)^{n-1}, \left(a - \frac{a}{2}\right) \text{ and } \left(a - \frac{a}{4}\right)^{n-1} \text{ are all } > 0$$

R1

Note: Do not award **A0R1**.

Accept equivalent reasoning on correct alternative expressions for $f'_n\left(\frac{a}{4}\right)$ and accept any logically equivalent conditions/statements on a and n .

Exceptions to the above are condone $n > 1$ and condone $n > 0$.

An alternative form for $f'_n\left(\frac{a}{4}\right)$ is $(2n)(3)^{n-1}\left(\frac{a}{4}\right)^{2n-1}$.

THEN

$$\text{hence } f'_n\left(\frac{a}{4}\right) > 0$$

AG

[2 marks]

METHOD 2

$$f_n(0) = 0 \text{ and } f'_n\left(\frac{a}{2}\right) > 0$$

A1

(since f_n is continuous and there are no stationary points between $x=0$ and $x=\frac{a}{2}$)

the gradient (of the curve) must be positive between $x=0$ and $x=\frac{a}{2}$

R1

Note: Do not award **A0R1**.

hence $f_n'\left(\frac{a}{4}\right) > 0$

AG

[2 marks]

(g) (i) $f_n'(-1) = n(-1)^{n-1}(a+2)(a+1)^{n-1}$

for n even:

$$n(-1)^{n-1} (= -n) < 0 \text{ (and } (a+2), (a+1)^{n-1} \text{ are both } > 0)$$

R1

$$f_n'(-1) < 0$$

A1

$$f_n'(0) = 0 \text{ and } f_n'\left(\frac{a}{4}\right) > 0 \text{ (seen anywhere)}$$

A1

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the **R** mark.

For example, award **R1** for the following:

If n is even, then $n-1$ is odd and hence $(-1)^{n-1} < 0$ ($= -1$).

Do not award **R0A1**.

The second **A1** is independent of the other two marks.

The **A** marks can be awarded for correct descriptions expressed in words.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or

show, state or explain (words or diagram) that $f_n'(0) = 0$. The last **A** mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last **A1** can be awarded for use of a specific case (e.g. $n=2$).

hence $(0,0)$ is a local minimum point

AG

[3 marks]

(ii) for n odd:

$$n(-1)^{n-1} (=n) > 0, \text{ (and } (a+2), (a+1)^{n-1} \text{ are both } > 0 \text{) so } f'_n(-1) > 0 \quad \mathbf{R1}$$

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the **R** mark.

For example, award **R1** for the following:

If n is odd, then $n-1$ is even and hence $(-1)^{n-1} > 0$ ($=1$).

$$f'_n(0) = 0 \text{ and } f'_n\left(\frac{a}{4}\right) > 0 \text{ (seen anywhere)} \quad \mathbf{A1}$$

Note: The **A1** is independent of the **R1**.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f'_n(0) = 0$. The last A mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last A1 can be awarded for use of a specific case (e.g. $n = 3$).

hence $(0,0)$ is a point of inflexion with zero gradient **AG**

[2 marks]

(h) considers the parity of n

(M1)

Note: Award **M1** for stating at least one specific even value of n .

n must be even (for four solutions)

A1

Note: The above 2 marks are independent of the 3 marks below.

$$0 < k < \left(\frac{a}{2}\right)^{2n}$$

A1A1A1

Note: Award **A1** for the correct lower endpoint, **A1** for the correct upper endpoint and **A1** for strict inequality signs.

The third **A1** (strict inequality signs) can only be awarded if **A1A1** has been awarded.

For example, award **A1A1A0** for $0 \leq k \leq \left(\frac{a}{2}\right)^{2n}$. Award **A1A0A0** for $k > 0$.

Award **A1A0A0** for $0 < k < f_n\left(\frac{a}{2}\right)$.

[5 marks]

Total [31 marks]

2. (a) (i) **METHOD 1**

attempts to expand $(\omega - 1)(\omega^2 + \omega + 1)$ **(M1)**

$$= \omega^3 + \omega^2 + \omega - \omega^2 - \omega - 1$$
A1

$$= \omega^3 - 1$$
AG

[2 marks]

METHOD 2

attempts polynomial division on $\frac{\omega^3 - 1}{\omega - 1}$ **M1**

$$= \omega^2 + \omega + 1$$
A1

$$\text{so } (\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$$
AG

[2 marks]

(ii) (since ω is a root of $z^3 = 1$) $\Rightarrow \omega^3 - 1 = 0$ **R1**

and $\omega \neq 1$ **R1**

$$\Rightarrow \omega^2 + \omega + 1 = 0$$
AG

Note: In part (a), award marks as appropriate where ω has been converted into Cartesian, modulus-argument (polar) or Euler form.

[2 marks]

(b) **METHOD 1**

attempts to find either P_0P_1 or P_0P_2

(M1)

accept any valid method

e.g. $2\sin\frac{\pi}{3}, 1^2+1^2-2\cos\frac{2\pi}{3}, \frac{1}{\sin\frac{\pi}{6}} = \frac{P_0P_1}{\sin\frac{2\pi}{3}}$ from either $\triangle OP_0P_1$ or $\triangle OP_0P_2$

e.g. use of Pythagoras' theorem

e.g. $\left|1 - e^{i\frac{2\pi}{3}}\right|, \left|1 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right|$ by calculating the distance between 2 points

$$P_0P_1 = \sqrt{3}$$

A1

$$P_0P_2 = \sqrt{3}$$

A1

Note: Award a maximum of **M1A1A0** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_2 or both.

so $P_0P_1 \times P_0P_2 = 3$

AG

METHOD 2

attempts to find $P_0P_1 \times P_0P_2 = |1 - \omega||1 - \omega^2|$

(M1)

$$P_0P_1 \times P_0P_2 = |\omega^3 - \omega^2 - \omega + 1|$$

A1

$$= |1 - (\omega^2 + \omega + 1) + 2| \text{ and since } \omega^2 + \omega + 1 = 0$$

R1

so $P_0P_1 \times P_0P_2 = 3$

AG

[3 marks]

(c) **METHOD 1**

$$z^4 - 1 = (z - 1)(z^3 + z^2 + z + 1) \quad \text{A1}$$

$$(\omega \text{ is a root hence}) \omega^4 - 1 = 0 \text{ and } \omega \neq 1 \quad \text{R1}$$

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0 \quad \text{AG}$$

Note: Condone the use of ω throughout.

[2 marks]

METHOD 2

$$\text{considers the sum of roots of } z^4 - 1 = 0 \quad \text{(M1)}$$

$$\text{the sum of roots is zero (there is no } z^3 \text{ term)} \quad \text{A1}$$

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0 \quad \text{AG}$$

[2 marks]

METHOD 3

$$\text{substitutes for } \omega \quad \text{(M1)}$$

$$\begin{aligned} \text{e.g. LHS} &= e^{i\frac{3\pi}{2}} + e^{\pi i} + e^{i\frac{\pi}{2}} + 1 \\ &= -i - 1 + i + 1 \quad \text{A1} \end{aligned}$$

Note: This can be demonstrated geometrically or by using vectors.
Accept Cartesian or modulus-argument (polar) form.

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0 \quad \text{AG}$$

[2 marks]

METHOD 4

$$\omega^3 + \omega^2 + \omega + 1 = \frac{\omega^4 - 1}{\omega - 1} \quad \text{A1}$$

$$= \frac{0}{\omega - 1} = 0 \text{ as } \omega \neq 1 \quad \text{R1}$$

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0 \quad \text{AG}$$

[2 marks]

(d) **METHOD 1**

$$P_0P_2 = 2$$

A1

attempts to find either P_0P_1 or P_0P_3

(M1)

Note: For example, $P_0P_1 = |1-i|$ and $P_0P_3 = |1+i|$.

Various geometric and trigonometric approaches can be used by candidates.

$$P_0P_1 = \sqrt{2}, P_0P_3 = \sqrt{2}$$

A1A1

Note: Award a maximum of **A1M1A1A0** if labels such as P_0P_1 are not clearly shown.

Award full marks if the lengths are shown on a clearly labelled diagram.

Award a maximum of **A1M1A1A0** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_3 or both.

$$P_0P_1 \times P_0P_2 \times P_0P_3 = 4$$

AG

[4 marks]

METHOD 2

$$\text{attempts to find } P_0P_1 \times P_0P_2 \times P_0P_3 = |1-\omega||1-\omega^2||1-\omega^3|$$

M1

$$P_0P_1 \times P_0P_2 \times P_0P_3 = |-\omega^6 + \omega^5 + \omega^4 - \omega^2 - \omega + 1|$$

A1

$$= | -(-1) + \omega^5 + 1 - (-1) - \omega + 1 | \text{ since } \omega^6 = \omega^2 = -1 \text{ and } \omega^4 = 1$$

A1

$$= | \omega^5 - \omega + 4 | \text{ and since } \omega^5 = \omega$$

R1

$$\text{so } P_0P_1 \times P_0P_2 \times P_0P_3 = 4$$

AG

[4 marks]

METHOD 3

$$P_0 P_2 = 2$$

A1

$$\text{attempts to find } P_0 P_1 \times P_0 P_3 = |1 - \omega| |1 - \omega^3|$$

M1

$$P_0 P_1 \times P_0 P_3 = |\omega^4 - \omega^3 - \omega + 1|$$

A1

$$= |2 - (-\omega) - \omega| \text{ since } \omega^4 = 1 \text{ and } \omega^3 = -\omega$$

R1

$$\text{so } P_0 P_1 \times P_0 P_2 \times P_0 P_3 = 4$$

AG

[4 marks]



(e) $(P_0P_1 \times P_0P_2 \times \dots \times P_0P_{n-1}) = n$

A1

[1 mark]

(f) (i) $P_0P_2 = |1 - \omega^2|$, $P_0P_3 = |1 - \omega^3|$

A1A1

[2 marks]

(ii) $P_0P_{n-1} = |1 - \omega^{n-1}|$

A1

Note: Accept $|1 - \omega|$ from symmetry.

[1 mark]



(g) (i) $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$

considers the equation $z^{n-1} + z^{n-2} + \dots + z + 1 = 0$ **(M1)**

the roots are $\omega, \omega^2, \dots, \omega^{n-1}$ **(A1)**

so $(z - \omega)(z - \omega^2) \dots (z - \omega^{n-1})$ **A1**

[3 marks]

(ii) **METHOD 1**

substitutes $z = 1$ into $(z - \omega)(z - \omega^2) \dots (z - \omega^{n-1}) \equiv z^{n-1} + z^{n-2} + \dots + z + 1$

M1

$(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1}) = n$ **(A1)**

takes modulus of both sides **M1**

$|(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1})| = |n|$

$|1 - \omega||1 - \omega^2| \dots |1 - \omega^{n-1}| = n$ **A1**

so $P_0 P_1 \times P_0 P_2 \times \dots \times P_0 P_{n-1} = n$ **AG**

Note: Award a maximum of **M1A1FTM1A0** from part (e).

[4 marks]

METHOD 2

$(1-\omega), (1-\omega^2), \dots, (1-\omega^{n-1})$ are the roots of $(1-v)^{n-1} + (1-v)^{n-2} + \dots + (1-v) + 1 = 0$ **M1**

coefficient of v^{n-1} is $(-1)^{n-1}$ and the coefficient of 1 is n **A1**

product of the roots is $\frac{(-1)^{n-1} n}{(-1)^{n-1}} = n$ **A1**

$|1-\omega||1-\omega^2|\dots|1-\omega^{n-1}| = n$ **A1**

so $P_0 P_1 \times P_0 P_2 \times \dots \times P_0 P_{n-1} = n$ **AG**

[4 marks]

Total[24 marks]



Markscheme

May 2021

**Mathematics:
analysis and approaches**

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, e.g. **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **A1** for using the **correct** values.
- Where there are two or more **A** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award **A0A1A1**.
- Where the markscheme specifies **A3**, **M2** etc., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the **AG** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award **FT** marks as appropriate but do not award the final **A1** in the first part. Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	No. Last part in question.	Award A1 for the final mark (condone the incorrect further working)
2.	$\frac{35}{72}$	0.468111... (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award A0 for the final mark (and full FT is available in subsequent parts)

3 Implied marks

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then **FT** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is **(M1)A1**, it is possible to award full marks for *their* correct answer, **without working being seen**. For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any **FT** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these **FT** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread and do not award the first mark, even if this is an **M** mark, but award all others as appropriate.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- **MR** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for parts of questions are indicated by **EITHER . . . OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, **M** marks and intermediate **A** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come “*from the use of 3 sf values*”.

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an **A** mark to be awarded, arithmetic should be completed,

and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written

as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required

(although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left

in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^x$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate **A** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

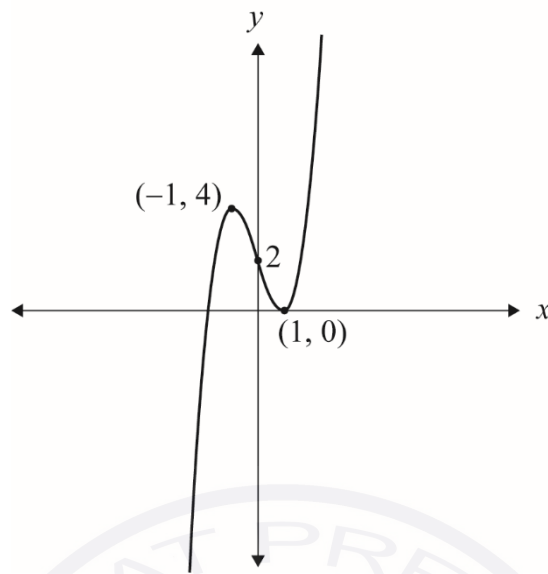
10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.



1. (a) (i)



$c = 1$: positive cubic with correct y-intercept labelled

A1

local maximum point correctly labelled

A1

local minimum point correctly labelled

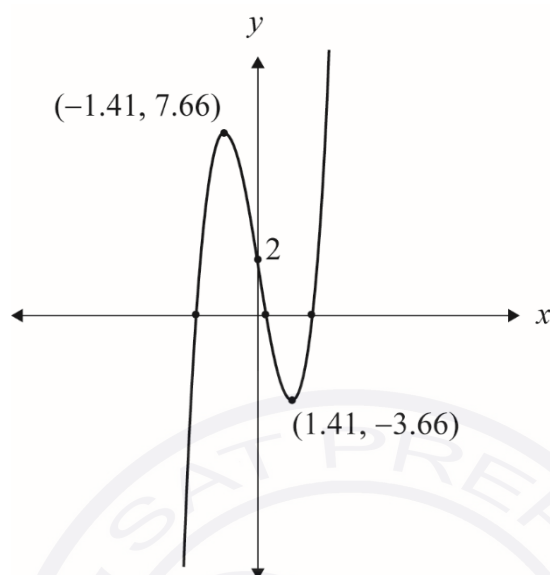
A1

[3 marks]

continued...

Question 1 continued

(ii)



$c = 2$: positive cubic with correct y -intercept labelled

A1

local maximum point correctly labelled

A1

local minimum point correctly labelled

A1

Note: Accept the following exact answers:

Local maximum point coordinates $(-\sqrt{2}, 2 + 4\sqrt{2})$.

Local minimum point coordinates $(\sqrt{2}, 2 - 4\sqrt{2})$.

[3 marks]

continued...

Question 1 continued

(b) $f'(x) = 3x^2 - 3c$

A1

Note: Accept $3x^2 - 3c$ (an expression).

[1 mark]

(c) (i) $c = 0$

A1

[1 mark]

(ii) considers the number of solutions to their $f'(x) = 0$

(M1)

$$3x^2 - 3c = 0$$

$$c > 0$$

A1

[2 marks]

(iii) $c < 0$

A1

Note: The **(M1)** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

[1 mark]

(d) attempts to solve their $f'(x) = 0$ for x **(M1)**

$x = \pm\sqrt{c}$ **(A1)**

Note: Award **(A1)** if either $x = -\sqrt{c}$ or $x = \sqrt{c}$ is subsequently considered.
Award the above **(M1)(A1)** if this work is seen in part (c).

(i) correctly evaluates $f(-\sqrt{c})$ **A1**

$$f(-\sqrt{c}) = -c^{\frac{3}{2}} + 3c^{\frac{3}{2}} + 2 \quad (= -c\sqrt{c} + 3c\sqrt{c} + 2)$$

the y -coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$ **AG**
[3 marks]

(ii) correctly evaluates $f(\sqrt{c})$ **A1**

$$f(\sqrt{c}) = c^{\frac{3}{2}} - 3c^{\frac{3}{2}} + 2 \quad (= c\sqrt{c} - 3c\sqrt{c} + 2)$$

the y -coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$ **AG**
[1 mark]
continued...

Question 1 continued

- (e) (i) the graph of $y = f(x)$ will have one x – axis intercept if

EITHER

$$-2c^{\frac{3}{2}} + 2 > 0 \text{ (or equivalent reasoning)}$$

R1

OR

the minimum point is above the x – axis

R1

Note: Award **R1** for a rigorous approach that does not (only) refer to sketched graphs.

THEN

$$0 < c < 1$$

A1

Note: Condone $c < 1$. The **A1** is independent of the **R1**.

[2 marks]

- (ii) the graph of $y = f(x)$ will have two x – axis intercepts if

EITHER

$$-2c^{\frac{3}{2}} + 2 = 0 \text{ (or equivalent reasoning)}$$

(M1)

OR

evidence from the graph in part(a)(i)

(M1)

THEN

$$c = 1$$

A1

[2 marks]

continued...

Question 1 continued

- (iii) the graph of $y = f(x)$ will have three x – axis intercepts if

EITHER

$$-2c^{\frac{3}{2}} + 2 < 0 \text{ (or equivalent reasoning)}$$

(M1)

OR

reasoning from the results in both parts (e)(i) and (e)(ii)

(M1)

THEN

$$c > 1$$

A1

[2 marks]

continued...



Question 1 continued

(f) case 1:

$c \leq 0$ (independent of the value of d)

A1

EITHER

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution)

R1

OR

$\Rightarrow g'(x) \geq 0$ for $x \in \sim$

R1

OR

the graph of $y = f(x)$ has no local maximum or local minimum points,
hence any vertical translation of this graph ($y = g(x)$) will also have
no local maximum or local minimum points

R1

THEN

therefore there is only one x -axis intercept

AG

Note: Award at most **A0R1** if only $c < 0$ is considered.

continued...

Question 1 continued

case 2

$$c > 0$$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a
local minimum point

(A1)

Note: Award (A1) for a correct y -coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point (M1)

EITHER

considers both points above the x -axis or both points below the x -axis

OR

considers either the local minimum point only above the x -axis OR
the local maximum point only below the x -axis

THEN

$$d > 2c^{\frac{3}{2}} \text{ (both points above the } x\text{-axis)}$$

A1

$$d < -2c^{\frac{3}{2}} \text{ (both points below the } x\text{-axis)}$$

A1

Note: Award at most (A1)(M1)A0A0 for case 2 if $c > 0$ is not clearly stated.

[6 marks]

Total [27 marks]

2. (a) $A = s^2$ and $P = 4s$ (A1)

$A = P \Rightarrow s^2 = 4s$ (M1)

$s(s - 4) = 0$

$\Rightarrow s = 4 (s > 0)$ A1

Note: Award **A1M1A0** if both $s = 4$ and $s = 0$ are stated as final answers.

[3 marks]

(b) $A_T = \frac{1}{2} x^2 \sin \frac{2\pi}{n}$ A1

Note: Award **A1** for a correct alternative form expressed in terms of x and n only.

For example, using Pythagoras' theorem, $A_T = x \sin \frac{\pi}{n} \sqrt{x^2 - x^2 \sin^2 \frac{\pi}{n}}$ or

$A_T = 2 \left(\frac{1}{2} \left(x \sin \frac{\pi}{n} \right) \left(x \cos \frac{\pi}{n} \right) \right)$ or $A_T = x^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$.

[1 mark]

continued...

Question 2 continued

(c) **METHOD 1**

uses $\sin \theta = \frac{\text{opp}}{\text{hyp}}$ **(M1)**

$$\frac{\frac{y}{2}}{x} = \sin \frac{\pi}{n} \quad \text{A1}$$

$$y = 2x \sin \frac{\pi}{n} \quad \text{AG}$$

[2 marks]

METHOD 2

uses Pythagoras' theorem $\left(\frac{y}{2}\right)^2 + h^2 = x^2$ and $h = x \cos \frac{\pi}{n}$ **(M1)**

$$\left(\frac{y}{2}\right)^2 + \left(x \cos \frac{\pi}{n}\right)^2 = x^2 \quad \left(y^2 = 4x^2 \left(1 - \cos^2 \frac{\pi}{n}\right)\right)$$

$$= 4x^2 \sin^2 \frac{\pi}{n} \quad \text{A1}$$

$$y = 2x \sin \frac{\pi}{n} \quad \text{AG}$$

[2 marks]

continued...

Question 2 continued

METHOD 3

uses the cosine rule

(M1)

$$y^2 = 2x^2 - 2x^2 \cos \frac{2\pi}{n} \left(= 2x^2 \left(1 - \cos \frac{2\pi}{n} \right) \right)$$

$$= 4x^2 \sin^2 \frac{\pi}{n}$$

A1

$$y = 2x \sin \frac{\pi}{n}$$

AG

[2 marks]

METHOD 4

uses the sine rule

(M1)

$$\frac{y}{\sin \frac{2\pi}{n}} = \frac{x}{\sin \left(\frac{\pi}{2} - \frac{\pi}{n} \right)}$$

$$y \cos \frac{\pi}{n} = 2x \sin \frac{\pi}{n} \cos \frac{\pi}{n}$$

A1

$$y = 2x \sin \frac{\pi}{n}$$

AG

[2 marks]

continued...

Question 2 continued

(d) $A = P \Rightarrow nA_T = ny$ (M1)

Note: Award **M1** for equating correct expressions for A and P .

$$\frac{1}{2}nx^2 \sin \frac{2\pi}{n} = 2nx \sin \frac{\pi}{n} \left(nx^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = 2nx \sin \frac{\pi}{n} \right)$$

$$\frac{1}{2}x^2 \sin \frac{2\pi}{n} = 2x \sin \frac{\pi}{n} \left(x^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = 2x \sin \frac{\pi}{n} \right)$$
A1

uses $\sin \frac{2\pi}{n} = 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$ (seen anywhere in part (d) or in part (b)) (M1)

$$x^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = 2x \sin \frac{\pi}{n}$$

attempts to either factorise or divide their expression (M1)

$$x \sin \frac{\pi}{n} \left(x \cos \frac{\pi}{n} - 2 \right) = 0$$

$$x = \frac{2}{\cos \frac{\pi}{n}}, \left(x \sin \frac{\pi}{n} \neq 0 \right) \text{ (or equivalent)}$$
A1

continued...

Question 2 continued

EITHER

substitutes $x = \frac{2}{\cos \frac{\pi}{n}}$ (or equivalent) into $P = ny$ **(M1)**

$$P = 2n \left(\frac{2}{\cos \frac{\pi}{n}} \right) \left(\sin \frac{\pi}{n} \right) \quad \text{A1}$$

Note: Other approaches are possible. For example, award **A1** for $P = 2nx \cos \frac{\pi}{n} \tan \frac{\pi}{n}$ and **M1** for substituting $x = \frac{2}{\cos \frac{\pi}{n}}$ into P .

OR

substitutes $x = \frac{2}{\cos \frac{\pi}{n}}$ (or equivalent) into $A = nA_T$ **(M1)**

$$A = \frac{1}{2}n \left(\frac{2}{\cos \frac{\pi}{n}} \right)^2 \left(\sin \frac{2\pi}{n} \right)$$

$$A = \frac{1}{2}n \left(\frac{2}{\cos \frac{\pi}{n}} \right)^2 \left(2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) \quad \text{A1}$$

THEN

$$A = P = 4n \tan \frac{\pi}{n} \quad \text{AG}$$

[7 marks]

continued...

Question 2 continued

- (e) (i) attempts to use the Maclaurin series for $\tan x$ with $x = \frac{\pi}{n}$ **(M1)**

$$\tan \frac{\pi}{n} = \frac{\pi}{n} + \frac{\left(\frac{\pi}{n}\right)^3}{3} + \frac{2\left(\frac{\pi}{n}\right)^5}{15} (+\dots)$$

$$4n \tan \frac{\pi}{n} = 4n \left(\frac{\pi}{n} + \frac{\pi^3}{3n^3} + \frac{2\pi^5}{15n^5} (+\dots) \right) \text{ (or equivalent)} \quad \textbf{A1}$$

$$= 4 \left(\pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + \dots \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(4n \tan \frac{\pi}{n} \right) = 4\pi \quad \textbf{A1}$$

Note: Award a maximum of **M1A1A0** if $\lim_{n \rightarrow \infty}$ is not stated anywhere.

[3 marks]

- (ii) (as $n \rightarrow \infty$, $P \rightarrow 4\pi$ and $A \rightarrow 4\pi$)

the polygon becomes a circle of radius 2

R1

Note: Award **R1** for alternative responses such as:

the polygon becomes a circle of area 4π OR

the polygon becomes a circle of perimeter 4π OR

the polygon becomes a circle with $A = P = 4\pi$.

Award **R0** for polygon becomes a circle.

[1 mark]

(f) $A = \frac{1}{2}ab$ and $P = a + b + \sqrt{a^2 + b^2}$ (A1)(A1)

equates their expressions for A and P M1

$$A = P \Rightarrow a + b + \sqrt{a^2 + b^2} = \frac{1}{2}ab$$

$$\sqrt{a^2 + b^2} = \frac{1}{2}ab - (a + b)$$
 M1

Note: Award **M1** for isolating $\sqrt{a^2 + b^2}$ or $\pm 2\sqrt{a^2 + b^2}$. This step may be seen later.

$$a^2 + b^2 = \left(\frac{1}{2}ab - (a + b) \right)^2$$

$$a^2 + b^2 = \frac{1}{4}a^2b^2 - 2\left(\frac{1}{2}ab\right)(a + b) + (a + b)^2$$
 M1

$$\left(= \frac{1}{4}a^2b^2 - a^2b - ab^2 + a^2 + 2ab + b^2 \right)$$

Note: Award **M1** for attempting to expand their RHS of either $a^2 + b^2 = \dots$
or $4(a^2 + b^2) = \dots$

EITHER

$$ab\left(\frac{1}{4}ab - a - b + 2\right) = 0 \quad (ab \neq 0)$$
 A1

$$\frac{1}{4}ab - a - b + 2 = 0$$

$$ab - 4a = 4b - 8$$

OR

$$\frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab = 0$$

$$a\left(\frac{1}{4}b^2 - b\right) + (2b - b^2) = 0 \quad (a(b^2 - 4b) + (8b - 4b^2) = 0)$$
 A1

$$a = \frac{4b^2 - 8b}{b^2 - 4b}$$

continued...

Question 2 continued

THEN

$$\Rightarrow a = \frac{4b-8}{b-4}$$

A1

$$a = \frac{4b-16+8}{b-4}$$

$$a = \frac{8}{b-4} + 4$$

AG

Note: Award a maximum of **A1A1M1M1M0A0A0** for attempting to verify.

For example, verifying that $A = P = \frac{16}{b-4} + 2b + 4$ gains 4 of the 7 marks.

[7 marks]

continued...



Question 2 continued

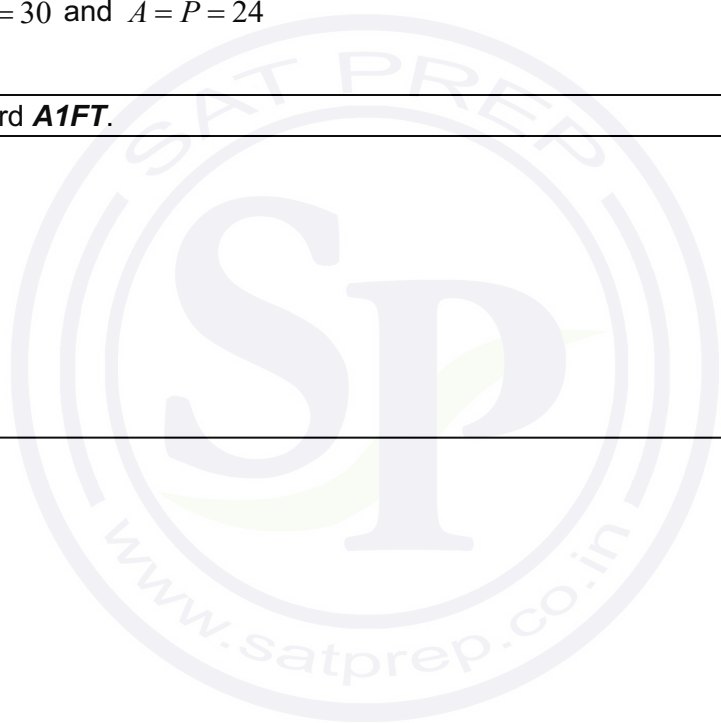
- (g) (i) using an appropriate method (M1)
 eg substituting values for b or using divisibility properties
 (5,12,13) and (6,8,10) A1A1

Note: Award **A1A0** for either one set of three correct side lengths or two sets of two correct side lengths.

- [3 marks]
- (ii) $A = P = 30$ and $A = P = 24$ A1

Note: Do not award **A1FT**.

[1 mark]
Total [28 marks]



Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 3

3 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) or subpart(s). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then **FT** marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further **A** marks can be awarded for work which uses the error, but **M** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (e.g. probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin \theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question*

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. probability greater than 1, $\sin \theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The **MR** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.

7 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy:** when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

9 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features/ CAS functionality are not allowed.

Calculator notation

The subject guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

1. (a) **METHOD 1**

consider right-angled triangle OCX where $CX = \frac{x}{2}$

$$\sin \frac{\pi}{3} = \frac{\frac{x}{2}}{1}$$

M1A1

$$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$$

A1

$$P_i = 3 \times x = 3\sqrt{3}$$

AG

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2(1)(1)\cos \frac{2\pi}{3}$

M1A1

$$x = \sqrt{3}$$

A1

$$P_i = 3 \times x = 3\sqrt{3}$$

AG

Note: Accept use of sine rule.

[3 marks]

(b) $\sin \frac{\pi}{4} = \frac{1}{x}$ where x = side of square

M1

$$x = \sqrt{2}$$

A1

$$P_i = 4\sqrt{2}$$

A1

[3 marks]

(c) 6 equilateral triangles $\Rightarrow x = 1$

A1

$$P_i = 6$$

A1

[2 marks]

(d) in right-angled triangle $\sin \left(\frac{\pi}{n} \right) = \frac{\frac{x}{2}}{1}$

M1

$$\Rightarrow x = 2 \sin \left(\frac{\pi}{n} \right)$$

A1

$$P_i = n \times x$$

$$P_i = n \times 2 \sin \left(\frac{\pi}{n} \right)$$

M1

$$P_i = 2n \sin \left(\frac{\pi}{n} \right)$$

AG

[3 marks]

continued...

Question 1 continued

(e) consider $\lim_{n \rightarrow \infty} 2n \sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

M1

$$2n \sin\left(\frac{\pi}{n}\right) = 2n \left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \dots \right)$$

(A1)

$$= 2 \left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \dots \right)$$

A1

$$\Rightarrow \lim_{n \rightarrow \infty} 2n \sin\left(\frac{\pi}{n}\right) = 2\pi$$

A1

as $n \rightarrow \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$

R1

[5 marks]

(f) consider an n -sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre

M1A1

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2 \tan\left(\frac{\pi}{n}\right)$

M1A1

Perimeter $P_c = 2n \tan\left(\frac{\pi}{n}\right)$

AG

[4 marks]

(g) (i) $\lim_{x \rightarrow \infty} P(x) = \lim_{x \rightarrow \infty} \frac{2 \tan\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$

R1

attempt to use L'Hôpital's rule

M1

$$\lim_{x \rightarrow \infty} P(x) = \lim_{x \rightarrow \infty} \frac{-\frac{2\pi}{x^2} \sec^2\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$$

A1

$$\lim_{x \rightarrow \infty} P(x) = 2\pi$$

A1

(ii) $\lim_{n \rightarrow \infty} P_c(n) = 2\pi$

A1

[5 marks]

continued...

Question 1 continued

(h) $P_i < 2\pi < P_c$

$$2n \sin\left(\frac{\pi}{n}\right) < 2\pi < 2n \tan\left(\frac{\pi}{n}\right)$$

M1

$$n \sin\left(\frac{\pi}{n}\right) < \pi < n \tan\left(\frac{\pi}{n}\right)$$

A1

[2 marks]

- (i) attempt to find the lower bound and upper bound approximations within 0.005 of π
 $n = 46$

(M1)

A2

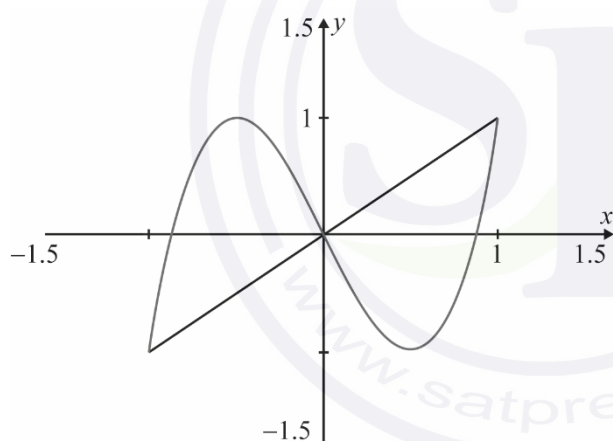
[3 marks]

Total [30 marks]

2. (a) correct graph of $y = f_1(x)$
 correct graph of $y = f_3(x)$

A1

A1



[2 marks]

- (b) (i) graphical or tabular evidence that n has been systematically varied
 eg $n = 3$, 1 local maximum point and 1 local minimum point
 $n = 5$, 2 local maximum points and 2 local minimum points
 $n = 7$, 3 local maximum points and 3 local minimum points

M1

(A1)

$$\frac{n-1}{2} \text{ local maximum points}$$

A1

- (ii) $\frac{n-1}{2}$ local minimum points

A1

Note: Allow follow through from an incorrect local maximum formula expression.

[4 marks]

continued...

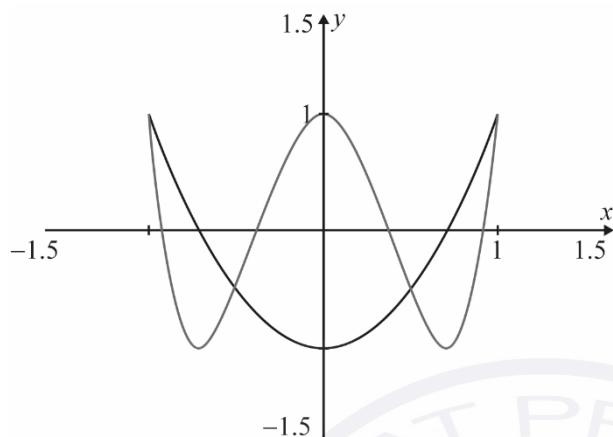
Question 2 continued

(c) correct graph of $y = f_2(x)$

A1

correct graph of $y = f_4(x)$

A1



[2 marks]

(d) (i) graphical or tabular evidence that n has been systematically varied

M1

eg $n = 2$, 0 local maximum point and 1 local minimum point

$n = 4$, 1 local maximum points and 2 local minimum points

$n = 6$, 2 local maximum points and 3 local minimum points

(A1)

$\frac{n-2}{2}$ local maximum points

A1

(ii) $\frac{n}{2}$ local minimum points

A1

[4 marks]

(e) $f_n(x) = \cos(n \arccos(x))$

$$f'_n(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$$

M1A1

Note: Award **M1** for attempting to use the chain rule.

$$f'_n(x) = 0 \Rightarrow n \sin(n \arccos(x)) = 0$$

M1

$$n \arccos(x) = k\pi \quad (k \in \mathbb{Z}^+)$$

A1

leading to

$$x = \cos \frac{k\pi}{n} \quad (k \in \mathbb{Z}^+ \text{ and } 0 < k < n)$$

AG

[4 marks]

continued...

Question 2 continued

- (f) $f_2(x) = \cos(2 \arccos x)$
 $= 2(\cos(\arccos x))^2 - 1$ **M1**
 stating that $(\cos(\arccos x)) = x$ **A1**
 so $f_2(x) = 2x^2 - 1$ **AG**
[2 marks]
- (g) $f_{n+1}(x) = \cos((n+1) \arccos x)$
 $= \cos(n \arccos x + \arccos x)$ **A1**
 use of $\cos(A+B) = \cos A \cos B - \sin A \sin B$ leading to **M1**
 $= \cos(n \arccos x) \cos(\arccos x) - \sin(n \arccos x) \sin(\arccos x)$ **AG**
[2 marks]
- (h) (i) $f_{n-1}(x) = \cos((n-1) \arccos x)$ **A1**
 $= \cos(n \arccos x) \cos(\arccos x) + \sin(n \arccos x) \sin(\arccos x)$ **M1**
 $f_{n+1}(x) + f_{n-1}(x) = 2 \cos(n \arccos x) \cos(\arccos x)$ **A1**
 $= 2xf_n(x)$ **AG**
- (ii) $f_3(x) = 2xf_2(x) - f_1(x)$ **(M1)**
 $= 2x(2x^2 - 1) - x$
 $= 4x^3 - 3x$ **A1**
[5 marks]

Total [25 marks]