Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

16 May 2025

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2225 – 7307

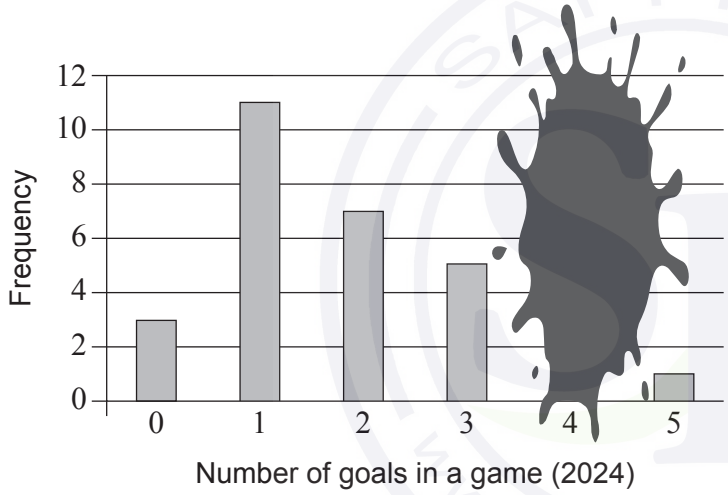
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

Paul has a bar graph for the total number of goals scored in each game of a soccer tournament in 2024. The bar graph is shown below, however the frequency of 4 goals in a game is unreadable.

Paul uses this bar graph to create a frequency table.

Bar graph



Frequency table

Number of goals in a game (2024)	Frequency
0	3
1	11
2	7
3	k
4	p
5	1

(a) Write down the value of k. [1]

Paul knows that the mean number of goals per game scored during the tournament was 2.2.

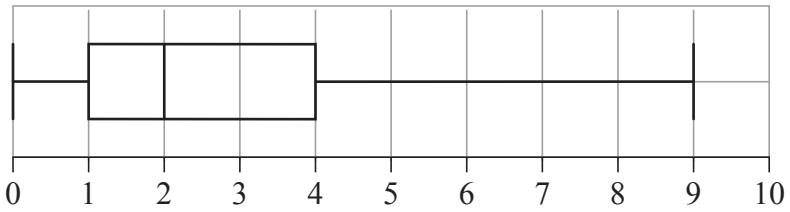
(b) (i) Write down an equation for the mean in terms of p.

(ii) Determine the value of p. [3]

(This question continues on the following page)

(Question 1 continued)

Data for the number of goals per game in the **2025 soccer tournament** are shown in the following box and whisker diagram.



Number of goals in a game (2025)

After comparing the box and whisker diagram from the 2025 tournament with the frequency table from the 2024 tournament, Paul concludes that the distribution of goals is consistent between the two tournaments.

(c) State two observations that support Paul's conclusion using values from the data to compare any **two** of:

range, symmetry, median, and interquartile range. [3]

Paul plans to watch all the games from the **2024** tournament in a random order.
He will watch each game once.

For the first game he watches, he defines event F as:

"scoring either 0 goals or exactly 1 goal".

(d) Write down the event(s) from the table that are equivalent to F'. There may be more than one correct event. [2]

Event	Description
A	Scoring exactly 2 goals in this game
B	Scoring more than 1 goal in this game
C	Scoring at least 2 goals in this game
D	Scoring either 0 goals or exactly 1 goal in all games except this game
E	Not scoring 0 goals or exactly 1 goal in any game

(e) If exactly 1 goal was scored in the first game Paul watches, write down the probability that exactly 1 goal was scored in the second game he watches. Give your answer as a fraction. [2]

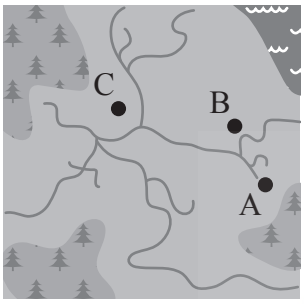
(f) Calculate the probability that 5 goals were scored in the first game that Paul watches **and** 0 goals were scored in the second game he watches. [4]

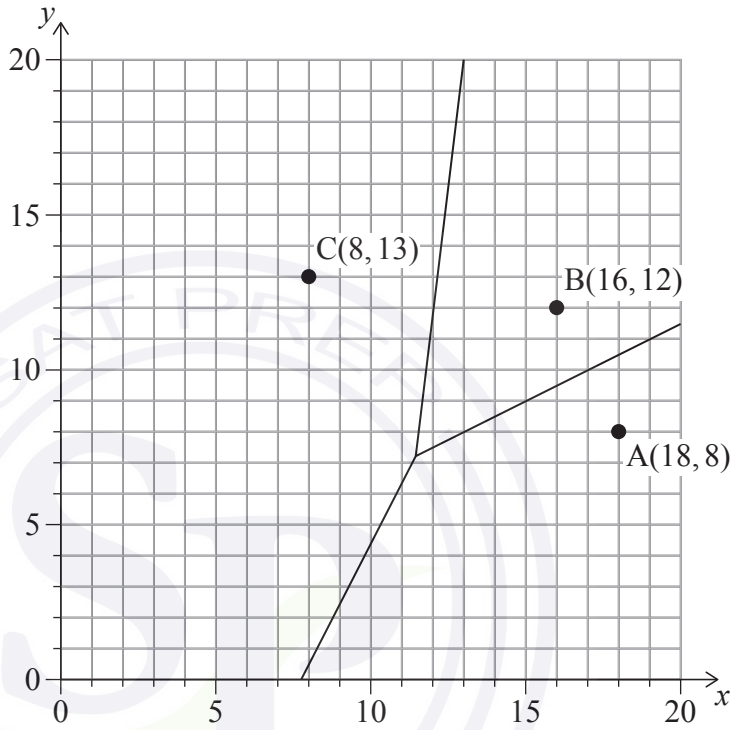
2. [Maximum mark: 19]

The locations of three fire stations within a 50 by 50 kilometre rural area of Japan are shown on the following map.

A Voronoi diagram can be used to determine the part of the rural area for which each fire station is responsible. The Voronoi diagram for the three fire stations is shown on the coordinate grid beside the map, where 1 unit represents $2.5\,\text{km}$.

Map





(a) (i) Find the midpoint of $[\text{BC}]$.

(ii) Find the gradient of the perpendicular bisector of $[\text{BC}]$.

(iii) Hence find the equation of the edge between sites B and C. [6]

(b) (i) Identify the fire station that is expected to respond, based on the Voronoi diagram, if a fire is reported at a location with coordinates $(14, 10)$. Justify your response.

(ii) Suggest a reason why a different fire station might respond to this fire. [2]

(This question continues on the following page)

(Question 2 continued)

A fire is reported at a location D with coordinates $\left(11\frac{1}{3}, 7\frac{1}{6}\right)$. The distance of D from A on the Voronoi diagram, to six significant figures, is 6.71855 units ($16.7964\,\mathrm{km}$ on the map).

(c) (i) Show that the distance of D from B, to six significant figures, is also 6.71855 units.

(ii) Show that any of the three fire stations would be expected to respond to the fire reported at D. [4]

The equation of the edge between fire stations A and B is $y = 0.5x + 1.5$. The edge between fire stations A and C has an x-intercept of 7.75.

(d) (i) Determine the area, to the nearest square unit, of the region within the given Voronoi diagram for which fire station A is responsible.

(ii) Hence, calculate the actual area, in square kilometres, of this region. [7]

3. [Maximum mark: 17]

Ethan and Avery are researching population data for the city of Los Angeles to create a model predicting population values. They collect the following data.

Population data for the city of Los Angeles

Year	Population (thousands)
1900	102
1920	577
1940	1504
1960	2479
1980	2967
2000	3685
2020	3899

Ethan proposes the population can be modelled using quadratic regression to find a function of the form $f(x) = ax^2 + bx + c$, where x is the number of years after 1900.

(a) Find the equation of Ethan's model. [3]

Ethan finds the coefficient of determination for his model is 0.98843 to five significant figures.

(b) State whether the coefficient of determination supports Ethan's proposal. Justify your answer. [2]

(c) Comment on the validity of Ethan's model with reference to one of the parameters in the equation. [1]

(d) (i) Find the value of $f'(110)$ and interpret this value in context.

(ii) By considering the population changes in the table, use the value found in part (d)(i) to comment on the validity of Ethan's model. [4]

(This question continues on the following page)

(Question 3 continued)

Avery proposes that the population instead follows a logistic model of the form

$$g(x) = \frac{4000}{1 + 14\ \mathrm{e}^{-0.05x}}$$

where x is the number of years after 1900.

(e) State a reason why it may be valid to use Avery's proposal to predict the future population. [1]

(f) (i) Find $g'(x)$.

(ii) Hence find the year, according to Avery's model, during which the greatest population growth rate occurred. [6]

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4. [Maximum mark: 17]

Lizjerika works in an air traffic control tower. At 10:00 am, she detects a passenger airplane and a flock of birds whose flight paths are given by the following equations:

$$\text{Airplane: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + t_1 \begin{pmatrix} -1.4 \\ 1.65 \\ 0 \end{pmatrix} \qquad \text{Birds: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \\ 6 \end{pmatrix} + t_1 \begin{pmatrix} -0.40 \\ -0.35 \\ 0.1 \end{pmatrix}$$

where the x-coordinate represents displacement east of the control tower, the y-coordinate represents displacement north of the control tower, and the z-coordinate represents height (above sea level).

Distances are measured in kilometres and t_1 is the time, in minutes, since 10:00am.

(a) Determine the speed, in kilometres per minute, at which the flock of birds is travelling. [2]

(b) By making two statements, describe the path of the airplane in context. [2]

(c) (i) Find the time at which the birds and the airplane will have the same height.

(ii) Using your answer to part (c)(i), explain why the birds will not collide with the airplane. [6]

At 10:20 am, Lizjerika detects a jet whose movement is given by the following equation,

$$\text{Jet: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -95 \\ 32 \\ 1.5 \end{pmatrix} + t_2 \begin{pmatrix} 1.3 \\ 1.45 \\ 0.2 \end{pmatrix}$$

where t_2 is the time, in minutes, since 10:20am.

(d) Find the vector equation of the motion of **the passenger airplane** in terms of t_2. [2]

An air traffic law requires that the jet and the airplane must be more than 5 kilometres from each other at all times.

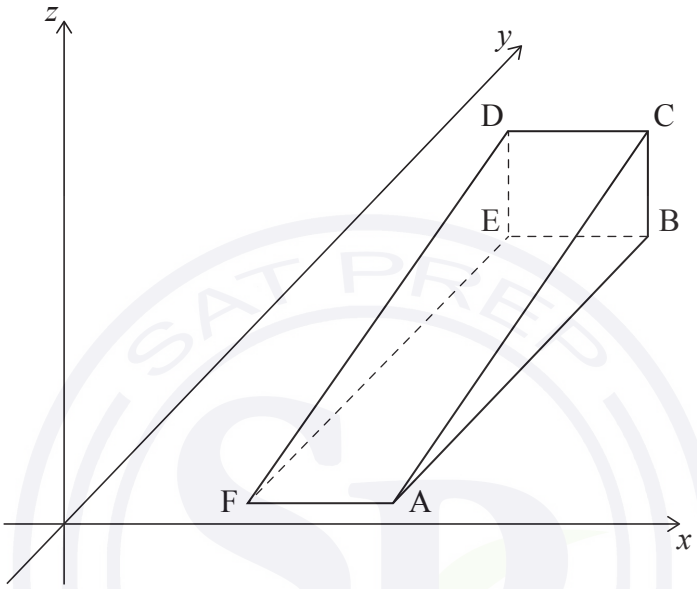
(e) Determine whether the jet and the airplane will break this law if they follow their current paths. [5]

5. [Maximum mark: 16]

AirFlow Industries is designing a doorstop in the shape of a triangular prism, $ABCDEF$.

The points $A(13, 1, 0)$, $B(13, 25, 0)$ and $C(13, 25, 7)$ are the coordinates of the vertices of one of the triangular faces of the prism. Point $D(4, 25, 7)$ is another vertex as shown in the following diagram. All the measurements are in centimetres.

diagram not to scale



(a) Find AC. [2]

(b) Show that triangle ABC is a right-angled triangle. [2]

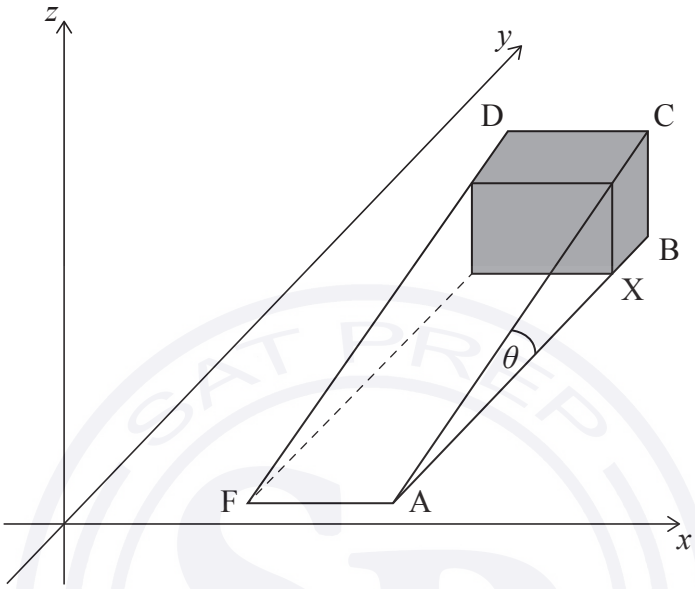
(c) Find the volume of material needed to make the doorstop. [3]

(This question continues on the following page)

(Question 5 continued)

To lower the cost of the doorstop, AirFlow decides to reduce its volume to $625\,\text{cm}^3$ by removing the shaded section shown in the following diagram. The remaining triangular faces are still right-angled triangles.

diagram not to scale



(d) (i) Find the value of θ.

(ii) Hence, or otherwise, find AX. [6]

The material needed to make the doorstop costs 0.025 US dollars (USD) per cubic centimetre. The production of each doorstop requires $10\,\%$ more material than its final volume, due to wastage in the production process.

To make a profit, the company will sell the doorstop for $20\,\%$ more than the cost of the material.

(e) Determine the price, to two decimal places, the company will charge for the **new** doorstop. [3]

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6. [Maximum mark: 14]

Rea is investigating how the number of healthy skin cells changes following infection by a virus. At the beginning of the investigation, she inserts $27\,000$ virus particles into a sample of 1350 healthy skin cells.

Let h be the number of healthy skin cells at time t, where t is the number of hours after the investigation has begun. Let v be the number of virus particles, **in thousands**, at time t.

Rea finds that the rate of change, $\frac{\mathrm{d}h}{\mathrm{d}t}$, is proportional to $\frac{h}{v}$.

(a) Given that $\frac{\mathrm{d}h}{\mathrm{d}t} = -5$ at time $t = 0$, show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{h}{10v}$. [3]

Rea models the number of virus particles, in thousands, after t hours as $v = 27 + 0.3t$.

(b) Find an expression for the number of healthy skin cells at time t. [7]

Rea asks her colleague Artem to attempt the same investigation. Artem's models differ from those of Rea.

Artem models the number of virus particles, in thousands, after t hours as $v = 27 + 0.28t$ and finds that $h = 4380\,(27 + 0.28t)^{-0.36}$.

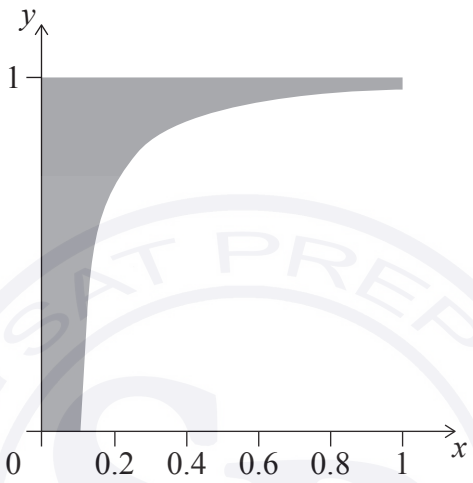
(c) Using **Artem's** models, predict how many hours it will take for the number of virus particles to be at least 100 times the number of healthy skin cells. [4]

7. [Maximum mark: 12]

Bobby is designing a table that will be 1 metre high. The table will have a flat circular top with a radius of 1 metre. To begin the design, he uses the curve of

$$y = \frac{2}{3}\cos^{-1}\left(\frac{1}{10x}\right), \text{ for } 0.1 \leq x \leq 1,$$

to shape the underside of the table, where x and y are measured in metres, as shown.



(a) Find the area of the shaded region. [4]

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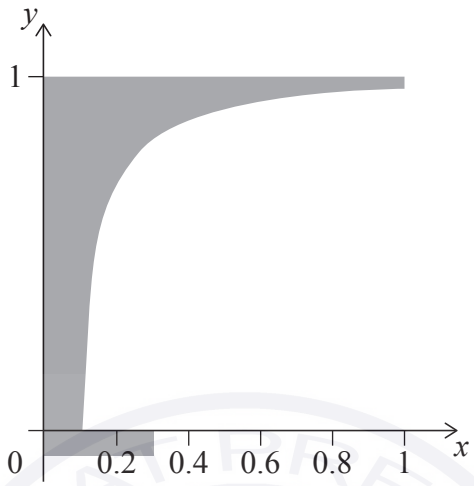
(Question 7 continued)

To create the base of the table, Bobby adds a rectangle with a height of 0.075 metres and a width of 0.3 metres to the bottom of the figure, as shown.



To finish the design of the table, he rotates the entire shaded region through 2π about the y-axis.

(b) Find the three expressions necessary to determine the total volume of this table. [6]

(c) Hence find the total volume of the table. [2]

References:

3. Los Angeles Almanac, n.d. *Historical General Population City & County of Los Angeles, 1850 to 2020* [online] Available at: https://www.laalmanac.com/population/po02.php [Accessed 4 June 2024]. Source adapted.

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Higher level
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2225 – 7312

12 pages

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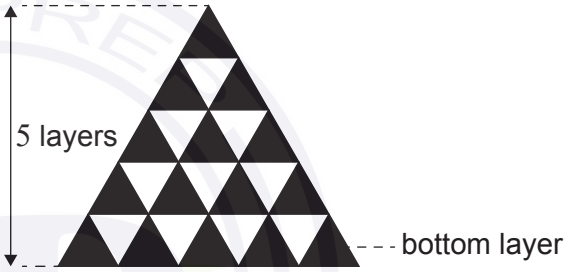
1. [Maximum mark: 16]

Thai cushions are designed with a triangular cross-section and are made from layers of smaller cushions. These cushions can be modelled as triangular prisms.

This is shown in the diagram.

diagram not to scale

Thai cushion with 4 layers



Cross-section of Thai cushion with 5 layers

(a) Write down the number of triangular prisms in the bottom layer of the cushion with

(i) 4 layers.

(ii) 5 layers. [2]

Mayumi notices that the number of triangular prisms in the bottom layer of the cushions forms an arithmetic sequence.

(b) (i) Write down the common difference of this sequence.

(ii) Find an expression for the number of triangular prisms in the bottom layer of a cushion with n layers. [3]

(This question continues on the following page)

(Question 1 continued)

Mayumi wants to extend this design to create a cushion with 9 layers.

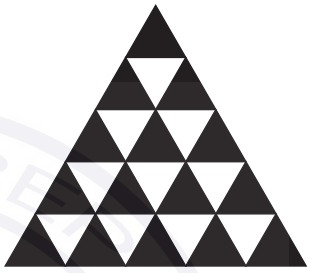
(c) (i) Find the number of triangular prisms in the bottom layer of Mayumi's cushion.

(ii) Calculate the **total** number of triangular prisms in Mayumi's cushion. [3]

(d) Find an expression for the **total** number of triangular prisms in a cushion with n layers, giving your answer in its simplest form. [2]

The cross-section of the cushion consists of black triangles and white triangles.

This cushion with 4 layers has a total of 6 white triangles.



This cushion with 5 layers has 4 white triangles in its bottom layer.

(e) Write down the total number of black triangles in a cushion with 4 layers. [1]

The number of black triangles in each layer forms an arithmetic sequence.

(f) Find and simplify an expression for the total number of black triangles in a cushion with n layers. [2]

The total number of white triangles in a cushion with n layers is $\frac{n(n-1)}{2}$.

(g) Using both the given expression and your answer to part (f), find and simplify an expression for the total number of black and white triangles in a cushion with n layers. [3]

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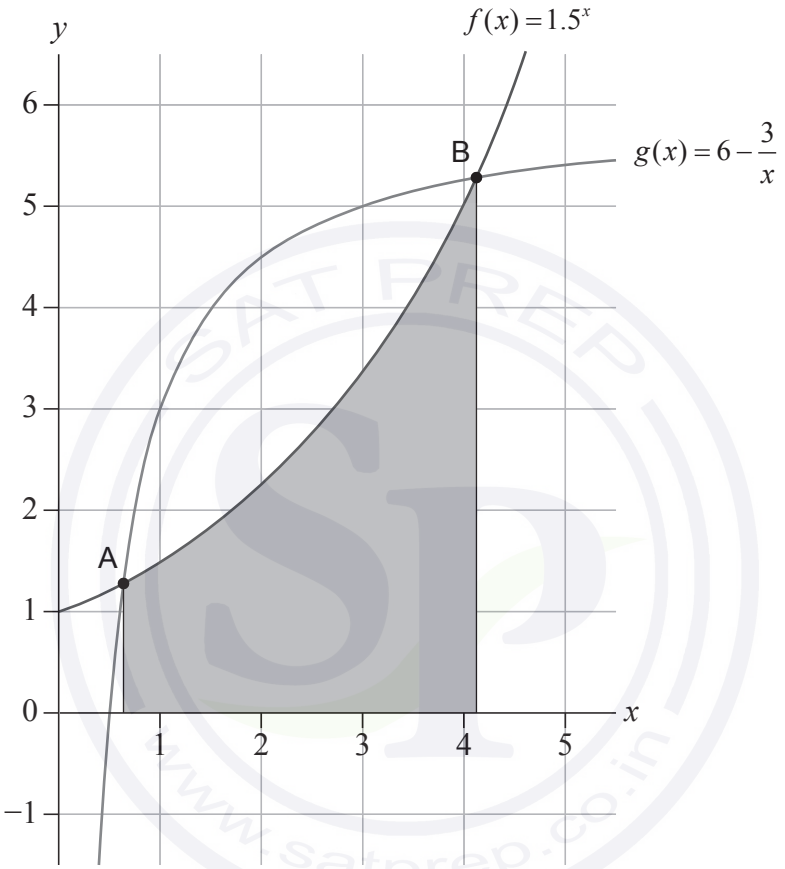
2. [Maximum mark: 12]

The diagram shows part of the graphs of the functions

$$f(x) = 1.5^x \qquad x \geq 0$$

$$g(x) = 6 - \frac{3}{x} \qquad x > 0\,.$$



(a) Solve $f(x) = g(x)$. [3]

(b) (i) Write down the integral that represents the area of the shaded region.

(ii) Calculate the area of this shaded region.

(iii) Hence, or otherwise, calculate the area of the region enclosed between the curves $y = f(x)$ and $y = g(x)$. [6]

The tangent to the graph of $y = f(x)$ is parallel to the tangent to the graph of $y = g(x)$ at $x = k$.

(c) Find the value of k. [3]

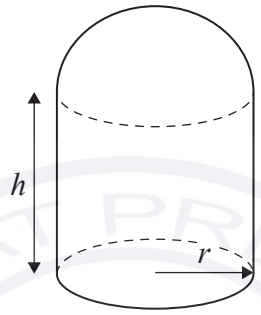
3. [Maximum mark: 16]

Ju Shen designs a plastic cover, in the shape of a cylinder combined with a hemisphere on top, as shown in the diagram.

The plastic used to make the cover forms the curved surface of both the hemisphere and the cylinder; there is no bottom to the cover, however it stands on a flat horizontal surface.

Let the height of the cylinder be $h\,\text{cm}$ and the internal radius of its base be $r\,\text{cm}$.

diagram not to scale



(a) Find an expression for the total internal surface area, $A\,\text{cm}^2$, of the cover in terms of r and h. [2]

(b) Show that the total volume, $V\,\text{cm}^3$, enclosed by the cover and the horizontal surface is given by the expression

$$V = \frac{2\pi r^3 + 3\pi r^2 h}{3}.$$ [2]

The total volume enclosed by the cover is $10\,000\,\text{cm}^3$.

(c) Hence show that $h = \frac{30\,000 - 2\pi r^3}{3\pi r^2}$. [2]

Ju Shen uses the total internal surface area to model the amount of plastic used to construct the cover.

(d) Show that A is given by the expression

$$A = \frac{2\pi r^2}{3} + \frac{20\,000}{r}.$$ [2]

(This question continues on the following page)

(Question 3 continued)

Ju Shen wants to use the minimum amount of plastic in the construction of the cover.

(e) Find an expression for $\frac{dA}{dr}$. [3]

(f) Find the value of r and the value of h that minimizes the use of plastic. [4]

(g) By interpreting your answer to part (f), suggest the best shape for Ju Shen's plastic cover. [1]

4. [Maximum mark: 13]

A wind farm consists of five wind turbines, located at points A to E.

The table below shows the distances, in kilometres, between each pair of turbines.

	A	B	C	D	E
A		0.90	0.88	1.56	0.86
B	0.90		0.74	0.94	1.28
C	0.88	0.74		0.78	0.62
D	1.56	0.94	0.78		1.36
E	0.86	1.28	0.62	1.36	

The turbines must all be connected by cables. However, there does not need to be a direct connection between every pair.

(a) Use Prim's algorithm, starting with vertex A, to find the minimum total length of cable required to connect the turbines. Show the order in which you added the vertices. [4]

The supervisor of the wind farm has a monitoring cabin located at point F. The distances from F to each turbine are shown in the table.

Turbine	A	B	C	D	E
Distance from F (km)	0.96	1.82	1.57	2.24	1.14

The supervisor wants to visit every turbine exactly once for inspection, starting and finishing at the cabin, and using the route of shortest possible length.

(b) By deleting vertex F, find a lower bound for the length of the shortest route. [2]

(c) Use the nearest neighbour algorithm starting at F to find an upper bound for the length of the shortest route. [3]

(This question continues on the following page)

(Question 4 continued)

The table below shows the lower bounds found by deleting each of the other five vertices, and the upper bounds found by starting at each of the other five vertices.

Vertex	A	B	C	D	E
Lower bound	5.02	4.86	5.02	4.90	4.84
Upper bound	6.36	6.36	7.13	7.22	6.82

The supervisor travels between the turbines at a constant speed of $28\,\mathrm{km/h}$ and spends 12 minutes inspecting each turbine.

(d) Based on the information given in the table, find the best possible upper and lower bounds for the shortest amount of time, T hours, required for the inspection. Write your answer as an inequality. [4]

5. [Maximum mark: 15]

A zoologist collects a sample of cane beetles. He measures their length and categorizes them as "small" meaning from 10 to $12\,\text{mm}$ long, "medium" meaning from 12 to $16\,\text{mm}$ long and "large" meaning from 16 to $18\,\text{mm}$ long. He also notes their sex and records the frequencies in the following table.

		Length, x mm		
		Small $10 < x \leq 12$	Medium $12 < x \leq 16$	Large $16 < x \leq 18$
Sex	Female	42	25	19
	Male	61	27	12

(a) Find how many cane beetles are in the zoologist's sample. [1]

(b) Based on this data set, estimate the mean length of a cane beetle. [2]

Two female beetles are chosen at random with replacement from the sample.

(c) Find the probability that they are both categorized as small. [3]

(d) Test, at the 5% significance level, the null hypothesis that length category and sex are independent. State the p-value of your test and write your conclusion in context. Justify your answer. [4]

Let ϕ be the population proportion of cane beetles that are male.

(e) Test, at the 5% significance level, the hypothesis that more than 45% of cane beetles are male. Write the null and alternative hypotheses. State the p-value of your test and write your conclusion in context. [5]

6. [Maximum mark: 21]

A financial analyst models the change in value of one share, x dollars at time t minutes, after a report is released. She uses the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0 .$$

This equation can be written as the coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x - 4y .$$

(a) Find the general solution for x. [5]

Initially $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -1$.

(b) (i) Find an expression for x in terms of t.

(ii) Sketch x against t in the interval $0 \le t \le 4$. [6]

Once the report has been released, the analyst is going to buy some shares and then sell them later.

(c) (i) Use your graph to find how long after the report is released the analyst should wait to buy the shares in order to maximize her profit.

(ii) Find the upper limit of the profit the analyst can make per share. [4]

An improved model is written as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = x\sin t .$$

The same initial conditions as above apply.

(d) Use Euler's method with a t-interval of 0.1 to predict the value of x when $t = 1$. [6]

7. [Maximum mark: 17]

On any given day, the probability that Emlyn charges his phone depends only on whether he charged it the previous day.

If he charged his phone the previous day, the probability he charges it today is 0.4.

If he did not charge his phone the previous day, the probability he charges it today is p.

On day n this can be represented using the vector $\boldsymbol{v}_n$ where

$$\boldsymbol{v}_n = \begin{pmatrix} \text{probability that Emlyn charges his phone on day } n \\ \text{probability that Emlyn does not charge his phone on day } n \end{pmatrix}$$

A Markov chain model is formed where

$$\boldsymbol{v}_{n+1} = \boldsymbol{M}\boldsymbol{v}_n.$$

Matrix $\boldsymbol{M}$ is of the form $\begin{pmatrix} a & p \\ b & 1-p \end{pmatrix}$.

(a) Write down the value of

(i) a.

(ii) b. [2]

(b) On day zero Emlyn charges his phone. Find the probability

(i) that Emlyn charges his phone on all days from $n = 1$ to $n = 4$.

(ii) that Emlyn charges his phone on day 4, when $p = 0.7$. [5]

(c) Demonstrate that, for all values of p, one eigenvector of $\boldsymbol{M}$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and hence state the associated eigenvalue. [4]

(d) Find, in terms of p, the steady state probability that Emlyn charges his phone on a given day. [4]

In the long term, Emlyn wants to charge his phone on at least 60% of days.

(e) Find the minimum value of p required for this to occur. [2]

References:

1. naisupakit, 2016. *Triangle Pillow tradition native Thai style pillow*. [image online] Available at: https://www.gettyimages.co.uk/detail/photo/triangle-pillow-tradition-native-thai-style-pillow-royalty-free-image/623127206 [Accessed 9 April 2024]. Source adapted.

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

16 May 2025

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2225 – 7317

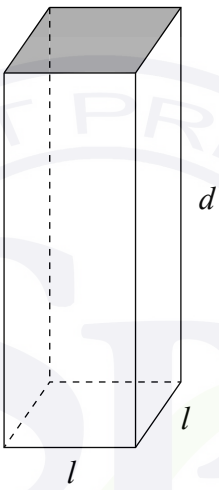
15 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 20]

Kailash manufactures drink containers in the shape of a cuboid. The container has a square top and a square base of length, $l\,\text{cm}$. Its height, $d\,\text{cm}$, is three times the length of the base.

diagram not to scale



(a) Write down an expression for d in terms of l. [1]

The container can hold $375\,\text{cm}^3$ of drink.

(b) Find the value of l and d. [3]

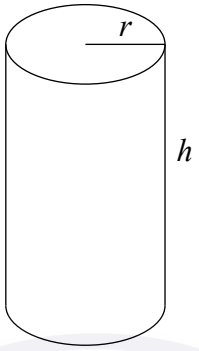
(c) Calculate the total external surface area of the container. [3]

(This question continues on the following page)

(Question 1 continued)

To reduce environmental impact, Kailash is trying to minimize the amount of material needed for the production of the $375\,\text{cm}^3$ container.

He is willing to change the shape to a cylinder with radius $r\,\text{cm}$, and height $h\,\text{cm}$, as shown below.



The cylindrical container of drink must also hold $375\,\text{cm}^3$.

(d) Find an expression for the height, h, of the container in terms of r. [2]

Let the total external surface area be $A\,\text{cm}^2$.

(e) Show that $A = 2\pi r^2 + \dfrac{750}{r}$. [2]

(f) Find $\dfrac{\mathrm{d}A}{\mathrm{d}r}$. [3]

(g) Hence or otherwise

(i) find the value of r that will minimize A.

(ii) find the minimum value of A needed for the cylinder. [3]

(h) (i) Find $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}$.

(ii) Hence determine whether the graph of A is concave-up or concave-down for $r > 0$. Justify your answer. [3]

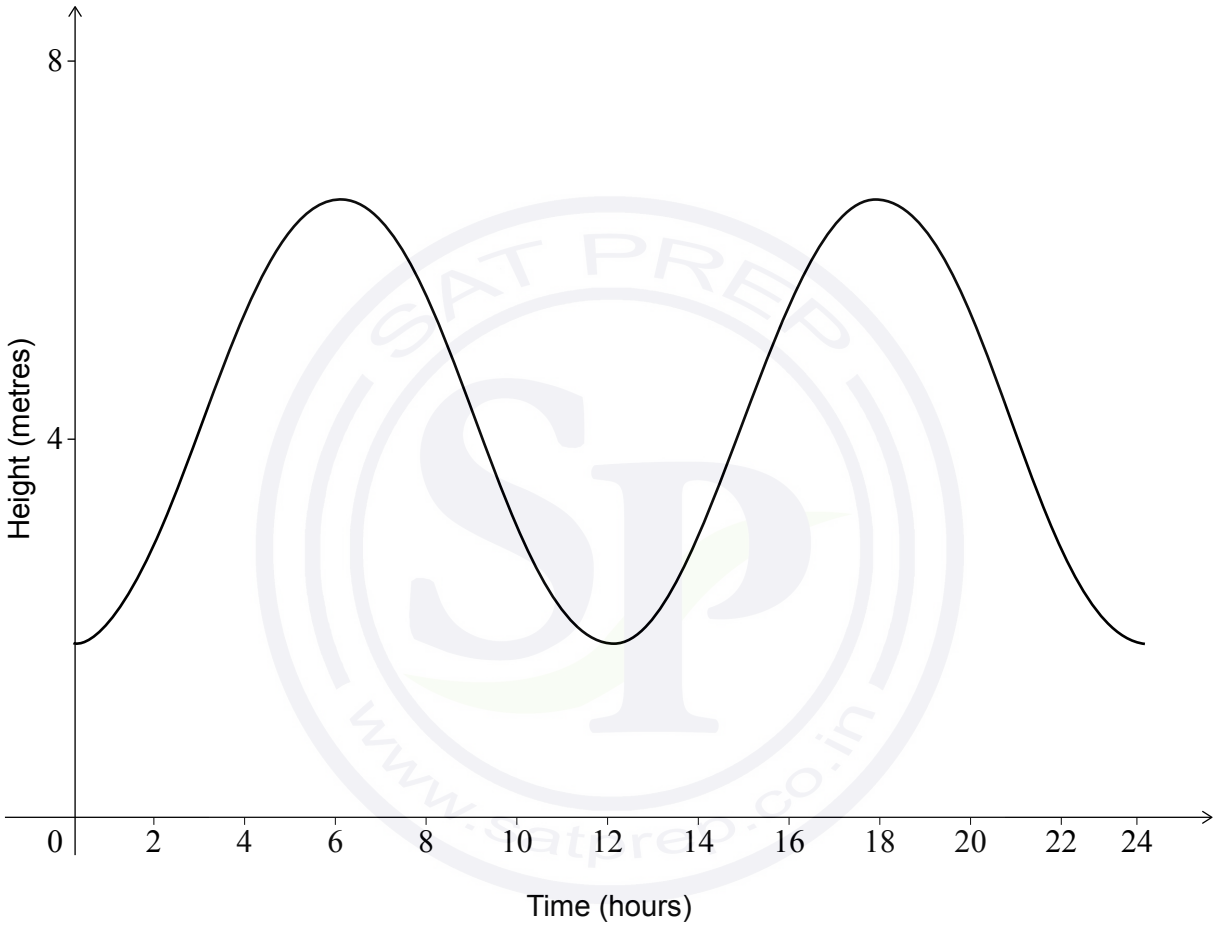
2. [Maximum mark: 15]

On a particular day the height of the tide, h, in metres, at Albion harbour can be modelled by the function

$$h(t) = -2.5\cos(bt°) + 4.5, \text{ where } b \in \mathbb{R}, 0 \leq t \leq 24$$

and t represents the number of hours after midnight.

The graph of h is shown in the following diagram.



(a) Show that the value of b is 30. [1]

(b) Find the height of the tide when $t = 5$. [2]

(c) Write down

(i) the amplitude of h.

(ii) the equation of the principal axis. [3]

(This question continues on the following page)

(Question 2 continued)

Boats can only leave or return to Albion harbour when the height of the tide is greater than $2.65\,\text{m}$.

Robin knows that due to local weather conditions, there could be possible errors in the height of the tide, h, predicted by the model. The actual height may vary by up to $10\,\text{cm}$ above and below the value of h.

Robin wants to leave the harbour to go fishing as soon as possible after the time is 12:00.

(d) Use the above information to determine the earliest time that it could be possible for Robin to leave the harbour. Give your answer to the nearest minute. [4]

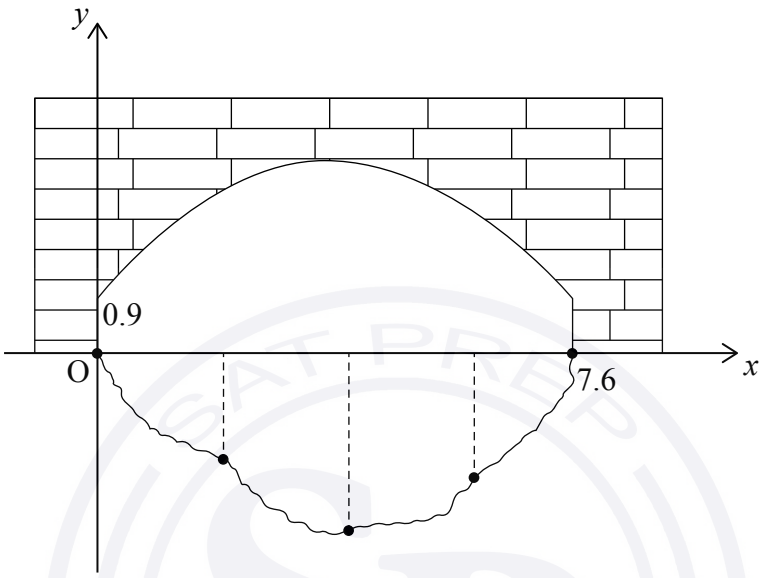
The boat will take 15 minutes to travel from the harbour to the fishing site. Robin intends to return to the harbour on the same day. He wants to be certain he will be able to get back into the harbour.

(e) Determine the maximum possible length of time he could spend at the fishing site, in hours, and still be certain he will be able to enter the harbour on his return. [5]

3. [Maximum mark: 12]

The diagram shows the cross-section of a bridge and a river. A coordinate system has been added with the origin, O, at the point where the bridge meets the water on one side. All units are in metres.

diagram not to scale



A researcher wants to calculate the volume of water that flows under the bridge. To do this he takes measurements of the depth every $1.9\,\text{m}$ from O. The depths are shown in the following table.

Horizontal distance from O in metres	0	1.9	3.8	5.7	7.6
Vertical depth of water in metres	0	1.68	2.81	2.32	0

(a) Use the trapezoidal rule to find the cross-sectional area of the river as it passes under the bridge. [3]

The water flows under the bridge at a rate of $0.3\,\text{m}\,\text{s}^{-1}$.

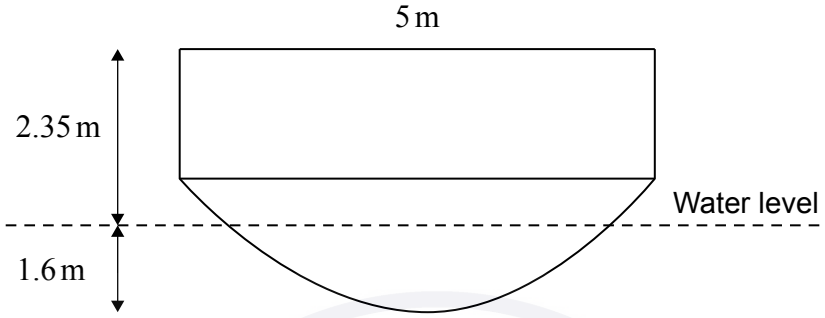
(b) Find the volume of water that passes under the bridge each second. [2]

(This question continues on the following page)

(Question 3 continued)

A boat is travelling along the river. The cross-section of the boat and the water level is shown in the following diagram.

The top of the boat is parallel to the water level and has a width of $5\,\text{m}$. The height of the boat is $2.35\,\text{m}$ above the water level and the lowest part of the boat is $1.6\,\text{m}$ below the water level.



The boat is travelling down the centre of the river.

(c) Find the vertical distance between the lowest part of the boat and the bottom of the river as it passes under the bridge. [1]

The curved arch of the bridge can be modelled by the equation

$$y = -0.15x^2 + 1.14x + 0.9,\ 0 \le x \le 7.6.$$

(d) Find the maximum height of the curved arch above the water level. [2]

(e) Determine whether the top of the boat will be able to pass under the bridge. [4]

4. [Maximum marks: 16]

The ferry routes between five ports, A to E, can be represented by a graph, G.

The adjacency matrix $\boldsymbol{M}$ for the graph G is shown below.

$$\boldsymbol{M} = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(a) State which direct route between ports can only be travelled in one direction. [1]

(b) By finding an appropriate matrix

(i) show that it is not possible to reach all of the ports from any other port in **exactly** three trips.

(ii) state whether it is possible to reach all of the ports from any other port in three trips or fewer. Justify your answer. [5]

In the summer there is a new timetable for ferry routes. The least cost, in euros (EUR), to travel between any two of the ports is shown in the table below.

Port	A	B	C	D	E
A	0	3	7	8	5
B	3	0	4	9	6
C	7	4	0	a	10
D	8	9	a	0	3
E	5	6	10	3	0

(c) Given there is no direct route between D and C or C and D, show that the value of a is $13\,\mathrm{EUR}$. In your answer ensure you have considered all possible routes. [3]

(d) Starting at A, use the nearest neighbour algorithm to find an upper bound for the cost of visiting all the ports and returning to your starting position. [3]

(e) By deleting the vertex A, use the deleted vertex algorithm to find a lower bound for the cost of visiting all the ports. [4]

5. [Maximum mark: 15]

The position of a particle, P_1, t seconds after leaving a point A, is given by

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 4 \\ 7 \end{pmatrix},\ t \geq 0.$$

The units of distance are metres.

(a) Write down the coordinates of A. [1]

Five seconds after leaving A, P_1 is at point B.

(b) Find

(i) $\vec{\mathrm{AB}}$.

(ii) $\left|\vec{\mathrm{AB}}\right|$. [4]

A second particle P_2 leaves A at the same time as P_1. P_2 is moving in the direction of the vector $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, and passes through a point C.

(c) Find $\mathrm{C\hat{A}B}$. [5]

The particle P_2 has a speed of $12\,\mathrm{m\,s^{-1}}$.

(d) Find the distance from P_1 to P_2 when $t = 5$. [5]

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6. [Maximum mark: 15]

Daphne runs a coffee shop and needs to decide whether she should hire more staff.

Over a long period of time, the staff collect data on the number of customers, C, who are queuing when a new customer enters the shop. The probability distribution of C is shown in the following table.

Number of customers in queue, c	0	1	2	3	≥ 4
$\mathrm{P}(C = c)$	0.21	0.34	0.28	0.17	0

(a) (i) Find the probability that there are at least two customers in the queue when a new customer arrives.

(ii) Find $\mathrm{E}(C)$. [3]

The time in seconds, T, taken to serve a single customer can be modelled by the normal distribution $T \sim \mathrm{N}(115, 28^2)$.

Daphne is told that an estimate for the expected time a customer will need to wait before being served can be found by calculating $\mathrm{E}(C) \times \mathrm{E}(T)$.

(b) Find the value of $\mathrm{E}(C) \times \mathrm{E}(T)$. [2]

Daphne now decides that she will employ a new member of staff if the probability that a customer has to wait more than three minutes before being served is greater than 0.4.

(c) Using the distribution of T given above, find the probability that it takes more than three minutes to serve a randomly selected customer. [2]

(d) Find the probability it takes more than three minutes in total to serve two randomly selected customers. You may assume all serving times are independent of all other serving times. [4]

Daphne assumes that when a new customer arrives the person at the front of the queue has only just reached the service counter. She also assumes that the probability that it takes more than three minutes to serve three customers is equal to 1.

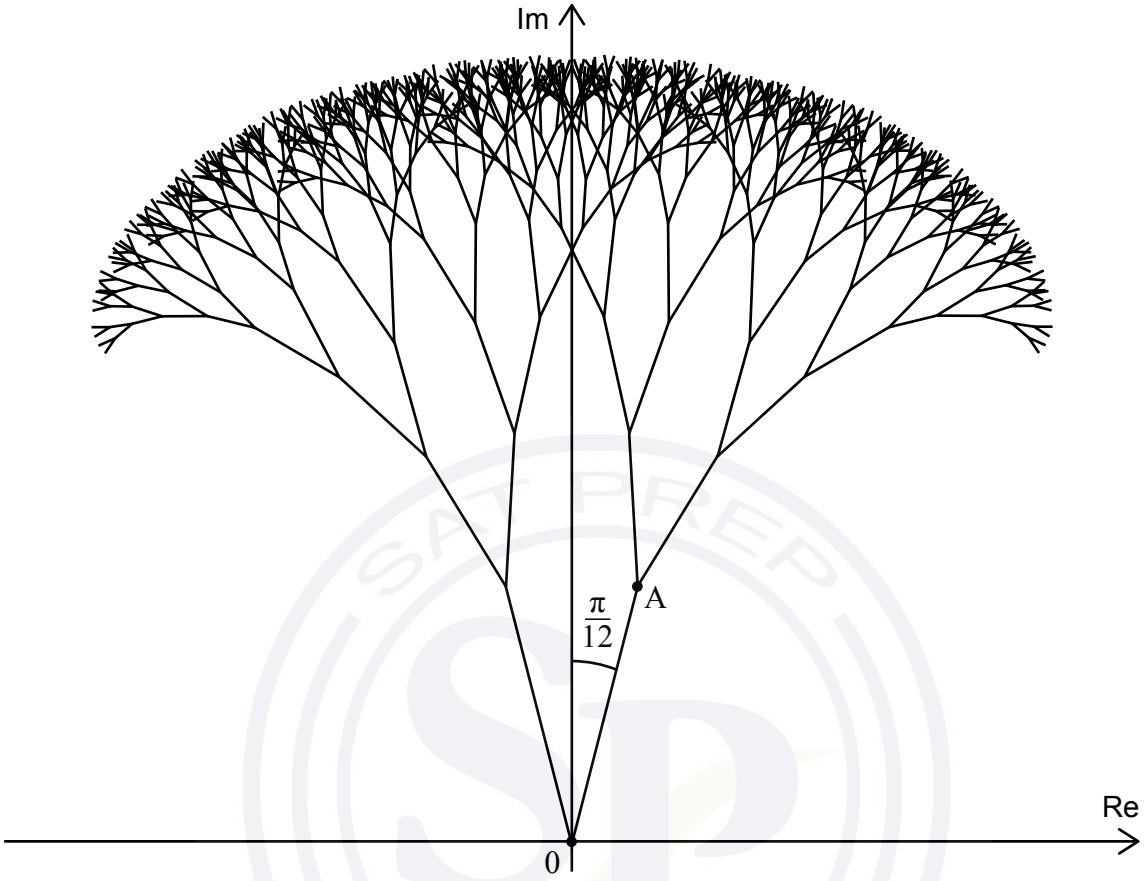
(e) Using Daphne's assumptions and the probabilities for C given in the table above

(i) find the probability a customer just arriving at the coffee shop will wait more than three minutes before being served.

(ii) Hence state whether Daphne will decide to employ more staff. [4]

7. [Maximum mark: 17]

A fractal tree is shown on the following Argand diagram.



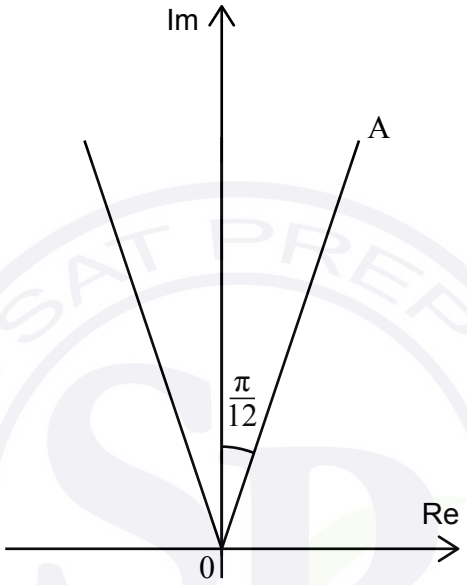
(This question continues on the following page)

(Question 7 continued)

The tree is formed in the following manner.

The line segment joining 0 to $5i$ is rotated clockwise by $\frac{\pi}{12}$ about 0 to form the first branch and then anti-clockwise by $\frac{\pi}{12}$ about 0 to form the second branch. The end of one of these branches is shown as A in **Diagram 1**.

Diagram 1



(a) Find the complex number represented by the point A, giving your answer in exponential form. [3]

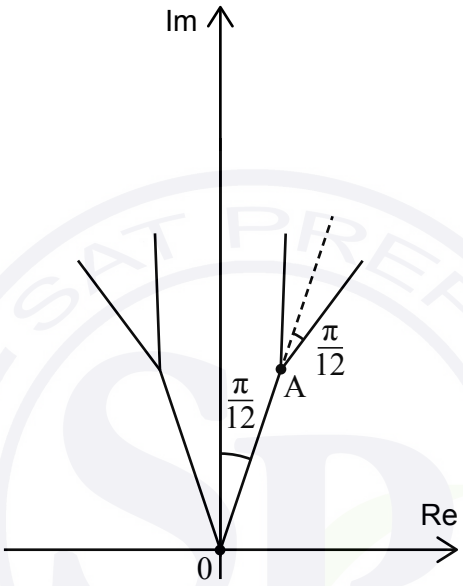
(This question continues on the following page)

(Question 7 continued)

Subsequent branches are created by adding two branches to the end of each branch created in the previous stage.

Each new branch is 70% the length of the previous one. One new branch is rotated $\frac{\pi}{12}$ clockwise from the direction of the previous branch and the other is rotated anticlockwise. This is shown in **Diagram 2**.

Diagram 2



This process is continued using the end of all the branches formed in the previous stage as the centres of rotation for the next stage.

(This question continues on the following page)

(Question 7 continued)

The trees after the 3rd and 10th stage are shown in **Diagrams 3** and **4** respectively.

Diagram 3

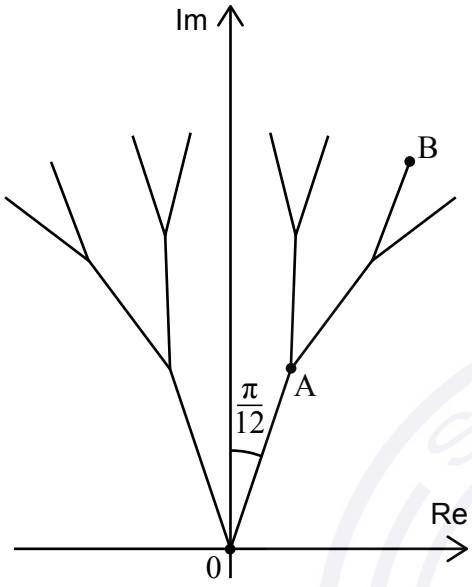
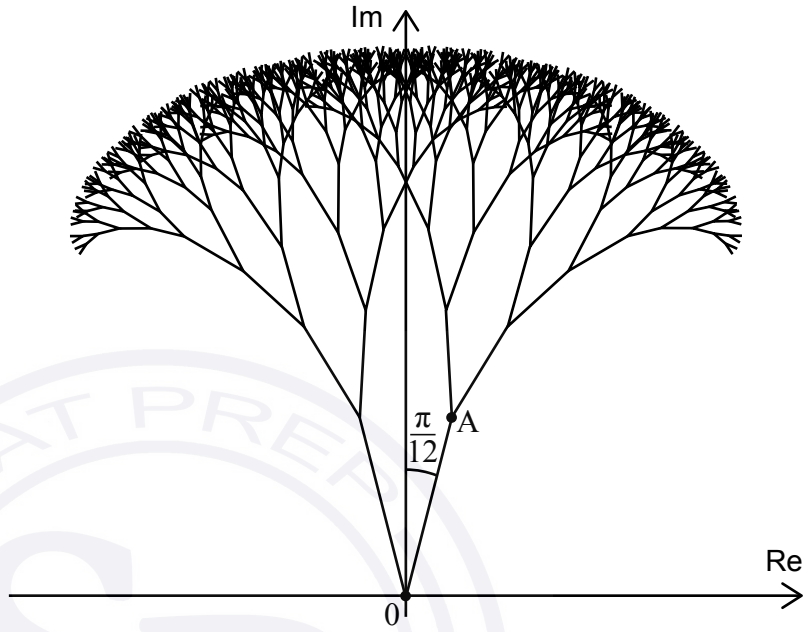


Diagram 4



(b) (i) Show that the total length of all branches in the tree after the second stage (**Diagram 2**) is equal to 24.

(ii) Find the total length of all branches after 10 stages (**Diagram 4**).

(iii) If this process continues indefinitely, state whether there is a limit to the total length of all the branches. Justify your answer. [9]

Let b be the complex number represented by the point B in **Diagram 3**.

(c) (i) Find an expression for b as the sum of three complex numbers, each written in exponential form.

(ii) Hence write down b in the form $a + b\mathrm{i}$, $a, b \in \mathbb{R}$. [5]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

25 October 2024

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8824 – 9741

15 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

A survey was answered by $20\,000$ expatriates (people living in a country that is not their own). The data ranked countries in order of the country they felt was best for expatriates. The highest-ranked country was Switzerland.

These results were compared to happiness scores taken from *The World Happiness Report 2022*. The following table shows this data for the top 10 expatriate countries.

Country	Switzerland	New Zealand	Spain	Australia	Cyprus	Portugal	Ireland	United Arab Emirates	France	Netherlands
Expatriate country rank	1	2	3	4	5	6	7	8	9	10
Happiness score	7.5	7.2	6.5	7.2	6.2	6.0	7.0	6.6	6.7	7.4

(a) For the **happiness score**, find

(i) the upper quartile

(ii) the interquartile range. [4]

(b) Show that Switzerland's happiness score is not an outlier for this data. [3]

(This question continues on the following page)

(Question 1 continued)

The happiness scores were ranked to calculate Spearman's rank correlation coefficient, r_s. These ranks are shown in the following table.

Country	Switzerland	New Zealand	Spain	Australia	Cyprus	Portugal	Ireland	United Arab Emirates	France	Netherlands
Happiness score	7.5	7.2	6.5	7.2	6.2	6.0	7.0	6.6	6.7	7.4
Happiness rank	1	a	b	c	9	10	5	7	6	2

(c) Write down the value of

(i) a

(ii) b

(iii) c. [3]

(d) (i) Find r_s.

(ii) If France's happiness score is upgraded to 6.9, explain why the value of r_s does not change. [3]

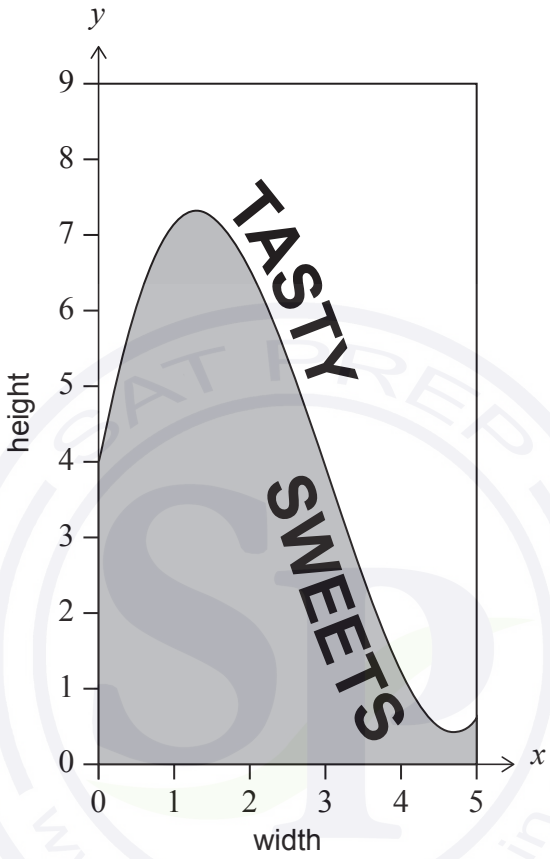
Jose concludes from this data that countries with high happiness scores are likely to be favourite expatriate countries.

(e) State, with a reason, whether Jose's conclusion is appropriate. [1]

2. [Maximum mark: 18]

Sweets are sold in cylindrical containers. A new label for the container is being considered. The label will be a rectangle that is $5\,\text{cm}$ wide and $9\,\text{cm}$ high.

The design on the label is a curve, as shown on the following axes, where one unit represents $1\,\text{cm}$ for both axes.



The values in the table approximate points on the curve, correct to one decimal place.

Width, x	0	1	2	3	4	5
Height, y	4	7.3	6.7	4.0	1.3	0.7

(a) Use the trapezoidal rule with five intervals, and the values given in the table, to estimate the shaded area below the curve. [3]

The curve used in the label design can be modelled by:

$$y = \frac{x^3}{3} - 3x^2 + 6x + 4 \text{, for } 0 \le x \le 5 .$$

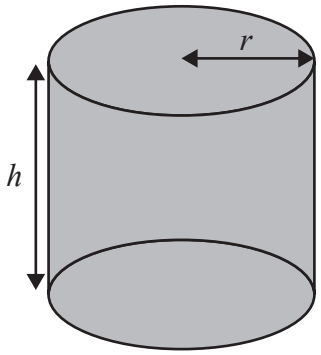
(b) Use this equation to find the area of the shaded region. [2]

(This question continues on the following page)

(Question 2 continued)

The sweets are sold in closed cylindrical containers, with radius r and height h.

diagram not to scale



The whole container is made from one type of material, and it is assumed that the thickness of the material is negligible.

Each container has a volume of $600\,\text{cm}^3$.

(c) Write down an equation, in terms of r and h, that shows this information. [1]

The amount of material used for each container can be modelled by the external surface area of the container.

The external surface area, A, of the container can be expressed as

$$A = 2\pi r^2 + \frac{k}{r}, \text{ where } r > 0.$$

(d) Find the value of k. [4]

(e) (i) Find $\frac{\mathrm{d}A}{\mathrm{d}r}$.

(ii) Given that A has a minimum value, find the value of r that will minimize the material used. [5]

The containers are made so that the surface area is minimized. The $5\,\text{cm}$ by $9\,\text{cm}$ rectangular label is to be glued to the curved surface of the container.

(f) Show that the label will fit on the container. [3]

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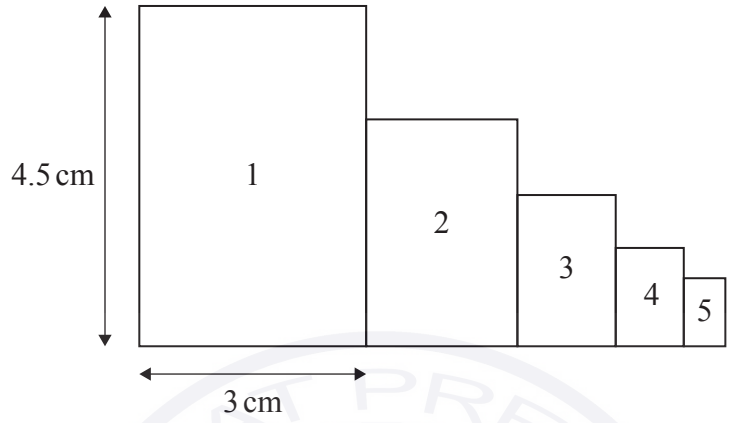
3. [Maximum mark: 15]

Ayaka is creating a design made from a sequence of rectangles. The diagram shows part of her design, using 5 rectangles.

diagram not to scale



The first rectangle has the following dimensions: height $4.5\,\text{cm}$ and width $3\,\text{cm}$.

The dimensions of each subsequent rectangle are $\frac{2}{3}$ of the dimensions of the previous rectangle.

(a) Calculate the width of the 5th rectangle. [2]

(b) Calculate the total width of the design that uses 5 rectangles. [2]

Ayaka continues to add rectangles to her design.

(c) Find the smallest total width that her design cannot exceed. [3]

The width of Ayaka's final design must be at least $8.5\,\text{cm}$ and use the least number of rectangles.

(d) Find the total number of rectangles in her final design. [3]

The decreasing **areas** of the rectangles form a geometric sequence.

(e) Find the common ratio for this sequence of areas. [2]

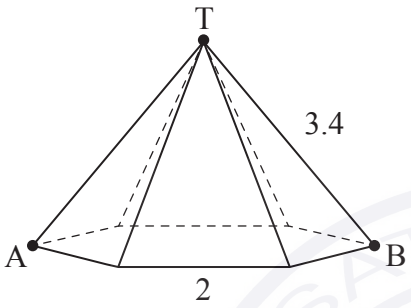
(f) Find the total area of Ayaka's final design. [3]

4. [Maximum mark: 17]

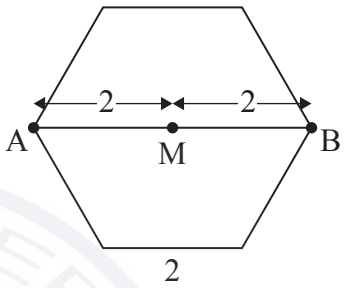
Gaurika is designing a tent in the shape of a right pyramid with a regular hexagonal base, centre M. The length of each side of the base is $2\,\text{m}$, the length of each sloping edge is $3.4\,\text{m}$, and the distance between each vertex on the base and M is $2\,\text{m}$, as shown in the diagrams.

diagrams not to scale

3D view of tent



2D view of base

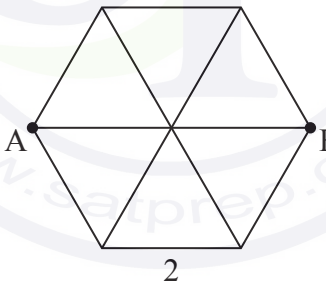


The top of the tent, T, will be supported by a vertical pole from M.

(a) Find the length of the pole, MT. [2]

The hexagonal base can be divided into six equilateral triangles.

diagram not to scale



(b) Find

(i) the area of the base

(ii) the volume of the tent. [5]

(c) Find the value of $M\hat{A}T$. [2]

(This question continues on the following page)

(Question 4 continued)

For extra support, Gaurika decides to attach a rope, with length $2.6\,\text{m}$, to the tent at a point, X, on the edge AT.

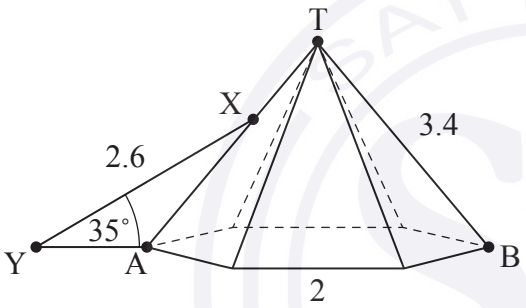
The rope will be fixed to the ground at point Y, such that:

- the rope, $[\text{XY}]$, is straight
- points Y, A and B lie on a straight line
- $\text{A}\hat{\text{Y}}\text{X} = 35^\circ$.

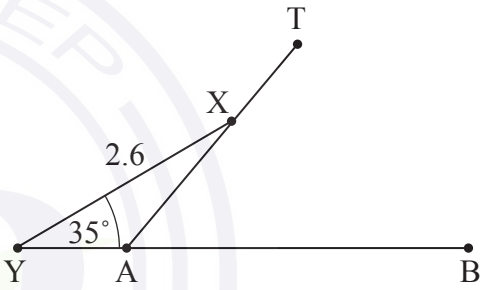
This is shown in the diagrams.

diagrams not to scale

3D view of tent and rope



2D view of triangle AXY



(d) Find AY. [4]

For decoration at night, Gaurika wants to fix a strip of lights from point A to a point, Z, along the rope $[\text{XY}]$.

The strip of lights, $[\text{AZ}]$, is straight and has length $0.9\,\text{m}$.

(e) Find the two possible values of YZ. [4]

5. [Maximum mark: 15]

In this question, all distances are in kilometres and t is in hours.

Let $\begin{pmatrix}1\\0\\0\end{pmatrix}$ be a displacement of $1\,\text{km}$ due east, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ be a displacement of $1\,\text{km}$ due north, and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ be a vertical displacement of $1\,\text{km}$.

Highway 85 in Saudi Arabia is a long, straight, flat road.

Relative to the centre of the town Arar, point O, the position vector of a car, C, travelling along this road is given by:

$$\overrightarrow{\text{OC}} = \begin{pmatrix}10\\-5\\0\end{pmatrix} + t\begin{pmatrix}50\\-33\\0\end{pmatrix}.$$

(a) Find the speed of the car. [2]

The police are testing a long-range drone, D, to monitor cars travelling along this road. The drone is launched at $t = 0$ from the point with position vector $\begin{pmatrix}200\\-100\\0.02\end{pmatrix}$ and flies in a straight line with a constant height of $0.02\,\text{km}$ and a constant velocity of $\begin{pmatrix}-15\\-20\\0\end{pmatrix}$.

(b) Find the angle between the path of the car and the path of the drone. [3]

(c) Write down the position vector, $\overrightarrow{\text{OD}}$, of the drone at time t. [1]

(This question continues on the following page)

(Question 5 continued)

(d) At time t_1, the drone passes through the point with position vector $\begin{pmatrix} 152 \\ p \\ 0.02 \end{pmatrix}$.

Find the value of

(i) t_1

(ii) p. [3]

(e) (i) Find an expression for $\overrightarrow{CD}$, the relative position of the drone from the car.

(ii) Hence, find the shortest distance between the car and the drone. [6]

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6. [Maximum mark: 13]

Juan is creating animations for a website. He uses matrices to transform objects relative to the origin, O.

One matrix that he uses is $\boldsymbol{A} = \begin{pmatrix} \cos(15°) & -\sin(15°) \\ \sin(15°) & \cos(15°) \end{pmatrix}$.

(a) Describe fully the transformation represented by matrix $\boldsymbol{A}$. [1]

(b) Find the smallest value of n such that $\boldsymbol{A}^n = I$, $n \in \mathbb{Z}^+$. [2]

Juan also uses matrix $\boldsymbol{B}$, which represents an enlargement with a scale factor of 1.05, centre $(0, 0)$.

(c) (i) Write down matrix $\boldsymbol{B}$.

(ii) Describe fully the transformation represented by $\boldsymbol{B}^n$, where n is the value found in part (b). [3]

Juan creates a new matrix, $\boldsymbol{C} = \boldsymbol{AB}$.

(d) Find matrix $\boldsymbol{C}$. [2]

Juan creates an animation by repeatedly transforming an object by $\boldsymbol{C}$.

A point, P, on the object is initially at $(1, 0)$. Juan sets the speed of the animation to 6 transformations per second.

(e) Sketch the path of P for the first 4 seconds of motion **and** label the coordinates of the start and end points. [2]

Juan uses a different transformation, $\boldsymbol{T}$, defined by

$$\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To create his animation, he repeatedly transforms an object by $\boldsymbol{T}$. After many transformations, he notices that all points, (x, y), on the object tend towards a single point, (p, q), such that

$$\lim_{a \to \infty} T^a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \text{ where } a \in \mathbb{Z}^+.$$

(f) Find $\begin{pmatrix} p \\ q \end{pmatrix}$, where $p, q \in \mathbb{R}$. [3]

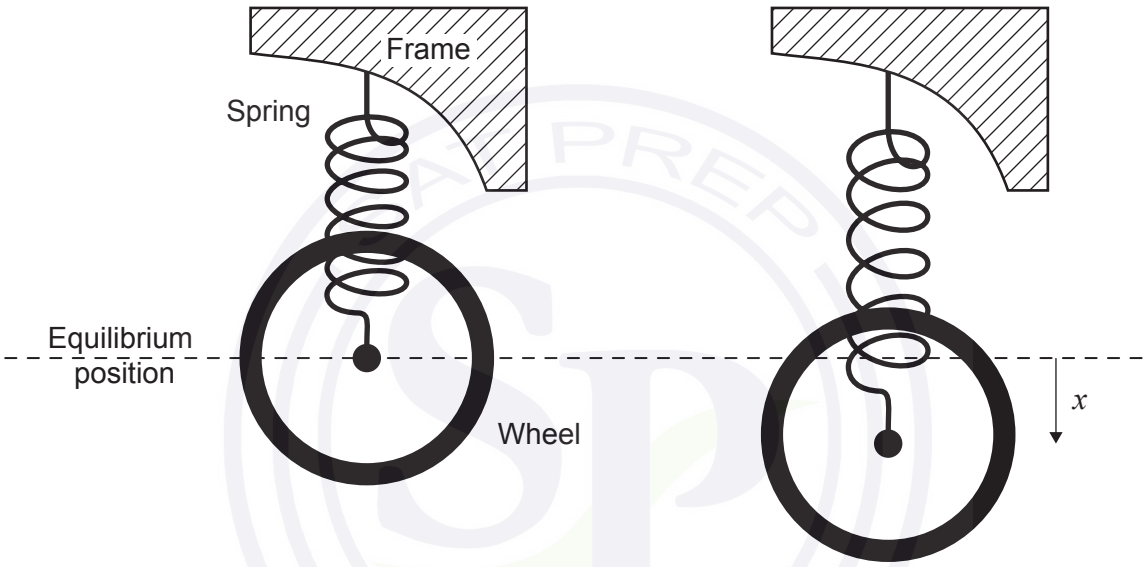
7. [Maximum mark: 18]

The wheel on a motorbike is attached to the frame by a spring. The movement of the spring acts as a **shock absorber**. When the rider sits on the motorbike, the spring compresses, and this position is called the equilibrium position.

When the wheel goes into holes or over bumps in the road, the spring will extend or compress to ensure a smooth ride.

Let x denote the vertical displacement, in centimetres, of the centre of the wheel below the equilibrium position, as shown in the diagram.

diagram not to scale



The vertical displacement of the centre of the wheel, at time t seconds, can be modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = 0 \text{, where } a, b \in \mathbb{R}.$$

Let $y = \frac{\mathrm{d}x}{\mathrm{d}t}$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay$. [2]

(This question continues on the following page)

(Question 7 continued)

The equations $y = \dfrac{\mathrm{d}x}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay$ can be written in matrix form as

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}.$$

A manufacturer wants to compare two springs, Spring 1 and Spring 2, that could be used as shock absorbers.

The differential equation for Spring 1 has $a = 18$ and $b = 77$.

For these values, the eigenvalues of $\boldsymbol{M}$ are -7 and -11.

(b) Find the corresponding eigenvectors. [3]

The manufacturer models **both** springs using the same initial conditions:

$$t = 0,\ x = 5\,\mathrm{cm} \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = 2\,\mathrm{cm\,s^{-1}}.$$

(c) Hence, for Spring 1

(i) find the exact solution for $x(t)$

(ii) sketch the graph of $x(t)$, for $0 \le t \le 1$. [7]

The differential equation for Spring 2 has $a = 18$ and $b = 85$.

For these values, the eigenvalues of $\boldsymbol{M}$ are $-9 \pm 2\mathrm{i}$.

(d) (i) Sketch the phase portrait for Spring 2, indicating the direction of the trajectory.

(ii) Hence, sketch the graph of x against t. [5]

(e) Using your answers to parts (c)(ii) and (d)(ii), give a reason why Spring 1 might make a better shock absorber than Spring 2. [1]

References:

1. InterNations. 2022. *Expat Insider 2022*. [online] Available at: https://www.internations.org/expat-insider/2022/. Source adapted.

Heliwell, J. F., Huang, H., Wang, S. and Norton, M., 2022. Statistical Appendix for "Happiness, benevolence, and trust during COVID-19 and beyond," Chapter 2 of *World Happiness Report 2022*. [pdf online] Available at: https://worldhappiness.report/ed/2022/happiness-benevolence-and-trust-during-covid-19-and-beyond/ [Accessed 13 November 2023]. Source adapted.

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

2 May 2024

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2224 – 7202

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

The company Fred Express delivers packages. From past experience, the time taken, T, to deliver a package follows a normal distribution with mean 64 hours and standard deviation 12 hours.

(a) State $\mathrm{P}(T < 64)$. [1]

(b) Find $\mathrm{P}(44 < T < 64)$. [2]

$30\,\%$ of packages are delivered in less than k hours.

(c) (i) Sketch a diagram of this normal distribution, shading the region that represents $\mathrm{P}(T < k)$.

(ii) Find the value of k. [4]

For quality control, the manager randomly selects five outgoing packages. These selections are independent.

(d) Find the probability that exactly two of these packages are delivered in less than k hours. [3]

Fred Express charges a fixed amount of $\$4.50$ for any package weighing $1\,\mathrm{kg}$ or less. Heavier packages are charged an additional fee of $\$2.00$ per kg. This fee is applied for any weight **in excess of** $1\,\mathrm{kg}$. For example, a $1.5\,\mathrm{kg}$ package is charged an additional $\$1.00$.

(e) Write down an expression for the amount charged to deliver a package of weight $x\,\mathrm{kg}$, where $x > 1$. [2]

(f) Find the amount Fred Express charges for a $5.3\,\mathrm{kg}$ package. [1]

Meiling is charged $\$7.20$ for the delivery of a package.

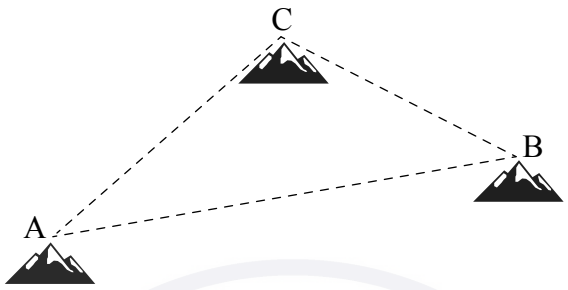
(g) Find the weight of Meiling's package. [2]

2. [Maximum mark: 12]

A national park contains three mountains whose summits are at points A, B and C.

According to a coordinate system, the position of A is $(0, 0, 2.8)$ and the position of B is $(7.2, 5.1, 2.4)$. All the values are in kilometres.

diagram not to scale



(a) (i) Find the vector $\vec{AB}$.

(ii) Hence find AB, the distance between A and B. [3]

The vector $\vec{AC}$ is parallel to the vector $\begin{pmatrix} 1.1 \\ 8.4 \\ 0.2 \end{pmatrix}$.

(b) Find the angle between $\begin{pmatrix} 1.1 \\ 8.4 \\ 0.2 \end{pmatrix}$ and $\vec{AB}$. [5]

The angle between $\vec{BA}$ and $\vec{BC}$ is 55.2°.

(c) Use the sine rule to find AC. [4]

3. [Maximum mark: 13]

A shop uses the following model to estimate n, the number of smoothies sold per day, in terms of x, the price of a single smoothie in pesos.

$$n = \frac{40\,000}{x^2}$$

The maximum number of smoothies the shop can make in a day is 400.

(a) Find the maximum price they could charge per smoothie for the shop to sell 400 in one day. [2]

(b) On a day when the shop sells smoothies at 50 pesos each, use the model to find

(i) the number of smoothies sold.

(ii) the total income from the smoothies sold. [2]

The cost of making each smoothie is 20 pesos. The profit per day (P) is the total income from the sale of smoothies that day minus the cost of making them.

(c) (i) Show that, according to the model, $P = \frac{40\,000}{x} - \frac{800\,000}{x^2}$.

(ii) Find $\frac{\mathrm{d}P}{\mathrm{d}x}$.

(iii) Find the value of x for which $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$.

(iv) Find the number of smoothies sold when the profit is maximized. [9]

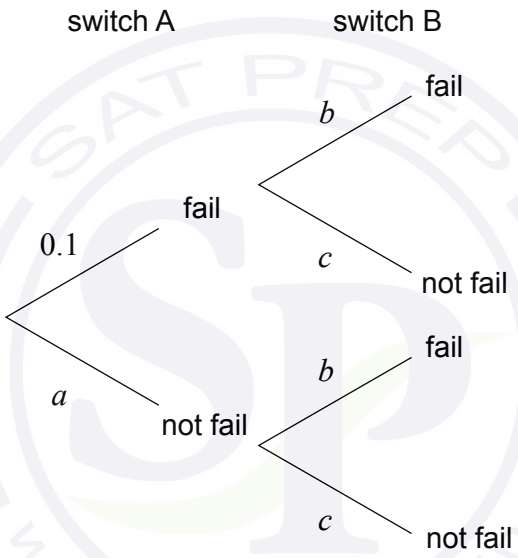
4. [Maximum mark: 12]

A type of generator will only function if a particular switch is working. The generator has a main switch, A, and a 'back up' switch, B.

The manufacturer claims the probability of switch A failing within one month of being fitted is 0.1 and the probability of the cheaper switch B failing within one month is 0.3. Whether or not a switch fails is independent of the state of the other switch.

If both switches fail, the generator needs to shut down to replace the switches. Both switches are replaced after a month of use (whether they have failed or not) or whenever the generator needs to be shut down.

The following tree diagram shows the probabilities of a switch failing within one month of them both being replaced, assuming the manufacturer's claim is correct.



(a) Write down the values of

(i) a.

(ii) b.

(iii) c. [2]

(b) Hence find the probability that the generator needs to shut down within one month of the switches being replaced. [1]

(This question continues on the following page)

(Question 4 continued)

The owner of the generator is suspicious of the switch manufacturer's claims, so they look back through the past 200 occasions when the switches were replaced. The records show whether no switches, one switch or two switches had failed.

The data the owner collected are shown in the following table.

No switch fails	One switch fails	Two switches fail
118	72	10

(c) Perform a χ^2 goodness of fit test at the 5% significance level to test whether the manufacturer's claims are correct using the following hypotheses.

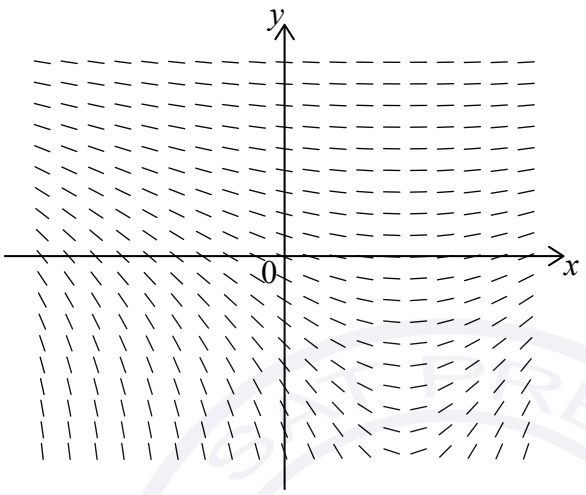
H_0: The manufacturer's claims are correct.
H_1: The manufacturer's claims are not both correct. [9]

5. [Maximum mark: 13]

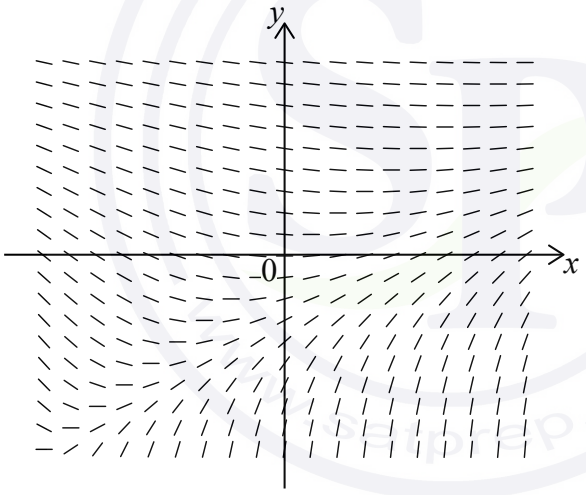
Consider the differential equation $\frac{dy}{dx} = \frac{x}{e^{2y}}$.

(a) Identify which of the following diagrams, **A**, **B** or **C**, represents the slope field for the differential equation. Give a reason for your answer. [2]

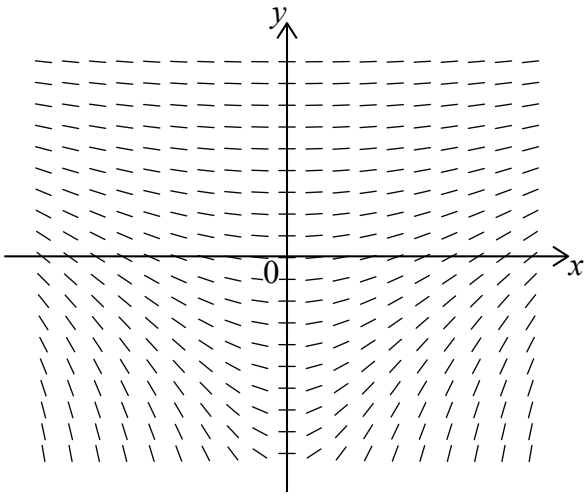
A.



B.



C.



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(Question 5 continued)

It is given that, for a particular solution, $x = 0$ and $y = 0$.

(b) Find an expression for y, in terms of x, for this solution. [7]

(c) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, in terms of x, by differentiating your answer from part (b). [2]

(d) Hence verify that your answer to part (b) is a solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\mathrm{e}^{2y}}$. [2]

6. [Maximum mark: 13]

Taylor is playing a computer game in which they shoot at spaceships and battleships. The number of spaceships they hit per minute can be modelled by a Poisson distribution with mean 4.2. The number of battleships they hit per minute can be modelled by a Poisson distribution with a mean of 2.3. Any single hit occurs independently of all others.

(a) Find the probability Taylor hits

(i) at most 10 spaceships in 2 minutes.

(ii) a total of more than 10 spaceships and battleships in one minute. [5]

Every spaceship that is hit earns Taylor 3 points and every battleship 5 points. Let T be the total points earned in one minute.

(b) Find

(i) $\mathrm{E}(T)$.

(ii) $\mathrm{Var}(T)$. [3]

(c) State one reason why the distribution of T cannot be Poisson. [1]

Taylor intends to play the game for one hour.

(d) Use the central limit theorem to find the probability that Taylor's mean score per minute is greater than 25. [4]

7. [Maximum mark: 14]

The interior of a vase is modelled by rotating the region bounded by the curve $y = \frac{1}{2}x^2 - 1$, and the lines $x = 0$, $y = 0$ and $y = 15$, through 2π radians about the y-axis. The values of x and y are measured in centimetres.

The vase is filled with water to a height of $h\,\text{cm}$.

(a) Find an explicit expression for the volume of water in terms of h. [5]

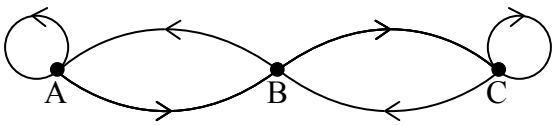
The vase is filled at a rate of $20\,\text{cm}^3\,\text{s}^{-1}$.

(b) Find the time taken to completely fill the vase. [2]

(c) Find the rate at which the height is changing when $h = 10\,\text{cm}$. [7]

8. [Maximum mark: 18]

(a) Write down the adjacency matrix for the directed graph shown below. [2]



(b) Find the total number of walks of length 5 from A to B. [3]

A bird sits on one of three posts, labelled A, B and C, with B between A and C. When the bird moves, it will either fly to an adjacent post or return to the same post according to the following pattern.

- If it is on B, it will fly to A or C, each with a probability of 0.5.
- If it is on A or C, it will return to the same post with a probability of 0.5 or fly to B with a probability of 0.5.

The possible flights of the bird can be represented by the graph in part (a).

(c) (i) Every possible sequence of 5 flights by the bird has the same probability of occurring. State this probability.

(ii) Use your answer to part (b) to find the probability that if the bird was initially on post A, it will be on post B after 5 flights. [3]

A second bird often joins the first on the posts. Their flights now follow the pattern given below.

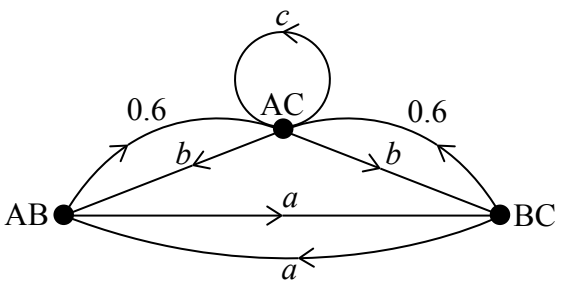
- The birds will never sit on the same post.
- They will always fly from the posts at the same time.
- If they are on adjacent posts, the bird on post B will **always** fly to the vacant end post. The other bird will fly to post B with a probability of 0.4 or return to the same post with a probability of 0.6.
- If they are each on one of the end posts, they will fly to post B with a probability of 0.5 or return to the same post with a probability of 0.5. However, if they both try to fly to post B at the same time, they will see the other one doing so and both will immediately return to the post they were previously on.

(This question continues on the following page)

(Question 8 continued)

The possible flights of the two birds can be represented by the following transition diagram, where the three vertices represent which posts are **occupied**.

diagram not to scale



(d) Write down the value of

(i) a.

(ii) b.

(iii) c. [3]

(e) Given that the birds are initially on posts A and B, find the probability they will be on posts B and C after 5 flights. [4]

The birds continue this pattern of flights for a long period.

(f) Given that the time between flights is always the same, find the post which is sat on least and the proportion of the time it is free. [3]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

2 May 2024

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

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2224 – 7207

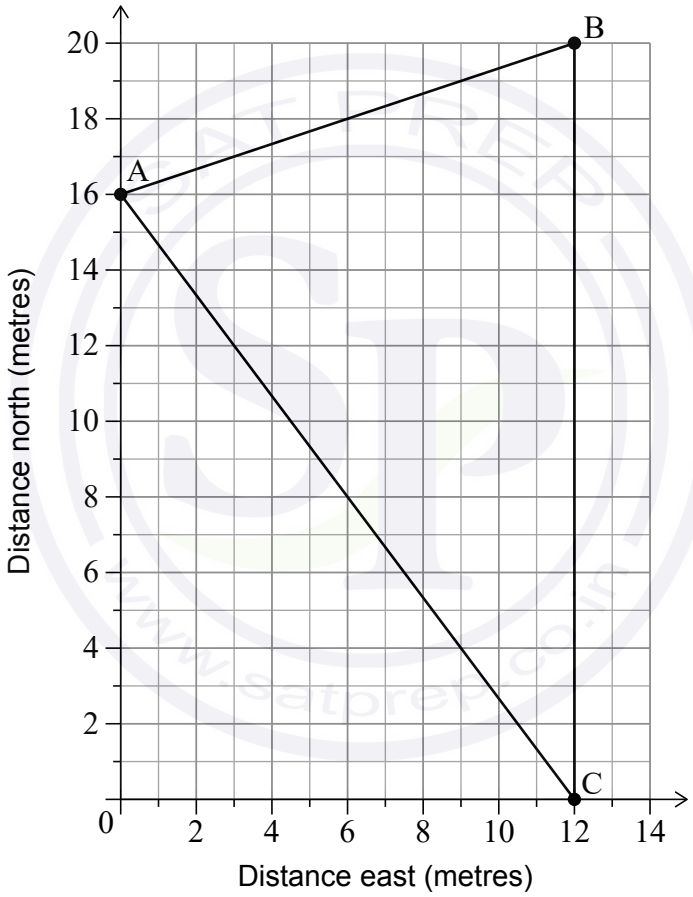
13 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

Mai is at an amusement park. A map of part of the amusement park is represented on the following coordinate axes.

Mai's favourite three attractions are positioned at $A(0, 16)$, $B(12, 20)$ and $C(12, 0)$.
All measurements are in metres.



(a) Write down the distance between B and C. [1]

(b) Calculate the distance between A and B. [2]

(This question continues on the following page)

(Question 1 continued)

Mai is standing at the attraction at B and wants to walk directly to the attraction at A.

(c) Calculate the bearing of A from B. [3]

A drinking fountain is to be installed at a point that is an equal distance from each of the attractions at A, B and C.

(d) (i) Write down the gradient of [AC].

(ii) Write down the mid-point of [AC].

(iii) Hence calculate the coordinates of the drinking fountain. [8]

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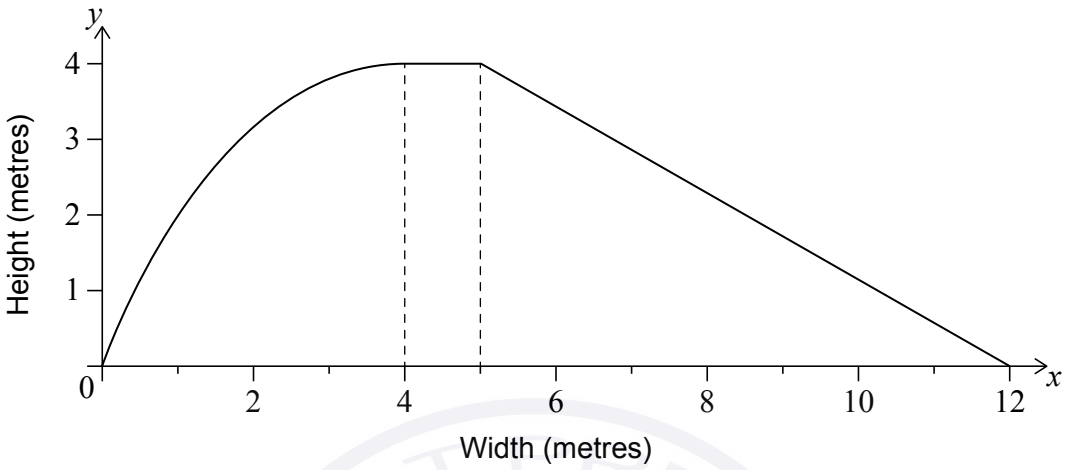
2. [Maximum mark: 15]

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$$f(x) = -\frac{1}{4}x^2 + 2x \text{ for } 0 \le x \le 4.$$

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, \text{ for } 4 \le x \le 5 \\ \frac{48}{7} - \frac{4x}{7}, \text{ for } 5 \le x \le 12 \end{cases}$$

(a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \le x \le 4$. [5]

(b) Find $\int\left(-\frac{1}{4}x^2 + 2x\right)\mathrm{d}x$. [3]

(c) Calculate the exact area under the entire model of the slide, for $0 \le x \le 12$. [4]

(d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a). [3]

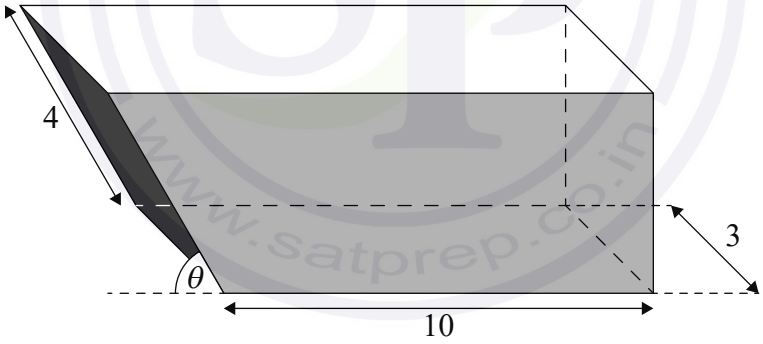
3. [Maximum mark: 22]

A skip is a container used to carry garbage away from a construction site. For safety reasons the garbage must not extend beyond the top of the skip. The maximum volume of garbage to be removed is therefore equal to the volume of the skip.

A particular design of skip can be modelled as a prism with a trapezoidal cross section. For the skip to be transported, it must have a rectangular base of length $10\,\text{m}$ and width $3\,\text{m}$. The length of the sloping edge is fixed at $4\,\text{m}$, and makes an angle of θ with the horizontal.

The following diagram shows such a skip.

diagram not to scale



(This question continues on the following page)

(Question 3 continued)

(a) Find the volume of this skip,

(i) if the length of the top edge of the skip is $11\,\text{m}$.

(ii) if the height of the skip is $3.2\,\text{m}$.

(iii) if θ is $\frac{\pi}{3}$. [9]

(b) Show that the volume, $V\,\text{m}^3$, of the skip is given by

$$24\sin(\theta)(5+\cos(\theta)).$$ [2]

(c) Explain, in context, why $\theta \neq 0$. [1]

(d) (i) Sketch the graph of $V = 24\sin(\theta)\left(5+\cos(\theta)\right),\ 0 < \theta < \frac{\pi}{2}$.

(ii) Find the maximum volume of the skip and the value of θ for which this maximum volume occurs. [4]

(e) Show, by differentiation, that the maximum volume occurs at a value of θ that satisfies the equation $2\cos^2\theta + 5\cos\theta - 1 = 0$. [6]

4. [Maximum mark: 12]

A hygiene inspector lives in Town A and must visit restaurants in five towns (B – F), before returning to A. The inspector must not repeat any of the towns. The distances, in km, between the six towns are shown in the table.

	A	**B**	**C**	**D**	**E**	**F**
A		31	28	26	22	23
B	31		25	20	27	25
C	28	25		19	22	24
D	26	20	19		21	22
E	22	27	22	21		24
F	23	25	24	22	24	

(a) Starting at A, use the nearest neighbour algorithm to find an upper bound for the length of the journey the inspector must take. State the order in which the towns are to be visited. [4]

(b) By deleting A, use Prim's algorithm starting at B to find a lower bound for the length of the inspector's journey. [5]

(c) By considering the minimum spanning tree found in part (b), determine whether the journey given by this lower bound is an achievable solution. [3]

5. [Maximum mark: 16]

The drivers of a delivery company can park their vans overnight either at its headquarters or at home.

Urvashi is a driver for the company. If Urvashi has parked her van overnight at headquarters on a given day, the probability that she parks her van at headquarters on the following day is 0.88. If Urvashi has parked her van overnight at her home on a given day, the probability that she parks her van at home on the following day is 0.92.

(a) Write down a transition matrix, $\boldsymbol{T}$, that shows the movement of Urvashi's van between headquarters and home. [2]

On Monday **morning** she collected her van from headquarters where it was parked overnight.

(b) Find the probability that Urvashi's van will be parked at home at the end of the week on Friday **evening**. [3]

(c) Write down the characteristic polynomial for the matrix $\boldsymbol{T}$. Give your answer in the form $\lambda^2 + b\lambda + c$. [2]

(d) Calculate eigenvectors for the matrix $\boldsymbol{T}$. [4]

(e) Write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [2]

(f) Hence find the long-term probability that Urvashi's van is parked at home. [3]

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6. [Maximum mark: 14]

The kth triangle number, T_k, is defined as $T_k = \sum_{r=1}^{k} r$.

(a) (i) Calculate the value of the fifth triangle number, T_5.

(ii) Determine the formula for T_k in the form $ak^2 + bk$. [4]

(b) (i) Find the value of $T_5 + T_4$.

(ii) Find the simplest expression for $T_k + T_{k-1}$. [3]

A bag contains 15 red discs and 10 blue discs, all identical except for colour. Two discs are chosen at random from the bag without replacement.

(c) Calculate the probability that the two discs are different colours. [3]

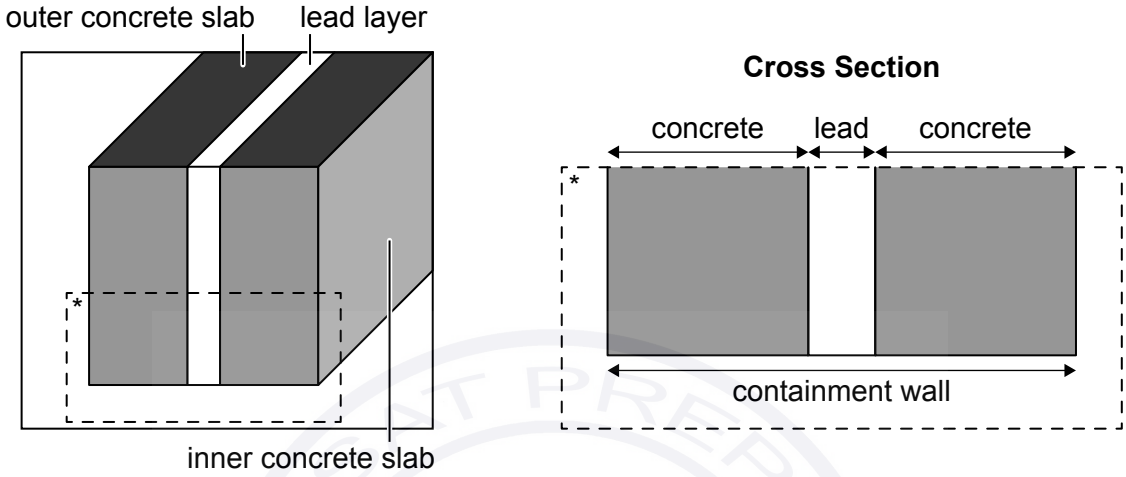
A bag contains T_k red discs and T_{k-1} blue discs, all identical except for colour.
Two discs are chosen at random from the bag without replacement.

(d) Show that the probability that the two discs are different colours is independent of k. [4]

7. [Maximum mark: 17]

Containment walls to protect against radiation are constructed from two parallel concrete slabs that have a layer of lead between them as shown in the diagram.

diagram not to scale



The width of a concrete slab is modelled by a normal distribution with mean $350\,\text{mm}$ and standard deviation $10\,\text{mm}$.

(a) Find the probability that a randomly chosen concrete slab is less than $340\,\text{mm}$ in width. [2]

(b) Find the endpoints of the interval, symmetric about the mean, such that $95\,\%$ of the slabs have a width that lies in this interval. [3]

Stephen assumes the lead layer is also modelled by a normal distribution, but with mean $100\,\text{mm}$ and standard deviation $5\,\text{mm}$ and is independent of the width of the slabs.

Let W be the random variable that represents the total width of the wall, measured in mm.

(c) (i) Given that the widths of any two concrete slabs are independent, calculate Stephen's value for the mean and standard deviation of W.

(ii) Hence find $\text{P}(780 < W < 810)$. [7]

(This question continues on the following page)

(Question 7 continued)

There are concerns that the mean and standard deviation for Stephen's model of the lead layer are incorrect. However, his assumption that the model is normal and the width of the lead is independent of the width of the concrete slabs still holds.

On investigation it is found that the total width of the containment wall is normally distributed with mean $810\,\text{mm}$ and standard deviation $16\,\text{mm}$. The model for the width of a concrete slab does not change.

(d) Use the results for the **sum** of independent random variables to find a revised value for

(i) the mean of the width of the lead layer.

(ii) the standard deviation of the width of the lead layer. [4]

Under this revised model, $80\,\%$ of the lead layers have a width less than $k\,\text{mm}$.

(e) Calculate the value of k. [1]

References:

3. Andyqwe, n.d. *Dumpster truck* [image online] Available at: https://www.gettyimages.co.uk/detail/photo/dumpster-truck-royalty-free-image/157611454 [Accessed 18 April 2023] Source adapted.

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8823 – 7202

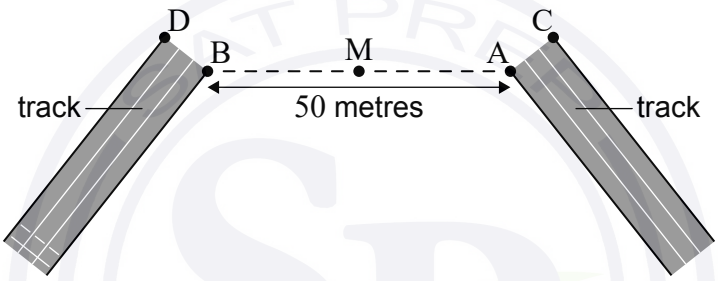
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

Madhu is designing a jogging track for the campus of her school. The following diagram shows an incomplete portion of the track.

Madhu wants to design the track such that the inner edge is a smooth curve from point A to point B, and the other edge is a smooth curve from point C to point D. The distance between points A and B is 50 metres.

diagram not to scale



To create a smooth curve, Madhu first walks to M, the midpoint of [AB].

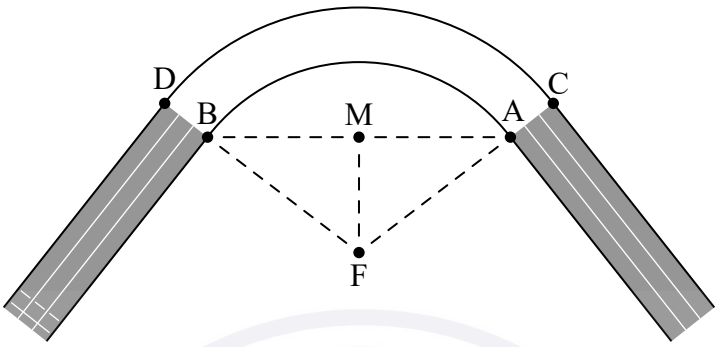
(a) Write down the length of [BM]. [1]

(This question continues on the following page)

(Question 1 continued)

Madhu then walks 20 metres in a direction perpendicular to $[AB]$ to get from point M to point F. Point F is the centre of a circle whose arc will form the smooth curve between points A and B on the track, as shown in the following diagram.

diagram not to scale



(b) (i) Find the length of $[BF]$.

(ii) Find $B\hat{F}M$. [4]

(c) Hence, find the length of arc AB. [3]

The outer edge of the track, from C to D, is also a circular arc with centre F, such that the track is 2 metres wide.

(d) Calculate the area of the curved portion of the track, $ABDC$. [4]

The base of the track will be made of concrete that is $12\,\text{cm}$ deep.

(e) Calculate the volume of concrete needed to create the curved portion of the track. [3]

2. [Maximum mark: 18]

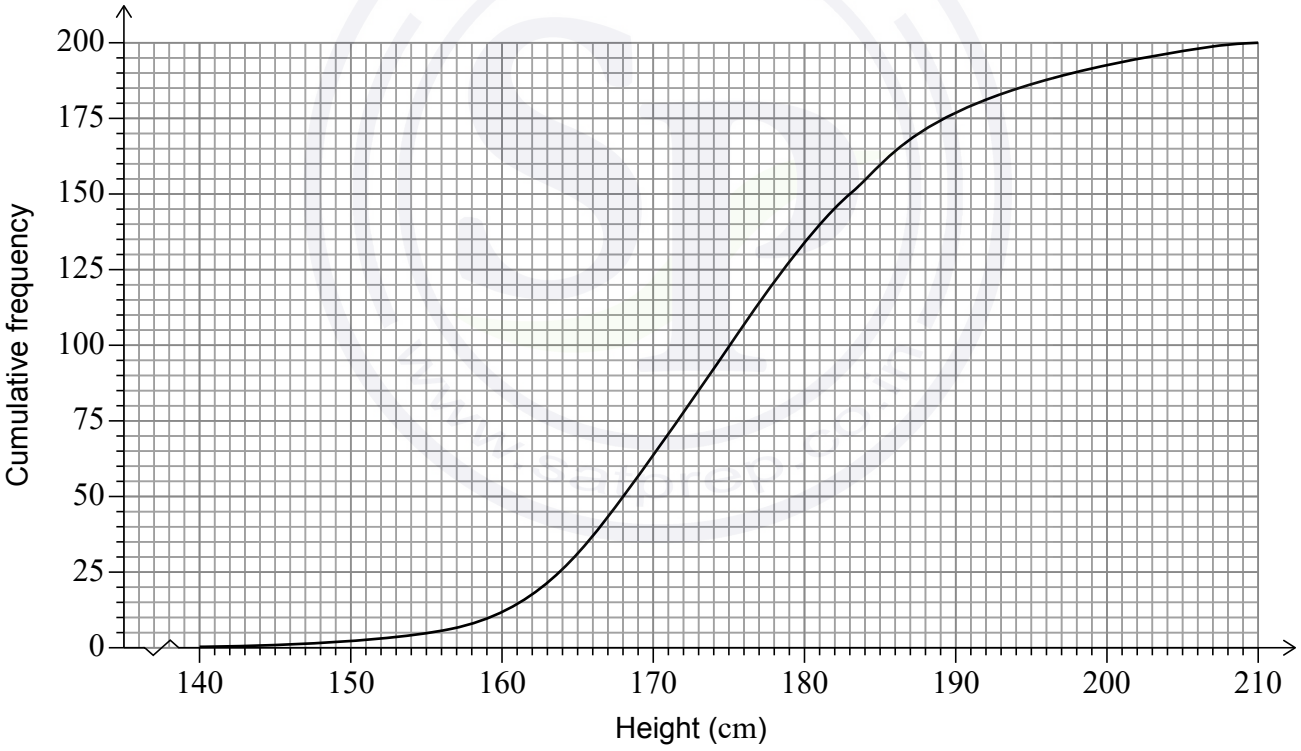
The heights, h, of 200 university students are recorded in the following table.

Height (cm)	Frequency
$140 \leq h < 160$	11
$160 \leq h < 170$	51
$170 \leq h < 180$	68
$180 \leq h < 190$	47
$190 \leq h < 210$	23

(a) (i) Write down the mid-interval value of $140 \leq h < 160$.

(ii) Calculate an estimate of the mean height of the 200 students. [3]

This table is used to create the following cumulative frequency graph.



(b) Use the cumulative frequency curve to estimate the interquartile range. [2]

Laszlo is a student in the data set and his height is $204\,\text{cm}$.

(c) Use your answer to part (b) to estimate whether Laszlo's height is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 2 continued)

It is believed that the heights of university students follow a normal distribution with mean $176\,\text{cm}$ and standard deviation $13.5\,\text{cm}$.

It is decided to perform a χ^2 goodness of fit test on the data to determine whether this sample of 200 students could have plausibly been drawn from an underlying distribution $\text{N}(176, 13.5^2)$.

(d) Write down the null and the alternative hypotheses for the test. [2]

As part of the test, the following table is created.

Height of student (cm)	Observed frequency	Expected frequency
$h < 160$	11	23.6
$160 \leq h < 170$	51	42.1
$170 \leq h < 180$	68	a
$180 \leq h < 190$	47	46.7
$190 \leq h$	23	b

(e) (i) Find the value of a and the value of b.

(ii) Hence, perform the test to a 5% significance level, clearly stating the conclusion in context. [8]

3. [Maximum mark: 16]

Tiffany wants to buy a house for a price of $285\,000$ US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to $15\,\%$ of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a $4\,\%$ nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

(a) (i) Find the original amount of the loan after the down payment is paid. Give the exact answer.

(ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]

(b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays $1300\,\text{USD}$ each month.

(c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than $1300\,\text{USD}$.

(d) Determine the amount of Tiffany's final payment, to two decimal places. [4]

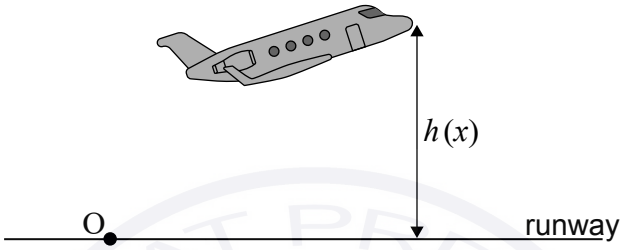
(e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1+150e^{-0.07x}} - 0.06,\ x \geq 0.$$

diagram not to scale



(a) (i) Find $h(0)$.

(ii) Interpret this value in terms of the context. [2]

(b) (i) Find the horizontal asymptote of the graph of $y = h(x)$.

(ii) Interpret this value in terms of the context. [2]

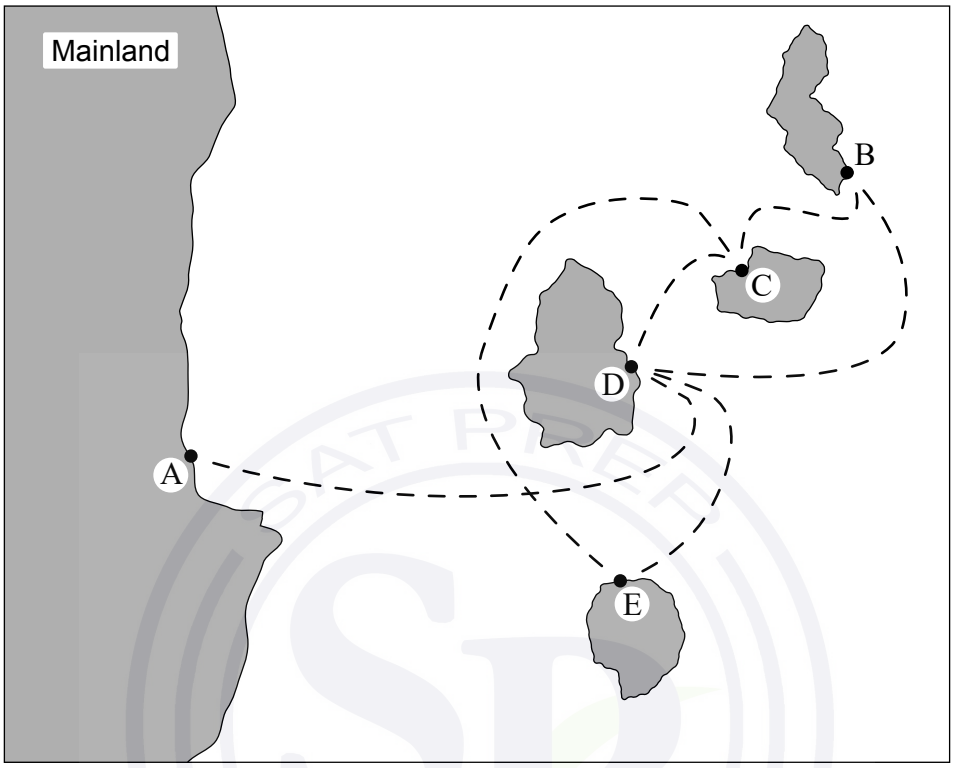
(c) Find $h'(x)$ in terms of x. [4]

A safety regulation recommends that $h'(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least $200\,\text{km}$ horizontally from point O, determine whether the plane is following this safety regulation. [4]

5. [Maximum mark: 18]

The following diagram is a map of a group of four islands and the closest mainland. Travel from the mainland and between the islands is by boat. The scheduled boat routes between the ports A, B, C, D and E are shown as dotted lines on the map.



Let the undirected graph G represent the boat routes between the ports A, B, C, D and E.

(a) Draw graph G. [1]

(b) Graph G can be represented by an adjacency matrix $\boldsymbol{P}$, where the rows and columns represent the ports in alphabetical order.

(i) Given that $\boldsymbol{P}^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$, find the value of a.

(ii) Hence, write down the number of different ways that someone could start at port B and end at port C, using three boat route journeys. [3]

(c) Find a possible Eulerian trail in G, starting at port A. [2]

(This question continues on the following page)

(Question 5 continued)

The cost of a journey on the different boat routes is given in the following table; all prices are given in USD. The cost of a journey is the same in either direction between two ports.

	A	**B**	**C**	**D**	**E**
A				10	
B			20	25	
C		20		50	45
D	10	25	50		30
E			45	30	

Sofia wants to make a trip where she travels on each of the boat routes at least once, beginning and ending at port A.

(d) Find the minimum cost of Sofia's trip. [3]

The boat company decides to add an additional boat route to make it possible to travel on each boat route **exactly** once, starting and ending at the same port.

(e) (i) Identify between which two ports the additional boat route should be added.

(ii) Determine the cost of the additional boat route such that the overall cost of the trip is the same as your answer to part (d). [2]

The boat company plans to redesign which ports are connected by boat routes. Their aim is to have a single boat trip that visits all the islands and minimizes the total distance travelled, starting and finishing at the mainland, A.

The following table shows the distances in kilometres between the ports A, B, C, D and E.

	A	**B**	**C**	**D**	**E**
A		80	90	50	60
B	80		30	70	120
C	90	30		45	100
D	50	70	45		55
E	60	120	100	55	

(f) (i) Use the nearest neighbour algorithm to find an upper bound for the minimum total distance.

(ii) Use the deleted vertex algorithm on port A to find a lower bound for the minimum total distance. [7]

6. [Maximum mark: 14]

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O. In one of his games, an object travels on a straight line L_1 with vector equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where l, $m \in \mathbb{Z}$. [1]

François uses the matrix $\boldsymbol{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2. The object will then travel along L_2.

(b) Find the vector equation of L_2. [4]

François knows that the transformation given by matrix $\boldsymbol{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan\alpha$, $0 \leq \alpha < \pi$

(c) Write down the matrix that represents

(i) the rotation.

(ii) the enlargement. [2]

(d) The matrix $\boldsymbol{R}$ represents the reflection. Write down $\boldsymbol{R}$ in terms of α. [1]

(e) Given that $\boldsymbol{T} = \boldsymbol{RX}$,

(i) use your answers to part (c) to find matrix $\boldsymbol{X}$.

(ii) hence, find the value of α. [6]

7. [Maximum mark: 17]

The city of Melba has an adult population of four million people. It is assumed that the weights of adults in Melba can be modelled by a normal distribution with mean $72\,\text{kg}$ and standard deviation $10\,\text{kg}$.

(a) If 10 adults in Melba are chosen independently and at random, find the probability that more than 3 of them have a weight greater than $85\,\text{kg}$. [4]

Laetitia runs a travel agency in Melba. The elevator to her office is designed to hold a maximum of 8 people.

(b) Write down a probability distribution that models the total weight of 8 adults chosen independently and at random from Melba. [3]

The total weight of 8 adults exceeds w on less than $1\,\%$ of all occasions that 8 adults enter the elevator.

(c) Find the value of w. [2]

A newspaper claims that $42\,\%$ of the adults in Melba who go on holiday choose to go abroad. Laetitia believes that this is an overestimation of the true number. During the past month, Laetitia found that 67 of her clients chose a holiday abroad, and 133 chose a holiday that was not abroad.

(d) Laetitia decides to perform a test using the binomial distribution on her data for the population proportion, p, that go on holiday abroad.

(i) State **two** assumptions that Laetitia makes in order to conduct the test.

(ii) Write down the null and the alternative hypotheses for Laetitia's test, in terms of p.

(iii) Using the data from Laetitia's sample, perform the test at a $5\,\%$ significance level to determine whether Laetitia's belief is reasonable. [8]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2223 – 7207

11 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Temperature °C (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates $(1708, 8.73)$ and $(1958, 9.45)$.

(a) Calculate the gradient of the straight line that passes through these two points. [2]

(b) (i) Interpret the meaning of the gradient in the context of the question.

(ii) State appropriate units for the gradient. [2]

(c) Find the equation of this line giving your answer in the form $y = mx + c$. [2]

(d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Thandizo uses linear regression to obtain a model for the data.

(e) (i) Find the equation of the regression line y on x.

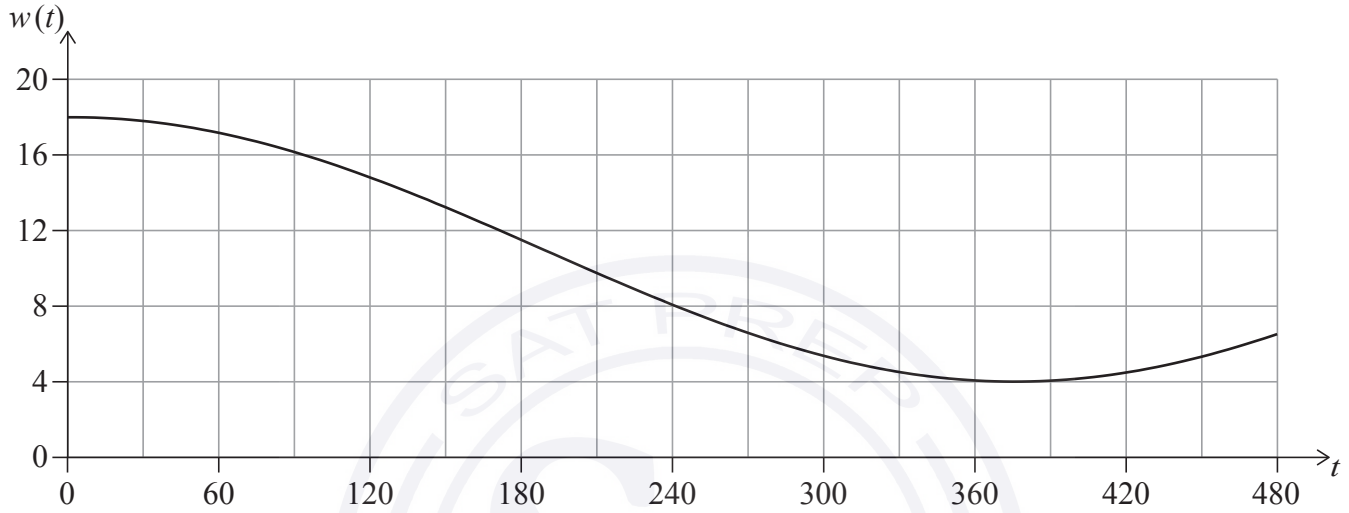
(ii) Find the value of r, the Pearson's product-moment correlation coefficient. [3]

(f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

2. [Maximum mark: 15]

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^\circ) + d$ where t is the length of time, in minutes, after $06{:}00$.

On 20 January, the first high tide occurs at $06{:}00$, at which time the depth of water is $18\,\text{m}$. The following low tide occurs at $12{:}15$ when the depth of water is $4\,\text{m}$. This is shown in the diagram.



(a) Find the value of a. [2]

(b) Find the value of d. [2]

(c) Find the period of the function in minutes. [3]

(d) Find the value of b. [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\text{m}$.

(e) Find the latest time before $12{:}00$, to the nearest minute, that Naomi can enter the harbour. [4]

(f) Find the length of time (in minutes) between $06{:}00$ and $15{:}00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

3. [Maximum mark: 17]

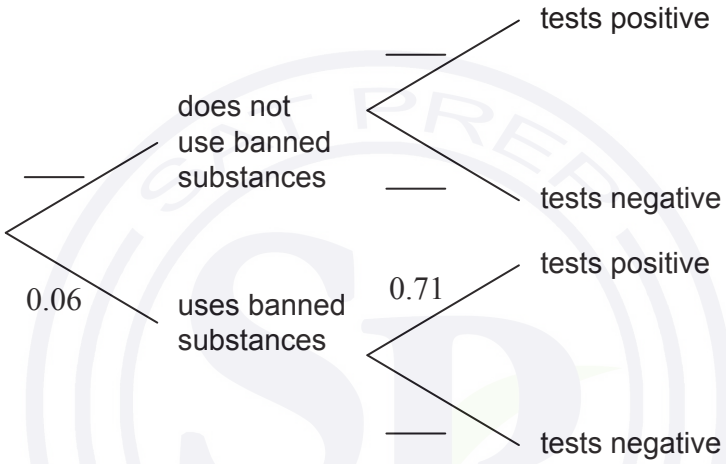
A large international sports tournament tests their athletes for banned substances.
They interpret a positive test result as meaning that the athlete uses banned substances.
A negative result means that they do not.

The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

(a) Using the information given, **copy** (into your answer booklet) and complete the following tree diagram. [2]



(b) (i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative.

(ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [4]

(c) (i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result.

(ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [5]

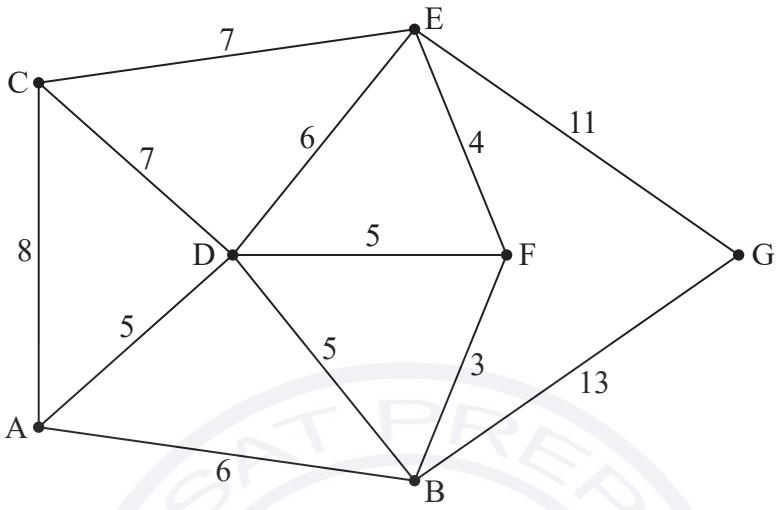
Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use banned substances.

(d) Calculate the probability that none of the athletes in Team X will test positive. [4]

(e) Determine the probability that more than 2 athletes in Team X will test positive. [2]

4. [Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	***a***	***b***
D	5	5	7		6	5	***c***
E	11	7	7	6		4	11
F	9	3	***a***	5	4		***d***
G	19	13	***b***	***c***	11	***d***	

(This question continues on the following page)

(Question 4 continued)

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

(c) Use the nearest neighbour algorithm, starting at vertex G, to find an upper bound for the travelling salesman problem. [3]

(d) (i) Sketch a minimum spanning tree for the subgraph with vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}$.

(ii) Write down the total weight of the minimum spanning tree. [4]

(e) Hence find a lower bound for the travelling salesman problem. [2]

(f) Explain one way in which an improved lower bound could be found. [1]

It is found that the optimum solution starting at A is actually $\mathrm{A–C–E–G–B–F–D–A}$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

5. [Maximum mark: 15]

Goran is interested in the number of sightings of a particular bird each week in the 50 weeks following the first day of September. He collects some data which is shown in the table.

Number of sightings	0	1	2	3	4	5	More than 5
Number of weeks	8	16	13	8	3	2	0

The sample mean number of sightings per week for this data is 1.76.

(a) Calculate the unbiased estimate of the population variance of sightings per week. [3]

Goran believes that the data follows a Poisson distribution.

(b) State why your answer to part (a) supports Goran's belief. [1]

Goran decides to test at the 5% significance level to see if his belief is correct.

His null hypothesis is $X \sim \text{Po}(m)$, where the random variable, X, is defined as the number of sightings per week.

Goran estimates parameter m to be the mean of the sample, 1.76. He calculates the expected frequencies for sightings per week in the 50 weeks after the first day of September. These are shown to two decimal places in the following table.

Number of sightings	0	1	2	3	4	5 or more
Expected frequencies	8.60	15.14	13.32	7.82	j	k

(c) Find the value of

(i) j;

(ii) k. [5]

(d) State a reason why Goran should combine groups to conduct his significance test. [1]

(e) Write down the degrees of freedom for the test. [1]

(f) Find the p-value for the test. [2]

(g) State the conclusion of the test. Justify your answer. [2]

6. [Maximum mark: 15]

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5y - 0.05x\,,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 0.05y\,,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $\boldsymbol{A} = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0$, $b \neq 0$. [4]

(b) (i) State what $a \neq 0$ indicates about the path of the speedboat.

(ii) State what the sign of a indicates about the path of the speedboat. [2]

At time $t = 0$, the speedboat has position $(20, 0)$.

(c) At time $t = 0$, find the value of

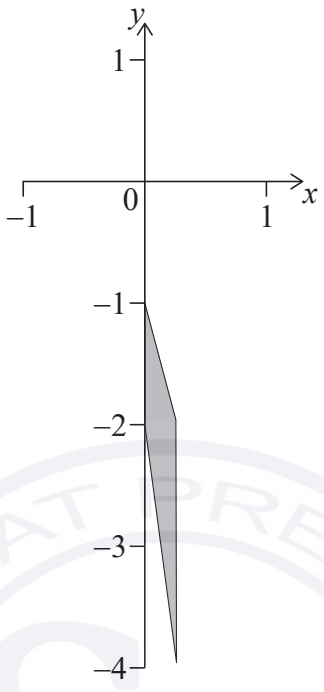
(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [5]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

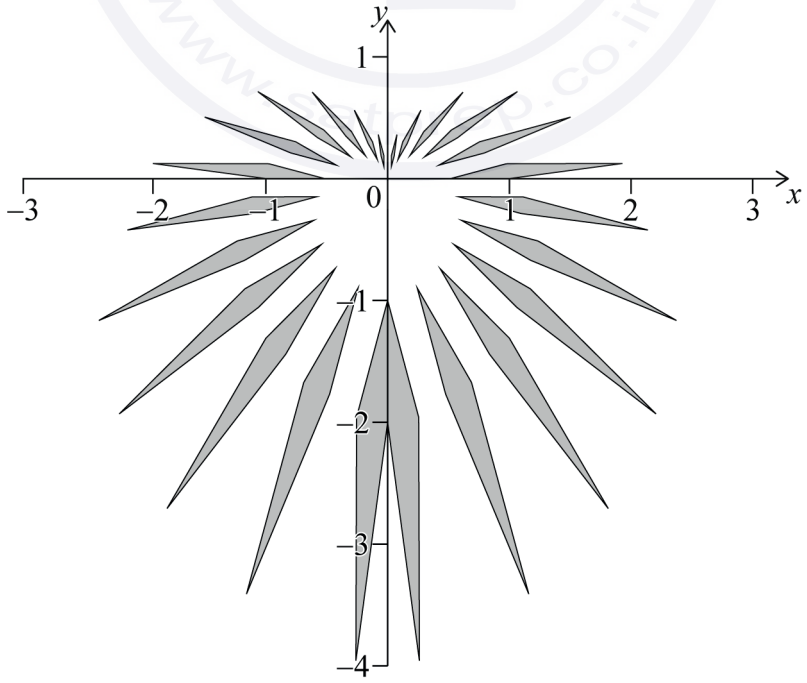
7. [Maximum mark: 18]

A trapezoid, Q, has vertices $(0, -1)$, $(0, -2)$, $(\sin 15°, -3-\cos 15°)$, $(\sin 15°, -1-\cos 15°)$ as shown.



(a) Show that the area of the trapezoid is $\frac{3}{2}\sin 15°$. [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the y-axis is a line of symmetry.



(This question continues on the following page)

(Question 7 continued)

The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$\boldsymbol{M}_k = \begin{pmatrix} \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) & -\left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) \\ \left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) & \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) \end{pmatrix}$$

where $k = 0, 1, 2, 3, \ldots, 11$.

(b) (i) Find the matrix $\boldsymbol{M}_6$. Give your answer in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Q}$.

(ii) Hence find the coordinates of the image of the vertex $(0, -1)$ after it is transformed by the matrix $\boldsymbol{M}_6$. [4]

The matrix $\boldsymbol{M}_k$ can be expressed as the product of a rotation matrix and an enlargement matrix.

(c) Write down, in terms of k,

(i) the rotation matrix;

(ii) the enlargement matrix;

(iii) the angle of the rotation;

(iv) the scale factor of the enlargement. [4]

(d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix $\boldsymbol{M}_k$ in terms of k. [2]

(e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix $\boldsymbol{N}_k$, where $k = 0, 1, 2, 3, \ldots, 11$.

(f) Write down the matrix $\boldsymbol{N}_k$ as a product of two matrices. [2]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2223 – 7212

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

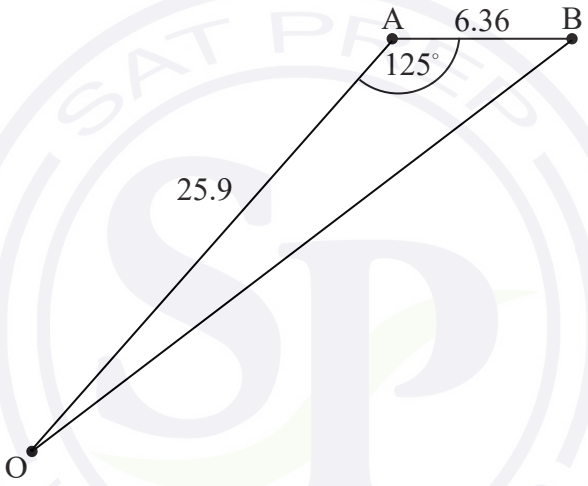
1. [Maximum mark: 13]

The diagram shows points in a park viewed from above, at a specific moment in time.

The distance between two trees, at points A and B, is $6.36\,\text{m}$.

Odette is playing football in the park and is standing at point O, such that $\text{OA} = 25.9\,\text{m}$ and $\text{O}\hat{\text{A}}\text{B} = 125^\circ$.

diagram not to scale



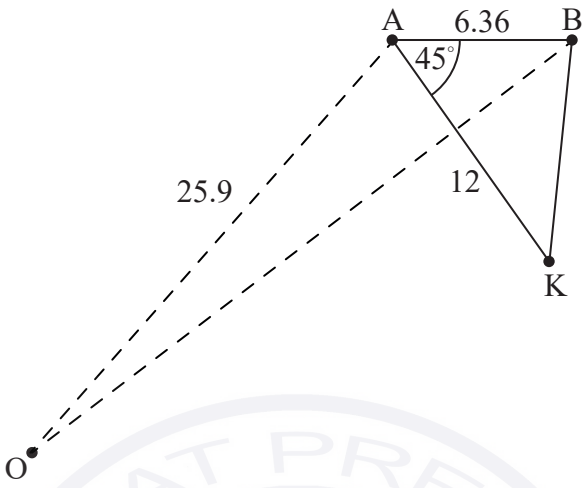
(a) Calculate the area of triangle AOB. [3]

(This question continues on the following page)

(Question 1 continued)

Odette's friend, Khemil, is standing at point K such that he is $12\,\text{m}$ from A and $\text{K}\hat{\text{A}}\text{B} = 45^\circ$.

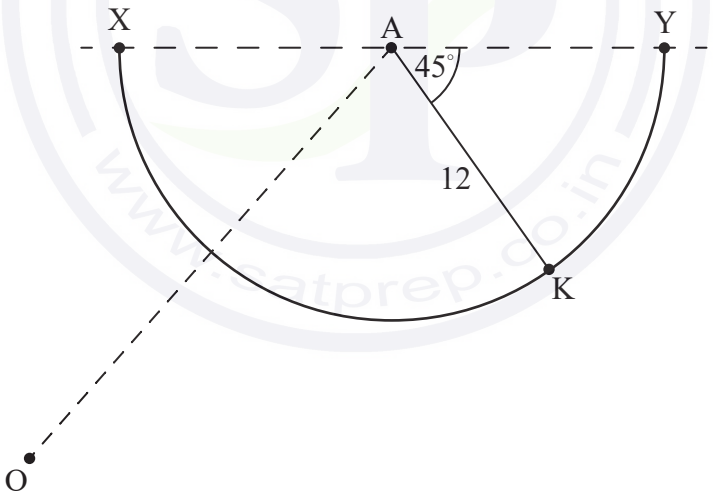
diagram not to scale



(b) Calculate Khemil's distance from B. [3]

XY is a semicircular path in the park with centre A, such that $\text{K}\hat{\text{A}}\text{Y} = 45^\circ$. Khemil is standing on the path and Odette's football is at point X. This is shown in the diagram below.

diagram not to scale



The length $\text{KX} = 22.2\,\text{m}$, $\text{K}\hat{\text{O}}\text{X} = 53.8^\circ$ and $\text{O}\hat{\text{K}}\text{X} = 51.1^\circ$.

(c) Find whether Odette or Khemil is closer to the football. [4]

Khemil runs along the semicircular path to pick up the football.

(d) Calculate the distance that Khemil runs. [3]

2. [Maximum mark: 12]

A scientist is conducting an experiment on the growth of a certain species of bacteria.

The population of the bacteria, P, can be modelled by the function

$$P(t) = 1200 \times k^t,\ t \geq 0,$$

where t is the number of hours since the experiment began, and k is a positive constant.

(a) (i) Write down the value of $P(0)$.

(ii) Interpret what this value means in this context. [2]

3 hours after the experiment began, the population of the bacteria is $18\,750$.

(b) Find the value of k. [2]

(c) Find the population of the bacteria 1 hour and 30 minutes after the experiment began. [2]

The scientist conducts a second experiment with a different species of bacteria.
The population of this bacteria, S, can be modelled by the function

$$S(t) = 5000 \times 1.65^t,\ t \geq 0,$$

where t is the number of hours since both experiments began.

(d) Find the value of t when the two populations of bacteria are equal. [2]

It takes 2 hours and m minutes for the number of bacteria in the second experiment to reach $19\,000$.

(e) Find the value of m, giving your answer as an integer value. [4]

3. [Maximum mark: 16]

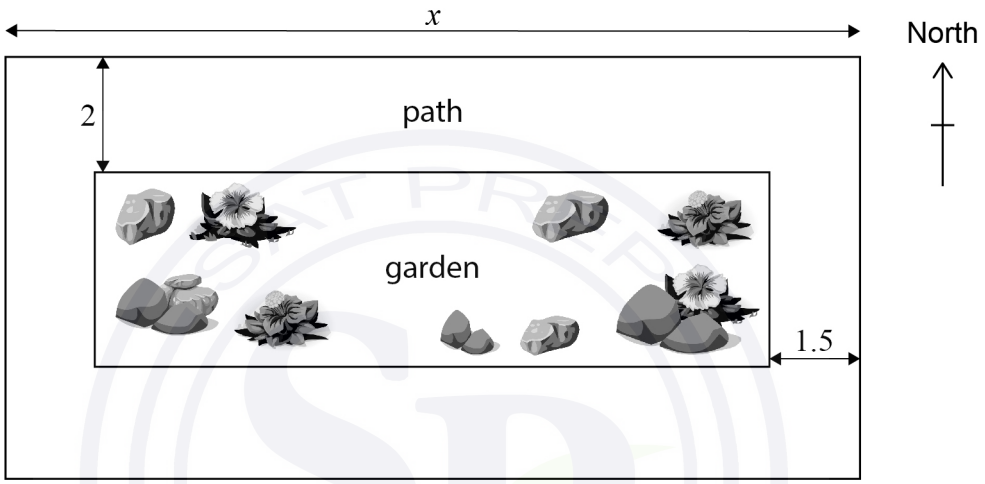
A particular park consists of a rectangular garden, of area $A\,\text{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\text{m}^2$.

The width of the path at the north and south side of the park is $2\,\text{m}$.

The width of the path at the west and east side of the park is $1.5\,\text{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\text{m}^2$. [4]

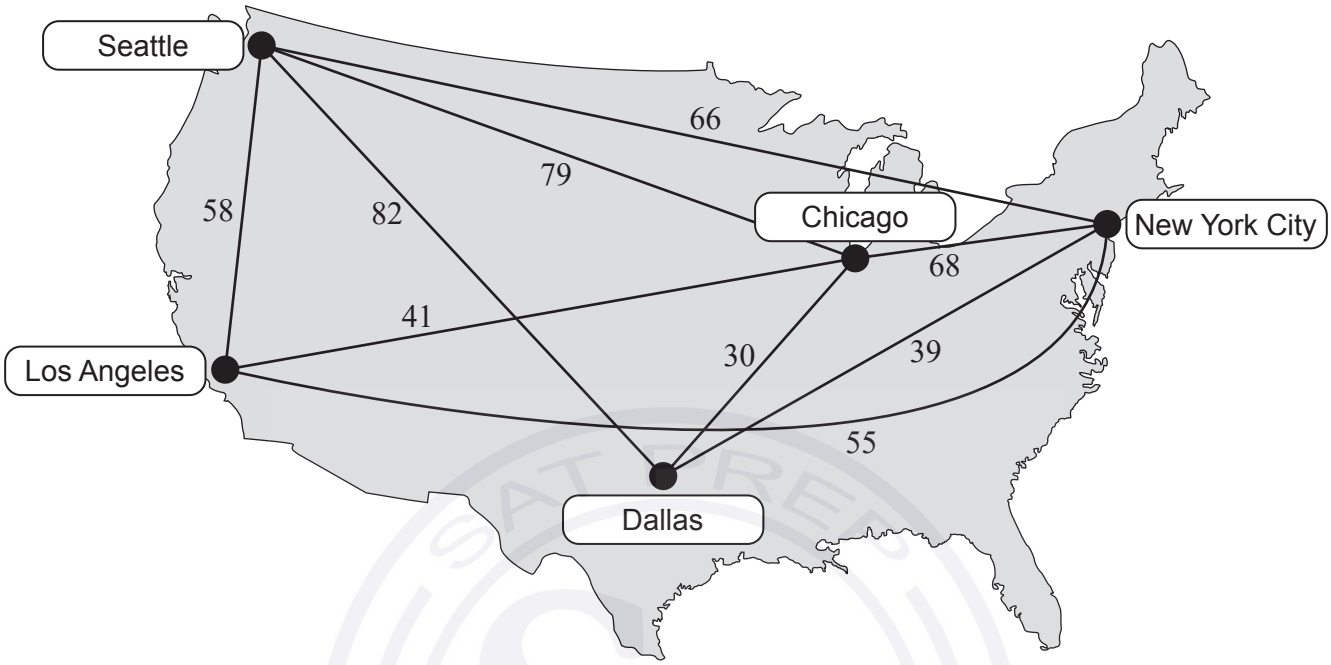
(c) Find an expression for $\frac{\text{d}A}{\text{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

4. [Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($) between cities.



(a) Explain why the graph can be described as "connected", but not "complete". [2]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $26. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

(This question continues on the following page)

(Question 4 continued)

(e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem.

(ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

5. [Maximum mark: 14]

The three countries of Belgium, Germany and The Netherlands meet at a single point called Vaalserberg.

To support future transport planning, a $10\,\text{km}$ circle was drawn around Vaalserberg on a map. A study was carried out over five years to determine what percentage of people living in each of these countries (within the $10\,\text{km}$ circular region) either stayed in their own country or moved to another country within the circle.

From this study, the following movements were observed during the five years.
- From Belgium, $5\,\%$ moved to Germany, and $0.5\,\%$ moved to The Netherlands.
- From Germany, $2\,\%$ moved to The Netherlands, and $1.5\,\%$ moved to Belgium.
- From The Netherlands, $3\,\%$ moved to Germany, and $2\,\%$ moved to Belgium.

All additional population changes within the circular region may be ignored.

(a) Represent the above information in a transition matrix $\boldsymbol{T}$. [3]

At the end of the study, the population of the Belgian side was $26\,000$, the population of the German side was $240\,000$, and the population of The Netherlands side was $50\,000$.

(b) By using $\boldsymbol{T}$, find the expected population of the German side of Vaalserberg 30 years after the end of the study. [4]

For matrix $\boldsymbol{T}$ there exists a steady state vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where u_1, u_2 and u_3 are the proportions of the total population on the Belgian side, the German side and The Netherlands side respectively.

The steady state vector $\boldsymbol{u}$ may be found by solving a system of equations.

(c) (i) Determine these equations that are to be solved.

(ii) By solving your system of equations, find $\boldsymbol{u}$. [3]

(d) Use your answer to part (c)(ii) to determine the long-term expected population of the German side. [2]

(e) Suggest two reasons why your answer to part (d) is not likely to be accurate. You may comment on both the model and the situation in context. [2]

6. [Maximum mark: 18]

The gardener in a local park suggested that the number of snails found in the park can be modelled by a Poisson distribution.

(a) Suggest two observations that the gardener may have made that led him to suggest this model. [2]

Now assume that the model is valid and that the mean number of snails per m^2 is 0.2.
The gardener inspects, at random, a $12\,\mathrm{m}^2$ area of the park.

(b) Find the probability that the gardener finds exactly four snails. [3]

(c) Find the probability that the gardener finds fewer than three snails. [2]

(d) Find the probability that, in three consecutive inspections, the gardener finds at least one snail per inspection. [3]

Following heavy rain overnight, the gardener wished to determine whether the number of snails found in a random $12\,\mathrm{m}^2$ area of the park had increased.

(e) State the hypotheses for the test. [2]

(f) Find the critical region for the test at the $1\,\%$ significance level. [3]

(g) Given that the mean number of snails per m^2 has actually risen to 0.75, find the probability that the gardener makes a Type II error. [3]

7. [Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 6y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 9x - y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{p}_1 \mathrm{e}^{\lambda_1 t} + B\boldsymbol{p}_2 \mathrm{e}^{\lambda_2 t}$, where A, B, λ_1, λ_2 $(\lambda_2 > \lambda_1)$ are scalar constants and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

(c) Determine the value of A and the value of B. [2]

(d) State the long-term ratio of male fruit flies to female fruit flies. [1]

(e) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at time $t = 0$. [3]

(f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Tuesday 1 November 2022 (morning)

2 hours

Instructions to candidates

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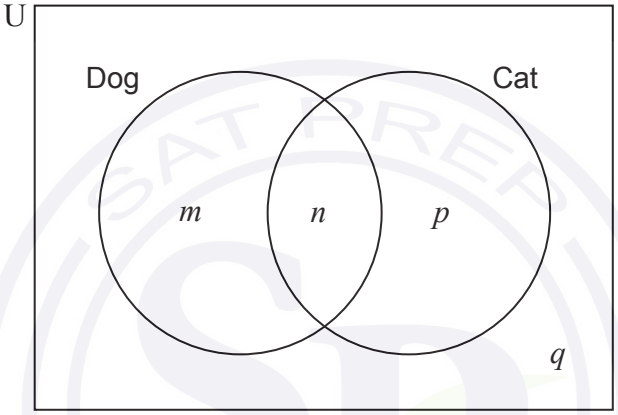
8822 – 7202

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

At Mirabooka Primary School, a survey found that 68% of students have a dog and 36% of students have a cat. 14% of students have both a dog and a cat.

This information can be represented in the following Venn diagram, where m, n, p and q represent the percentage of students within each region.



(a) Find the value of

(i) m.

(ii) n.

(iii) p.

(iv) q. [4]

(b) Find the probability that a randomly chosen student

(i) has a dog but does not have a cat.

(ii) has a dog given that they do not have a cat. [3]

(This question continues on the following page)

(Question 1 continued)

Each year, one student is chosen randomly to be the school captain of Mirabooka Primary School.

Tim is using a binomial distribution to make predictions about how many of the next 10 school captains will own a dog. He assumes that the percentages found in the survey will remain constant for future years and that the events "being a school captain" and "having a dog" are independent.

Use Tim's model to find the probability that in the next 10 years

(c) (i) 5 school captains have a dog.

(ii) more than 3 school captains have a dog.

(iii) exactly 9 school captains in succession have a dog. [7]

John randomly chooses 10 students from the survey.

(d) State why John should not use the binomial distribution to find the probability that 5 of these students have a dog. [1]

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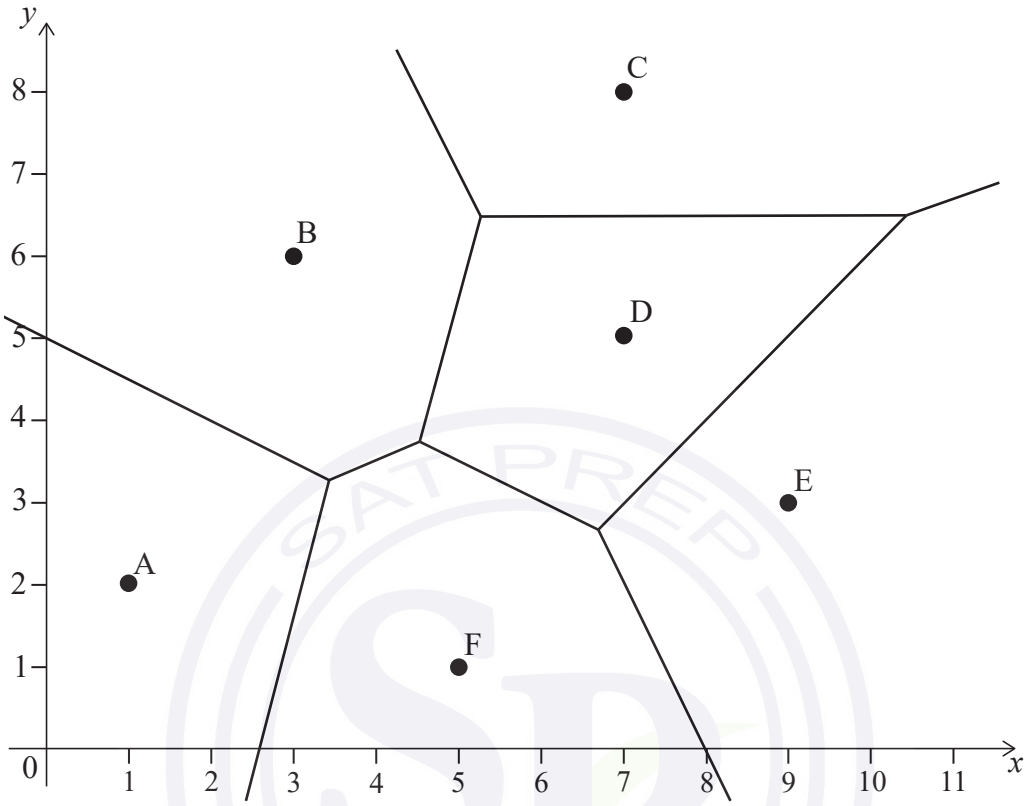
2. [Maximum mark: 13]

Six restaurant locations (labelled A, B, C, D, E and F) are shown, together with their Voronoi diagram. All distances are measured in kilometres.



(a) Elena wants to eat at the closest restaurant to her. Write down the restaurant she should go to, if she is at

(i) $(2, 7)$.

(ii) $(0, 1)$, when restaurant A is closed. [2]

Restaurant C is at $(7, 8)$ and restaurant D is at $(7, 5)$.

(b) Find the equation of the perpendicular bisector of CD. [2]

Restaurant B is at $(3, 6)$.

(c) Find the equation of the perpendicular bisector of BC. [5]

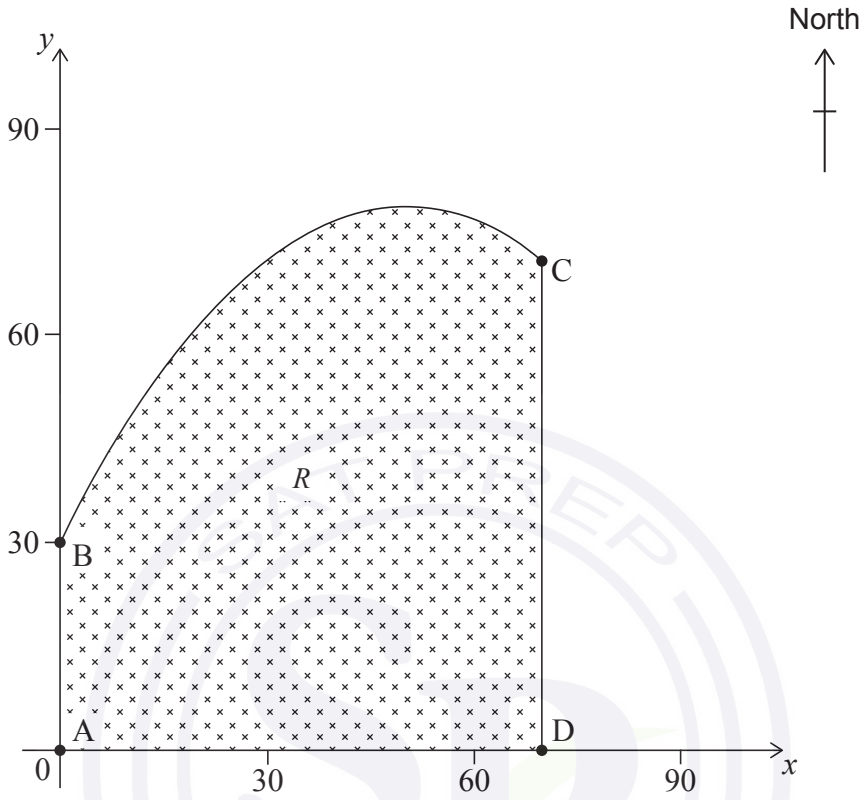
(d) Hence find

(i) the coordinates of the point which is of equal distance from B, C and D.

(ii) the distance of this point from D. [4]

3. [Maximum mark: 17]

Linda owns a field, represented by the shaded region R. The plan view of the field is shown in the following diagram, where both axes represent distance and are measured in metres.



The segments [AB], [CD] and [AD] respectively represent the western, eastern and southern boundaries of the field. The function, $f(x)$, models the northern boundary of the field between points B and C and is given by

$$f(x) = \frac{-x^2}{50} + 2x + 30, \text{ for } 0 \le x \le 70.$$

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the point on the field that is furthest north. [5]

Point A has coordinates $(0, 0)$, point B has coordinates $(0, 30)$, point C has coordinates $(70, 72)$ and point D has coordinates $(70, 0)$.

(b) (i) Write down the integral which can be used to find the area of the shaded region R.

(ii) Find the area of Linda's field. [4]

(This question continues on the following page)

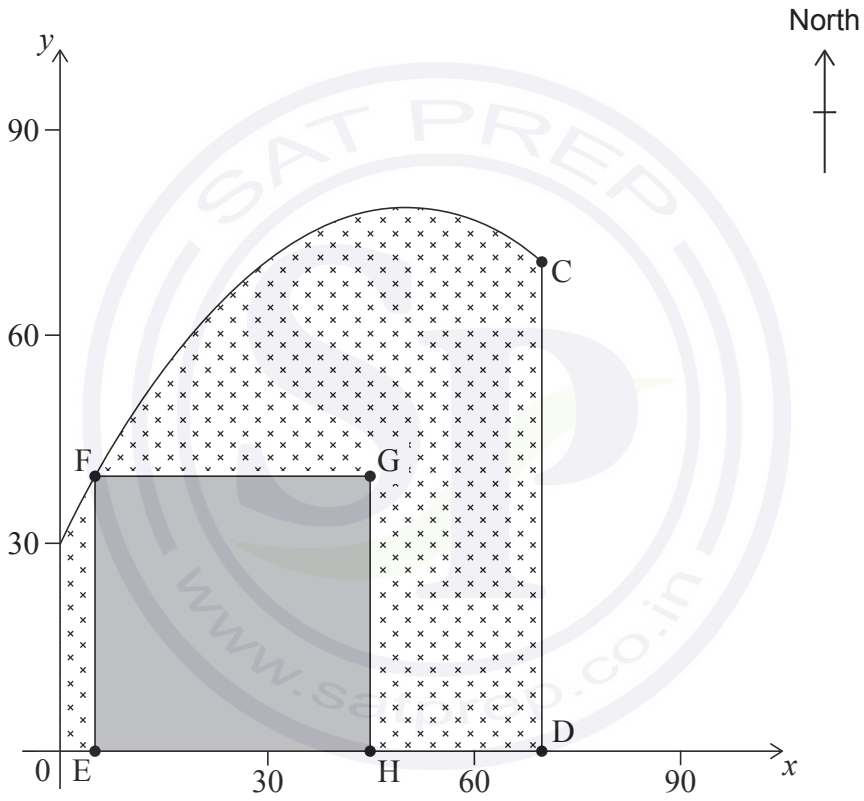
(Question 3 continued)

Linda used the trapezoidal rule with ten intervals to estimate the area. This calculation underestimated the area by $11.4\,\text{m}^2$.

(c) (i) Calculate the percentage error in Linda's estimate.

(ii) Suggest how Linda might be able to reduce the error whilst still using the trapezoidal rule. [3]

Linda would like to construct a building on her field. The **square** foundation of the building, EFGH, will be located such that [EH] is on the southern boundary and point F is on the northern boundary of the property. A possible location of the foundation of the building is shown in the following diagram.



The area of the square foundation will be largest when [GH] lies on [CD].

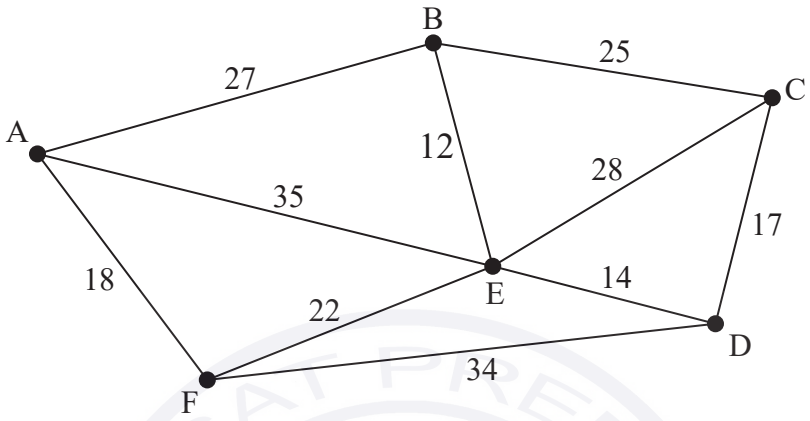
(d) (i) Find the x-coordinate of point E for the largest area of the square foundation of building EFGH.

(ii) Find the largest area of the foundation. [5]

4. [Maximum mark: 14]

A company has six offices, A, B, C, D, E and F. One of the company managers, Nanako, needs to visit the offices. She creates the following graph that shows the distances, in kilometres, between some of the offices.

diagram not to scale



(a) Write down a Hamiltonian cycle for this graph. [1]

(b) State, with a reason, whether the graph contains an Eulerian circuit. [1]

Nanako wishes to find the shortest cycle to visit all the offices. She decides to complete a weighted adjacency table, showing the least distance between each pair of offices.

	A	B	C	D	E	F
A		27	52	p	35	18
B			25	26	12	q
C				17	28	r
D					14	34
E						22
F						

(c) Write down the value of

(i) p.

(ii) q.

(iii) r. [3]

(This question continues on the following page)

(Question 4 continued)

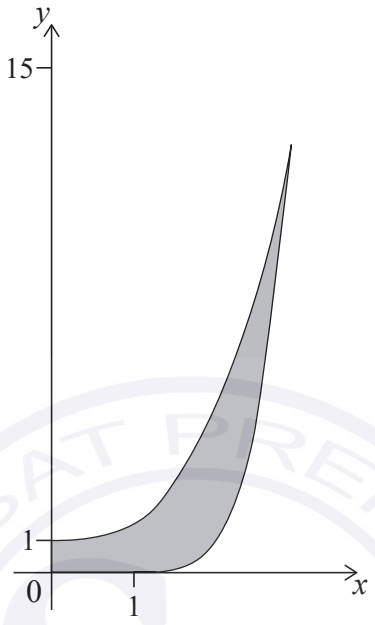
(d) Starting at vertex E, use the nearest neighbour algorithm to find an upper bound for Nanako's cycle. [3]

(e) By deleting vertex F, find a lower bound for Nanako's cycle. [4]

(f) Explain, with a reason, why the answer to part (e) might not be the best lower bound. [2]

5. [Maximum mark: 13]

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\text{cm}$.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

6. [Maximum mark: 18]

A company makes doors for kitchen cupboards from two layers. The inside layer is wood, and its thickness is normally distributed with mean $7\,\text{mm}$ and standard deviation $0.3\,\text{mm}$. The outside layer is plastic, and its thickness is normally distributed with mean $3\,\text{mm}$ and standard deviation $0.16\,\text{mm}$. The thickness of the plastic is independent of the thickness of the wood.

(a) Find the probability that a randomly chosen door has a total thickness of less than $9.5\,\text{mm}$. [5]

Eight doors are to be packed into a box to send to a customer. The width of the box is $82\,\text{mm}$. The thickness of each door is independent.

(b) Find the probability that the total thickness of the eight doors is greater than the width of the box. [4]

The company buys two new machines, A and B, to make the wooden layers. An employee claims that the layers from machine B are thinner than the layers from machine A. In order to test this claim, a random sample is taken from each machine.

The seven layers in the sample from machine A have a thickness, in mm, of

$$6.23,\ 7.04,\ 7.31,\ 6.79,\ 6.91,\ 6.79,\ 7.47.$$

(c) Find the

(i) mean.

(ii) unbiased estimate of the population variance. [3]

The eight layers in the sample from machine B have a mean thickness of $6.89\,\text{mm}$ and $s_{n-1} = 0.31$.

(d) Perform a suitable test, at the 5% significance level, to test the employee's claim. You may assume the thickness of the wooden layers from each machine are normally distributed with equal population variance. [6]

7. [Maximum mark: 20]

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$.
Displacement is measured in metres and time is measured in seconds.

(a) (i) Find an expression for the velocity of the particle at time t.

(ii) Hence find the speed when $t = 3$. [4]

(b) (i) Find an expression for the acceleration of the particle at time t.

(ii) Hence show that the acceleration is always directed towards the origin. [4]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \leq t \leq 10$, find the time when the two particles are closest to each other. [5]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

(d) (i) Find the value of k.

(ii) At time k, show that the two particles are moving in the opposite direction. [7]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Monday 9 May 2022 (morning)

2 hours

Instructions to candidates

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2222–7212

8 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

Scott purchases food for his dog in large bags and feeds the dog the same amount of dog food each day. The amount of dog food left in the bag at the end of each day can be modelled by an arithmetic sequence.

On a particular day, Scott opened a new bag of dog food and fed his dog. By the end of the third day there were 115.5 cups of dog food remaining in the bag and at the end of the eighth day there were 108 cups of dog food remaining in the bag.

(a) Find the number of cups of dog food

(i) fed to the dog per day;

(ii) remaining in the bag at the end of the first day. [4]

(b) Calculate the number of days that Scott can feed his dog with one bag of food. [2]

In 2021, Scott spent $\$625$ on dog food. Scott expects that the amount he spends on dog food will increase at an annual rate of 6.4%.

(c) Determine the amount that Scott expects to spend on dog food in 2025. Round your answer to the nearest dollar. [3]

(d) (i) Calculate the value of $\sum_{n=1}^{10}(625 \times 1.064^{(n-1)})$.

(ii) Describe what the value in part (d)(i) represents in this context. [3]

(e) Comment on the appropriateness of modelling this scenario with a geometric sequence. [1]

2. [Maximum mark: 15]

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{dC}{dx}$ in terms of k and x. [3]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c) (i) Find the value of k.

(ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [3]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

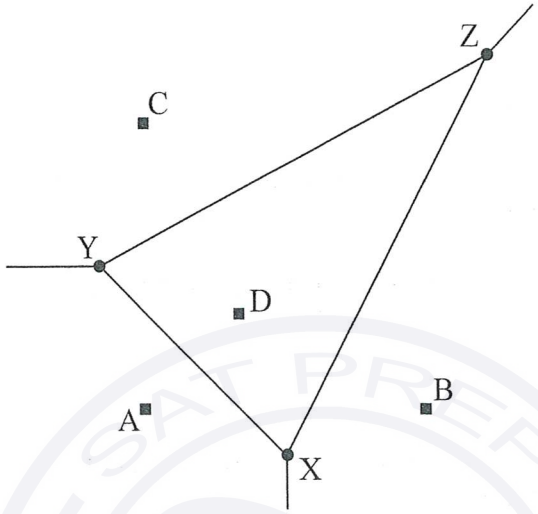
The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

3. [Maximum mark: 18]

The Voronoi diagram below shows four supermarkets represented by points with coordinates $A(0, 0)$, $B(6, 0)$, $C(0, 6)$ and $D(2, 2)$. The vertices X, Y, Z are also shown. All distances are measured in kilometres.

diagram not to scale



(a) Find the midpoint of $[BD]$. [2]

(b) Find the equation of (XZ). [4]

The equation of (XY) is $y = 2 - x$ and the equation of (YZ) is $y = 0.5x + 3.5$.

(c) Find the coordinates of X. [3]

The coordinates of Y are $(-1, 3)$ and the coordinates of Z are $(7, 7)$.

(d) Determine the exact length of $[YZ]$. [2]

(e) Given that the exact length of $[XY]$ is $\sqrt{32}$, find the size of $X\hat{Y}Z$ in degrees. [4]

(f) Hence find the area of triangle XYZ. [2]

A town planner believes that the larger the area of the Voronoi cell XYZ, the more people will shop at supermarket D.

(g) State one criticism of this interpretation. [1]

4. [Maximum mark: 15]

A student investigating the relationship between chemical reactions and temperature finds the Arrhenius equation on the internet.

$$k = Ae^{-\frac{c}{T}}$$

This equation links a variable k with the temperature T, where A and c are positive constants and $T > 0$.

(a) Show that $\frac{dk}{dT}$ is always positive. [3]

(b) Given that $\lim_{T \to \infty} k = A$ and $\lim_{T \to 0} k = 0$, sketch the graph of k against T. [3]

The Arrhenius equation predicts that the graph of $\ln k$ against $\frac{1}{T}$ is a straight line.

(c) Write down

(i) the gradient of this line in terms of c;

(ii) the y-intercept of this line in terms of A. [4]

The following data are found for a particular reaction, where T is measured in Kelvin and k is measured in $\text{cm}^3\ \text{mol}^{-1}\ \text{s}^{-1}$:

T	k
590	5×10^{-4}
600	6×10^{-4}
610	10×10^{-4}
620	14×10^{-4}
630	20×10^{-4}
640	29×10^{-4}
650	36×10^{-4}

(d) Find the equation of the regression line for $\ln k$ on $\frac{1}{T}$. [2]

(e) Find an estimate of

(i) c;

(ii) A.

It is not required to state units for these values. [3]

5. [Maximum mark: 12]

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the nth time, and Z_n is the probability of it being in another state after dividing for the nth time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a) (i) Write down the value of b.

(ii) What does b represent in this context? [2]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

(d) The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

(i) when $n = 5$;

(ii) in the long term. [4]

6. [Maximum mark: 21]

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a) (i) Find the initial speed of the ball.

(ii) Find the angle of elevation of the ball as it is launched. [4]

(b) Find the maximum height reached by the ball. [3]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

(d) For the path of the ball, find an expression for y in terms of x. [3]

An archer releases an arrow from the point $(0, 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\text{m s}^{-1}$ and an angle of elevation of $10°$.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

7. [Maximum mark: 16]

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Monday 9 May 2022 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2222 – 7207

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 16]

The scores of the eight highest scoring countries in the 2019 Eurovision song contest are shown in the following table.

	Eurovision score
Netherlands	498
Italy	472
Russia	370
Switzerland	364
Sweden	334
Norway	331
North Macedonia	305
Azerbaijan	302

(a) For this data, find

(i) the upper quartile.

(ii) the interquartile range. [4]

(b) Determine if the Netherlands' score is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 1 continued)

Chester is investigating the relationship between the highest-scoring countries' Eurovision score and their population size to determine whether population size can reasonably be used to predict a country's score.

The populations of the countries, to the nearest million, are shown in the table.

	Population (x) (millions)	**Eurovision score (y)**
Netherlands	17	498
Italy	60	472
Russia	145	370
Switzerland	9	364
Sweden	10	334
Norway	5	331
North Macedonia	2	305
Azerbaijan	10	302

Chester finds that, for this data, the Pearson's product moment correlation coefficient is $r = 0.249$.

(c) State whether it would be appropriate for Chester to use the equation of a regression line for y on x to predict a country's Eurovision score. Justify your answer. [2]

Chester then decides to find the Spearman's rank correlation coefficient for this data, and creates a table of ranks.

	Population rank (to the nearest million)	**Eurovision score rank**
Netherlands	3	1
Italy	2	2
Russia	1	3
Switzerland	a	4
Sweden	b	5
Norway	7	6
North Macedonia	8	7
Azerbaijan	c	8

(This question continues on the following page)

(Question 1 continued)

(d) Write down the value of:

(i) a,

(ii) b,

(iii) c. [3]

(e) (i) Find the value of the Spearman's rank correlation coefficient r_s.

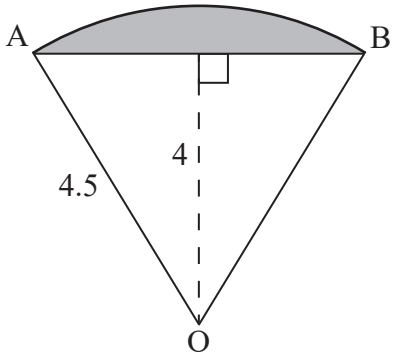
(ii) Interpret the value obtained for r_s. [3]

(f) When calculating the ranks, Chester incorrectly read the Netherlands' score as 478. Explain why the value of the Spearman's rank correlation r_s does not change despite this error. [1]

2. [Maximum mark: 17]

A sector of a circle, centre O and radius $4.5\,\text{m}$, is shown in the following diagram.

diagram not to scale

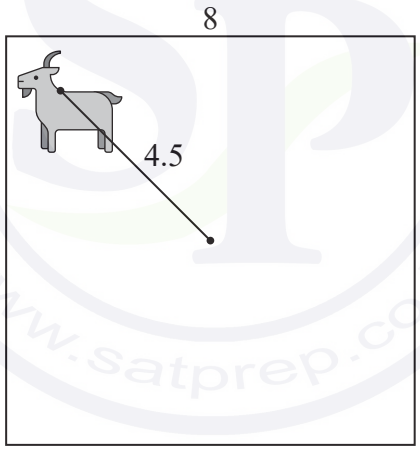


(a) (i) Find the angle $\text{A}\hat{\text{O}}\text{B}$.

(ii) Find the area of the shaded segment. [8]

A square field with side $8\,\text{m}$ has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to $4.5\,\text{m}$ from the post.

diagram not to scale



(b) Find the area of the field that can be reached by the goat. [4]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{\text{d}V}{\text{d}t} = 0.3\,t\text{e}^{-t}$.

(c) Find the value of t at which the goat is eating grass at the greatest rate. [2]

The goat is tied in the field for 8 hours.

(d) Find the total volume of grass eaten by the goat during this time. [3]

3. [Maximum mark: 13]

A Principal would like to compare the students in his school with a national standard. He decides to give a test to eight students made up of four boys and four girls. One of the teachers offers to find the volunteers from his class.

(a) Name the type of sampling that best describes the method used by the Principal. [1]

The marks out of 40, for the students who took the test, are:

$$25,\ 29,\ 38,\ 37,\ 12,\ 18,\ 27,\ 31.$$

(b) For the eight students find

(i) the mean mark.

(ii) the standard deviation of the marks. [3]

The national standard mark is 25.2 out of 40.

(c) Perform an appropriate test at the 5% significance level to see if the mean marks achieved by the students in the school are higher than the national standard. It can be assumed that the marks come from a normal population. [5]

(d) State one reason why the test might not be valid. [1]

Two additional students take the test at a later date and the mean mark for all ten students is 28.1 and the standard deviation is 8.4.

For further analysis, a standardized score out of 100 for the ten students is obtained by multiplying the scores by 2 and adding 20.

(e) For the ten students, find

(i) their mean standardized score.

(ii) the standard deviation of their standardized score. [3]

4. [Maximum mark: 13]

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0$$

(a) (i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

(iii) Hence state the long-term velocity of the particle. [5]

The equation for the motion of the particle is amended to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 3t + 4.$$

(b) (i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to write the differential equation as a system of coupled, first order differential equations.

When $t = 0$ the particle is stationary at O.

(ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$.

(iii) Find the long-term velocity of the particle. [8]

5. [Maximum mark: 15]

The aircraft for a particular flight has 72 seats. The airline's records show that historically for this flight only 90% of the people who purchase a ticket arrive to board the flight. They assume this trend will continue and decide to sell extra tickets and hope that no more than 72 passengers will arrive.

The number of passengers that arrive to board this flight is assumed to follow a binomial distribution with a probability of 0.9.

(a) The airline sells 74 tickets for this flight. Find the probability that more than 72 passengers arrive to board the flight. [3]

(b) (i) Write down the expected number of passengers who will arrive to board the flight if 72 tickets are sold. [2]

(ii) Find the maximum number of tickets that could be sold if the expected number of passengers who arrive to board the flight must be less than or equal to 72. [2]

Each passenger pays $\$150$ for a ticket. If too many passengers arrive, then the airline will give $\$300$ in compensation to each passenger that cannot board.

(c) Find, to the nearest integer, the expected increase or decrease in the money made by the airline if they decide to sell 74 tickets rather than 72. [8]

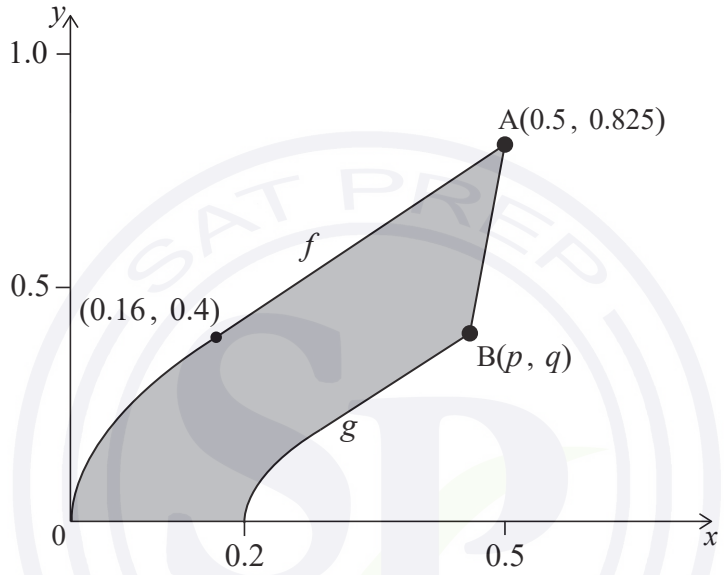
6. [Maximum mark: 18]

Consider the curve $y = \sqrt{x}$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence show that the equation of the tangent to the curve at the point $(0.16,\ 0.4)$ is $y = 1.25x + 0.2$. [4]

The shape of a piece of metal can be modelled by the region bounded by the functions f, g, the x-axis and the line segment $[AB]$, as shown in the following diagram. The units on the x and y axes are measured in metres.



The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 0.16 \\ 1.25x + 0.2 & 0.16 < x \le 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5,\ 0.825)$. Point B is the image of A under the given transformations and has coordinates $(p,\ q)$.

(b) Find the value of p and the value of q. [2]

(This question continues on the following page)

(Question 6 continued)

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \le x \le a \\ 1.25x + b & a < x \le p \end{cases}$$

(c) Find

(i) an expression for $h(x)$.

(ii) the value of a.

(iii) the value of b. [5]

(d) (i) Find the area enclosed by $y = f(x)$, the x-axis and the line $x = 0.5$.

The area enclosed by $y = g(x)$, the x-axis and the line $x = p$ is $0.0627292\,\text{m}^2$ correct to six significant figures.

(ii) Find the area of the shaded region on the diagram. [7]

7. [Maximum mark: 18]

A transformation, T, of a plane is represented by $\boldsymbol{r}' = \boldsymbol{P}\boldsymbol{r} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}'$ the position vector of its image under T.

The triangle OAB has coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$. Under T, these points are transformed to $(0, 1)$, $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ respectively.

(a) (i) By considering the image of $(0, 0)$, find $\boldsymbol{q}$.

(ii) By considering the image of $(1, 0)$ and $(0, 1)$, show that [6]

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{S}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0, 0)$.

$\boldsymbol{R}$ represents a rotation about $(0, 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

(c) (i) Use $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{S}$ to find the matrix $\boldsymbol{R}$.

(ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [7]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre (a, b), followed by a rotation about the same centre (a, b).

(d) (i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(ii) Find the value of a and the value of b. [4]

References:

2. mynamepong, n.d. Goat [image online] Available at: https://thenounproject.com/term/goat/1761571/ This file is licensed under the Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0) https://creativecommons.org/licenses/by-sa/3.0/deed.en [Accessed 22 April 2010] Source adapted.

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Tuesday 2 November 2021 (morning)

2 hours

Instructions to candidates

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- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
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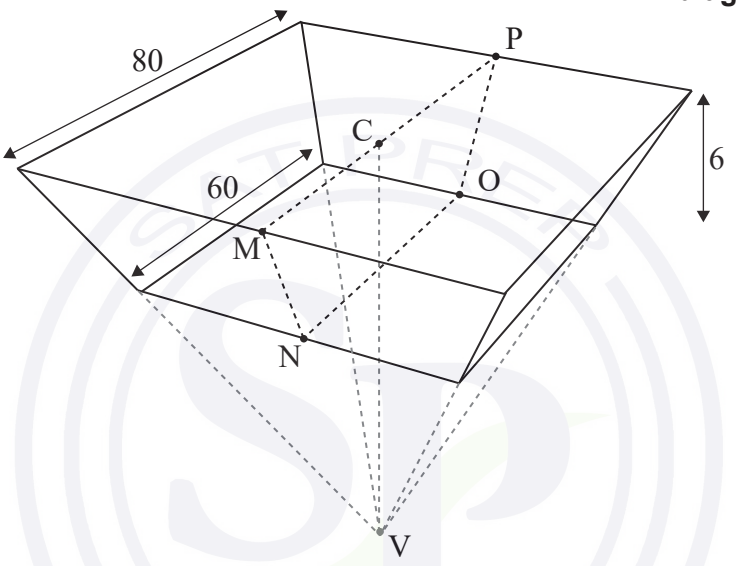
8821 – 7202

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

A large water reservoir is built in the form of part of an upside-down right pyramid with a horizontal square base of length 80 metres. The point C is the centre of the square base and point V is the vertex of the pyramid.

diagram not to scale



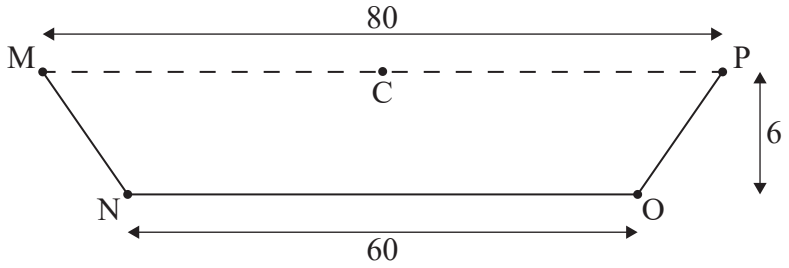
The bottom of the reservoir is a square of length 60 metres that is parallel to the base of the pyramid, such that the depth of the reservoir is 6 metres as shown in the diagram.

(This question continues on the following page)

(Question 1 continued)

The second diagram shows a vertical cross section, MNOPC, of the reservoir.

diagram not to scale



(a) Find the angle of depression from M to N. [2]

(b) (i) Find CV.

(ii) Hence or otherwise, show that the volume of the reservoir is $29\,600\,\text{m}^3$. [5]

Every day $80\,\text{m}^3$ of water from the reservoir is used for irrigation.

Joshua states that, if no other water enters or leaves the reservoir, then when it is full there is enough irrigation water for at least one year.

(c) By finding an appropriate value, determine whether Joshua is correct. [2]

To avoid water leaking into the ground, the five interior sides of the reservoir have been painted with a watertight material.

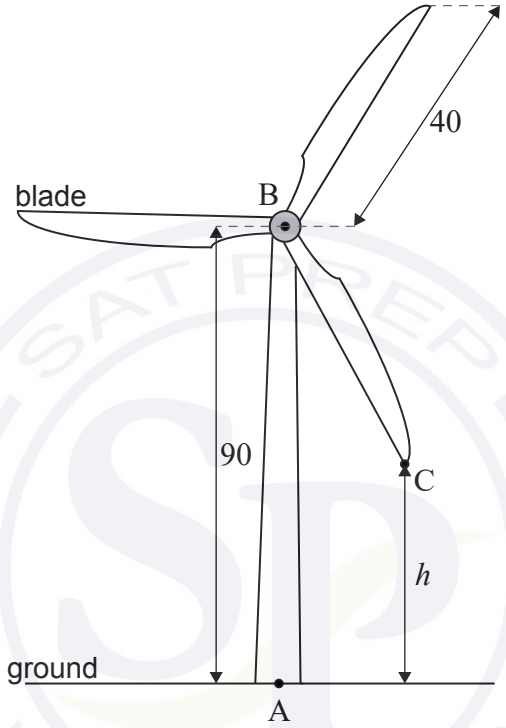
(d) Find the area that was painted. [5]

2. [Maximum mark: 20]

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point A is on the base of the tower directly below point B at the top of the tower. The height of the tower, AB, is $90\,\text{m}$. The blades of the turbine are centred at B and are each of length $40\,\text{m}$. This is shown in the following diagram.

diagram not to scale



The end of one of the blades of the turbine is represented by point C on the diagram. Let h be the height of C above the ground, measured in metres, where h varies as the blade rotates.

(a) Find the

(i) maximum value of h.

(ii) minimum value of h. [2]

The blades of the turbine complete 12 rotations per minute under normal conditions, moving at a constant rate.

(b) (i) Find the time, in seconds, it takes for the blade [BC] to make one complete rotation under these conditions.

(ii) Calculate the angle, in degrees, that the blade [BC] turns through in one second. [3]

(This question continues on the following page)

(Question 2 continued)

The height, h, of point C can be modelled by the following function. Time, t, is measured from the instant when the blade $[\mathrm{BC}]$ first passes $[\mathrm{AB}]$ and is measured in seconds.

$$h(t) = 90 - 40\cos(72t^{\circ}),\ t \geq 0$$

(c) (i) Write down the amplitude of the function.

(ii) Find the period of the function. [2]

(d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]

(e) (i) Find the height of C above the ground when $t = 2$.

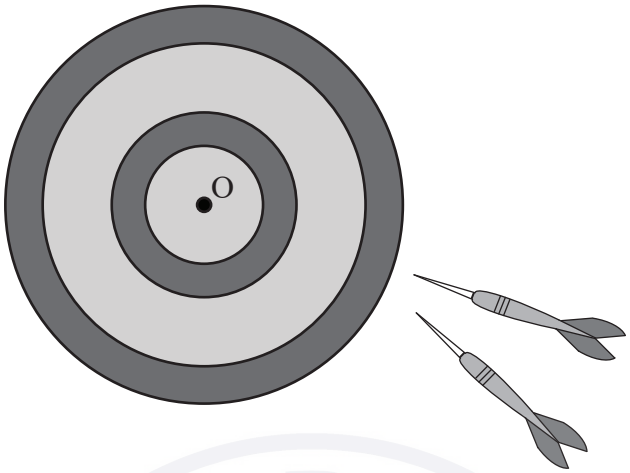
(ii) Find the time, in seconds, that point C is above a height of $100\,\mathrm{m}$, during each complete rotation. [5]

The wind speed increases and the blades rotate faster, but still at a constant rate.

(f) Given that point C is now higher than $110\,\mathrm{m}$ for 1 second during each complete rotation, find the time for one complete rotation. [5]

3. [Maximum mark: 16]

Arianne plays a game of darts.



The distance that her darts land from the centre, O, of the board can be modelled by a normal distribution with mean $10\,\text{cm}$ and standard deviation $3\,\text{cm}$.

(a) Find the probability that

(i) a dart lands less than $13\,\text{cm}$ from O.

(ii) a dart lands more than $15\,\text{cm}$ from O. [3]

Each of Arianne's throws is independent of her previous throws.

(b) Find the probability that Arianne throws two consecutive darts that land more than $15\,\text{cm}$ from O. [2]

In a competition a player has three darts to throw on each turn. A point is scored if a player throws **all** three darts to land within a central area around O. When Arianne throws a dart the probability that it lands within this area is 0.8143.

(c) Find the probability that Arianne does **not** score a point on a turn of three darts. [2]

In the competition Arianne has ten turns, each with three darts.

(d) (i) Find Arianne's expected score in the competition.

(ii) Find the probability that Arianne scores at least 5 points in the competition.

(iii) Find the probability that Arianne scores at least 5 points and less than 8 points.

(iv) Given that Arianne scores at least 5 points, find the probability that Arianne scores less than 8 points. [9]

4. [Maximum mark: 18]

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$.
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a) (i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation.

(ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position.

(iii) Find $\boldsymbol{P}^2$.

(iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [12]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points A′, B′ and C′ respectively.

Show that the area of triangle ABC is equal to the area of triangle A′B′C′. [2]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

5. [Maximum mark: 13]

(a) Let $z = 1 - \mathrm{i}$.

(i) Plot the position of z on an Argand Diagram.

(ii) Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where a, $b \in \mathbb{R}$, giving the exact value of a and the exact value of b. [3]

(b) Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x-\frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(i) Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$.

(ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \leq \frac{\pi}{2}$. [6]

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_A and I_B. The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_A	12 amps	0
I_B	12 amps	$\frac{\pi}{2}$

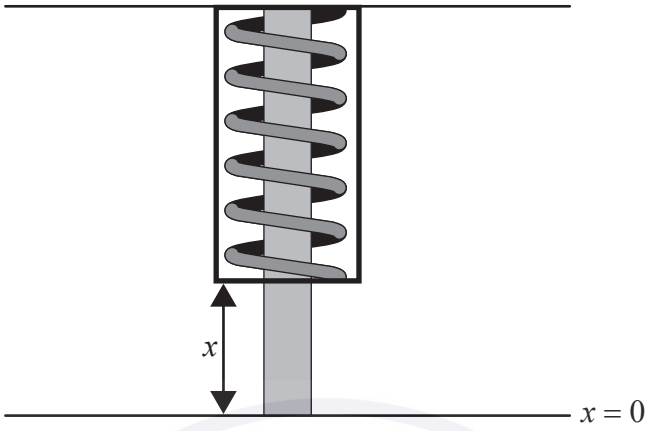
When the two voltage sources are connected to the circuit at the same time, the total current I_T can be expressed as $I_\mathrm{A} + I_\mathrm{B}$.

(c) (i) Find the maximum value of I_T.

(ii) Find the phase shift of I_T. [4]

6. [Maximum mark: 15]

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

(c) (i) Find the eigenvalues of matrix $\boldsymbol{A}$.

(ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [6]

(d) Given that when $t = 0$ the shock absorber is displaced $8\,\text{cm}$ and its velocity is zero, find an expression for x in terms of t. [6]

7. [Maximum mark: 14]

Loreto is a manager at the Da Vinci health centre. If the mean rate of patients arriving at the health centre exceeds 1.5 per minute then Loreto will employ extra staff. It is assumed that the number of patients arriving in any given time period follows a Poisson distribution.

Loreto performs a hypothesis test to determine whether she should employ extra staff. She finds that 320 patients arrived during a randomly selected 3-hour clinic.

(a) (i) Write down null and alternative hypotheses for Loreto's test.

(ii) Using the data from Loreto's sample, perform the hypothesis test at a 5% significance level to determine if Loreto should employ extra staff. [7]

Loreto is also concerned about the average waiting time for patients to see a nurse. The health centre aims for at least 95% of patients to see a nurse in under 20 minutes.

Loreto assumes that the waiting times for patients are independent of each other and decides to perform a hypothesis test at a 10% significance level to determine whether the health centre is meeting its target.

Loreto surveys 150 patients and finds that 11 of them waited more than 20 minutes.

(b) (i) Write down null and alternative hypotheses for this test.

(ii) Perform the test, clearly stating the conclusion in context. [7]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

Instructions to candidates

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2221 – 7212

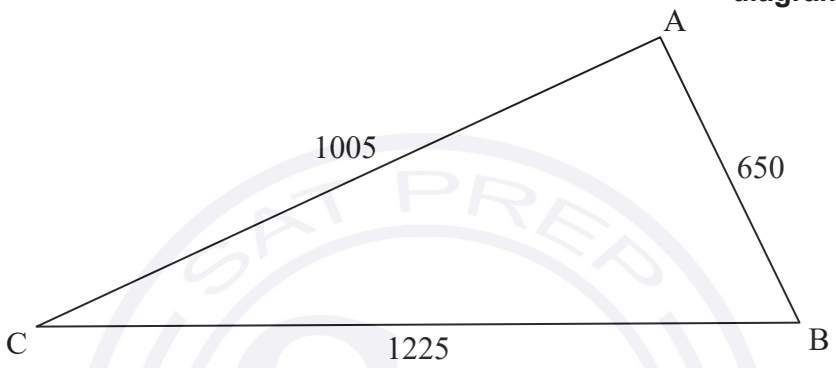
8 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

A farmer owns a field in the shape of a triangle ABC such that $AB = 650\,\text{m}$, $AC = 1005\,\text{m}$ and $BC = 1225\,\text{m}$.

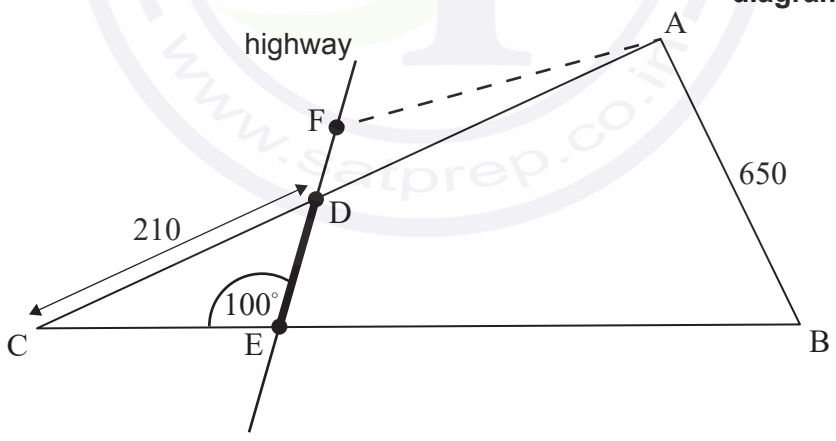
diagram not to scale



(a) Find the size of $A\hat{C}B$. [3]

The local town is planning to build a highway that will intersect the borders of the field at points D and E, where $DC = 210\,\text{m}$ and $C\hat{E}D = 100^\circ$, as shown in the diagram below.

diagram not to scale



(b) Find DE. [3]

The town wishes to build a carpark here. They ask the farmer to exchange the part of the field represented by triangle DCE. In return the farmer will get a triangle of equal area ADF, where F lies on the same line as D and E, as shown in the diagram above.

(c) Find the area of triangle DCE. [5]

(d) Estimate DF. You may assume the highway has a width of zero. [4]

2. [Maximum mark: 16]

It is known that the weights of male Persian cats are normally distributed with mean $6.1\,\text{kg}$ and variance $0.5^2\,\text{kg}^2$.

(a) Sketch a diagram showing the above information. [2]

(b) Find the proportion of male Persian cats weighing between $5.5\,\text{kg}$ and $6.5\,\text{kg}$. [2]

A group of 80 male Persian cats are drawn from this population.

(c) Determine the expected number of cats in this group that have a weight of less than $5.3\,\text{kg}$. [3]

The male cats are now joined by 80 female Persian cats. The female cats are drawn from a population whose weights are normally distributed with mean $4.5\,\text{kg}$ and standard deviation $0.45\,\text{kg}$.

(d) Ten female cats are chosen at random.

(i) Find the probability that exactly one of them weighs over $4.62\,\text{kg}$.

(ii) Let N be the number of cats weighing over $4.62\,\text{kg}$.

Find the variance of N. [5]

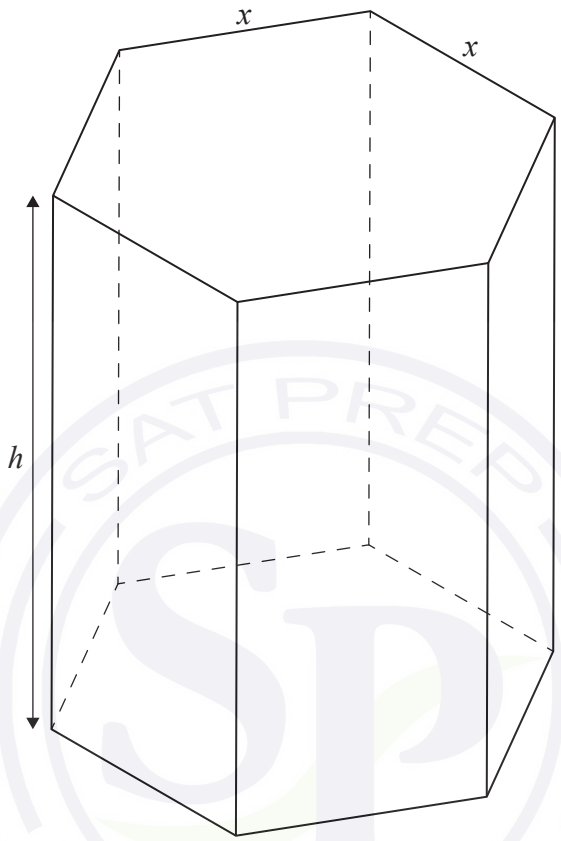
A cat is selected at random from all 160 cats.

(e) Find the probability that the cat was female, given that its weight was over $4.7\,\text{kg}$. [4]

3. [Maximum mark: 15]

A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is $h\,\text{cm}$, and the top and base of the prism have sides of length $x\,\text{cm}$.

diagram not to scale



(a) Given that $\sin 60^\circ = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]

(b) Given that the total external surface area of the box is $1200\,\text{cm}^2$, show that the volume of the box may be expressed as $V = 300\sqrt{3}\,x - \frac{9}{4}x^3$. [5]

(c) Sketch the graph of $V = 300\sqrt{3}\,x - \frac{9}{4}x^3$, for $0 \le x \le 16$. [2]

(d) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}x}$. [2]

(e) Find the value of x which maximizes the volume of the box. [2]

(f) Hence, or otherwise, find the maximum possible volume of the box. [2]

4. [Maximum mark: 18]

In a small village there are two doctors' clinics, one owned by Doctor Black and the other owned by Doctor Green. It was noted after each year that 3.5% of Doctor Black's patients moved to Doctor Green's clinic and 5% of Doctor Green's patients moved to Doctor Black's clinic. All additional losses and gains of patients by the clinics may be ignored.

At the start of a particular year, it was noted that Doctor Black had 2100 patients on their register, compared to Doctor Green's 3500 patients.

(a) Write down a transition matrix $\boldsymbol{T}$ indicating the annual population movement between clinics. [2]

(b) Find a prediction for the ratio of the number of patients Doctor Black will have, compared to Doctor Green, after two years. [2]

(c) Find a matrix $\boldsymbol{P}$, with integer elements, such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [6]

(d) Hence, show that the long-term transition matrix $\boldsymbol{T}^{\infty}$ is given by $\boldsymbol{T}^{\infty} = \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$. [6]

(e) Hence, or otherwise, determine the expected ratio of the number of patients Doctor Black would have compared to Doctor Green in the long term. [2]

5. [Maximum mark: 14]

Hank sets up a bird table in his garden to provide the local birds with some food. Hank notices that a specific bird, a large magpie, visits several times per month and he names him Bill. Hank models the number of times per month that Bill visits his garden as a Poisson distribution with mean 3.1.

(a) Using Hank's model, find the probability that Bill visits the garden on exactly four occasions during one particular month. [1]

(b) Over the course of 3 consecutive months, find the probability that Bill visits the garden:

(i) on exactly 12 occasions.

(ii) during the first and third month only. [5]

(c) Find the probability that over a 12-month period, there will be exactly 3 months when Bill does not visit the garden. [4]

After the first year, a number of baby magpies start to visit Hank's garden. It may be assumed that each of these baby magpies visits the garden randomly and independently, and that the number of times each baby magpie visits the garden per month is modelled by a Poisson distribution with mean 2.1.

(d) Determine the least number of magpies required, including Bill, in order that the probability of Hank's garden having at least 30 magpie visits per month is greater than 0.2. [4]

6. [Maximum mark: 15]

A particle P moves along the x-axis. The velocity of P is $v\,\text{m s}^{-1}$ at time t seconds, where $v = -2t^2 + 16t - 24$ for $t \geq 0$.

(a) Find the times when P is at instantaneous rest. [2]

(b) Find the magnitude of the particle's acceleration at 6 seconds. [4]

(c) Find the greatest speed of P in the interval $0 \leq t \leq 6$. [2]

(d) The particle starts from the origin O. Find an expression for the displacement of P from O at time t seconds. [4]

(e) Find the total distance travelled by P in the interval $0 \leq t \leq 4$. [3]

7. [Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]

(d) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$

(i) at $(4, 0)$.

(ii) at $(-4, 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \le x \le 6$, $-6 \le y \le 6$. [4]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

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2221 – 7207

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested $37\,000$ Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

(a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than $50\,000$ AUD.

(b) Find the minimum value of m, where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is $200\,000$ AUD.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

(c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of 1700 AUD made by Daisy at the end of each month.

(d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

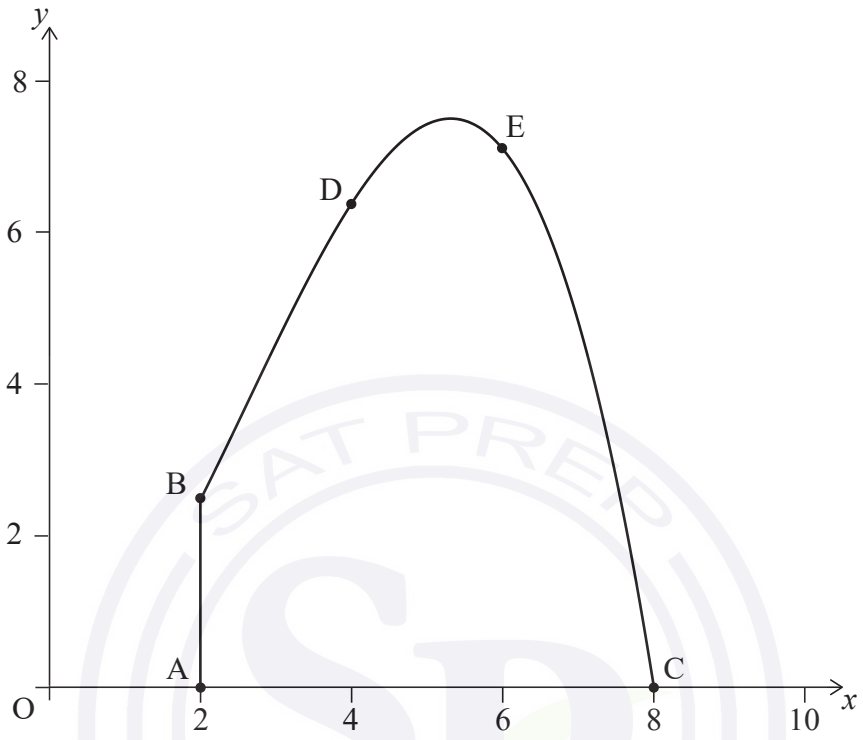
After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

(e) Find the amount of Daisy's final payment. [3]

(f) Find how much money Daisy saved by making one final payment after 5 years. [3]

2. [Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line $[AB]$ represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2,\ 2 \le x \le 8$, relative to an origin O.



Point A has coordinates $(2, 0)$, point B has coordinates $(2, 2.4)$, and point C has coordinates $(8, 0)$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence find the maximum height of the tunnel. [6]

(b) Find the height of the tunnel when

(i) $x = 4$.

(ii) $x = 6$. [3]

(c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]

(d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.

(ii) Hence find the cross-sectional area of the tunnel. [4]

3. [Maximum mark: 13]

The stopping distances for bicycles travelling at $20\,\text{km}\,\text{h}^{-1}$ are assumed to follow a normal distribution with mean $6.76\,\text{m}$ and standard deviation $0.12\,\text{m}$.

(a) Under this assumption, find, correct to four decimal places, the probability that a bicycle chosen at random travelling at $20\,\text{km}\,\text{h}^{-1}$ manages to stop

(i) in less than $6.5\,\text{m}$.

(ii) in more than $7\,\text{m}$. [3]

1000 randomly selected bicycles are tested and their stopping distances when travelling at $20\,\text{km}\,\text{h}^{-1}$ are measured.

(b) Find, correct to four significant figures, the expected number of bicycles tested that stop between

(i) $6.5\,\text{m}$ and $6.75\,\text{m}$.

(ii) $6.75\,\text{m}$ and $7\,\text{m}$. [3]

The measured stopping distances of the 1000 bicycles are given in the table.

Measured stopping distance	**Number of bicycles**
Less than $6.5\,\text{m}$	12
Between $6.5\,\text{m}$ and $6.75\,\text{m}$	428
Between $6.75\,\text{m}$ and $7\,\text{m}$	527
More than $7\,\text{m}$	33

It is decided to perform a χ^2 goodness of fit test at the $5\,\%$ level of significance to decide whether the stopping distances of bicycles travelling at $20\,\text{km}\,\text{h}^{-1}$ can be modelled by a normal distribution with mean $6.76\,\text{m}$ and standard deviation $0.12\,\text{m}$.

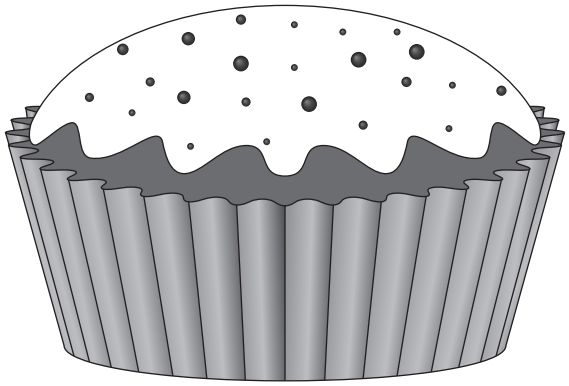
(c) State the null and alternative hypotheses. [2]

(d) Find the p-value for the test. [3]

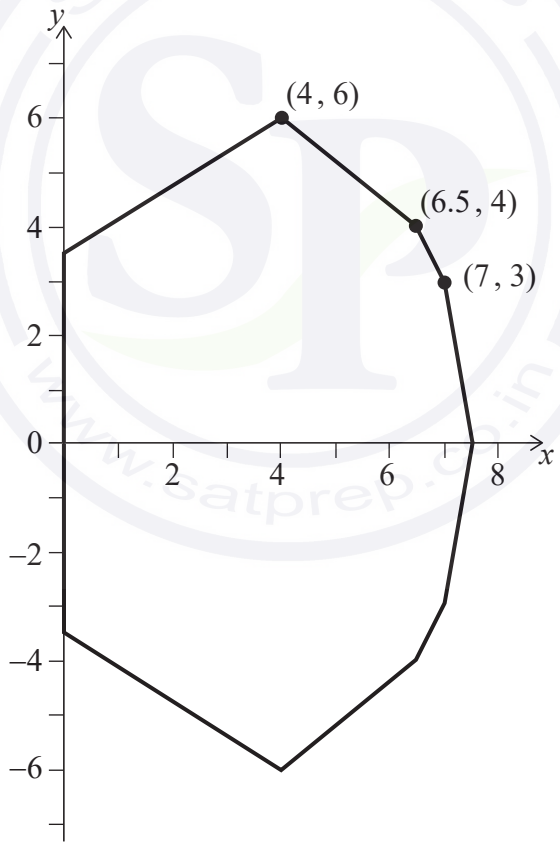
(e) State the conclusion of the test. Give a reason for your answer. [2]

4. [Maximum mark: 14]

Charlotte decides to model the shape of a cupcake to calculate its volume.



From rotating a photograph of her cupcake she estimates that its cross-section passes through the points $(0, 3.5)$, $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$, where all units are in centimetres. The cross-section is symmetrical in the x-axis, as shown below:



She models the section from $(0, 3.5)$ to $(4, 6)$ as a straight line.

(a) Find the equation of the line passing through these two points. [2]

(This question continues on the following page)

(Question 4 continued)

Charlotte models the section of the cupcake that passes through the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$ with a quadratic curve.

(b) (i) Find the equation of the least squares regression quadratic curve for these four points.

(ii) By considering the gradient of this curve when $x = 4$, explain why it may not be a good model. [3]

Charlotte thinks that a quadratic with a maximum point at $(4, 6)$ and that passes through the point $(7.5, 0)$ would be a better fit.

(c) Find the equation of the new model. [4]

Believing this to be a better model for her cupcake, Charlotte finds the volume of revolution about the x-axis to estimate the volume of the cupcake.

(d) (i) Write down an expression for her estimate of the volume as a sum of two integrals.

(ii) Find the value of Charlotte's estimate. [5]

5. [Maximum mark: 13]

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where $\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c) (i) Write down the matrix $\boldsymbol{P}$.

(ii) Write down the matrix $\boldsymbol{D}$. [2]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

6. [Maximum mark: 18]

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\,\mathrm{e}^{bt}\cos t \\ a\,\mathrm{e}^{bt}\sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t. [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a\,\mathrm{e}^{bt}\sqrt{(1+b^2)}\,.$$ [4]

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b. [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

At a point P, the ice-skater is skating parallel to the y-axis for the first time.

(e) Find OP. [6]

7. [Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$$

(a) Find the population of rabbits 1 year after they were introduced. [5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t, can be approximately modelled by the coupled equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(2 - 0.01y)$$

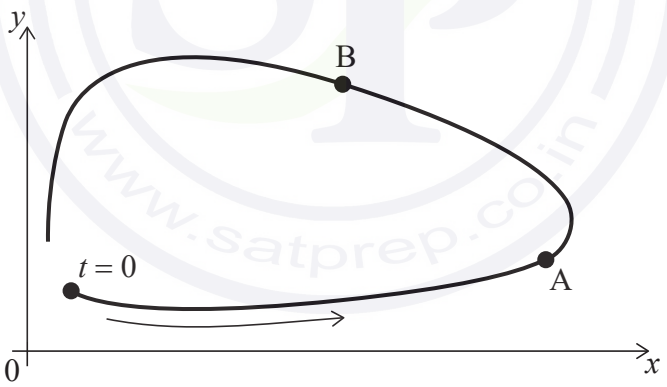
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(0.0002x - 0.8)$$

(b) Use Euler's method with a step size of 0.25, to find

(i) the population of rabbits 1 year after the foxes were introduced.

(ii) the population of foxes 1 year after the foxes were introduced. [6]

(c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.



Describe the changes in the populations of rabbits and foxes for these 4 years at

(i) point A.

(ii) point B. [3]

(d) Find the non-zero equilibrium point for the populations of rabbits and foxes. [3]

References:

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Mathematics: applications and interpretation
Higher level
Paper 2

Specimen paper

2 hours

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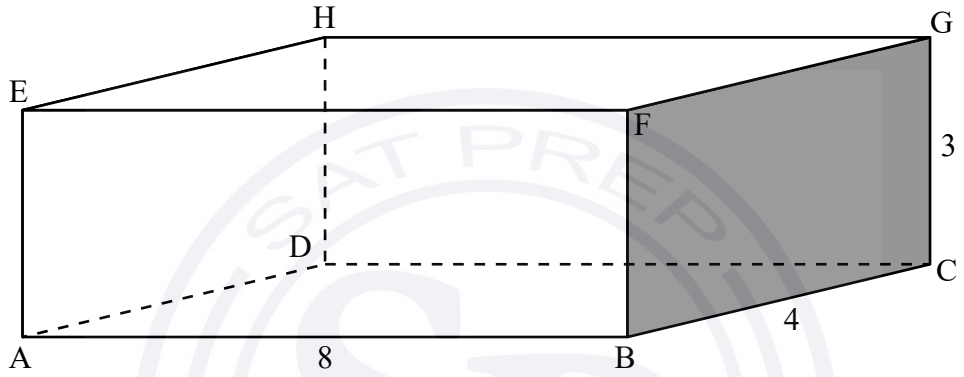
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length $8\,\text{cm}$, width $4\,\text{cm}$ and height $3\,\text{cm}$. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

(b) Calculate the length AG. [2]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{\text{d}P}{\text{d}x} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

The profit from the sale of $20\,000$ boxes is $\$1700$.

(d) Find $P(x)$. [5]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

2. [Maximum mark: 12]

Slugworth Candy Company sell a variety pack of colourful, shaped sweets.

The sweets are produced such that 80% are star shaped and 20% are shaped like a crescent moon. It is known that 10% of the stars and 30% of the crescent moons are coloured yellow.

(a) A sweet is selected at random.

(i) Find the probability that the sweet is yellow.

(ii) Given that the sweet is yellow, find the probability it is star shaped. [4]

According to manufacturer specifications, the colours in each variety pack should be distributed as follows.

Colour	Brown	Red	Green	Orange	Yellow	Purple
Percentage (%)	15	25	20	20	10	10

Mr Slugworth opens a pack of 80 sweets and records the frequency of each colour.

Colour	Brown	Red	Green	Orange	Yellow	Purple
Observed Frequency	10	20	16	18	12	4

To investigate if the sample is consistent with manufacturer specifications, Mr Slugworth conducts a χ^2 goodness of fit test. The test is carried out at a 5% significance level.

(b) Write down the null hypothesis for this test. [1]

(c) **Copy** and complete the following table in your answer booklet. [2]

Colour	Brown	Red	Green	Orange	Yellow	Purple
Expected Frequency						

(d) Write down the number of degrees of freedom. [1]

(e) Find the p-value for the test. [2]

(f) State the conclusion of the test. Give a reason for your answer. [2]

3. [Maximum mark: 18]

In this question, give all answers to two decimal places.

Bryan decides to purchase a new car with a price of €14 000, but cannot afford the full amount. The car dealership offers two options to finance a loan.

Finance option A:

A 6 year loan at a nominal annual interest rate of 14% **compounded quarterly**.
No deposit required and repayments are made each quarter.

(a) (i) Find the repayment made each quarter.

(ii) Find the total amount paid for the car.

(iii) Find the interest paid on the loan. [7]

Finance option B:

A 6 year loan at a nominal annual interest rate of r% **compounded monthly**. Terms of the loan require a 10% deposit and monthly repayments of €250.

(b) (i) Find the amount to be borrowed for this option.

(ii) Find the annual interest rate, r. [5]

(c) State which option Bryan should choose. Justify your answer. [2]

Bryan chooses option B. The car dealership invests the money Bryan pays as soon as they receive it.

(d) If they invest it in an account paying 0.4% interest per month and inflation is 0.1% per month, calculate the real amount of money the car dealership has received by the end of the 6 year period. [4]

4. [Maximum mark: 14]

An aircraft's position is given by the coordinates (x, y, z), where x and y are the aircraft's displacement east and north of an airport, and z is the height of the aircraft above the ground. All displacements are given in kilometres.

The velocity of the aircraft is given as $\begin{pmatrix} -150 \\ -50 \\ -20 \end{pmatrix} \text{km h}^{-1}$.

At $13{:}00$ it is detected at a position $30\,\text{km}$ east and $10\,\text{km}$ north of the airport, and at a height of $5\,\text{km}$. Let t be the length of time in hours from $13{:}00$.

(a) Write down a vector equation for the displacement, $\boldsymbol{r}$ of the aircraft in terms of t. [2]

(b) If the aircraft continued to fly with the velocity given

(i) verify that it would pass directly over the airport;

(ii) state the height of the aircraft at this point;

(iii) find the time at which it would fly directly over the airport. [4]

When the aircraft is $4\,\text{km}$ above the ground it continues to fly on the same bearing but adjusts the angle of its descent so that it will land at the point $(0, 0, 0)$.

(c) (i) Find the time at which the aircraft is $4\,\text{km}$ above the ground.

(ii) Find the direct distance of the aircraft from the airport at this point. [5]

(d) Given that the velocity of the aircraft, after the adjustment of the angle of descent, is $\begin{pmatrix} -150 \\ -50 \\ a \end{pmatrix} \text{km h}^{-1}$, find the value of a. [3]

5. [Maximum mark: 17]

The following table shows the costs in US dollars (US$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

(a) Show the direct flights between the cities as a graph. [2]

(b) Write down the adjacency matrix for this graph. [2]

(c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]

(d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

The following table shows the least cost to travel between the cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

(e) Find the values of a and b. [2]

A travelling salesman has to visit each of the cities, starting and finishing at city A.

(f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]

(g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

6. [Maximum mark: 14]

A city has two cable companies, X and Y. Each year $20\,\%$ of the customers using company X move to company Y and $10\,\%$ of the customers using company Y move to company X. All additional losses and gains of customers by the companies may be ignored.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movements between the two companies in a particular year. [2]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$. [4]

(c) Hence write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$. [2]

Initially company X and company Y both have 1200 customers.

(d) Find an expression for the number of customers company X has after n years, where $n \in \mathbb{N}$. [5]

(e) Hence write down the number of customers that company X can expect to have in the long term. [1]

7. [Maximum mark: 20]

An object is placed into the top of a long vertical tube, filled with a thick viscous fluid, at time $t = 0$.

Initially it is thought that the resistance of the fluid would be proportional to the velocity of the object. The following model was proposed, where the object's displacement, x, from the top of the tube, measured in metres, is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9.81 - 0.9\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right).$$

(a) By substituting $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ into the equation, find an expression for the velocity of the particle at time t. Give your answer in the form $v = f(t)$. [7]

The maximum velocity approached by the object as it falls is known as the terminal velocity.

(b) From your solution to part (a), or otherwise, find the terminal velocity of the object predicted by this model. [2]

An experiment is performed in which the object is placed in the fluid on a number of occasions and its terminal velocity recorded. It is found that the terminal velocity was consistently smaller than that predicted by the model used. It was suggested that the resistance to motion is actually proportional to the velocity squared and so the following model was set up.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9.81 - 0.9\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$$

(c) Write down the differential equation as a system of first order differential equations. [2]

(d) Use Euler's method, with a step length of 0.2, to find the displacement and velocity of the object when $t = 0.6$. [4]

(e) By repeated application of Euler's method, find an approximation for the terminal velocity, to five significant figures. [1]

At terminal velocity the acceleration of an object is equal to zero.

(f) Use the differential equation to find the terminal velocity for the object. [2]

(g) Use your answers to parts (d), (e) and (f) to comment on the accuracy of the Euler approximation to this model. [2]