Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8823 – 7202

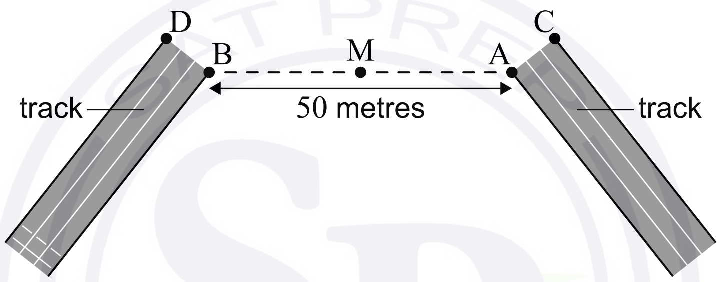
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

Madhu is designing a jogging track for the campus of her school. The following diagram shows an incomplete portion of the track.

Madhu wants to design the track such that the inner edge is a smooth curve from point A to point B, and the other edge is a smooth curve from point C to point D. The distance between points A and B is 50 metres.

diagram not to scale



To create a smooth curve, Madhu first walks to M, the midpoint of $[\mathrm{AB}]$.

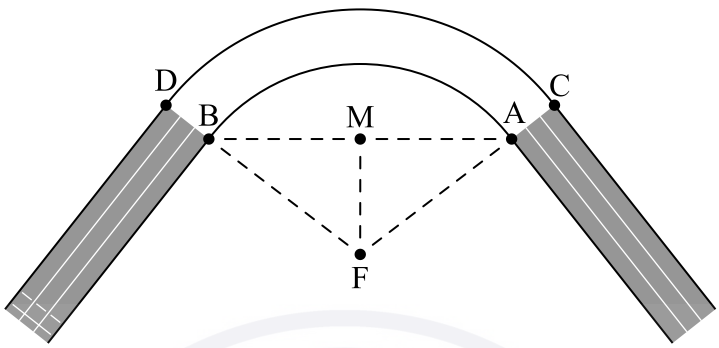
(a) Write down the length of $[\mathrm{BM}]$. [1]

(This question continues on the following page)

(Question 1 continued)

Madhu then walks 20 metres in a direction perpendicular to $[AB]$ to get from point M to point F. Point F is the centre of a circle whose arc will form the smooth curve between points A and B on the track, as shown in the following diagram.

diagram not to scale



(b) (i) Find the length of $[BF]$.

(ii) Find $B\hat{F}M$. [4]

(c) Hence, find the length of arc AB. [3]

The outer edge of the track, from C to D, is also a circular arc with centre F, such that the track is 2 metres wide.

(d) Calculate the area of the curved portion of the track, $ABDC$. [4]

The base of the track will be made of concrete that is $12\,\text{cm}$ deep.

(e) Calculate the volume of concrete needed to create the curved portion of the track. [3]

2. [Maximum mark: 18]

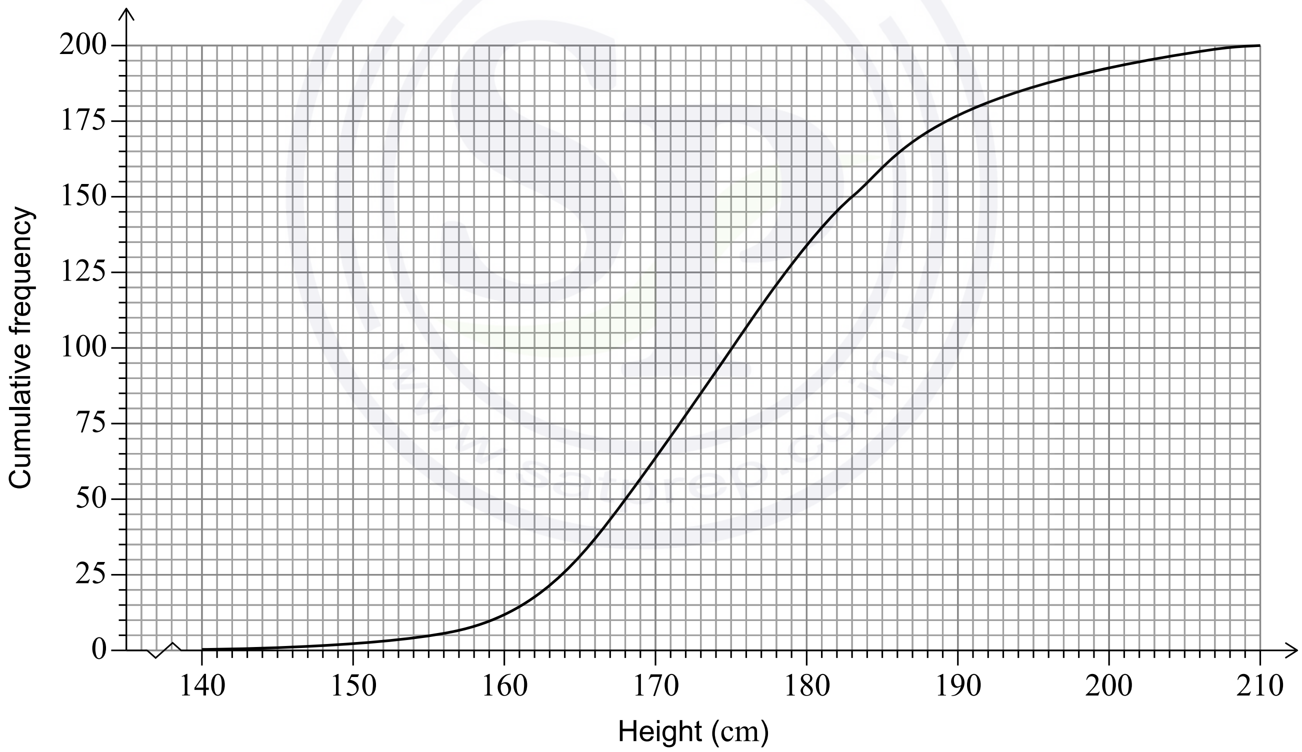
The heights, h, of 200 university students are recorded in the following table.

Height (cm)	**Frequency**
$140 \le h < 160$	11
$160 \le h < 170$	51
$170 \le h < 180$	68
$180 \le h < 190$	47
$190 \le h < 210$	23

(a) (i) Write down the mid-interval value of $140 \le h < 160$.

(ii) Calculate an estimate of the mean height of the 200 students. [3]

This table is used to create the following cumulative frequency graph.



(b) Use the cumulative frequency curve to estimate the interquartile range. [2]

Laszlo is a student in the data set and his height is $204\,\text{cm}$.

(c) Use your answer to part (b) to estimate whether Laszlo's height is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 2 continued)

It is believed that the heights of university students follow a normal distribution with mean $176\,\text{cm}$ and standard deviation $13.5\,\text{cm}$.

It is decided to perform a χ^2 goodness of fit test on the data to determine whether this sample of 200 students could have plausibly been drawn from an underlying distribution $\text{N}(176,\, 13.5^2)$.

(d) Write down the null and the alternative hypotheses for the test. [2]

As part of the test, the following table is created.

Height of student (cm)	Observed frequency	Expected frequency
$h < 160$	11	23.6
$160 \le h < 170$	51	42.1
$170 \le h < 180$	68	a
$180 \le h < 190$	47	46.7
$190 \le h$	23	b

(e) (i) Find the value of a and the value of b.

(ii) Hence, perform the test to a 5% significance level, clearly stating the conclusion in context. [8]

3. [Maximum mark: 16]

Tiffany wants to buy a house for a price of $285\,000$ US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to $15\,\%$ of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a $4\,\%$ nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

(a) (i) Find the original amount of the loan after the down payment is paid. Give the exact answer.

(ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]

(b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays $1300\,\text{USD}$ each month.

(c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than $1300\,\text{USD}$.

(d) Determine the amount of Tiffany's final payment, to two decimal places. [4]

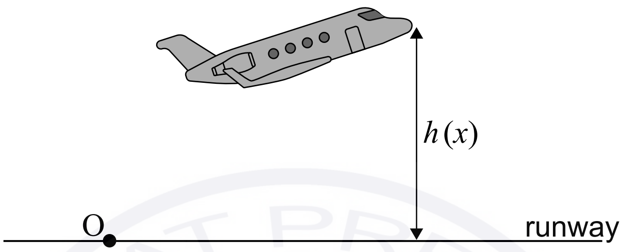
(e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1 + 150\mathrm{e}^{-0.07x}} - 0.06, \ x \geq 0.$$

diagram not to scale



(a) (i) Find $h(0)$.

(ii) Interpret this value in terms of the context. [2]

(b) (i) Find the horizontal asymptote of the graph of $y = h(x)$.

(ii) Interpret this value in terms of the context. [2]

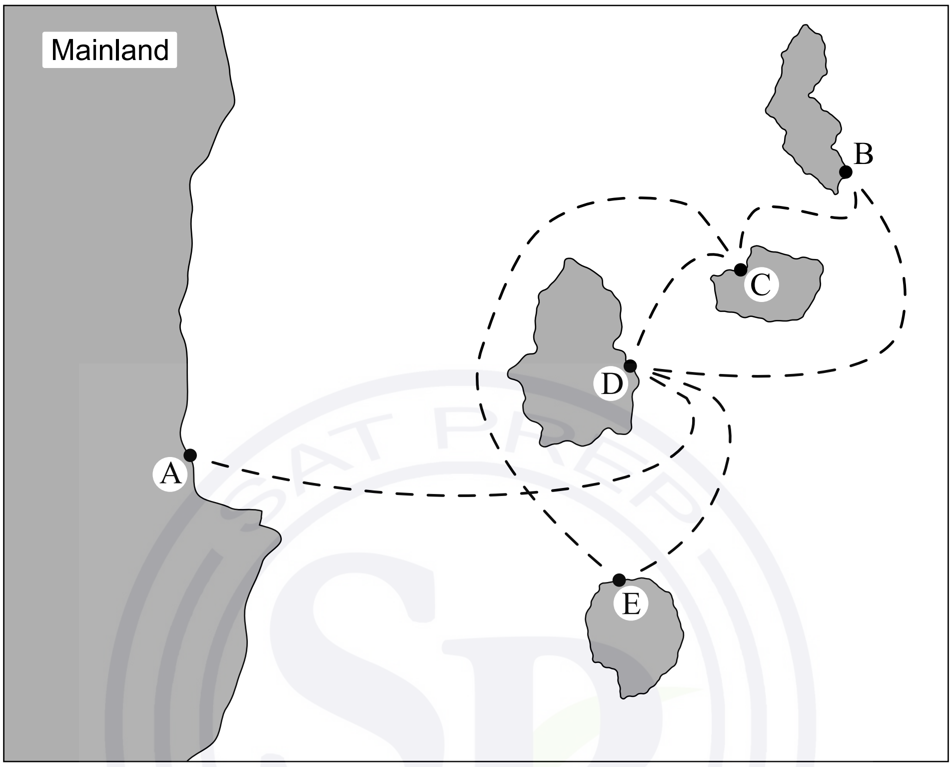
(c) Find $h'(x)$ in terms of x. [4]

A safety regulation recommends that $h'(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least $200\,\text{km}$ horizontally from point O, determine whether the plane is following this safety regulation. [4]

5. [Maximum mark: 18]

The following diagram is a map of a group of four islands and the closest mainland. Travel from the mainland and between the islands is by boat. The scheduled boat routes between the ports A, B, C, D and E are shown as dotted lines on the map.



Let the undirected graph G represent the boat routes between the ports A, B, C, D and E.

(a) Draw graph G. [1]

(b) Graph G can be represented by an adjacency matrix $\boldsymbol{P}$, where the rows and columns represent the ports in alphabetical order.

(i) Given that $\boldsymbol{P}^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$, find the value of a.

(ii) Hence, write down the number of different ways that someone could start at port B and end at port C, using three boat route journeys. [3]

(c) Find a possible Eulerian trail in G, starting at port A. [2]

(This question continues on the following page)

(Question 5 continued)

The cost of a journey on the different boat routes is given in the following table; all prices are given in USD. The cost of a journey is the same in either direction between two ports.

	A	B	C	D	E
A				10	
B			20	25	
C		20		50	45
D	10	25	50		30
E			45	30	

Sofia wants to make a trip where she travels on each of the boat routes at least once, beginning and ending at port A.

(d) Find the minimum cost of Sofia's trip. [3]

The boat company decides to add an additional boat route to make it possible to travel on each boat route **exactly** once, starting and ending at the same port.

(e) (i) Identify between which two ports the additional boat route should be added.

(ii) Determine the cost of the additional boat route such that the overall cost of the trip is the same as your answer to part (d). [2]

The boat company plans to redesign which ports are connected by boat routes. Their aim is to have a single boat trip that visits all the islands and minimizes the total distance travelled, starting and finishing at the mainland, A.

The following table shows the distances in kilometres between the ports A, B, C, D and E.

	A	B	C	D	E
A		80	90	50	60
B	80		30	70	120
C	90	30		45	100
D	50	70	45		55
E	60	120	100	55	

(f) (i) Use the nearest neighbour algorithm to find an upper bound for the minimum total distance.

(ii) Use the deleted vertex algorithm on port A to find a lower bound for the minimum total distance. [7]

Turn over

6. [Maximum mark: 14]

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O. In one of his games, an object travels on a straight line L_1 with vector equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where $l,\ m \in \mathbb{Z}$. [1]

François uses the matrix $\boldsymbol{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2. The object will then travel along L_2.

(b) Find the vector equation of L_2. [4]

François knows that the transformation given by matrix $\boldsymbol{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan\alpha$, $0 \le \alpha < \pi$

(c) Write down the matrix that represents

(i) the rotation.

(ii) the enlargement. [2]

(d) The matrix $\boldsymbol{R}$ represents the reflection. Write down $\boldsymbol{R}$ in terms of α. [1]

(e) Given that $\boldsymbol{T} = \boldsymbol{RX}$,

(i) use your answers to part (c) to find matrix $\boldsymbol{X}$.

(ii) hence, find the value of α. [6]

7. [Maximum mark: 17]

The city of Melba has an adult population of four million people. It is assumed that the weights of adults in Melba can be modelled by a normal distribution with mean $72\,\text{kg}$ and standard deviation $10\,\text{kg}$.

(a) If 10 adults in Melba are chosen independently and at random, find the probability that more than 3 of them have a weight greater than $85\,\text{kg}$. [4]

Laetitia runs a travel agency in Melba. The elevator to her office is designed to hold a maximum of 8 people.

(b) Write down a probability distribution that models the total weight of 8 adults chosen independently and at random from Melba. [3]

The total weight of 8 adults exceeds w on less than $1\,\%$ of all occasions that 8 adults enter the elevator.

(c) Find the value of w. [2]

A newspaper claims that $42\,\%$ of the adults in Melba who go on holiday choose to go abroad. Laetitia believes that this is an overestimation of the true number. During the past month, Laetitia found that 67 of her clients chose a holiday abroad, and 133 chose a holiday that was not abroad.

(d) Laetitia decides to perform a test using the binomial distribution on her data for the population proportion, p, that go on holiday abroad.

(i) State **two** assumptions that Laetitia makes in order to conduct the test.

(ii) Write down the null and the alternative hypotheses for Laetitia's test, in terms of p.

(iii) Using the data from Laetitia's sample, perform the test at a $5\,\%$ significance level to determine whether Laetitia's belief is reasonable. [8]

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1. [Maximum mark: 13]

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Temperature °C (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates $(1708, 8.73)$ and $(1958, 9.45)$.

(a) Calculate the gradient of the straight line that passes through these two points. [2]

(b) (i) Interpret the meaning of the gradient in the context of the question.

(ii) State appropriate units for the gradient. [2]

(c) Find the equation of this line giving your answer in the form $y = mx + c$. [2]

(d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Thandizo uses linear regression to obtain a model for the data.

(e) (i) Find the equation of the regression line y on x.

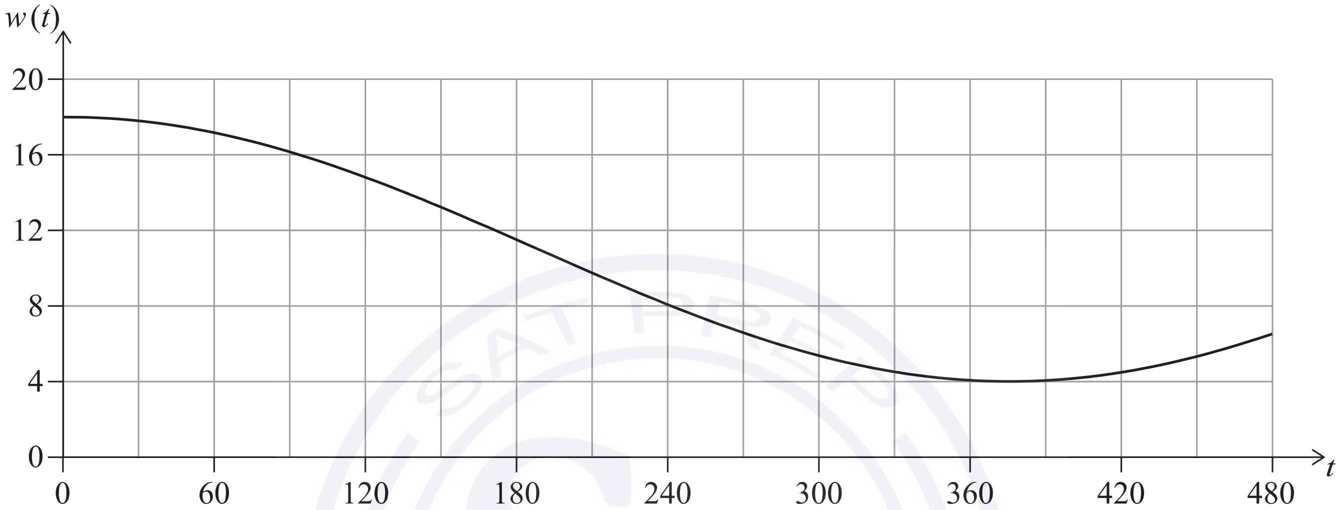
(ii) Find the value of r, the Pearson's product-moment correlation coefficient. [3]

(f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

2. [Maximum mark: 15]

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^\circ) + d$ where t is the length of time, in minutes, after $06{:}00$.

On 20 January, the first high tide occurs at $06{:}00$, at which time the depth of water is $18\,\text{m}$. The following low tide occurs at $12{:}15$ when the depth of water is $4\,\text{m}$. This is shown in the diagram.



(a) Find the value of a. [2]

(b) Find the value of d. [2]

(c) Find the period of the function in minutes. [3]

(d) Find the value of b. [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\text{m}$.

(e) Find the latest time before $12{:}00$, to the nearest minute, that Naomi can enter the harbour. [4]

(f) Find the length of time (in minutes) between $06{:}00$ and $15{:}00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

3. [Maximum mark: 17]

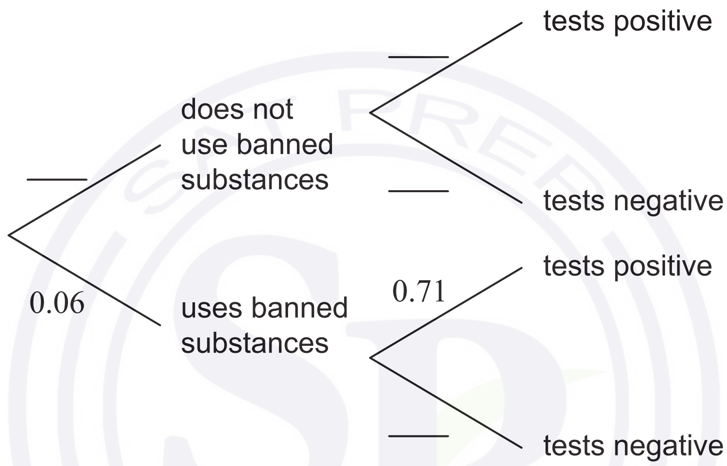
A large international sports tournament tests their athletes for banned substances.
They interpret a positive test result as meaning that the athlete uses banned substances.
A negative result means that they do not.

The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

(a) Using the information given, **copy** (into your answer booklet) and complete the following tree diagram. [2]



(b) (i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative.

(ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [4]

(c) (i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result.

(ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [5]

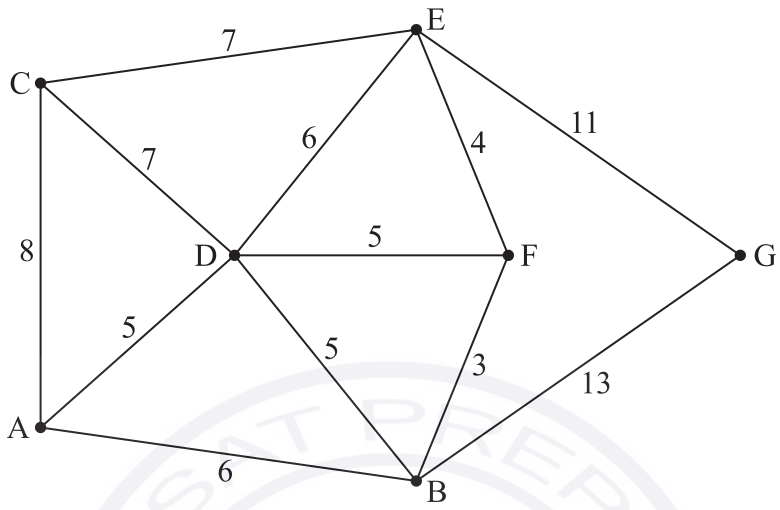
Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use banned substances.

(d) Calculate the probability that none of the athletes in Team X will test positive. [4]

(e) Determine the probability that more than 2 athletes in Team X will test positive. [2]

4. [Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	***a***	***b***
D	5	5	7		6	5	***c***
E	11	7	7	6		4	11
F	9	3	***a***	5	4		***d***
G	19	13	***b***	***c***	11	***d***	

(This question continues on the following page)

(Question 4 continued)

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

(c) Use the nearest neighbour algorithm, starting at vertex G, to find an upper bound for the travelling salesman problem. [3]

(d) (i) Sketch a minimum spanning tree for the subgraph with vertices A, B, C, D, E, F.

(ii) Write down the total weight of the minimum spanning tree. [4]

(e) Hence find a lower bound for the travelling salesman problem. [2]

(f) Explain one way in which an improved lower bound could be found. [1]

It is found that the optimum solution starting at A is actually $\mathrm{A–C–E–G–B–F–D–A}$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

5. [Maximum mark: 15]

Goran is interested in the number of sightings of a particular bird each week in the 50 weeks following the first day of September. He collects some data which is shown in the table.

Number of sightings	0	1	2	3	4	5	More than 5
Number of weeks	8	16	13	8	3	2	0

The sample mean number of sightings per week for this data is 1.76.

(a) Calculate the unbiased estimate of the population variance of sightings per week. [3]

Goran believes that the data follows a Poisson distribution.

(b) State why your answer to part (a) supports Goran's belief. [1]

Goran decides to test at the 5% significance level to see if his belief is correct.

His null hypothesis is $X \sim \mathrm{Po}(m)$, where the random variable, X, is defined as the number of sightings per week.

Goran estimates parameter m to be the mean of the sample, 1.76. He calculates the expected frequencies for sightings per week in the 50 weeks after the first day of September. These are shown to two decimal places in the following table.

Number of sightings	0	1	2	3	4	5 or more
Expected frequencies	8.60	15.14	13.32	7.82	j	k

(c) Find the value of

(i) j;

(ii) k. [5]

(d) State a reason why Goran should combine groups to conduct his significance test. [1]

(e) Write down the degrees of freedom for the test. [1]

(f) Find the p-value for the test. [2]

(g) State the conclusion of the test. Justify your answer. [2]

6. [Maximum mark: 15]

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{dx}{dt} = 5y - 0.05x\,,\ \frac{dy}{dt} = -5x - 0.05y\,,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $\boldsymbol{A} = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0$, $b \neq 0$. [4]

(b) (i) State what $a \neq 0$ indicates about the path of the speedboat.

(ii) State what the sign of a indicates about the path of the speedboat. [2]

At time $t = 0$, the speedboat has position $(20, 0)$.

(c) At time $t = 0$, find the value of

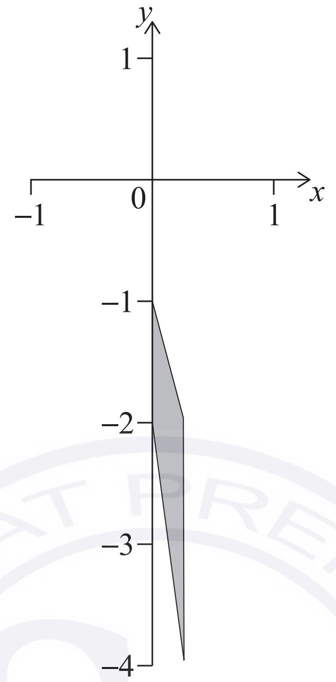
(i) $\dfrac{dy}{dt}$.

(ii) $\dfrac{dy}{dx}$. [5]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

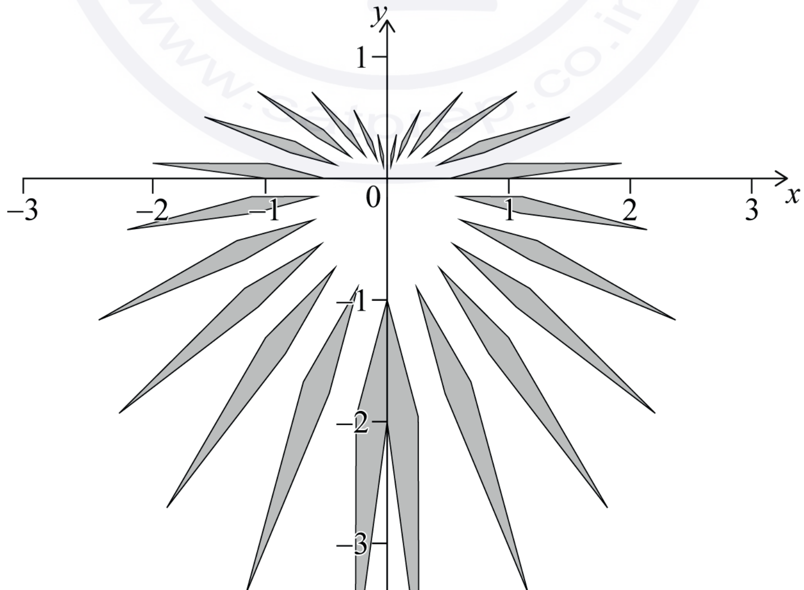
7. [Maximum mark: 18]

A trapezoid, Q, has vertices $(0, -1)$, $(0, -2)$, $(\sin 15°, -3-\cos 15°)$, $(\sin 15°, -1-\cos 15°)$ as shown.



(a) Show that the area of the trapezoid is $\frac{3}{2}\sin 15^{\circ}$. [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the y-axis is a line of symmetry.



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(Question 7 continued)

The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$\boldsymbol{M}_k = \begin{pmatrix} \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) & -\left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) \\ \left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) & \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) \end{pmatrix}$$

where $k = 0, 1, 2, 3, \ldots, 11$.

(b) (i) Find the matrix $\boldsymbol{M}_6$. Give your answer in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Q}$.

(ii) Hence find the coordinates of the image of the vertex $(0, -1)$ after it is transformed by the matrix $\boldsymbol{M}_6$. [4]

The matrix $\boldsymbol{M}_k$ can be expressed as the product of a rotation matrix and an enlargement matrix.

(c) Write down, in terms of k,

(i) the rotation matrix;

(ii) the enlargement matrix;

(iii) the angle of the rotation;

(iv) the scale factor of the enlargement. [4]

(d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix $\boldsymbol{M}_k$ in terms of k. [2]

(e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix $\boldsymbol{N}_k$, where $k = 0, 1, 2, 3, \ldots, 11$.

(f) Write down the matrix $\boldsymbol{N}_k$ as a product of two matrices. [2]

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2223 – 7212

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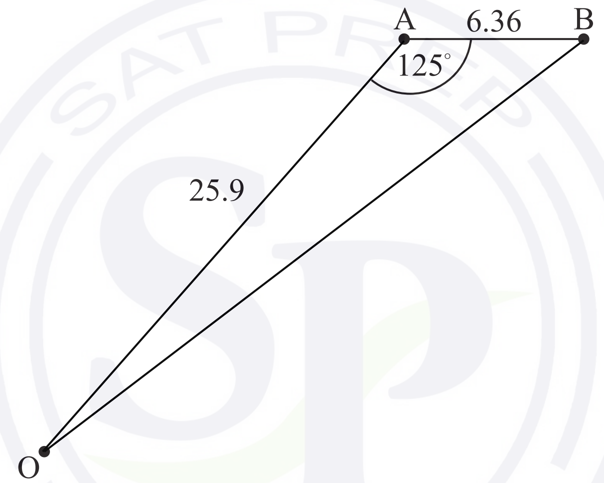
1. [Maximum mark: 13]

The diagram shows points in a park viewed from above, at a specific moment in time.

The distance between two trees, at points A and B, is $6.36\,\text{m}$.

Odette is playing football in the park and is standing at point O, such that $\text{OA} = 25.9\,\text{m}$ and $\text{O}\hat{\text{A}}\text{B} = 125^\circ$.

diagram not to scale



(a) Calculate the area of triangle AOB. [3]

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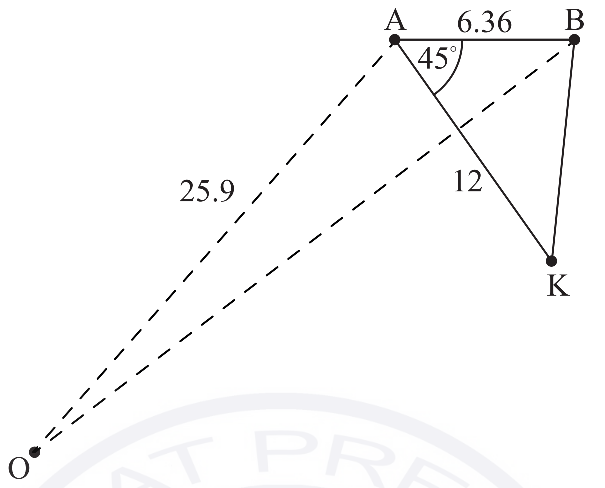
(Question 1 continued)

Odette's friend, Khemil, is standing at point K such that he is $12\,\mathrm{m}$ from A and $\mathrm{K\hat{A}B} = 45^\circ$.

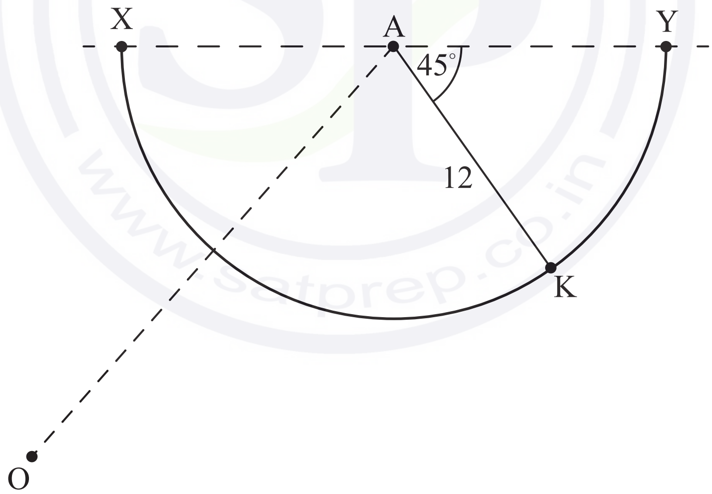
diagram not to scale



(b) Calculate Khemil's distance from B. [3]

XY is a semicircular path in the park with centre A, such that $\mathrm{K\hat{A}Y} = 45^\circ$. Khemil is standing on the path and Odette's football is at point X. This is shown in the diagram below.

diagram not to scale



The length $\mathrm{KX} = 22.2\,\mathrm{m}$, $\mathrm{K\hat{O}X} = 53.8^\circ$ and $\mathrm{O\hat{K}X} = 51.1^\circ$.

(c) Find whether Odette or Khemil is closer to the football. [4]

Khemil runs along the semicircular path to pick up the football.

(d) Calculate the distance that Khemil runs. [3]

2. [Maximum mark: 12]

A scientist is conducting an experiment on the growth of a certain species of bacteria.

The population of the bacteria, P, can be modelled by the function

$$P(t) = 1200 \times k^t,\ t \geq 0,$$

where t is the number of hours since the experiment began, and k is a positive constant.

(a) (i) Write down the value of $P(0)$.

(ii) Interpret what this value means in this context. [2]

3 hours after the experiment began, the population of the bacteria is $18\,750$.

(b) Find the value of k. [2]

(c) Find the population of the bacteria 1 hour and 30 minutes after the experiment began. [2]

The scientist conducts a second experiment with a different species of bacteria.
The population of this bacteria, S, can be modelled by the function

$$S(t) = 5000 \times 1.65^t,\ t \geq 0,$$

where t is the number of hours since both experiments began.

(d) Find the value of t when the two populations of bacteria are equal. [2]

It takes 2 hours and m minutes for the number of bacteria in the second experiment to reach $19\,000$.

(e) Find the value of m, giving your answer as an integer value. [4]

3. [Maximum mark: 16]

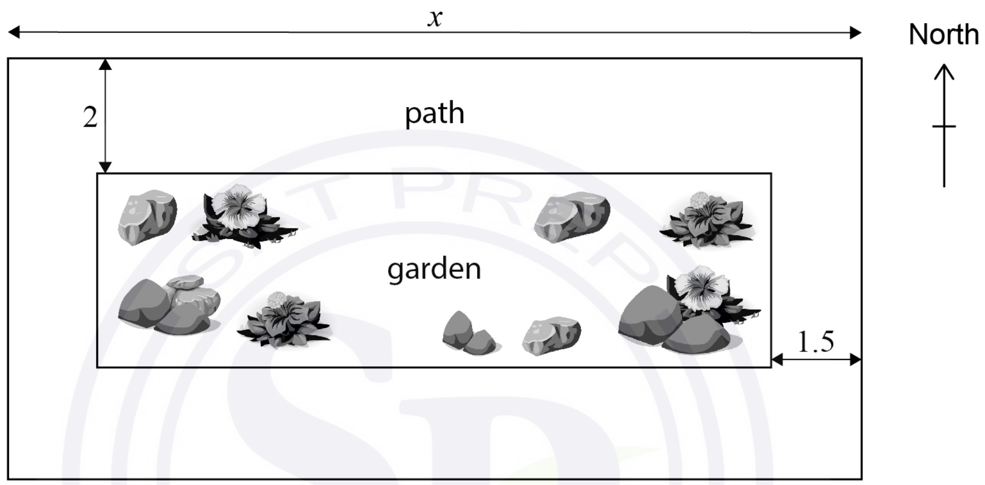
A particular park consists of a rectangular garden, of area $A\,\text{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\text{m}^2$.

The width of the path at the north and south side of the park is $2\,\text{m}$.

The width of the path at the west and east side of the park is $1.5\,\text{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \dfrac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\text{m}^2$. [4]

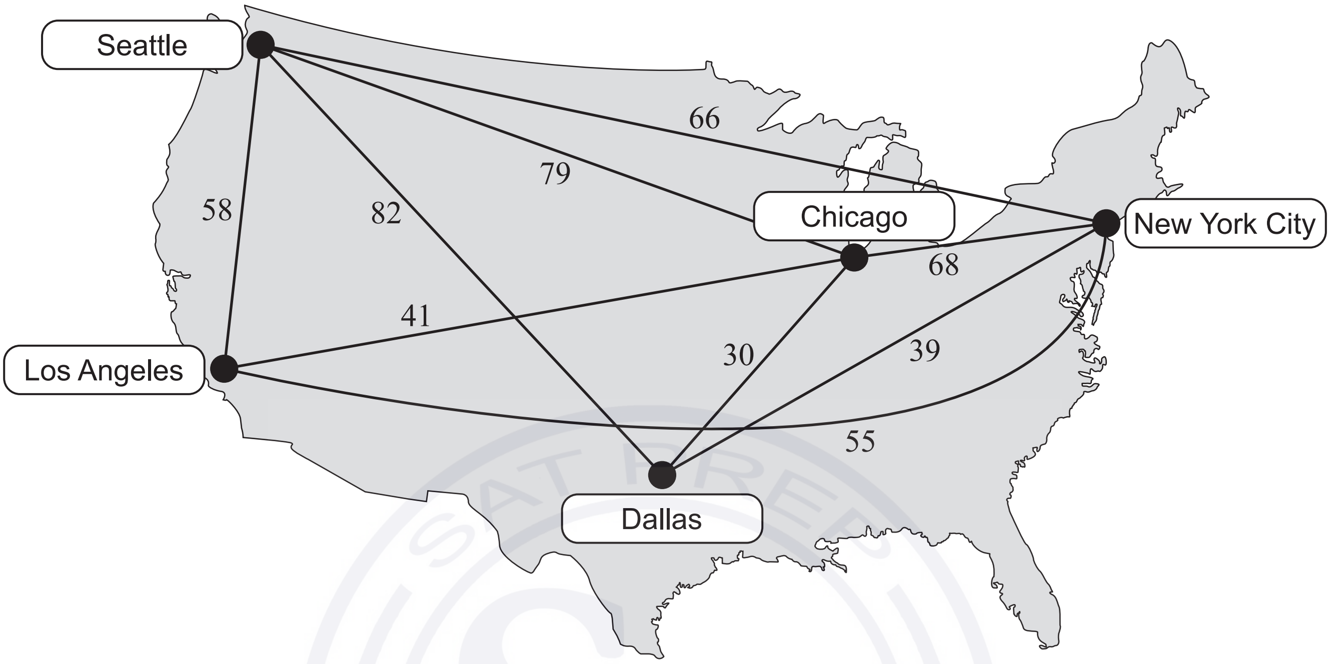
(c) Find an expression for $\dfrac{dA}{dx}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

4. [Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($) between cities.



(a) Explain why the graph can be described as “connected”, but not “complete”. [2]

(b) Find a minimum spanning tree for the graph using Kruskal’s algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

(This question continues on the following page)

(Question 4 continued)

(e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem.

(ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

5. [Maximum mark: 14]

The three countries of Belgium, Germany and The Netherlands meet at a single point called Vaalserberg.

To support future transport planning, a $10\,\text{km}$ circle was drawn around Vaalserberg on a map. A study was carried out over five years to determine what percentage of people living in each of these countries (within the $10\,\text{km}$ circular region) either stayed in their own country or moved to another country within the circle.

From this study, the following movements were observed during the five years.

- From Belgium, 5% moved to Germany, and 0.5% moved to The Netherlands.
- From Germany, 2% moved to The Netherlands, and 1.5% moved to Belgium.
- From The Netherlands, 3% moved to Germany, and 2% moved to Belgium.

All additional population changes within the circular region may be ignored.

(a) Represent the above information in a transition matrix $\boldsymbol{T}$. [3]

At the end of the study, the population of the Belgian side was $26\,000$, the population of the German side was $240\,000$, and the population of The Netherlands side was $50\,000$.

(b) By using $\boldsymbol{T}$, find the expected population of the German side of Vaalserberg 30 years after the end of the study. [4]

For matrix $\boldsymbol{T}$ there exists a steady state vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where u_1, u_2 and u_3 are the proportions of the total population on the Belgian side, the German side and The Netherlands side respectively.

The steady state vector $\boldsymbol{u}$ may be found by solving a system of equations.

(c) (i) Determine these equations that are to be solved.

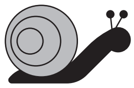
(ii) By solving your system of equations, find $\boldsymbol{u}$. [3]

(d) Use your answer to part (c)(ii) to determine the long-term expected population of the German side. [2]

(e) Suggest two reasons why your answer to part (d) is not likely to be accurate. You may comment on both the model and the situation in context. [2]

6. [Maximum mark: 18]

The gardener in a local park suggested that the number of snails found in the park can be modelled by a Poisson distribution.



(a) Suggest two observations that the gardener may have made that led him to suggest this model. [2]

Now assume that the model is valid and that the mean number of snails per m^2 is 0.2. The gardener inspects, at random, a $12\,m^2$ area of the park.

(b) Find the probability that the gardener finds exactly four snails. [3]

(c) Find the probability that the gardener finds fewer than three snails. [2]

(d) Find the probability that, in three consecutive inspections, the gardener finds at least one snail per inspection. [3]

Following heavy rain overnight, the gardener wished to determine whether the number of snails found in a random $12\,m^2$ area of the park had increased.

(e) State the hypotheses for the test. [2]

(f) Find the critical region for the test at the $1\,\%$ significance level. [3]

(g) Given that the mean number of snails per m^2 has actually risen to 0.75, find the probability that the gardener makes a Type II error. [3]

7. [Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{dx}{dt} = -4x + 6y$$

$$\frac{dy}{dt} = 9x - y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{p}_1 e^{\lambda_1 t} + B\boldsymbol{p}_2 e^{\lambda_2 t}$, where A, B, λ_1, λ_2 $(\lambda_2 > \lambda_1)$ are scalar constants and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

(c) Determine the value of A and the value of B. [2]

(d) State the long-term ratio of male fruit flies to female fruit flies. [1]

(e) Find the value of $\frac{dy}{dx}$ at time $t = 0$. [3]

(f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]

References:

Mathematics: applications and interpretation
Higher level
Paper 2

Tuesday 1 November 2022 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

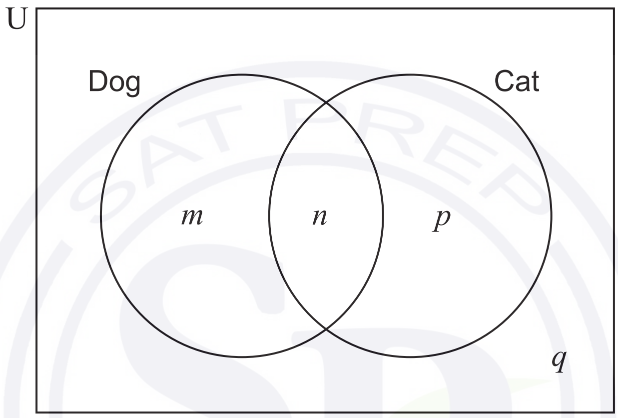
8822 – 7202

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

At Mirabooka Primary School, a survey found that 68% of students have a dog and 36% of students have a cat. 14% of students have both a dog and a cat.

This information can be represented in the following Venn diagram, where m, n, p and q represent the percentage of students within each region.



(a) Find the value of

(i) m.

(ii) n.

(iii) p.

(iv) q. [4]

(b) Find the probability that a randomly chosen student

(i) has a dog but does not have a cat.

(ii) has a dog given that they do not have a cat. [3]

(This question continues on the following page)

(Question 1 continued)

Each year, one student is chosen randomly to be the school captain of Mirabooka Primary School.

Tim is using a binomial distribution to make predictions about how many of the next 10 school captains will own a dog. He assumes that the percentages found in the survey will remain constant for future years and that the events “being a school captain” and “having a dog” are independent.

Use Tim’s model to find the probability that in the next 10 years

(c) (i) 5 school captains have a dog.

(ii) more than 3 school captains have a dog.

(iii) exactly 9 school captains in succession have a dog. [7]

John randomly chooses 10 students from the survey.

(d) State why John should not use the binomial distribution to find the probability that 5 of these students have a dog. [1]

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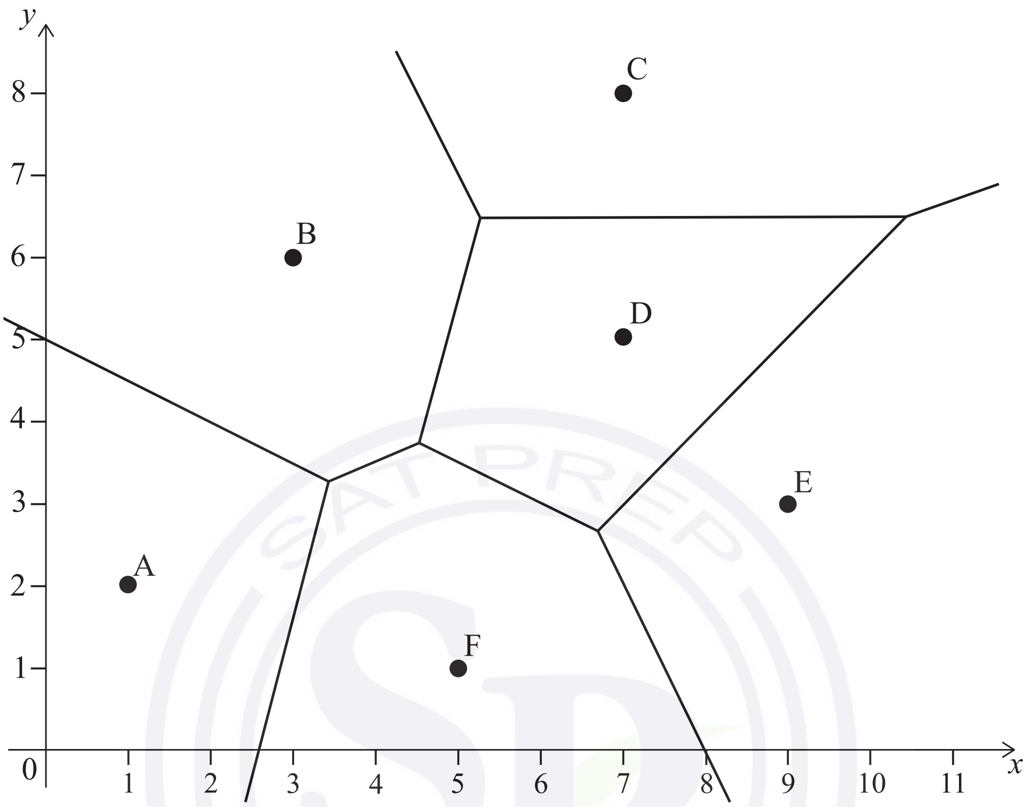
2. [Maximum mark: 13]

Six restaurant locations (labelled A, B, C, D, E and F) are shown, together with their Voronoi diagram. All distances are measured in kilometres.



(a) Elena wants to eat at the closest restaurant to her. Write down the restaurant she should go to, if she is at

(i) $(2, 7)$.

(ii) $(0, 1)$, when restaurant A is closed. [2]

Restaurant C is at $(7, 8)$ and restaurant D is at $(7, 5)$.

(b) Find the equation of the perpendicular bisector of CD. [2]

Restaurant B is at $(3, 6)$.

(c) Find the equation of the perpendicular bisector of BC. [5]

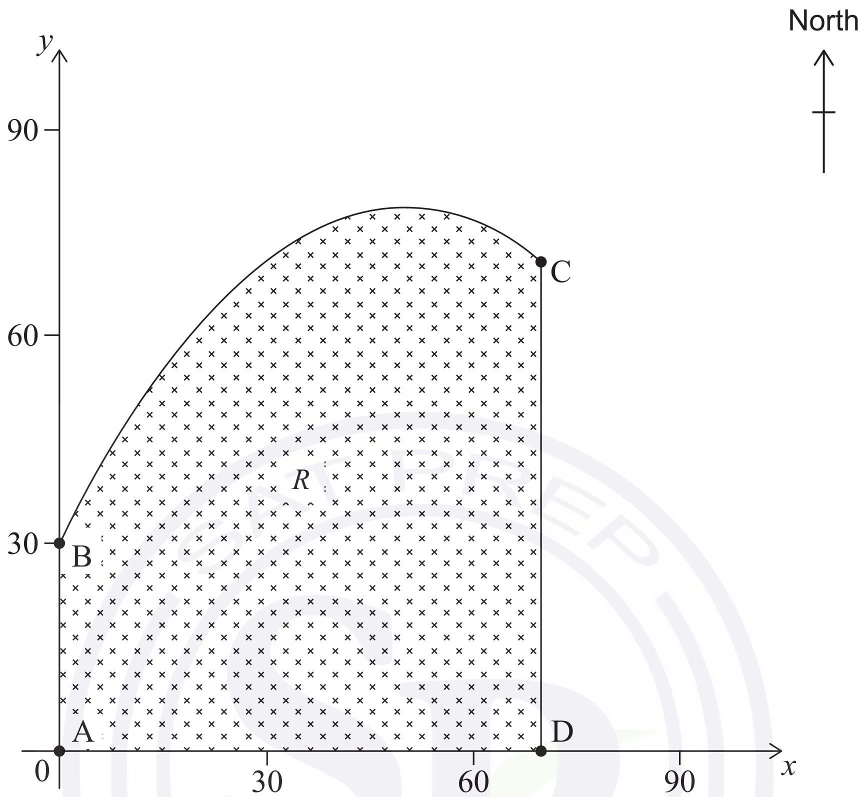
(d) Hence find

(i) the coordinates of the point which is of equal distance from B, C and D.

(ii) the distance of this point from D. [4]

3. [Maximum mark: 17]

Linda owns a field, represented by the shaded region R. The plan view of the field is shown in the following diagram, where both axes represent distance and are measured in metres.



The segments [AB], [CD] and [AD] respectively represent the western, eastern and southern boundaries of the field. The function, $f(x)$, models the northern boundary of the field between points B and C and is given by

$$f(x) = \frac{-x^2}{50} + 2x + 30 \text{, for } 0 \le x \le 70.$$

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the point on the field that is furthest north. [5]

Point A has coordinates $(0, 0)$, point B has coordinates $(0, 30)$, point C has coordinates $(70, 72)$ and point D has coordinates $(70, 0)$.

(b) (i) Write down the integral which can be used to find the area of the shaded region R.

(ii) Find the area of Linda's field. [4]

(This question continues on the following page)

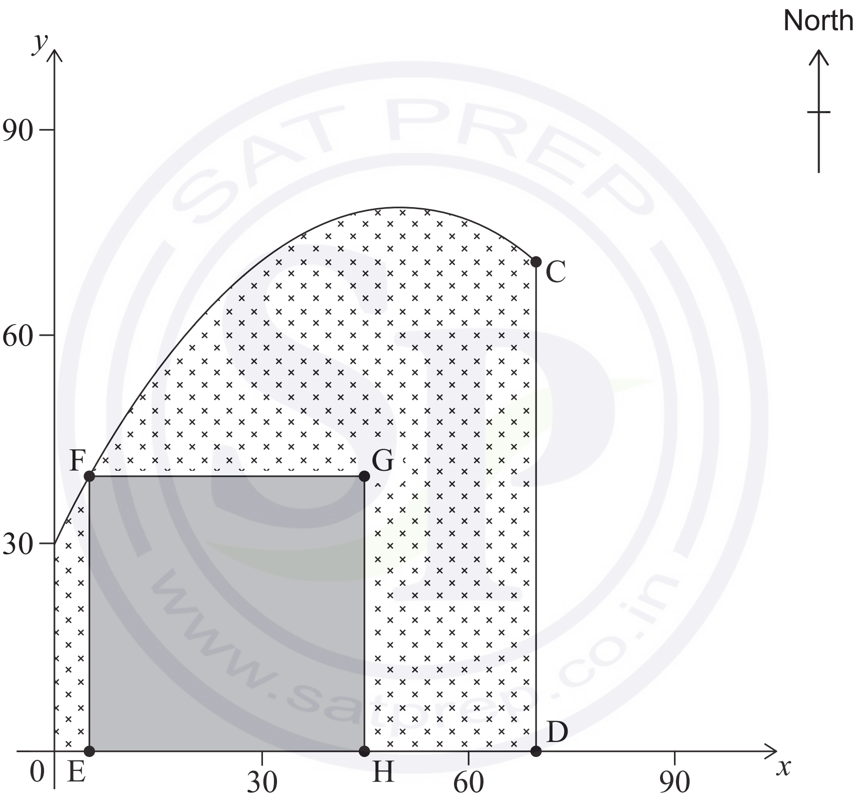
(Question 3 continued)

Linda used the trapezoidal rule with ten intervals to estimate the area. This calculation underestimated the area by $11.4\,\mathrm{m}^2$.

(c) (i) Calculate the percentage error in Linda's estimate.

(ii) Suggest how Linda might be able to reduce the error whilst still using the trapezoidal rule. [3]

Linda would like to construct a building on her field. The **square** foundation of the building, EFGH, will be located such that [EH] is on the southern boundary and point F is on the northern boundary of the property. A possible location of the foundation of the building is shown in the following diagram.



The area of the square foundation will be largest when [GH] lies on [CD].

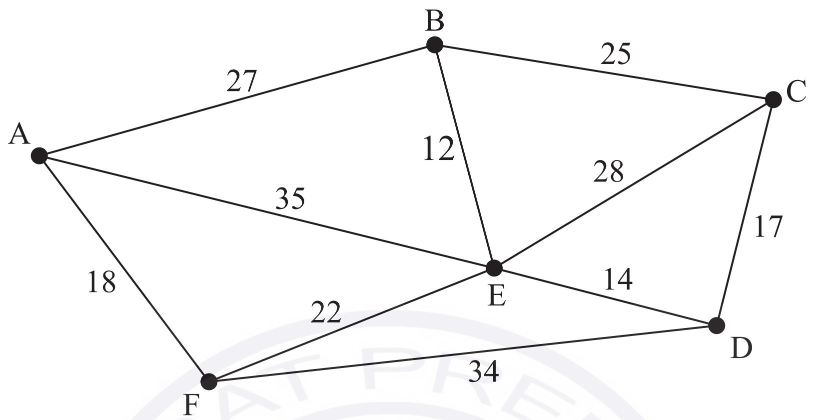
(d) (i) Find the x-coordinate of point E for the largest area of the square foundation of building EFGH.

(ii) Find the largest area of the foundation. [5]

4. [Maximum mark: 14]

A company has six offices, A, B, C, D, E and F. One of the company managers, Nanako, needs to visit the offices. She creates the following graph that shows the distances, in kilometres, between some of the offices.

diagram not to scale



(a) Write down a Hamiltonian cycle for this graph. [1]

(b) State, with a reason, whether the graph contains an Eulerian circuit. [1]

Nanako wishes to find the shortest cycle to visit all the offices. She decides to complete a weighted adjacency table, showing the least distance between each pair of offices.

	A	B	C	D	E	F
A		27	52	p	35	18
B			25	26	12	q
C				17	28	r
D					14	34
E						22
F						

(c) Write down the value of

(i) p.

(ii) q.

(iii) r. [3]

(This question continues on the following page)

(Question 4 continued)

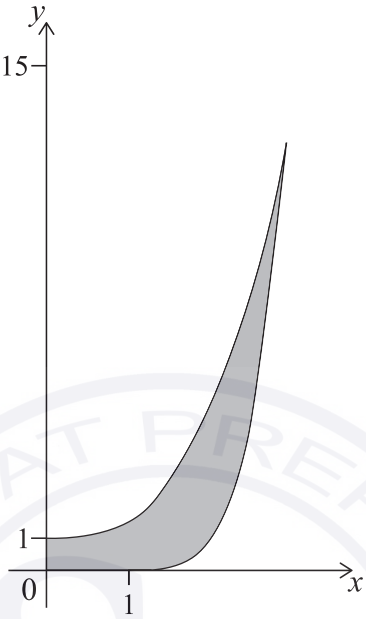
(d) Starting at vertex E, use the nearest neighbour algorithm to find an upper bound for Nanako's cycle. [3]

(e) By deleting vertex F, find a lower bound for Nanako's cycle. [4]

(f) Explain, with a reason, why the answer to part (e) might not be the best lower bound. [2]

5. [Maximum mark: 13]

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\text{cm}$.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

6. [Maximum mark: 18]

A company makes doors for kitchen cupboards from two layers. The inside layer is wood, and its thickness is normally distributed with mean $7\,\text{mm}$ and standard deviation $0.3\,\text{mm}$. The outside layer is plastic, and its thickness is normally distributed with mean $3\,\text{mm}$ and standard deviation $0.16\,\text{mm}$. The thickness of the plastic is independent of the thickness of the wood.

(a) Find the probability that a randomly chosen door has a total thickness of less than $9.5\,\text{mm}$. [5]

Eight doors are to be packed into a box to send to a customer. The width of the box is $82\,\text{mm}$. The thickness of each door is independent.

(b) Find the probability that the total thickness of the eight doors is greater than the width of the box. [4]

The company buys two new machines, A and B, to make the wooden layers. An employee claims that the layers from machine B are thinner than the layers from machine A. In order to test this claim, a random sample is taken from each machine.

The seven layers in the sample from machine A have a thickness, in mm, of

$$6.23, 7.04, 7.31, 6.79, 6.91, 6.79, 7.47.$$

(c) Find the

(i) mean.

(ii) unbiased estimate of the population variance. [3]

The eight layers in the sample from machine B have a mean thickness of $6.89\,\text{mm}$ and $s_{n-1} = 0.31$.

(d) Perform a suitable test, at the 5% significance level, to test the employee's claim. You may assume the thickness of the wooden layers from each machine are normally distributed with equal population variance. [6]

7. [Maximum mark: 20]

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$.
Displacement is measured in metres and time is measured in seconds.

(a) (i) Find an expression for the velocity of the particle at time t.

(ii) Hence find the speed when $t = 3$. [4]

(b) (i) Find an expression for the acceleration of the particle at time t.

(ii) Hence show that the acceleration is always directed towards the origin. [4]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \le t \le 10$, find the time when the two particles are closest to each other. [5]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

(d) (i) Find the value of k.

(ii) At time k, show that the two particles are moving in the opposite direction. [7]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Monday 9 May 2022 (morning)

2 hours

Instructions to candidates

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- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
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- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2222 – 7212

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

Scott purchases food for his dog in large bags and feeds the dog the same amount of dog food each day. The amount of dog food left in the bag at the end of each day can be modelled by an arithmetic sequence.

On a particular day, Scott opened a new bag of dog food and fed his dog. By the end of the third day there were 115.5 cups of dog food remaining in the bag and at the end of the eighth day there were 108 cups of dog food remaining in the bag.

(a) Find the number of cups of dog food

(i) fed to the dog per day;

(ii) remaining in the bag at the end of the first day. [4]

(b) Calculate the number of days that Scott can feed his dog with one bag of food. [2]

In 2021, Scott spent $\$625$ on dog food. Scott expects that the amount he spends on dog food will increase at an annual rate of 6.4%.

(c) Determine the amount that Scott expects to spend on dog food in 2025. Round your answer to the nearest dollar. [3]

(d) (i) Calculate the value of $\sum_{n=1}^{10}(625 \times 1.064^{(n-1)})$.

(ii) Describe what the value in part (d)(i) represents in this context. [3]

(e) Comment on the appropriateness of modelling this scenario with a geometric sequence. [1]

2. [Maximum mark: 15]

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{\mathrm{d}C}{\mathrm{d}x}$ in terms of k and x. [3]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c) (i) Find the value of k.

(ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [3]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

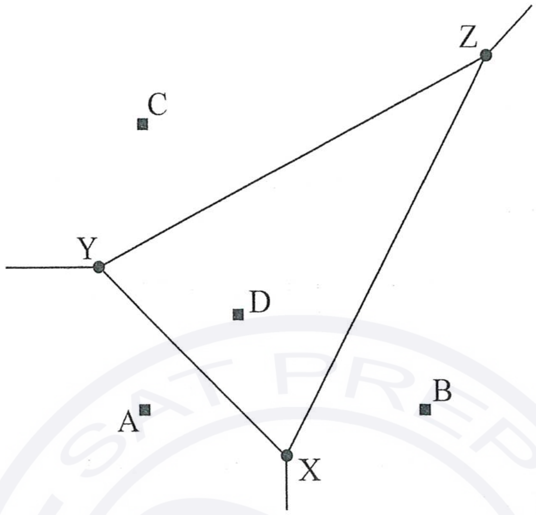
The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

3. [Maximum mark: 18]

The Voronoi diagram below shows four supermarkets represented by points with coordinates $A(0, 0)$, $B(6, 0)$, $C(0, 6)$ and $D(2, 2)$. The vertices X, Y, Z are also shown. All distances are measured in kilometres.

diagram not to scale



(a) Find the midpoint of [BD]. [2]

(b) Find the equation of (XZ). [4]

The equation of (XY) is $y = 2 - x$ and the equation of (YZ) is $y = 0.5x + 3.5$.

(c) Find the coordinates of X. [3]

The coordinates of Y are $(-1, 3)$ and the coordinates of Z are $(7, 7)$.

(d) Determine the exact length of [YZ]. [2]

(e) Given that the exact length of [XY] is $\sqrt{32}$, find the size of $X\hat{Y}Z$ in degrees. [4]

(f) Hence find the area of triangle XYZ. [2]

A town planner believes that the larger the area of the Voronoi cell XYZ, the more people will shop at supermarket D.

(g) State one criticism of this interpretation. [1]

4. [Maximum mark: 15]

A student investigating the relationship between chemical reactions and temperature finds the Arrhenius equation on the internet.

$$k = Ae^{-\frac{c}{T}}$$

This equation links a variable k with the temperature T, where A and c are positive constants and $T > 0$.

(a) Show that $\frac{dk}{dT}$ is always positive. [3]

(b) Given that $\lim_{T\to\infty} k = A$ and $\lim_{T\to 0} k = 0$, sketch the graph of k against T. [3]

The Arrhenius equation predicts that the graph of $\ln k$ against $\frac{1}{T}$ is a straight line.

(c) Write down

(i) the gradient of this line in terms of c;

(ii) the y-intercept of this line in terms of A. [4]

The following data are found for a particular reaction, where T is measured in Kelvin and k is measured in $\text{cm}^3\ \text{mol}^{-1}\ \text{s}^{-1}$:

T	k
590	5×10^{-4}
600	6×10^{-4}
610	10×10^{-4}
620	14×10^{-4}
630	20×10^{-4}
640	29×10^{-4}
650	36×10^{-4}

(d) Find the equation of the regression line for $\ln k$ on $\frac{1}{T}$. [2]

(e) Find an estimate of

(i) c;

(ii) A.

It is not required to state units for these values. [3]

5. [Maximum mark: 12]

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the nth time, and Z_n is the probability of it being in another state after dividing for the nth time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a) (i) Write down the value of b.

(ii) What does b represent in this context? [2]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

(d) The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

(i) when $n = 5$;

(ii) in the long term. [4]

6. [Maximum mark: 21]

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a) (i) Find the initial speed of the ball.

(ii) Find the angle of elevation of the ball as it is launched. [4]

(b) Find the maximum height reached by the ball. [3]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

(d) For the path of the ball, find an expression for y in terms of x. [3]

An archer releases an arrow from the point $(0,\ 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\mathrm{m\,s^{-1}}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

7. [Maximum mark: 16]

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Monday 9 May 2022 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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International Baccalaureate
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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 16]

The scores of the eight highest scoring countries in the 2019 Eurovision song contest are shown in the following table.

	Eurovision score
Netherlands	498
Italy	472
Russia	370
Switzerland	364
Sweden	334
Norway	331
North Macedonia	305
Azerbaijan	302

(a) For this data, find

(i) the upper quartile.

(ii) the interquartile range. [4]

(b) Determine if the Netherlands' score is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 1 continued)

Chester is investigating the relationship between the highest-scoring countries' Eurovision score and their population size to determine whether population size can reasonably be used to predict a country's score.

The populations of the countries, to the nearest million, are shown in the table.

	Population (x) (millions)	Eurovision score (y)
Netherlands	17	498
Italy	60	472
Russia	145	370
Switzerland	9	364
Sweden	10	334
Norway	5	331
North Macedonia	2	305
Azerbaijan	10	302

Chester finds that, for this data, the Pearson's product moment correlation coefficient is $r = 0.249$.

(c) State whether it would be appropriate for Chester to use the equation of a regression line for y on x to predict a country's Eurovision score. Justify your answer. [2]

Chester then decides to find the Spearman's rank correlation coefficient for this data, and creates a table of ranks.

	Population rank (to the nearest million)	Eurovision score rank
Netherlands	3	1
Italy	2	2
Russia	1	3
Switzerland	a	4
Sweden	b	5
Norway	7	6
North Macedonia	8	7
Azerbaijan	c	8

(This question continues on the following page)

(Question 1 continued)

(d) Write down the value of:

(i) a,

(ii) b,

(iii) c. [3]

(e) (i) Find the value of the Spearman's rank correlation coefficient r_s.

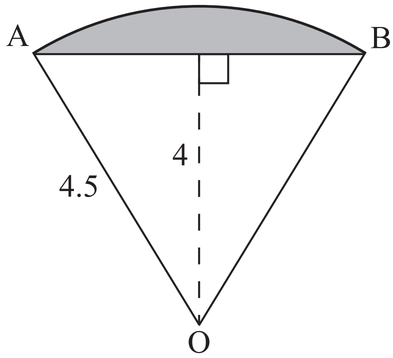
(ii) Interpret the value obtained for r_s. [3]

(f) When calculating the ranks, Chester incorrectly read the Netherlands' score as 478. Explain why the value of the Spearman's rank correlation r_s does not change despite this error. [1]

2. [Maximum mark: 17]

A sector of a circle, centre O and radius $4.5\,\text{m}$, is shown in the following diagram.

diagram not to scale

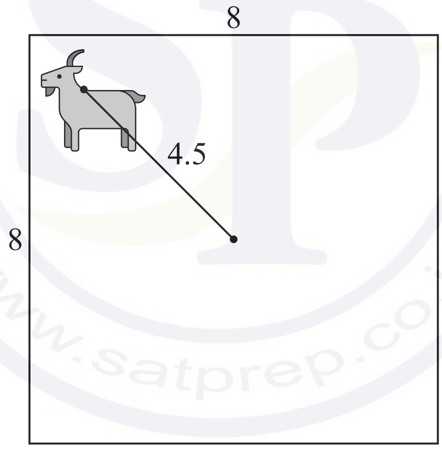


(a) (i) Find the angle $\text{A}\hat{\text{O}}\text{B}$.

(ii) Find the area of the shaded segment. [8]

A square field with side $8\,\text{m}$ has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to $4.5\,\text{m}$ from the post.

diagram not to scale



(b) Find the area of the field that can be reached by the goat. [4]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.3\,te^{-t}$.

(c) Find the value of t at which the goat is eating grass at the greatest rate. [2]

The goat is tied in the field for 8 hours.

(d) Find the total volume of grass eaten by the goat during this time. [3]

3. [Maximum mark: 13]

A Principal would like to compare the students in his school with a national standard. He decides to give a test to eight students made up of four boys and four girls. One of the teachers offers to find the volunteers from his class.

(a) Name the type of sampling that best describes the method used by the Principal. [1]

The marks out of 40, for the students who took the test, are:

$$25,\ 29,\ 38,\ 37,\ 12,\ 18,\ 27,\ 31.$$

(b) For the eight students find

(i) the mean mark.

(ii) the standard deviation of the marks. [3]

The national standard mark is 25.2 out of 40.

(c) Perform an appropriate test at the 5% significance level to see if the mean marks achieved by the students in the school are higher than the national standard. It can be assumed that the marks come from a normal population. [5]

(d) State one reason why the test might not be valid. [1]

Two additional students take the test at a later date and the mean mark for all ten students is 28.1 and the standard deviation is 8.4.

For further analysis, a standardized score out of 100 for the ten students is obtained by multiplying the scores by 2 and adding 20.

(e) For the ten students, find

(i) their mean standardized score.

(ii) the standard deviation of their standardized score. [3]

4. [Maximum mark: 13]

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0$$

(a) (i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

(iii) Hence state the long-term velocity of the particle. [5]

The equation for the motion of the particle is amended to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 3t + 4.$$

(b) (i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to write the differential equation as a system of coupled, first order differential equations.

When $t = 0$ the particle is stationary at O.

(ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$.

(iii) Find the long-term velocity of the particle. [8]

5. [Maximum mark: 15]

The aircraft for a particular flight has 72 seats. The airline's records show that historically for this flight only 90% of the people who purchase a ticket arrive to board the flight. They assume this trend will continue and decide to sell extra tickets and hope that no more than 72 passengers will arrive.

The number of passengers that arrive to board this flight is assumed to follow a binomial distribution with a probability of 0.9.

(a) The airline sells 74 tickets for this flight. Find the probability that more than 72 passengers arrive to board the flight. [3]

(b) (i) Write down the expected number of passengers who will arrive to board the flight if 72 tickets are sold. [2]

(ii) Find the maximum number of tickets that could be sold if the expected number of passengers who arrive to board the flight must be less than or equal to 72. [2]

Each passenger pays $\$150$ for a ticket. If too many passengers arrive, then the airline will give $\$300$ in compensation to each passenger that cannot board.

(c) Find, to the nearest integer, the expected increase or decrease in the money made by the airline if they decide to sell 74 tickets rather than 72. [8]

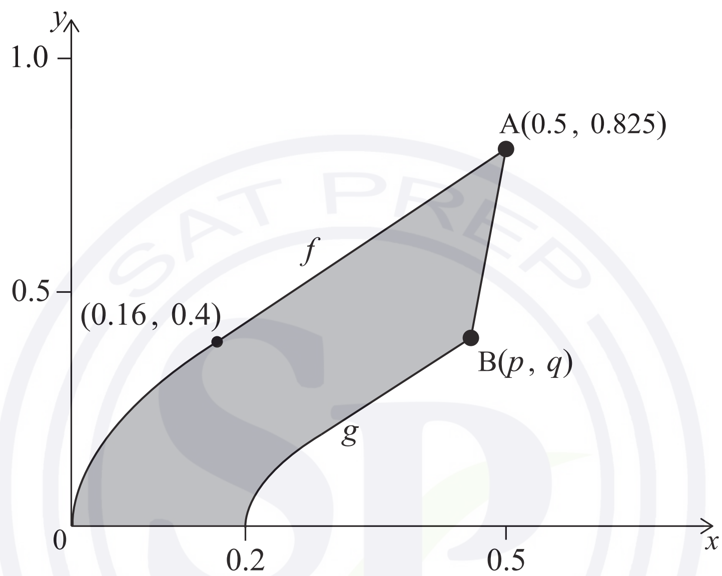
6. [Maximum mark: 18]

Consider the curve $y = \sqrt{x}$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence show that the equation of the tangent to the curve at the point $(0.16,\ 0.4)$ is $y = 1.25x + 0.2$. [4]

The shape of a piece of metal can be modelled by the region bounded by the functions f, g, the x-axis and the line segment [AB], as shown in the following diagram. The units on the x and y axes are measured in metres.



The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 0.16 \\ 1.25x + 0.2 & 0.16 < x \le 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5,\ 0.825)$. Point B is the image of A under the given transformations and has coordinates $(p,\ q)$.

(b) Find the value of p and the value of q. [2]

(This question continues on the following page)

(Question 6 continued)

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \le x \le a \\ 1.25x + b & a < x \le p \end{cases}$$

(c) Find

(i) an expression for $h(x)$.

(ii) the value of a.

(iii) the value of b. [5]

(d) (i) Find the area enclosed by $y = f(x)$, the x-axis and the line $x = 0.5$.

The area enclosed by $y = g(x)$, the x-axis and the line $x = p$ is $0.0627292\,\mathrm{m}^2$ correct to six significant figures.

(ii) Find the area of the shaded region on the diagram. [7]

7. [Maximum mark: 18]

A transformation, T, of a plane is represented by $\boldsymbol{r}' = \boldsymbol{Pr} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}'$ the position vector of its image under T.

The triangle OAB has coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$. Under T, these points are transformed to $(0, 1)$, $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ respectively.

(a) (i) By considering the image of $(0, 0)$, find $\boldsymbol{q}$.

(ii) By considering the image of $(1, 0)$ and $(0, 1)$, show that [6]

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{RS}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0, 0)$.

$\boldsymbol{R}$ represents a rotation about $(0, 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

(c) (i) Use $\boldsymbol{P} = \boldsymbol{RS}$ to find the matrix $\boldsymbol{R}$.

(ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [7]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre (a, b), followed by a rotation about the same centre (a, b).

(d) (i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(ii) Find the value of a and the value of b. [4]

References:

2. mynamepong, n.d. Goat [image online] Available at: https://thenounproject.com/term/goat/1761571/
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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Tuesday 2 November 2021 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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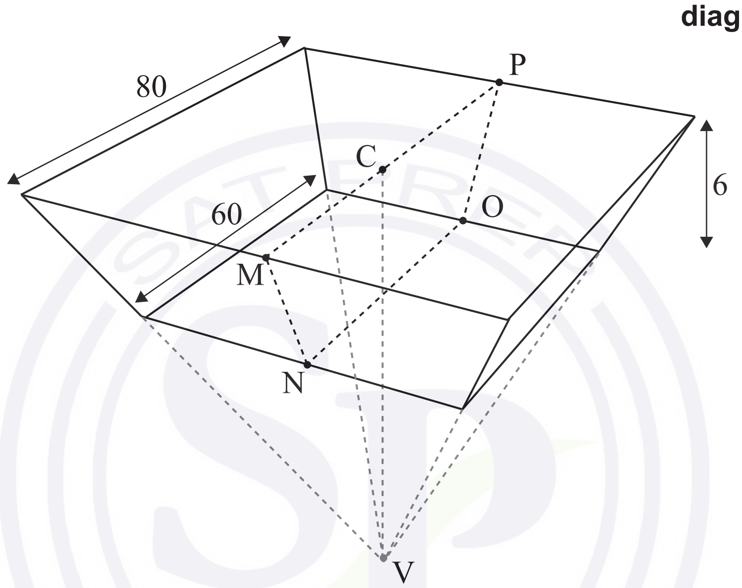
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

A large water reservoir is built in the form of part of an upside-down right pyramid with a horizontal square base of length 80 metres. The point C is the centre of the square base and point V is the vertex of the pyramid.

diagram not to scale



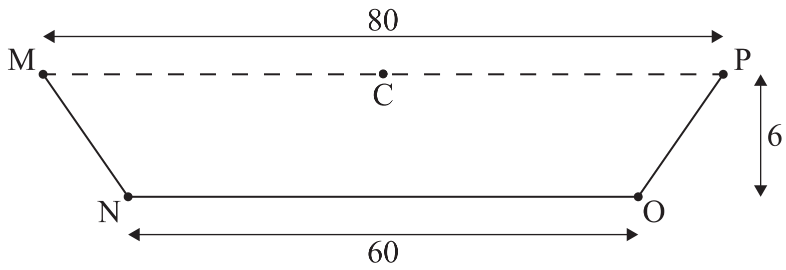
The bottom of the reservoir is a square of length 60 metres that is parallel to the base of the pyramid, such that the depth of the reservoir is 6 metres as shown in the diagram.

(This question continues on the following page)

(Question 1 continued)

The second diagram shows a vertical cross section, MNOPC, of the reservoir.

diagram not to scale



(a) Find the angle of depression from M to N. [2]

(b) (i) Find CV.

(ii) Hence or otherwise, show that the volume of the reservoir is $29\,600\,\mathrm{m}^3$. [5]

Every day $80\,\mathrm{m}^3$ of water from the reservoir is used for irrigation.

Joshua states that, if no other water enters or leaves the reservoir, then when it is full there is enough irrigation water for at least one year.

(c) By finding an appropriate value, determine whether Joshua is correct. [2]

To avoid water leaking into the ground, the five interior sides of the reservoir have been painted with a watertight material.

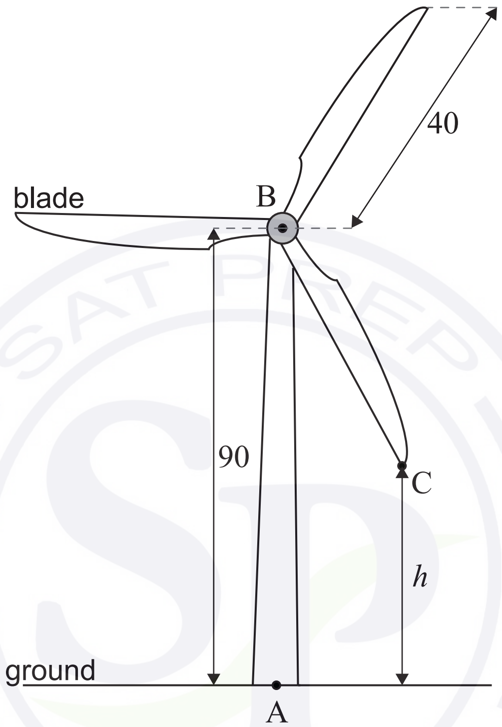
(d) Find the area that was painted. [5]

2. [Maximum mark: 20]

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point A is on the base of the tower directly below point B at the top of the tower. The height of the tower, AB, is $90\,\text{m}$. The blades of the turbine are centred at B and are each of length $40\,\text{m}$. This is shown in the following diagram.

diagram not to scale



The end of one of the blades of the turbine is represented by point C on the diagram. Let h be the height of C above the ground, measured in metres, where h varies as the blade rotates.

(a) Find the

(i) maximum value of h.

(ii) minimum value of h. [2]

The blades of the turbine complete 12 rotations per minute under normal conditions, moving at a constant rate.

(b) (i) Find the time, in seconds, it takes for the blade $[BC]$ to make one complete rotation under these conditions.

(ii) Calculate the angle, in degrees, that the blade $[BC]$ turns through in one second. [3]

(This question continues on the following page)

(Question 2 continued)

The height, h, of point C can be modelled by the following function. Time, t, is measured from the instant when the blade $[\mathrm{BC}]$ first passes $[\mathrm{AB}]$ and is measured in seconds.

$$h(t) = 90 - 40\cos(72t^{\circ}),\ t \geq 0$$

(c) (i) Write down the amplitude of the function.

(ii) Find the period of the function. [2]

(d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]

(e) (i) Find the height of C above the ground when $t = 2$.

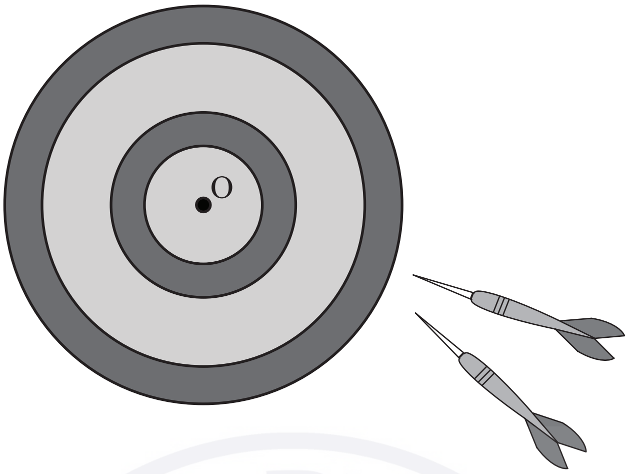
(ii) Find the time, in seconds, that point C is above a height of $100\,\mathrm{m}$, during each complete rotation. [5]

The wind speed increases and the blades rotate faster, but still at a constant rate.

(f) Given that point C is now higher than $110\,\mathrm{m}$ for 1 second during each complete rotation, find the time for one complete rotation. [5]

3. [Maximum mark: 16]

Arianne plays a game of darts.



The distance that her darts land from the centre, O, of the board can be modelled by a normal distribution with mean $10\,\mathrm{cm}$ and standard deviation $3\,\mathrm{cm}$.

(a) Find the probability that

(i) a dart lands less than $13\,\mathrm{cm}$ from O.

(ii) a dart lands more than $15\,\mathrm{cm}$ from O. [3]

Each of Arianne's throws is independent of her previous throws.

(b) Find the probability that Arianne throws two consecutive darts that land more than $15\,\mathrm{cm}$ from O. [2]

In a competition a player has three darts to throw on each turn. A point is scored if a player throws **all** three darts to land within a central area around O. When Arianne throws a dart the probability that it lands within this area is 0.8143.

(c) Find the probability that Arianne does **not** score a point on a turn of three darts. [2]

In the competition Arianne has ten turns, each with three darts.

(d) (i) Find Arianne's expected score in the competition.

(ii) Find the probability that Arianne scores at least 5 points in the competition.

(iii) Find the probability that Arianne scores at least 5 points and less than 8 points.

(iv) Given that Arianne scores at least 5 points, find the probability that Arianne scores less than 8 points. [9]

4. [Maximum mark: 18]

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$.
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a) (i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation.

(ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position.

(iii) Find $\boldsymbol{P}^2$.

(iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [12]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points A′, B′ and C′ respectively.

Show that the area of triangle ABC is equal to the area of triangle A′B′C′. [2]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

5. [Maximum mark: 13]

(a) Let $z = 1 - \mathrm{i}$.

(i) Plot the position of z on an Argand Diagram.

(ii) Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where a, $b \in \mathbb{R}$, giving the exact value of a and the exact value of b. [3]

(b) Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(i) Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$.

(ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \le \frac{\pi}{2}$. [6]

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_{A} and I_{B}. The maximum value and the phase shift of each current is shown in the following table.

Current	**Maximum value**	**Phase shift**
I_{A}	12 amps	0
I_{B}	12 amps	$\frac{\pi}{2}$

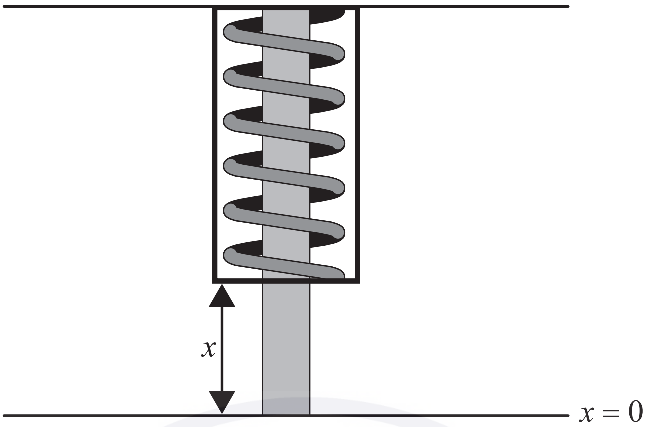
When the two voltage sources are connected to the circuit at the same time, the total current I_{T} can be expressed as $I_{\mathrm{A}} + I_{\mathrm{B}}$.

(c) (i) Find the maximum value of I_{T}.

(ii) Find the phase shift of I_{T}. [4]

6. [Maximum mark: 15]

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}x\\ y\end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

(c) (i) Find the eigenvalues of matrix $\boldsymbol{A}$.

(ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [6]

(d) Given that when $t = 0$ the shock absorber is displaced $8\,\text{cm}$ and its velocity is zero, find an expression for x in terms of t. [6]

7. [Maximum mark: 14]

Loreto is a manager at the Da Vinci health centre. If the mean rate of patients arriving at the health centre exceeds 1.5 per minute then Loreto will employ extra staff. It is assumed that the number of patients arriving in any given time period follows a Poisson distribution.

Loreto performs a hypothesis test to determine whether she should employ extra staff. She finds that 320 patients arrived during a randomly selected 3-hour clinic.

(a) (i) Write down null and alternative hypotheses for Loreto's test.

(ii) Using the data from Loreto's sample, perform the hypothesis test at a 5% significance level to determine if Loreto should employ extra staff. [7]

Loreto is also concerned about the average waiting time for patients to see a nurse. The health centre aims for at least 95% of patients to see a nurse in under 20 minutes.

Loreto assumes that the waiting times for patients are independent of each other and decides to perform a hypothesis test at a 10% significance level to determine whether the health centre is meeting its target.

Loreto surveys 150 patients and finds that 11 of them waited more than 20 minutes.

(b) (i) Write down null and alternative hypotheses for this test.

(ii) Perform the test, clearly stating the conclusion in context. [7]

References:

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

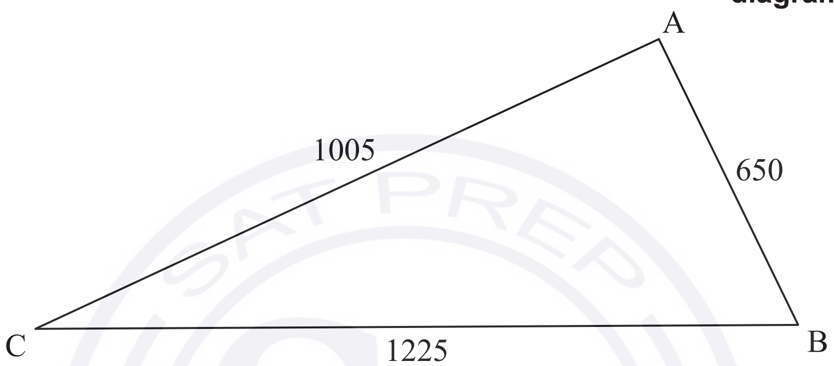
2221 – 7212

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

A farmer owns a field in the shape of a triangle ABC such that $AB = 650\,m$, $AC = 1005\,m$ and $BC = 1225\,m$.

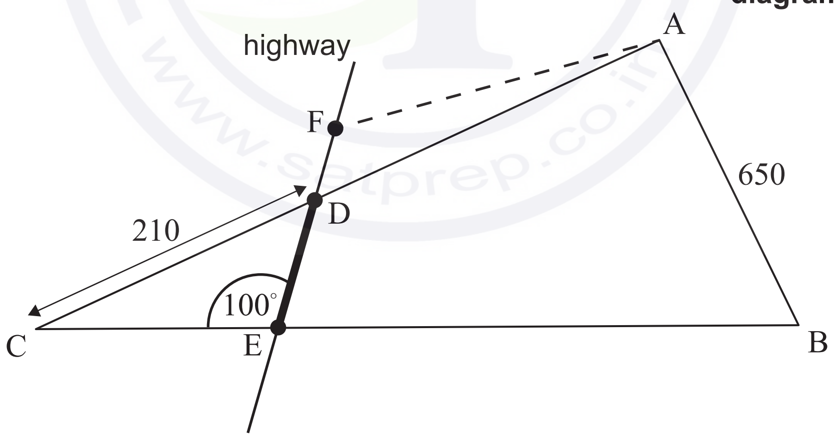
diagram not to scale



(a) Find the size of $A\hat{C}B$. [3]

The local town is planning to build a highway that will intersect the borders of the field at points D and E, where $DC = 210\,m$ and $C\hat{E}D = 100^\circ$, as shown in the diagram below.

diagram not to scale



(b) Find DE. [3]

The town wishes to build a carpark here. They ask the farmer to exchange the part of the field represented by triangle DCE. In return the farmer will get a triangle of equal area ADF, where F lies on the same line as D and E, as shown in the diagram above.

(c) Find the area of triangle DCE. [5]

(d) Estimate DF. You may assume the highway has a width of zero. [4]

2. [Maximum mark: 16]

It is known that the weights of male Persian cats are normally distributed with mean $6.1\,\text{kg}$ and variance $0.5^2\,\text{kg}^2$.

(a) Sketch a diagram showing the above information. [2]

(b) Find the proportion of male Persian cats weighing between $5.5\,\text{kg}$ and $6.5\,\text{kg}$. [2]

A group of 80 male Persian cats are drawn from this population.

(c) Determine the expected number of cats in this group that have a weight of less than $5.3\,\text{kg}$. [3]

The male cats are now joined by 80 female Persian cats. The female cats are drawn from a population whose weights are normally distributed with mean $4.5\,\text{kg}$ and standard deviation $0.45\,\text{kg}$.

(d) Ten female cats are chosen at random.

(i) Find the probability that exactly one of them weighs over $4.62\,\text{kg}$.

(ii) Let N be the number of cats weighing over $4.62\,\text{kg}$.

Find the variance of N. [5]

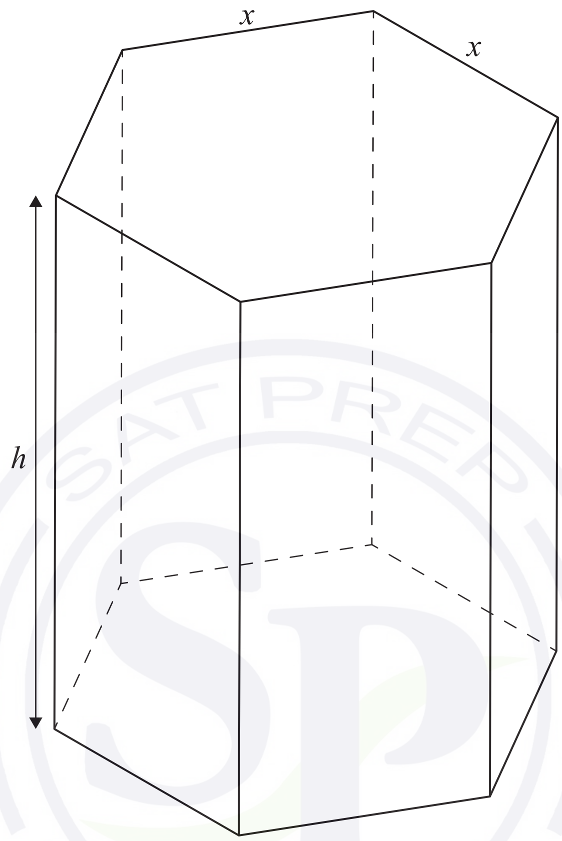
A cat is selected at random from all 160 cats.

(e) Find the probability that the cat was female, given that its weight was over $4.7\,\text{kg}$. [4]

3. [Maximum mark: 15]

A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is h cm, and the top and base of the prism have sides of length x cm.

diagram not to scale



(a) Given that $\sin 60^\circ = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]

(b) Given that the total external surface area of the box is $1200\,\text{cm}^2$, show that the volume of the box may be expressed as $V = 300\sqrt{3}x - \frac{9}{4}x^3$. [5]

(c) Sketch the graph of $V = 300\sqrt{3}x - \frac{9}{4}x^3$, for $0 \le x \le 16$. [2]

(d) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}x}$. [2]

(e) Find the value of x which maximizes the volume of the box. [2]

(f) Hence, or otherwise, find the maximum possible volume of the box. [2]

4. [Maximum mark: 18]

In a small village there are two doctors' clinics, one owned by Doctor Black and the other owned by Doctor Green. It was noted after each year that 3.5% of Doctor Black's patients moved to Doctor Green's clinic and 5% of Doctor Green's patients moved to Doctor Black's clinic. All additional losses and gains of patients by the clinics may be ignored.

At the start of a particular year, it was noted that Doctor Black had 2100 patients on their register, compared to Doctor Green's 3500 patients.

(a) Write down a transition matrix $\boldsymbol{T}$ indicating the annual population movement between clinics. [2]

(b) Find a prediction for the ratio of the number of patients Doctor Black will have, compared to Doctor Green, after two years. [2]

(c) Find a matrix $\boldsymbol{P}$, with integer elements, such that $\boldsymbol{T} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [6]

(d) Hence, show that the long-term transition matrix $\boldsymbol{T}^{\infty}$ is given by $\boldsymbol{T}^{\infty} = \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$. [6]

(e) Hence, or otherwise, determine the expected ratio of the number of patients Doctor Black would have compared to Doctor Green in the long term. [2]

5. [Maximum mark: 14]

Hank sets up a bird table in his garden to provide the local birds with some food. Hank notices that a specific bird, a large magpie, visits several times per month and he names him Bill. Hank models the number of times per month that Bill visits his garden as a Poisson distribution with mean 3.1.

(a) Using Hank's model, find the probability that Bill visits the garden on exactly four occasions during one particular month. [1]

(b) Over the course of 3 consecutive months, find the probability that Bill visits the garden:

(i) on exactly 12 occasions.

(ii) during the first and third month only. [5]

(c) Find the probability that over a 12-month period, there will be exactly 3 months when Bill does not visit the garden. [4]

After the first year, a number of baby magpies start to visit Hank's garden. It may be assumed that each of these baby magpies visits the garden randomly and independently, and that the number of times each baby magpie visits the garden per month is modelled by a Poisson distribution with mean 2.1.

(d) Determine the least number of magpies required, including Bill, in order that the probability of Hank's garden having at least 30 magpie visits per month is greater than 0.2. [4]

6. [Maximum mark: 15]

A particle P moves along the x-axis. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ at time t seconds, where $v = -2t^2 + 16t - 24$ for $t \geq 0$.

(a) Find the times when P is at instantaneous rest. [2]

(b) Find the magnitude of the particle's acceleration at 6 seconds. [4]

(c) Find the greatest speed of P in the interval $0 \leq t \leq 6$. [2]

(d) The particle starts from the origin O. Find an expression for the displacement of P from O at time t seconds. [4]

(e) Find the total distance travelled by P in the interval $0 \leq t \leq 4$. [3]

7. [Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]

(d) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$

(i) at $(4, 0)$.

(ii) at $(-4, 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \le x \le 6$, $-6 \le y \le 6$. [4]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

Instructions to candidates

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2221 – 7207

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested $37\,000$ Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

(a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than $50\ 000\ \text{AUD}$.

(b) Find the minimum value of m, where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is $200\,000\ \text{AUD}$.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

(c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of $1700\ \text{AUD}$ made by Daisy at the end of each month.

(d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

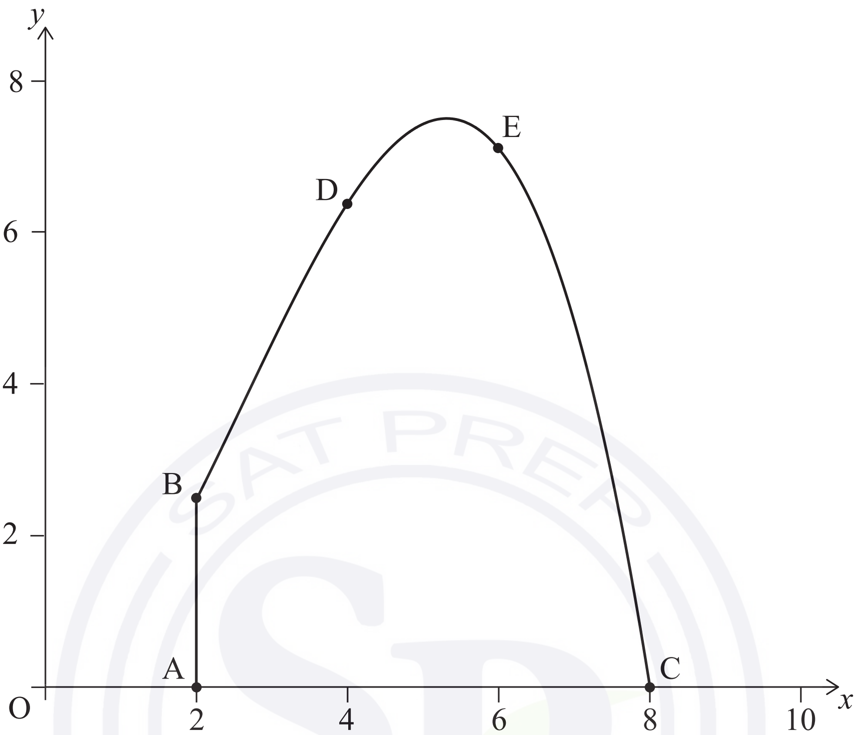
After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

(e) Find the amount of Daisy's final payment. [3]

(f) Find how much money Daisy saved by making one final payment after 5 years. [3]

2. [Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line $[AB]$ represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2$, $2 \le x \le 8$, relative to an origin O.



Point A has coordinates $(2, 0)$, point B has coordinates $(2, 2.4)$, and point C has coordinates $(8, 0)$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence find the maximum height of the tunnel. [6]

(b) Find the height of the tunnel when

(i) $x = 4$.

(ii) $x = 6$. [3]

(c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]

(d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.

(ii) Hence find the cross-sectional area of the tunnel. [4]

3. [Maximum mark: 13]

The stopping distances for bicycles travelling at $20\,\mathrm{km\,h^{-1}}$ are assumed to follow a normal distribution with mean $6.76\,\mathrm{m}$ and standard deviation $0.12\,\mathrm{m}$.

(a) Under this assumption, find, correct to four decimal places, the probability that a bicycle chosen at random travelling at $20\,\mathrm{km\,h^{-1}}$ manages to stop

(i) in less than $6.5\,\mathrm{m}$.

(ii) in more than $7\,\mathrm{m}$. [3]

1000 randomly selected bicycles are tested and their stopping distances when travelling at $20\,\mathrm{km\,h^{-1}}$ are measured.

(b) Find, correct to four significant figures, the expected number of bicycles tested that stop between

(i) $6.5\,\mathrm{m}$ and $6.75\,\mathrm{m}$.

(ii) $6.75\,\mathrm{m}$ and $7\,\mathrm{m}$. [3]

The measured stopping distances of the 1000 bicycles are given in the table.

Measured stopping distance	**Number of bicycles**
Less than $6.5\,\mathrm{m}$	12
Between $6.5\,\mathrm{m}$ and $6.75\,\mathrm{m}$	428
Between $6.75\,\mathrm{m}$ and $7\,\mathrm{m}$	527
More than $7\,\mathrm{m}$	33

It is decided to perform a χ^2 goodness of fit test at the $5\,\%$ level of significance to decide whether the stopping distances of bicycles travelling at $20\,\mathrm{km\,h^{-1}}$ can be modelled by a normal distribution with mean $6.76\,\mathrm{m}$ and standard deviation $0.12\,\mathrm{m}$.

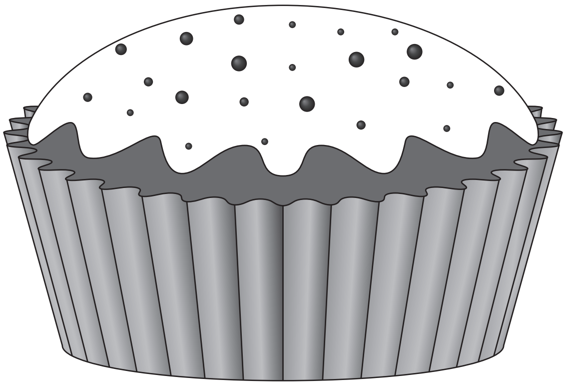
(c) State the null and alternative hypotheses. [2]

(d) Find the p-value for the test. [3]

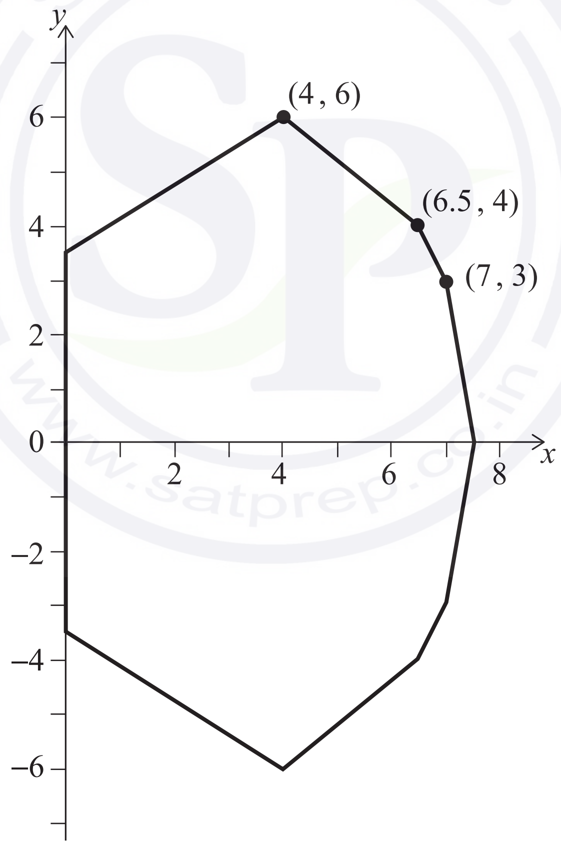
(e) State the conclusion of the test. Give a reason for your answer. [2]

4. [Maximum mark: 14]

Charlotte decides to model the shape of a cupcake to calculate its volume.



From rotating a photograph of her cupcake she estimates that its cross-section passes through the points $(0, 3.5)$, $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$, where all units are in centimetres. The cross-section is symmetrical in the x-axis, as shown below:



She models the section from $(0, 3.5)$ to $(4, 6)$ as a straight line.

(a) Find the equation of the line passing through these two points. [2]

(This question continues on the following page)

(Question 4 continued)

Charlotte models the section of the cupcake that passes through the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$ with a quadratic curve.

(b) (i) Find the equation of the least squares regression quadratic curve for these four points.

(ii) By considering the gradient of this curve when $x = 4$, explain why it may not be a good model. [3]

Charlotte thinks that a quadratic with a maximum point at $(4, 6)$ and that passes through the point $(7.5, 0)$ would be a better fit.

(c) Find the equation of the new model. [4]

Believing this to be a better model for her cupcake, Charlotte finds the volume of revolution about the x-axis to estimate the volume of the cupcake.

(d) (i) Write down an expression for her estimate of the volume as a sum of two integrals.

(ii) Find the value of Charlotte's estimate. [5]

5. [Maximum mark: 13]

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where $\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c) (i) Write down the matrix $\boldsymbol{P}$.

(ii) Write down the matrix $\boldsymbol{D}$. [2]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

6. [Maximum mark: 18]

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\mathrm{e}^{bt}\cos t \\ a\mathrm{e}^{bt}\sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t. [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a\mathrm{e}^{bt}\sqrt{(1+b^2)}.$$ [4]

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b. [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

At a point P, the ice-skater is skating parallel to the y-axis for the first time.

(e) Find OP. [6]

7. [Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$$

(a) Find the population of rabbits 1 year after they were introduced. [5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t, can be approximately modelled by the coupled equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(2 - 0.01y)$$

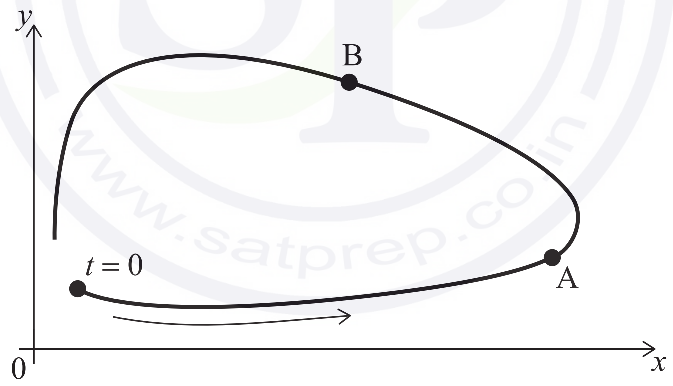
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(0.0002x - 0.8)$$

(b) Use Euler's method with a step size of 0.25, to find

(i) the population of rabbits 1 year after the foxes were introduced.

(ii) the population of foxes 1 year after the foxes were introduced. [6]

(c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.



Describe the changes in the populations of rabbits and foxes for these 4 years at

(i) point A.

(ii) point B. [3]

(d) Find the non-zero equilibrium point for the populations of rabbits and foxes. [3]

References:

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

Specimen

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
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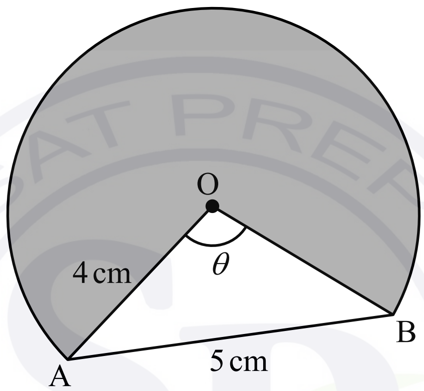
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius $4\,\mathrm{cm}$.



Chord AB has a length of $5\,\mathrm{cm}$ and $\mathrm{A\hat{O}B} = \theta$.

(a) Find the value of θ, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

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2. [Maximum mark: 6]

On 1st January 2020, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of 5.5%, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, r.

(a) Find the value of r, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

3. [Maximum mark: 6]

A six-sided biased die is weighted in such a way that the probability of obtaining a "six" is $\frac{7}{10}$.

The die is tossed five times. Find the probability of obtaining

(a) at most three "sixes". [3]

(b) the third "six" on the fifth toss. [3]

4. [Maximum mark: 7]

The following table below shows the marks scored by seven students on two different mathematics tests.

Test 1 (x)	15	23	25	30	34	34	40
Test 2 (y)	20	26	27	32	35	37	35

Let L_1 be the regression line of x on y. The equation of the line L_1 can be written in the form $x = ay + b$.

(a) Find the value of a and the value of b. [2]

Let L_2 be the regression line of y on x. The lines L_1 and L_2 pass through the same point with coordinates (p, q).

(b) Find the value of p and the value of q. [3]

(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's mark on the first test. [2]

5. [Maximum mark: 7]

The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t + 2t \sin t$, $0 \le t \le 5$.

(a) Find the maximum distance of the particle from O. [3]

(b) Find the acceleration of the particle at the instant it first changes direction. [4]

6. [Maximum mark: 6]

In a city, the number of passengers, X, who ride in a taxi has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X=x)$	0.60	0.30	0.03	0.05	0.02

After the opening of a new highway that charges a toll, a taxi company introduces a charge for passengers who use the highway. The charge is \$2.40 per taxi plus \$1.20 per passenger. Let T represent the amount, in dollars, that is charged by the taxi company per ride.

(a) Find $\mathrm{E}(T)$. [4]

(b) Given that $\mathrm{Var}(X) = 0.8419$, find $\mathrm{Var}(T)$. [2]

7. [Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$
$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

8. [Maximum mark: 7]

The complex numbers w and z satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

9. [Maximum mark: 5]

Consider the graphs of $y = \dfrac{x^2}{x-3}$ and $y = m(x+3)$, $m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The length, $X\,\text{mm}$, of a certain species of seashell is normally distributed with mean 25 and variance, σ^2.

The probability that X is less than 24.15 is 0.1446.

(a) Find $\mathrm{P}(24.15 < X < 25)$. [2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a seashell selected at random has a length greater than $26\,\text{mm}$. [5]

A random sample of 10 seashells is collected on a beach. Let Y represent the number of seashells with lengths greater than $26\,\text{mm}$.

(c) Find $\mathrm{E}(Y)$. [3]

(d) Find the probability that exactly three of these seashells have a length greater than $26\,\text{mm}$. [2]

A seashell selected at random has a length less than $26\,\text{mm}$.

(e) Find the probability that its length is between $24.15\,\text{mm}$ and $25\,\text{mm}$. [3]

Do **not** write solutions on this page.

11. [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let x grams represent the amount of salt in the tank and let t minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{dx}{dt}$, is described by the differential equation $\frac{dx}{dt} = 10e^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t + 1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40e^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of x versus t for $0 \le t \le 60$ and hence find the maximum amount of salt in the tank and the value of t at which this occurs. [5]

(d) Find the value of t at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first 60 minutes. [4]

12. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1 - \tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]