8824 – 9741M

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2024

## Mathematics: applications and interpretation

## Higher level

## Paper 2

21 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg **M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | Correct answer seen | Further working seen | Any FT issues? | Action |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

**1.** (a) (i) $(Q_3 =)\ 7.2$ ***A2***

**Note:** Award ***A1A0*** if the lower quartile, $6.5$, is given as the answer.
Award ***A1A0*** for a correct ordered list of happiness scores, when the correct $Q_3$ is not seen.

(ii) $Q_1 = 6.5$ ***(A1)***

$IQR = 7.2 - 6.5$

$= 0.7$ ***A1***

***[4 marks]***

(b) $Q_3 + 1.5 \times IQR$ ***(A1)***

$(7.2 + 1.5 \times 0.7 =)\ 8.25$ ***A1***

since $7.5 < 8.25$ ***R1***

Switzerland is not an outlier ***AG***

**Note:** Do not award ***A0A0R1***.

***[3 marks]***

(c) (i) $a = 3.5$ ***A1***

(ii) $b = 8$ ***A1***

(iii) $c = 3.5$ ***A1***

***[3 marks]***

(d) (i) $(r_s =)\ 0.164\ (0.164134\ldots)$ ***A2***

(ii) France rank (of sixth) is unchanged (so the $r_s$ is unchanged) ***R1***

***[3 marks]***

(e) Because $r_s$ is too close to zero and hence Jose's conclusion is not appropriate ***R1***

**Note:** Award ***R1*** for a comment on the value $r_s$ and "not appropriate" as a conclusion.
Accept "$r_s$ indicates a weak correlation of the ranks".
Do not accept "$r_s$ indicates a weak correlation of happiness score and country rank".

***[1 mark]***

***[Total 14 marks]***

**2.** (a) $\frac{1}{2}\times 1\times\big((4+0.7)+2(7.3+6.7+4+1.3)\big)$ ***(A1)(A1)***

$=21.7\ \ (21.65)\ \ (\text{cm}^2)$ ***A1***

***[3 marks]***

(b) $22.1\ (22.08333\ldots,\frac{265}{12})\ \ (\text{cm}^2)$ ***A2***

***[2 marks]***

(c) $600=\pi r^2 h$ ***A1***

***[1 mark]***

(d) **METHOD 1** (Substitution of $h=\frac{600}{\pi r^2}$ in $A=2\pi r^2+2\pi rh$)

$A=2\pi r^2+2\pi rh$ ***(A1)***

attempt to isolate $h$ or $\pi rh$ ***(M1)***

$h=\frac{600}{\pi r^2}$ **OR** $\pi rh=\frac{600}{r}$

correct substitution of their $h$ into correct expression ***(M1)***

$A=2\pi r^2+2\pi r\left(\frac{600}{\pi r^2}\right)\quad\left(A=2\pi r^2+\frac{1200}{r}\right)$

$k=1200$ ***A1***

**METHOD 2** (Equating $2\pi rh$ and $\frac{k}{r}$)

$A=2\pi r^2+2\pi rh$ **OR** $2\pi rh=\frac{k}{r}$ ***(A1)***

attempt to isolate $h$ or $(\pi)rh$ or $r$ or $r^2$ ***(M1)***

$h=\frac{600}{\pi r^2}$ **OR** $\pi rh=\frac{600}{r}$ **OR** $r=\sqrt{\frac{600}{\pi h}}$ **OR** $r^2=\frac{600}{\pi h}$

correct substitution of their $h$ or $\pi rh$ or $r$ or $r^2$ into $2\pi r^2+2\pi rh=2\pi r^2+\frac{k}{r}$ ***(M1)***

$2\pi r\left(\frac{600}{\pi r^2}\right)=\frac{k}{r}$ **OR** $2\left(\frac{600}{r}\right)=\frac{k}{r}$ **OR** $2(600)=k$ **OR** $2\pi\left(\frac{600}{\pi h}\right)h=k$

$k=1200$ ***A1***

***[4 marks]***

*continue…*

*Question 2 continued.*

(e) (i) $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - 1200r^{-2}$ ***A1(M1)A1***

**Note:** Award ***A1*** for $4\pi r$ seen, and ***(M1)*** for expressing $\dfrac{1200}{r}$ as $1200r^{-1}$ (can be implied through $\mp\dfrac{1200}{r^2}$ seen), ***A1*** for $-1200r^{-2}$. Award at most ***A1(M1)A0*** if any additional terms are seen.

(ii) $0 = 4\pi r - 1200r^{-2}$ **OR** $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 0$ ***(M1)***

$r = 4.57\left(4.570781\ldots,\ \sqrt[3]{\dfrac{300}{\pi}}\right)$ (cm) ***A1***

**Note:** Award at most ***M1A0*** if the final answer is in terms of $k$.

***[5 marks]***

(f) $(h =)\ 9.14\ (9.14156\ldots..)$ (cm) ***A1***

$(C =)\ 2\pi\ (4.570781\ldots.) \approx 28.7\ \left(28.7190\ldots\right)$(cm) ***A1***

**EITHER**

the longest dimension of the label (9 cm) is less than both values and hence the label will fit (in any rotation) ***R1***

**OR**

$9 < 9.14$ and $5 < 28.7$ ***R1***

**Note:** Do not accept an argument based on the comparison of areas.

***[3 marks]***

***[Total 18 marks]***

**3.** (a) recognition of a geometric sequence ***(M1)***

$3\left(\frac{2}{3}\right)^4$ **OR** $3, 2, 4/3\ldots$

$= 0.593\ (0.592592\ldots, 16/27)$ (cm) ***A1***

***[2 marks]***

(b) attempt to sum a geometric sequence ***(M1)***

$\frac{3\left(1-\left(\frac{2}{3}\right)^5\right)}{1-\frac{2}{3}}$ **OR** $3+2+\frac{4}{3}\ldots$

$= 7.81\ (7.81481\ldots, 211/27)$ (cm) ***A1***

***[2 marks]***

(c) recognition of need to find sum to infinity ***(M1)***

$\frac{3}{1-\frac{2}{3}}$ ***(A1)***

$= 9$ (cm) ***A1***

***[3 marks]***

(d) Comparing the sum of the widths greater than (or equal to) $8.5$ ***(M1)***

e.g. $\frac{3\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} \geq 8.5$ **OR** sketch **OR** list of values with cross-over values

$7.13\ \ (7.1285338740543\ldots)$ seen ***(A1)***

$n \geq 7.13$

$n = 8$ ***A1***

***[3 marks]***

(e) attempt to divide two adjacent areas **OR** $\left(\frac{2}{3}\right)^2$ ***(M1)***

$\frac{4}{9}\ (0.444, 0.444444\ldots)$ ***A1***

***[2 marks]***

*continue…*

*Question 3 continued.*

(f) $u_1 = 13.5$ (may be seen in part (e)) ***(A1)***

attempt to find the sum of their $n$ terms, with their $u_1$ and their $r$ ***M1***

$$S_8 = \frac{13.5\left(1-\left(\frac{4}{9}\right)^8\right)}{1-\frac{4}{9}}$$

**Note:** Do not award ***M1*** if $5$ or infinity used for $n$ or if $\frac{2}{3}$ used for $r$.

$= 24.3\ (24.2630\ldots)$ (cm$^2$) ***A1***

**Note:** Accept $24.2\ (24.2439\ldots)$ from using $0.444$.

***[3 marks]***
***[Total: 15 marks]***

**4.** (a) attempt to use Pythagoras' theorem ***(M1)***

$\sqrt{3.4^2 - 2^2}$

$= 2.75\ (2.74954\ldots)$ (m) ***A1***

***[2 marks]***

(b) (i) **METHOD 1** (Use of $\frac{1}{2} \times a \times b \times \sin(\theta)$)

$60°$ ***(A1)***

attempt to find area of one triangle using $\frac{1}{2} \times a \times b \times \sin(\theta)$ ***(M1)***

$\frac{1}{2} \times 2 \times 2 \times \sin(60°)$

$\left(6 \times \frac{1}{2} \times 2 \times 2 \times \sin(60^\circ)\right) = 10.4\ \ (10.3923..., 6\sqrt{3})\ (\text{m}^2)$ ***A1***

**Note:** Award ***A0M0A0*** for $\frac{1}{2} \times 2 \times 2$ or equivalent.

**METHOD 2** (Use of altitude)

(altitude is) $\sqrt{3}$ ***(A1)***

attempt to find the area of one triangle using $\frac{1}{2} \times b \times h$ with their altitude. ***(M1)***

$\frac{1}{2} \times 2 \times \sqrt{3}$

$\left(6 \times \frac{1}{2} \times 2 \times \sqrt{3}\right) = 10.4\ \ (10.3923..., 6\sqrt{3})\ (\text{m}^2)$ ***A1***

**Note:** Award ***A0M0A0*** for $\frac{1}{2} \times 2 \times 2$ or equivalent.

**METHOD 3** (Finding the area of a trapezoid)

(altitude of one trapezoid is) $\sqrt{3}$ ***(A1)***

attempt to find area of one trapezoid using $\frac{1}{2} \times (a + b)h$ ***(M1)***

$\frac{1}{2} \times (2 + 4)\sqrt{3} \quad \left(3\sqrt{3}\right)$

$\left(2 \times \frac{1}{2} \times (2 + 4)\sqrt{3}\ \right) = 10.4\ \ (10.3923..., 6\sqrt{3})\ (\text{m}^2)$ ***A1***

*continue…*

*Question 4 continued.*

(ii) $\frac{1}{3} \times 10.3923... \times 2.74954...$ ***(A1)***

$= 9.52 \text{ m}^3 \ (9.52470\ldots)$ ***A1***

**Note:** Units must be seen.

***[5 marks]***

(c) $\cos(\text{M}\hat{\text{A}}\text{T}) = \frac{2}{3.4}$ or correct equivalent ***(A1)***

$(\text{MAT} =) \ 54.0^\circ$ (53.9681…, 0.941921… radians) ***A1***

***[2 marks]***

(d) Angle $\text{YAX} = 180 - 53.9681... = 126.031\ldots^\circ$ ***(A1)***

Angle $\text{YXA} = 180 - 35 - 126.031... = 18.9681\ldots^\circ$ ***(A1)***

**Note:** These angles may be seen in the sine rule.

Attempt to substitute into sine rule ***(M1)***

$$\frac{\text{AY}}{\sin(18.9681...)} = \frac{2.6}{\sin(126.031)}$$

$\text{AY} = 1.05 \ \ (1.04503\ldots) \ (\text{m})$ ***A1***

***[4 marks]***

(e) **METHOD 1 COSINE RULE**

attempt to substitute into cosine rule to form a quadratic for YZ ***(M1)***

$0.9^2 = \text{YZ}^2 + 1.04503...^2 - 2 \times 1.04503... \times \text{YZ} \times \cos(35)$ ***(A1)***

$\text{YZ} = 0.185 \ (0.184692\ldots) \ (\text{m}) \ , \ 1.53 \ \ (1.52739\ldots) \ (\text{m})$ ***A1A1***

**Note:** Accept $0.191 \ (0.191313\ldots)$ from use of 3 s.f. values.

**METHOD 2 SINE RULE**

attempt to substitute into sine rule to find $\text{Y}\hat{\text{Z}}\text{A}$

$$\frac{\sin(\text{Y}\hat{\text{Z}}\text{A})}{1.04503\ldots} = \frac{\sin(35^\circ)}{0.9}$$ ***(M1)***

$\text{Y}\hat{\text{Z}}\text{A} = 41.7597\ldots$ or $\text{Y}\hat{\text{Z}}\text{A} = 138.240\ldots$

$\text{Z}\hat{\text{A}}\text{Y} = 103.240\ldots$ or $\text{Z}\hat{\text{A}}\text{Y} = 6.75972\ldots$ ***(A1)***

**Note:** Award ***A1*** for any of these angles seen.

$$\frac{\text{YZ}}{\sin(\text{Z}\hat{\text{A}}\text{Y})} = \frac{0.9}{\sin(35^\circ)}$$

$\text{YZ} = 0.185 \ (0.184692\ldots) \ (\text{m}) \ , \ 1.53 \ \ (1.52739\ldots) \ (\text{m})$ ***A1A1***

***[4 marks]***

***[Total: 17 marks]***

**5.** (a) recognition that speed is the magnitude of $\begin{pmatrix} 50 \\ -33 \\ 0 \end{pmatrix}$ ***(M1)***

$\sqrt{50^2 + (-33)^2}$ **OR** $\left\| \begin{pmatrix} 50 \\ -33 \\ 0 \end{pmatrix} \right\|$

$= 59.9$ (km h$^{-1}$). $(59.9082\ldots)$ ***A1***

***[2 marks]***

(b) $\begin{pmatrix} 50 \\ -33 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -15 \\ -20 \\ 0 \end{pmatrix} = -90$ ***(A1)***

$\cos\theta = \dfrac{-90}{59.9082... \times 25}$ ***(M1)***

$\theta = 93.4^\circ \left(93.4450\ldots^\circ, 1.63092\ldots\right)$ ***A1***

***[3 marks]***

(c) $\overrightarrow{OD} = \begin{pmatrix} 200 \\ -100 \\ 0.02 \end{pmatrix} + t \begin{pmatrix} -15 \\ -20 \\ 0 \end{pmatrix}$ ***A1***

***[1 mark]***

(d) (i) $\begin{pmatrix} 200 \\ -100 \\ 0.02 \end{pmatrix} + t \begin{pmatrix} -15 \\ -20 \\ 0 \end{pmatrix} = \begin{pmatrix} 152 \\ p \\ 0.02 \end{pmatrix}$ **OR** $200 = -15t + 152$ ***(M1)***

$t_1 = 3.2 \left(\dfrac{16}{5}\right)$ ***A1***

(ii) $p = -164$ ***A1***

***[3 marks]***

*continue…*

*Question 5 continued.*

(e) (i) attempt to find difference between the two position vectors ***(M1)***

$\begin{pmatrix} 190-65t \\ -95+13t \\ 0.02 \end{pmatrix}$ ***A1***

**Note:** Award ***A1M1A0*** for $\begin{pmatrix} -190+65t \\ 95-13t \\ -0.02 \end{pmatrix}$.

(ii) attempt to find $\left|\begin{pmatrix} 190-65t \\ -95+13t \\ 0.02 \end{pmatrix}\right|$ ***(M1)***

$\sqrt{(190-65t)^2+(13t-95)^2+0.02^2}$ ***(A1)***

attempt to find minimum. (e.g. $t = 3.09$ hours) ***(M1)***

closest distance = 55.9 (55.8931…) (km) ***A1***

***[6 marks]***

***[Total: 15 marks]***

**6.** (a) anticlockwise rotation of $15°$ about the origin ***A1***
***[1 mark]***

(b) recognizing that *I* is equivalent to one rotation of $360°$ ***(M1)***

e.g. $\frac{360}{15}$

$= 24$ ***A1***
***[2 marks]***

(c) (i) $(\boldsymbol{B} =) \begin{pmatrix} 1.05 & 0 \\ 0 & 1.05 \end{pmatrix}$ ***A1***

(ii) $(\boldsymbol{B}^{24} =) \begin{pmatrix} 3.23 & 0 \\ 0 & 3.23 \end{pmatrix}$ ***(A1)***

**Note:** Award ***A1*** for $3.23$ $(3.22509\ldots)$ **OR** $1.05^{24}$.

enlargement, with a scale factor of $3.23$ $(1.05^{24})$, (centre $(0, 0)$) ***A1***
***[3 marks]***

(d) $\boldsymbol{C} = \begin{pmatrix} 1.05\cos(15^\circ) & -1.05\sin(15^\circ) \\ 1.05\sin(15^\circ) & 1.05\cos(15^\circ) \end{pmatrix}$

$\left( = \begin{pmatrix} 1.01\ (1.01422...) & -0.272\ (-0.271759...) \\ 0.272\ (0.271759...) & 1.01\ (1.01422...) \end{pmatrix} = \frac{21}{80}\begin{pmatrix} \sqrt{6}+\sqrt{2} & -\sqrt{6}+\sqrt{2} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{pmatrix} \right)$ ***A2***

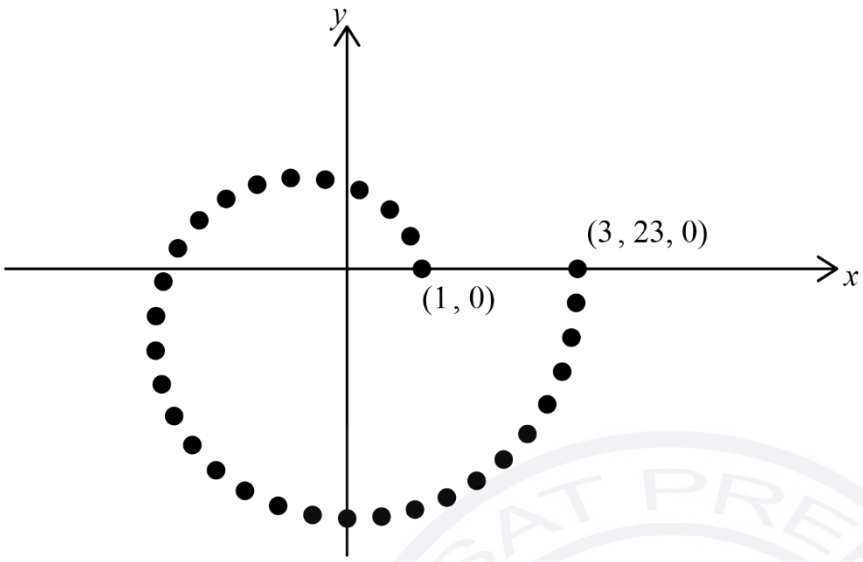
**Note:** Award ***A1*** for at least two correct elements in the matrix.

***[2 marks]***

*continue…*

*Question 6 continued.*

(e)




anticlockwise spiral ***A1***

starting at $(1, 0)$ and ending at $(3.23, 0)$ ***A1***

**Note:** Accept a continuous curve instead of discrete points. The two ***A1*** marks can be awarded independently

***[2 marks]***

(f) $T\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ ***(M1)***

$0.9p + 2 = p$ **OR** $0.8q + 1 = q$ ***(A1)***

$p = 20, q = 5$ **OR** $\begin{pmatrix} 20 \\ 5 \end{pmatrix}$ ***A1***

***[3 marks]***

***[Total: 13 marks]***

**7.** (a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$ ***A1***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} + ay + bx = 0$ ***A1***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay$ ***AG***

***[2 marks]***

(b) $\begin{pmatrix} 0 & 1 \\ -77 & -18 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -7\begin{pmatrix} x \\ y \end{pmatrix}$ **OR** $\begin{pmatrix} 0 & 1 \\ -77 & -18 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -11\begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

$y = -7x$

eigenvector is $\begin{pmatrix} 1 \\ -7 \end{pmatrix}$ (or any multiple) ***A1***

$y = -11x$

eigenvector is $\begin{pmatrix} 1 \\ -11 \end{pmatrix}$ (or any multiple) ***A1***

***[3 marks]***

(c) (i) $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-7t}\begin{pmatrix} 1 \\ -7 \end{pmatrix} + B\mathrm{e}^{-11t}\begin{pmatrix} 1 \\ -11 \end{pmatrix}$ ***(A1)***

substitution of initial values ***(M1)***

two correct equations (not in vector form) ***(A1)***

$5 = A + B$, $2 = -7A - 11B$

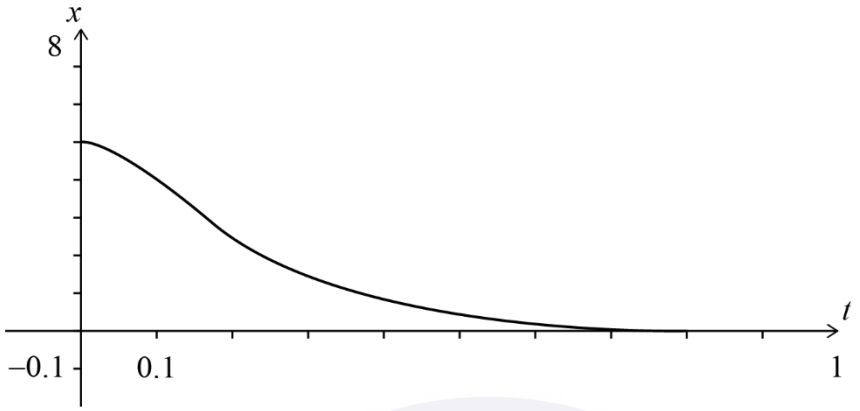
$A = \dfrac{57}{4}$, $B = -\dfrac{37}{4}$ ***(A1)***

$x = \dfrac{57}{4}\mathrm{e}^{-7t} - \dfrac{37}{4}\mathrm{e}^{-11t}$ ( $x = 14.25\mathrm{e}^{-7t} - 9.25\mathrm{e}^{-11t}$ ) ***A1***

*continue…*

*Question 7 continued.*

(ii)



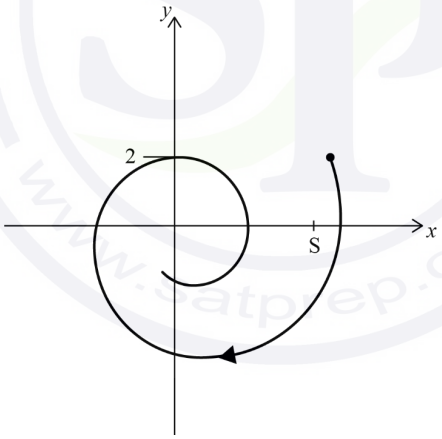

decreasing curve ***A1***

**Note:** There is a maximum at $t = 0.00496$, but this does not need to be shown.

starting at $x=5$ and asymptote at $x = 0$ ***A1***

***[7 marks]***

(d) (i)




spiral towards origin ***A1***

starting at $(5, 2)$ ***A1***

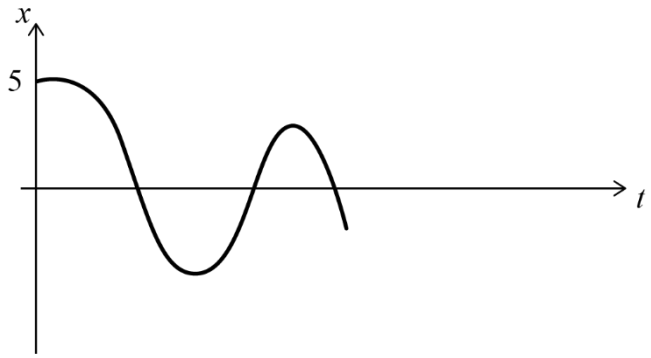
$$\frac{dy}{dt} = -85(5) - 18(2) = -461 < 0$$

Clockwise spiral ***A1***

*continue…*

*Question 7 continued.*

(ii)




oscillations ***A1***
starting at $x = 5$, decreasing amplitude ***A1***
***[5 marks]***

(e) spring 1 is better as it gives a smoother ride (less oscillations, less bouncy) ***R1***
***[1 mark]***
***[Total:18 marks]***

2224 – 7202M

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2024

## Mathematics: Applications and interpretation

## Higher level

## Paper 2

16 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2****.*

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |

| | | | | |
|---|---|---|---|---|
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

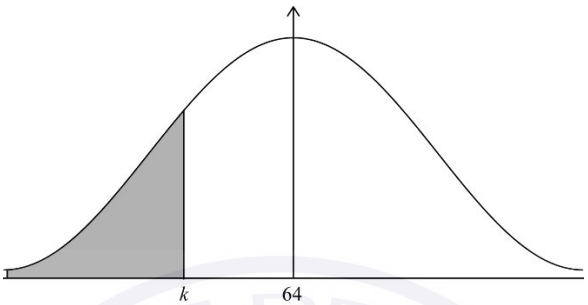
**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) $0.5$ ***A1***
***[1 mark]***

(b) $0.452 \ (0.452209\ldots)$ ***A2***
***[2 marks]***

(c) (i)




***A1A1***

**Note:** Award ***A1*** for a normal curve (with symmetry and some evidence of change of curvature towards the extreme values).
Award ***A1*** for a shaded region $x < k$, where $k$<mean.

(ii) $\mathrm{P}(T < k) = 0.3$
solving a cumulative distribution function **OR**
use of inverse function on GDC ***(M1)***
$k = 57.7 \ \ (57.7071\ldots)$ ***A1***
***[4 marks]***

(d) recognizing binomial distribution ***(M1)***
$\mathrm{B}(5,\ 0.3) \ \ \left(\mathrm{P}(X = 2)\right)$ ***(A1)***

$0.309 \ \ (0.3087)$ ***A1***
***[3 marks]***

(e) $2(x-1)+4.5$ **OR** $2x+2.5$ ***A1A1***

**Note:** Award ***A1*** for a linear expression with a gradient of $2$,
***A1*** for a completely correct expression in $x$.

***[2 marks]***

(f) $(\$)13.10$ (accept $13.1$) ***A1***

***[1 mark]***

(g) attempt to solve $2(x-1)+4.5=7.2$ **OR** $2x+2.5=7.2$ ***(M1)***
$2.35$ (kg) ***A1***

**Note:** Award ***M1A1FT*** for an answer of $1.35$ (kg) from $2x+4.5$ seen in (e).

***[2 marks]***
***[Total 15 marks]***

**2.** (a) (i) $\begin{pmatrix}7.2\\5.1\\2.4\end{pmatrix}-\begin{pmatrix}0\\0\\2.8\end{pmatrix}=\begin{pmatrix}7.2\\5.1\\-0.4\end{pmatrix}$ ***A1***

**Note:** Accept alternate vector notation, e.g. $(7.2, 5.1, -0.4)$ or $\langle 7.2, 5.1, -0.4\rangle$

(ii) use of correct formula to find $\left|\vec{AB}\right|$ ***(M1)***

$\sqrt{7.2^2+5.1^2+(-0.4)^2}$

$8.83$ (km) $(8.83232\ldots)$ ***A1***

***[3 marks]***

(b) magnitude of $\begin{pmatrix}1.1\\8.4\\0.2\end{pmatrix}$ is

$\sqrt{1.1^2+8.4^2+0.2^2}$ $(=8.47407\ldots)$ ***(A1)***

**EITHER**

$\begin{pmatrix}1.1\\8.4\\0.2\end{pmatrix}\bullet\begin{pmatrix}7.2\\5.1\\-0.4\end{pmatrix}$ ***(M1)***

$1.1\times 7.2+8.4\times 5.1-0.2\times 0.4$ $(=50.68)$ ***(A1)***

**Note:** The ***M*** mark can be implied by a partially correct ***A1*** line.

angle = $\arccos\left(\dfrac{50.68}{8.83232...\times 8.47407...}\right)$ ***(M1)***

**OR**

Attempt to find $\begin{pmatrix}1.1\\8.4\\0.2\end{pmatrix}\times\begin{pmatrix}7.2\\5.1\\-0.4\end{pmatrix}$ ***(M1)***

$\left|\begin{pmatrix}1.1\\8.4\\0.2\end{pmatrix}\times\begin{pmatrix}7.2\\5.1\\-0.4\end{pmatrix}\right|=\sqrt{4.38^2+1.88^2+54.87^2}$ $(=55.0766\ldots)$ ***(A1)***

angle = $\arcsin\left(\dfrac{55.0766...}{8.83232...\times 8.47407...}\right)$ ***(M1)***

**THEN**

$47.4^\circ$ $(47.3805\ldots)$ **OR** $0.827$ $(0.826947\ldots)$ ***A1***

***[5 marks]***

*continued...*

*Question 2 continued*

(c) using sum of angles in a triangle equals $180$ ***(M1)***

$\hat{ACB} = 180 - 47.3805 - 55.2 \ (= 77.4194\ldots^\circ)$ ***(A1)***

$\frac{AC}{\sin(55.2)} = \frac{8.83232...}{\sin(77.4194...)}$ ***(A1)***

$7.43$ (km) $(7.43107\ldots)$ ***A1***

***[4 marks]***

***[Total 12 marks]***

**3.** (a) $\frac{40000}{x^2} = 400$ ***(M1)***

$x = 10$ (pesos) (since $x$ is positive) ***A1***

***[2 marks]***

(b) (i) $\left(\frac{40000}{50^2} =\right) 16$ ***A1***

(ii) $(50 \times 16 =)\ 800$ (pesos) ***A1***

***[2 marks]***

(c) (i) **EITHER**

profit for each smoothie = $x - 20$ ***(M1)***

$P = \frac{40000}{x^2} \times (x - 20)$ ***A1***

**OR**

profit = revenue – costs = $nx - 20n$ ***(M1)***

$P = x \times \frac{40000}{x^2} - 20 \times \frac{40000}{x^2}$ ***A1***

**Note:** Do not award ***A1*** if $\frac{40000}{x}$ seen as first term unless explained (in part (a) or (b)), as it is given in question.

**THEN**

$P = \frac{40000}{x} - \frac{800000}{x^2}$ ***AG***

*continued...*

*Question 3 continued*

(ii) attempt to express $P$ ready for power rule ***(M1)***

$P = 40000x^{-1} - 800000x^{-2}$

$\frac{dP}{dx} = -\frac{40000}{x^2} + \frac{1600000}{x^3}$ **OR** $\frac{dP}{dx} = -40000\,x^{-2} + 1600000\,x^{-3}$ ***A1A1***

**Note:** The ***(M1)*** can be awarded for either of the correct terms seen.
***A1*** for each correct term.
At most ***M1A1A0*** if additional terms seen.

(iii) attempt to find $x$-value ***(M1)***

e.g. sketch of $\frac{dP}{dx}$ with $x$-intercept indicated **OR** recognition that it occurs at the maximum of P **OR** algebraic approach (requires multiplication by $x^3$)

$x = 40$ ***A1***

**Note:** $\frac{-40000}{x^2} + \frac{1600000}{x^3} = 0$ is insufficient to award ***M1***, this is given in the question. There must be an "attempt to find $x$-value".
Award ***M1A0*** for a coordinate pair $(40, 500)$.

(iv) attempt to substitue their $x$-value into equation for $n$ ***(M1)***

$n = \frac{40000}{40^2}$

$= 25$ ***A1***

**Note:** Given the nature of the function $P$, the local maximum is also the global maximum. This is often the case in examinations, but should not always be assumed.

***[9 marks]***
***[Total 13 marks]***

**4.** (a) (i) 0.9 (ii) 0.3 (iii) 0.7 ***A2***

**Note:** Award ***A1A0*** if one of the values is incorrect, ***A0A0*** otherwise.

***[2 marks]***

(b) $(0.1 \times 0.3 =)$ 0.03 ***A1***

***[1 mark]***

(c) P(no fail) = 0.63 ***(A1)***
P(one fails) = 0.34 ***(A1)***
P(two fail) = 0.03 ***(A1)***

**Note:** The three ***A1's*** can be awarded independently

*continued...*

*Question 4 continued*

multiplying by $200$ ***(M1)***

| No switch fails | One switch fails | Two switches fail |
|---|---|---|
| 126 | 68 | 6 |

***(A1)***

degrees of freedom $= 2$ ***(A1)***

**Note*:*** Award ***A1*** for df = $2$ seen anywhere and may be awarded independent of the ***M1*** mark.
The df=2 cannot be implied from chi squared statistic = $3.40989$

$p$-value $= 0.182$ $(0.181781\ldots)$ ***A1***

$0.182 > 0.05$ ***R1***

hence insufficient evidence to reject $H_0$ (that the manufacturers claims are correct) ***A1***

**Note:** The ***R1A1*** can be awarded as follow through within part (d) from their (explicitly labelled) incorrect p-value.
An unrealistic p-value should preclude awarding the final ***R1A1***.
Accept either a conclusion to not reject the null hypothesis or the manufacturers claims are correct.
Do not award ***R0A1***.

***[9 marks]***
***[Total 12 marks]***

**5.** (a) C. ***A1***

Any valid reason for accepting C. or rejecting A. and B. ***R1***

*for example:*

- when $x = 0$ slopes have (or appear to have) zero gradient
- (slope field is) always positive for $x > 0$

**Note:** Allow ***A1R0***.

***[2 marks]***

(b) $\int e^{2y} dy = \int x dx$ ***(M1)***

$\frac{1}{2}e^{2y} = \frac{1}{2}x^2 \ (+c)$ ***(A1)(A1)***

**Note:** ***A1*** for left hand side, ***A1*** for right hand side.

substituting in $x = 0$, $y = 0$ ***(M1)***

$\frac{1}{2} = c$ ***(A1)***

**Note:** The substitution may be seen and credited later, however at that point the constant term may be $1$.

*continued...*

*Question 5 continued*

$$e^{2y} = x^2 + 1$$

$$y = \frac{1}{2}\ln\left(x^2 + 1\right) \qquad \textbf{M1A1}$$

**Note:** Award ***M1*** for use of log law.

***[7 marks]***

(c) $\dfrac{dy}{dx} = \dfrac{1}{2} \times 2x \times \dfrac{1}{x^2+1} \left(= \dfrac{x}{x^2+1}\right)$ ***M1A1***

**Note:** Award ***M1*** for use of chain rule, or use of implicit differentiation of the penultimate line of the answer to (b).

***[2 marks]***

(d) substitution of $e^{2y} = x^2 + 1$ from part (b) into part(c)(i) or original differential equation ***M1***

$$\frac{dy}{dx} = \frac{x}{x^2+1} = \frac{x}{e^{2y}} \qquad \textbf{A1}$$

and hence $y = \dfrac{1}{2}\ln\left(x^2 + 1\right)$ is a solution for the differential equation ***AG***

**Note:** Only award the ***A1*** as follow-through if their $\dfrac{dy}{dx}$ is of the form $\dfrac{x}{x^2 + c}$.

***[2 marks]***
***[Total 13 marks]***

**6.** (a) let $S$ be the number of spaceships hit and $B$ the number of battleships

(i) mean = $8.4$ ***(A1)***

$P(S \leq 10) = 0.774 \ \ (0.774301...)$ ***A1***

(ii) attempt to add two means ***(M1)***

$4.2 + 2.3 = 6.5$

$P(S + B > 10) = P(S + B \geq 11)$ ***(M1)***

$0.0668 \ \ (0.0668387\ldots)$ ***A1***

***[5 marks]***

(b) (i) $E(T) = 3 \times 4.2 + 5 \times 2.3 = 24.1$ ***A1***

(ii) $\text{Var}(T) = 3^2 \times 4.2 + 5^2 \times 2.3 = 95.3$ ***(M1)A1***

***[3 marks]***

*continued...*

*Question 6 continued*

(c) any valid reason ***R1***
for example:
mean is not equal to variance **OR** $T$ cannot take all integer values

***[1 mark]***

(d) distribution of mean score is $\mathrm{N}\left(24.1, \frac{95.3}{60}\right)$ $\left(\mathrm{N}(24.1, 1.58833...)\right)$ ***(A1)(A1)***

**Note:** Award ***A1*** for normal distribution with mean $24.1$, and ***A1*** for variance $\frac{95.3}{60}$.

$\mathrm{P}(\overline{T} > 25) = 0.238$ $(0.237576...)$ ***A2***

***[4 marks]***
***[Total 13 marks]***

**7.** (a) attempt to use $V = \pi\int x^2 \,\mathrm{d}y$ ***(M1)***

$x^2 = 2y + 2$ or any reasonable attempt to find $x$ in terms of $y$ ***(M1)***

$V = \pi\int_0^h 2y + 2 \,\mathrm{d}y$ ***(A1)***

**Note:** Correct limits must be seen for the ***A1*** to be awarded however the $\mathrm{d}y$ may be omitted (as not a final answer).
If this is given as the final answer to this part the remaining marks can be awarded if seen in part (b).

$\int 2y + 2 \,\mathrm{d}y = y^2 + 2y$ ***(A1)***

**Note:** Accept equivalent with alternate variable

$V = \pi\left[y^2 + 2y\right]_0^h$

$= \pi(h^2 + 2h)$ ***A1***

**Note:** The final two ***A1*** marks can be awarded independently of the first ***A1***.
If $h^2 + 2h$ or $y^2 + 2y$ is the final (unsupported) answer award at most ***(M1)(M1)(A0)(A1)A0***.

***[5 marks]***

(b) volume of vase $= \pi(15^2 + 2\times 15)$ $(= 801.106...)$ ***(A1)***

(time to fill vase $= \frac{801.106...}{20}$ =) $40.1$ $(40.0553...)$ (seconds) ***A1***

**Note:** Accept exact answers in terms of $\pi$, e.g. $12.75\pi$ or $\frac{51\pi}{4}$

***[2 marks]***

*continued...*

*Question 7 continued*

(c) **EITHER**

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ ***(M1)***

$\frac{dV}{dh} = \pi(2h + 2)$ ***(A1)***

**OR**

differentiating $V = \pi(h^2 + 2h)$ implicitly ***(M1)***

$\frac{dV}{dt} = \pi(2h + 2)\frac{dh}{dt}$ ***(A1)***

**THEN**

$\frac{dh}{dt} = 20 \times \frac{1}{\pi(2h + 2)}$ ***(M1)(A1)***

**Note:** Award ***M1*** for attempting to solve for $\frac{dh}{dt}$, ***A1*** for a correct expression.

substituting $h = 10$ seen anywhere ***(M1)***

$0.289 \quad (0.289372\ldots)$ cm $s^{-1}$ ***A1A1***

**Note:** Award ***A1*** for the correct value. Award ***A1*** for the correct units, independent of other marks.

***[7 marks]***
***[Total 14 marks]***

**8.** (a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ ***M1A1***

**Note:** Award ***M1*** is for a 3x3 matrix with at least one column correct.
Column order is not explicit in question and may not be labelled in candidate response; accept their correct adjacency matrix.

***[2 marks]***

*continued...*

*Question 8 continued*

(b) **EITHER**

$$\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}^5$$ ***(M1)***

$$=\begin{pmatrix}11 & 11 & 10\\ 11 & 10 & 11\\ 10 & 11 & 11\end{pmatrix}$$ ***(A1)***

**OR**

Listing at least $8$ possible walks ***(M1)***

AAAAAB, AAABAB, AAABCB, AABAAB, AABCCB,
ABABAB, ABACAB, ABAAAB, ABCCCB, ABCBAB, ABCBCB ***(A1)***

**THEN**

$11$ different routes ***A1***

***[3 marks]***

(c) (i) $0.5^5$ $\left(\frac{1}{32}, 0.03125\right)$ ***A1***

(ii) **EITHER**

there are $11$ possible walks so probability is $11\times 0.5^5$ ***M1***

**OR**

total number of (equally likely) walks from $\mathrm{A}$ is $32$, $11$ end up at $\mathrm{B}$ ***M1***

**THEN**

$\frac{11}{32}$ **OR** $0.344$ $(0.34375)$ ***A1***

| **Note:** Solutions to this part must be using the value $(11)$ obtained from part (b) to be awarded any marks |
|---|

***[3 marks]***

(d) (i) $(1\times 0.4 =)\ 0.4$ ***A1***

(ii) $(0.5\times 0.5 =)\ 0.25$ ***A1***

(iii) $(0.5\times 0.5 + 0.5\times 0.5 =)\ 0.5$ ***A1***

***[3 marks]***

*continued...*

*Question 8 continued*

(e) transition matrix is $\begin{pmatrix} 0 & 0.25 & 0.4 \\ 0.6 & 0.5 & 0.6 \\ 0.4 & 0.25 & 0 \end{pmatrix}$ (with order AB, AC and BC) ***(M1)(A1)***

**Note:** Column order is not explicit in question and may not be labelled in candidate response; accept their correct transition matrix.
Accept the transposed matrix.

$$\begin{pmatrix} 0 & 0.25 & 0.4 \\ 0.6 & 0.5 & 0.6 \\ 0.4 & 0.25 & 0 \end{pmatrix}^5$$ ***(M1)***

$$= \begin{pmatrix} 0.22215 & 0.227275 & 0.23239 \\ 0.54546 & 0.54545 & 0.54546 \\ 0.23239 & 0.227275 & 0.22215 \end{pmatrix}$$

$0.232 \quad (0.23239)$ ***A1***

***[4 marks]***

(f) (Taking a high power of a matrix)
long term probabilities are $0.227275$, $0.545455$ and $0.227275$ ***(M1)***

B and $0.545$ $(54.5\% \ \frac{6}{11})$ ***A1A1***

**Note:** Award ***(M0)A0A0*** for an unsupported answer of "B" (with either no probability or an incorrect probability).

***[3 marks]***
***[Total 18 marks]***

2224 – 7207M

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2024

## Mathematics: applications and interpretation

## Higher level

## Paper 2

16 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

**1 General**

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | Correct answer seen | Further working seen | Any FT issues? | Action |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to a "correct" level of accuracy (e.g 3 sf) in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) $\text{BC} = 20\ (\text{m})$ ***A1***

***[1 mark]***

(b) use of Pythagoras ***(M1)***

$\text{AB} = \sqrt{12^2 + 4^2}$

$= 12.6\ (\text{m})\quad (12.6491\ldots,\ \sqrt{160})$ ***A1***

***[2 marks]***

(c) **METHOD 1 – finding angle ABC**

correct use of a trig ratio to find $\text{A}\hat{\text{B}}\text{C}$ (or finding the bearing of B from A) ***(A1)***

e.g. $\tan\left(\text{A}\hat{\text{B}}\text{C}\right) = \frac{12}{4},\ \cos\text{A}\hat{\text{B}}\text{C} = \frac{20^2 + 12.649^2 - 20^2}{2 \times 20 \times 12.649},\ \cos\text{A}\hat{\text{B}}\text{C} = \frac{6.3245}{20}$

$\text{A}\hat{\text{B}}\text{C} = 71.6\ (71.5650\ldots)$ ***(A1)***

**Note:** Angle $\text{A}\hat{\text{B}}\text{C}$ can be $71.5$ or $72.2$ depending on their working out. Bearings should be given in degrees.

$180 + 71.5650\ldots = 252°\ (251.565\ldots)$ ***A1***

**Note:** The final ***A1*** can be awarded for $180$ plus their $71.6$. If radians used, award ***A1A1*** for $1.24904\ldots$ or $4.39063\ldots$ seen, and then ***A0*** for the radian answer.

**METHOD 2 – finding angle that AB makes with the horizontal (angle *H*)**

correct use of a trig ratio to find $H$, the angle $\text{AB}$ makes with horizontal ***(A1)***

e.g. $\tan\hat{H} = \frac{4}{12},\ \cos\hat{H} = \frac{12^2 + 12.649^2 - 4^2}{2 \times 12 \times 12.649}$

$\hat{H} = 18.4\ (18.4349...)$ ***(A1)***

**Note:** Accept $18.5\ (18.5078\ldots)$ from use of 3sf answer from part (b). Bearings should be given in degrees.

$270 - 18.4348... = 252°\ (251.565\ldots)$ ***A1***

**Note:** The final ***A1*** can be awarded for $270$ minus their $18.4$. If radians used, award ***A1A1*** for $0.321750\ldots$ or $4.39063\ldots$ seen, and then ***A0*** for the radian answer.

***[3 marks]***

(d) (i) $-\frac{4}{3}$ $\left(-\frac{16}{12}\right)$ ***A1***

(ii) $(6, 8)$ ***A1A1***

**Note:** Award ***A1A0*** if parentheses are missing.

(iii) gradient of (their) perp line = $\frac{3}{4}$ ***(M1)***

equation of perpendicular bisector of $\mathrm{AC}$ ***(A1)***

e.g. $(y-8)=\frac{3}{4}(x-6)$ **OR** $y=\frac{3}{4}x+3.5$

**EITHER**

equation of perpendicular bisector of $\mathrm{BC}$ is $y=10$ ***(A1)***

**OR**

equation of perpendicular bisector of $\mathrm{AB}$ is $y=-3x+36$ ***(A1)***

**Note:** The ***A1*** is for either equation of perpendicular bisector of $\mathrm{BC}$ or $\mathrm{AB}$.

point of intersection $\left(8\frac{2}{3}, 10\right)$ **OR** $(8.67, 10)$ $\left[(8.666\ldots, 10)\right]$ ***(M1)A1***

**Note:** Award ***M1*** for an attempt to equate their perpendicular bisectors
Award the final ***A1*** for the correct coordinate pair – parentheses omitted or not.

***[8 marks]***

***[Total: 14 marks]***

**2.** (a) heights, $0$, $4$, $1.75$, $3$ and $3.75$ seen ***(A2)***

**Note:** Award ***A1A0*** if **two** of $1.75$, $3$ or $3.75$ are seen.

attempt to use trapezoidal rule formula for their heights ***(M1)***

$\frac{1}{2}\times 1\times\{0+4+2(1.75+3+3.75)\}$ ***(A1)***

**Note:** Award ***(M1)(A1)*** for correctly expressing this as $3$ trapezoids and a triangle. The "$\times 1$" need not be seen.

$=10.5\ (\text{m}^2)$ ***A1***

***[5 marks]***

(b) $-\frac{1}{12}x^3+x^2+c$ ***A1A1A1***

***[3 marks]***

(c) $\int_0^4\left(-\frac{1}{4}x^2+2x\right)\text{d}x+1\times 4+\frac{1}{2}\times 7\times 4$ ***(A1)(M1)(A1)***

**Note:** Award ***A1*** for correct area of rectangle **OR** triangle, ***M1*** for substituting correct limits into given integral (may be seen in part (b)), and ***A1*** for entire expression correct.

$=10.6666\ldots. + 4 + 14$

$= 28\frac{2}{3}\ (\text{m}^2)\left(\frac{86}{3}\right)$ ***A1***

**Note:** The answer must be **exact** for the ***A1*** to be awarded. For an answer of $28.7$ or $28.66$ award ***(A1)(M1)(A1)A0***.

***[4 marks]***

(d) (Total area using part (a) =) $28.5$ ***(A1)***

Percentage error $=\left|\frac{28.5-28.6666\ldots}{28.6666\ldots}\right|\times 100$ ***(M1)***

**Note:** if their trapezoid value is incorrect but is used correctly in the percentage error formula, award at most ***A0M1A0***. If it is clear from the answer that ×100 has been used, then condone the omission and award the ***M*** mark.

$=0.581\ (\%)\ (0.581395\ldots)$ ***A1***

(accept $0.697$ from use of $28.7$)

***[3 marks]***

***[Total: 15 marks]***

**3.** (a) (i) correct approach to find missing length ***(A1)***

$\sqrt{4^2-1^2}$ $(=\sqrt{15})$

attempt to find cross-section ***(M1)***

e.g. use of area of trapezoid formula or rectangle+triangle or rectangle – triangle

use of volume of prism formula ***(M1)***

(their cross-section multiplied by 3)

$3\left[\frac{1}{2}(10+11)\left(\sqrt{4^2-1^2}\right)\right]$

$=122\,(\text{m}^3)$ $(121.998\ldots)$ ***A1***

(ii) correct approach to find missing height ***(A1)***

$\sqrt{4^2-3.2^2}$ $(=2.4)$

attempt to find volume ***(M1)***

(multiplication by $3.2$ and $3$ seen)

$3\left[\frac{1}{2}\left(10+10+\sqrt{4^2-3.2^2}\right)(3.2)\right]$

$=108\,(\text{m}^3)$ $(107.52...)$ ***A1***

(iii) correct approach to find missing lengths ***(A1)***

$\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ **OR** $\sin\left(\frac{\pi}{3}\right)$ and Pythagoras etc seen in work

$3\left[\frac{1}{2}(10+10+4\cos\left(\frac{\pi}{3}\right))4\sin\left(\frac{\pi}{3}\right)\right]$

$=114\,(\text{m}^3)$ $(114.315\ldots)$ ***A1***

***[9 marks]***

(b) $V=3\left[\frac{1}{2}(10+10+4\cos(\theta))4\sin(\theta)\right]$ ***A1***

all correct intermediate working leading to given answer ***A1***

e.g. $V=6\sin(\theta)(20+4\cos(\theta))$

$V=24\sin(\theta)(5+\cos(\theta))$ ***AG***

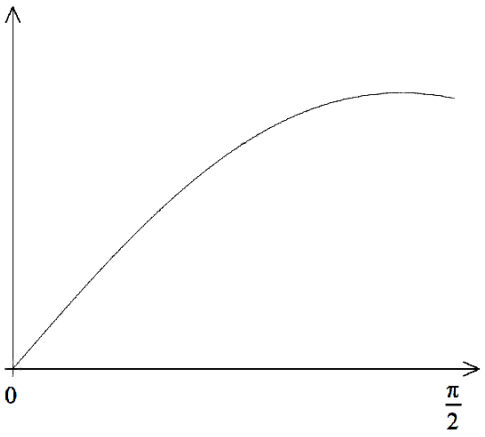
**Note:** The ***AG*** line must be seen for the final ***A1*** to be awarded.

***[2 marks]***

(c) *accept any reasoning along the lines:* "skip would have zero volume" or "if the angle is zero, then the contents would fall out" ***R1***

***[1 mark]***

(d) (i)




***A1A1***

**Note:** Award ***A1*** for the correct shape and ***A1*** for the graph on the correct, labelled, domain. Condone omission of $\theta / V$ labels (or $x/y$).

(ii) $\theta = 1.38 \ (1.38356...) \ (79.3^\circ \ (79.2723...^\circ))$ ***A1***

$V_{\max} = 122 \ (122.292...)$ ***A1***

**Note:** Award ***A0A1*** if values are reversed and ***A0A0*** for a coordinate pair.

***[4 marks]***

(e) recognizing that derivative is equal to zero (seen at any stage) ***M1***

$\frac{dV}{d\theta} = 0$ (accept $\frac{dy}{dx} = 0$)

(from graph, turning point is a global maximum)

use of product rule ***M1***

$\left(\frac{dV}{d\theta} =\right) 24\cos(\theta)(5 + \cos(\theta)) + 24\sin(\theta)(-\sin(\theta))$ ***A1***

$= 120\cos(\theta) + 24\cos^2(\theta) - 24\sin^2(\theta) \ (= 0)$ (or equivalent) ***A1***

substituting $1 - \cos^2(\theta)$ for $\sin^2(\theta)$ ***M1***

e.g $120\cos(\theta) + 24\cos^2(\theta) - 24(1 - \cos^2(\theta)) \ (= 0)$

correct intermediate steps leading to given answer ***A1***

$2\cos^2(\theta) + 5\cos(\theta) - 1 = 0$ ***AG***

***[6 marks]***

***[Total: 22 marks]***

**4.** (a) AEDCFBA ***A1A1***

**Note:** Award ***A1*** for AE at start, ***A1*** for correct completed route.

attempt to find the length of their route
length $22+21+19+24+25+31$ ***(M1)***
$=142$ (km) ***A1***

**Note:** Award ***A1A0M1A0*** for omitted final edge and their sum.

***[4 marks]***

(b) attempt to form MST without vertex A ***(M1)***

**Note:** Exactly 4 edges that form a spanning tree are required.

BD DC DE DF **OR** 20, 19, 21, 22 seen in that order ***A1***

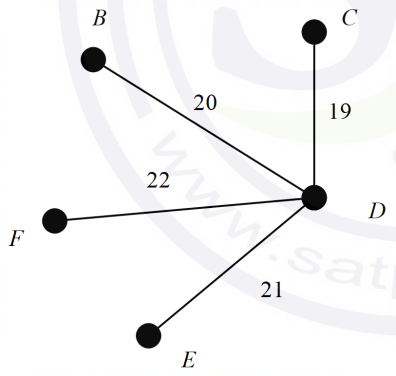
**Note:** Award ***M1A0*** for diagram of MST.

attempt to reconnect vertex A (one edge is sufficient) ***(M1)***
reconnecting A: AE (22) and AF (23) ***(A1)***
lower bound: $20+19+21+22+22+23$
$=127$ ***A1***

**Note:** If 127 seen, unsupported or without the explicit evidence of Prim's algorithm, award ***M1A0M1A1A1***.

***[5 marks]***

(c)




***(A1)***

**Note:** Condone the omission of the weights from their diagram. The diagram may include A with its two edges.

correct reasoning based on lack of cycle (once A is reattached) ***R1***
*e.g. edges BD and CD would be repeated*
this lower bound is not achievable (in this way) ***A1***

**Note:** Do not award ***R0A1***.

***[3 marks]***
***[Total: 12 marks]***

**5.** (a) $\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$ ***M1A1***

**Note:** Award ***M1*** for correct values used, ***A1*** if in correct positions.
Accept alternative consistent matrix (e.g. the transpose or diagonal elements exchanged) and follow through to eigenvectors and initial state vector.

***[2 marks]***

(b) 5 (seen) ***(A1)***

$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.596608 \\ 0.403392 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}^5 = \begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}$ ***(M1)***

$\mathrm{P}(\text{Friday evening}) = 0.403 \ \ (0.403392)$ ***A1***

**Note:** Award ***A0M1A0*** for use of $4$ (and resulting probability $0.354$).

***[3 marks]***

(c) attempt to find $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ ***(M1)***

$\begin{vmatrix} 0.88-\lambda & 0.08 \\ 0.12 & 0.92-\lambda \end{vmatrix}$ **OR** $(0.88-\lambda)(0.92-\lambda) - (0.12)(0.08)$

$\lambda^2 - 1.8\lambda + 0.8$ ***A1***

***[2 marks]***

(d) eigenvalues are $0.8$ and $1$ ***(A1)***

**Note:** If no attempt is made to find eigenvectors, do not award ***A1*** for finding eigenvalues.

$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.8 \begin{pmatrix} x \\ y \end{pmatrix}$

$0.88x + 0.08y = 0.8x$

eigenvector = eg. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

**EITHER**

$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

$0.88x + 0.08y = x$

$0.08y = 0.12x$

**OR**

eigenvalue $1$ gives

$\begin{pmatrix} -0.12 & 0.08 \\ 0.12 & -0.08 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***(M1)***

$-0.12x + 0.08y = 0$

$0.08y = 0.12x$

**Note:** Award ***M1*** for an attempt to find the eigenvector with eigenvalue $1$.

**THEN**

eigenvector = eg. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ***A1***

**Note:** Award ***A0A1M0A0*** if only $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is seen and no eigenvalues are found.

***[4 marks]***

(e) $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ **OR** $\boldsymbol{D} = \begin{pmatrix} 0.8 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ ***A1A1***

**Note:** Award ***A1*** for one of $\boldsymbol{P}$ or $\boldsymbol{D}$ correct. Do not award the second ***A1*** unless $\boldsymbol{P}$ and $\boldsymbol{D}$ are consistent.

***[2 marks]***

(f) **EITHER**

attempt to use $\boldsymbol{T}^n = (\boldsymbol{PDP}^{-1})^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$ ***M1***

**Note:** Award ***M1*** for their $\boldsymbol{D}^n$ seen.

limit of $\boldsymbol{D}^n$ calculated ***A1***

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}^{-1}$$

**Note:** $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ must be seen to award ***A1***.

**OR**

attempt to expand their $\boldsymbol{PD}^n\boldsymbol{P}^{-1}$ using explicit $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ ***M1***

$$(\boldsymbol{T}^n =)\ \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$$

$$(\boldsymbol{T}^n =)\ \frac{1}{5}\begin{pmatrix} 2+3(0.8^n) & 2-2(0.8^n) \\ 3-3(0.8^n) & 3+2(0.8^n) \end{pmatrix}$$ ***A1***

**Note:** Using this method, the limit of $0.8^n$ may be inferred and ***M1A1*** awarded.

**THEN**

$0.6$ ***A1***

**Note:** Multiplication by initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ may be seen at any point as part of their method.

For an answer of $0.6$ from incomplete methods award a maximum of ***M1A0A0***, or if no working is seen, award ***M0A0A1***.

***[3 marks]***

***[Total: 16 marks]***

**6.** (a) (i) $15$ ***A1***

(ii) **EITHER**
attempt to use arithmetic series formula ***(M1)***
**OR**
attempt to set up simultaneous equations ***(M1)***
**OR**
attempt to use quadratic regression ***(M1)***
$(T_k =)\frac{1}{2}k^2+\frac{1}{2}k$ ***A1A1***

**Note:** Condone variable change (eg in quadratic regression).
Accept $a=\frac{1}{2}$, $b=\frac{1}{2}$.

***[4 marks]***

(b) (i) $(15+10=)\ 25$ ***A1***

(ii) $\frac{k(k+1)}{2}+\frac{(k-1)((k-1)+1)}{2}$ **OR** $\frac{1}{2}k^2+\frac{1}{2}k+\frac{1}{2}(k-1)^2+\frac{1}{2}(k-1)$ ***(A1)***
$=k^2$ ***A1***

***[3 marks]***

(c) one correct product of probabilities seen: $\frac{15}{25}\times\frac{10}{24}$ **OR** $\frac{10}{25}\times\frac{15}{24}$ ***(A1)***
adding their products ***(M1)***
$\frac{15}{25}\times\frac{10}{24}+\frac{10}{25}\times\frac{15}{24}$
$=\frac{1}{2}$ ***A1***

***[3 marks]***

(d) attempt to add two products of probabilities involving $k$ only ***M1***
(these may be incorrect or in terms of $T_k$)
$\frac{\frac{k}{2}(k+1)}{k^2}\times\frac{\frac{k}{2}(k-1)}{k^2-1}+\frac{\frac{k}{2}(k-1)}{k^2}\times\frac{\frac{k}{2}(k+1)}{k^2-1}$ ***A1***

further simplification consistent with given answer ***A1***
$=\frac{1}{2}$ ***A1***
hence independent of $k$ ***AG***

***[4 marks]***
***[Total: 14 marks]***

**7.** (a) $\mathrm{P}(X < 340)$ **OR** labelled sketch of region **OR** calc syntax with correct bounds ***(M1)***

$= 0.159 \quad (0.158656\ldots)$ ***A1***

***[2 marks]***

(b) recognizing endpoint occurs at either $0.975$ or $0.025$ ***(M1)***

$\mathrm{P}(X < k) = 0.975$ **OR** $\mathrm{P}(X < m) = 0.025$

$330 < X < 370 \quad (330.400... < X < 369.599...)$ ***A1A1***

***[3 marks]***

(c) (i) recognizing mean of $W$ is sum of individual means within wall ***(M1)***

$W = C_1 + C_2 + L$ may be seen

$\mathrm{E}(W) = 2\mathrm{E}(C) + \mathrm{E}(L)$

$= 800$ ***A1***

recognizing **variance** of $W$ is sum of individual **variances** within wall ***(M1)***

$\mathrm{Var}(W) = 2\mathrm{Var}(C) + \mathrm{Var}(L)$ **OR** $225$ seen ***(A1)***

$\left(\mathrm{SD}(W) =\right) \ 15$ ***A1***

**Note:** Award ***M1A0A0*** for an answer of $20.6$ from using $\mathrm{Var}(2C)$ in place of $2\mathrm{Var}(C)$.

(ii) recognizing that $W$ is modelled by a normal distribution ***(M1)***

$\left(\mathrm{P}(780 < W < 810) =\right) \ 0.656 \ (0.656296...)$ ***A1***

**Note:** The answer is $0.521 \ (0.520)$ from using $\mathrm{SD} = 20.6 \ (5\sqrt{17})$. Follow through from part (c)(i) without working seen.

***[7 marks]***

(d) $810 = 350 + 350 + \mathrm{E}(L)$ (or equivalent) ***(A1)***

$\left(\mathrm{E}(L) =\right) \ 110$ ***A1***

$\mathrm{Var}(W) = 2\mathrm{Var}(C) + \mathrm{Var}(L)$ **OR** $256 = 2(100) + \mathrm{Var}(L)$ ***(A1)***

$\left(\mathrm{SD}(L) =\right) \ 7.48 \ (7.48331..., \sqrt{56})$ ***A1***

***[4 marks]***

(e) $116 \ (116.298)$ ***A1***

**Note:** Do not follow through from either a negative variance or a negative SD.

***[1 mark]***

***[Total: 17 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2023

## Mathematics: applications and interpretation

## Higher level

## Paper 2

16 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | Correct answer seen | Further working seen | Any FT issues? | Action |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to a "correct" level of accuracy (e.g 3 sf) in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) $25$ (m) ***A1***

***[1 mark]***

(b) (i) recognition of need to use Pythagoras theorem ***(M1)***

$BF^2 = 20^2 + 25^2$

$(BF =)\ 32.0\ \ (32.0156\ldots,\ \sqrt{1025},\ 5\sqrt{41})$ (m) ***A1***

(ii) correct use of trig ratio for $B\hat{F}M$ ***(M1)***

$(B\hat{F}M =)\ \tan^{-1}\left(\frac{25}{20}\right)$ or equivalent

$(B\hat{F}M =)\ 51.3\ (51.3401...)$ ***A1***

**Note:** Accept radian answer of $0.896\ \ (0.896055\ldots)$ Accept an answer of $51.4$ from use of 3sf answer to part (b)(i) and then either cosine rule or inverse sine.

***[4 marks]***

(c) attempt to use arc length formula ***(M1)***

(arc length =) $\frac{2\times 51.3401...}{360}\times 2\pi(32.0156\ldots)$ ***(A1)***

(arc length =) $57.4\ \ (57.3755\ldots)$ (m) ***A1***

**Note:** Accept $57.3$ from use of 3 sf. values of their answers from parts (b)(i) and (b)(ii).

***[3 marks]***

(d) $34.0156\ldots$ (seen anywhere) ***(A1)***

use of area of sector formula ***(M1)***

recognition of subtracting areas of two sectors ***(M1)***

(area =) $\frac{102.680\ldots}{360}\times\pi\left((34.0156\ldots)^2-(32.0156\ldots)^2\right)$

(area =) $118\ (m^2)\ \ (118.335\ldots)$ ***A1***

***[4 marks]***

(e) multiplying their area from part (d) by $0.12$ or $12$ ***(M1)***

$0.12$ (m) seen **OR** $1183350\ (cm^2)$ seen ***(A1)***

$118.335\ldots\times 0.12$ **OR** $1183350\times 12$

$14.2\ \ (14.2002\ldots)\ m^3$ **OR** $14200000\ (14200236)\ cm^3$ ***A1***

***[3 marks]***

***[Total 15 marks]***

**2.** (a) (i) $150$ (cm) ***A1***

(ii) attempt to substitute values in the mean formula with at least one mid-interval value multiplied by a corresponding frequency ***(M1)***

(mean $=$) $176$ $(176.3)$ (cm) ***A1***

***[3 marks]***

(b) $183$ **OR** $168$ seen ***(A1)***

**Note:** These values may be seen in the working for part (c).

$(\text{IQR} = 183 - 168 =)$ $15$ (cm) ***A1***

***[2 marks]***

(c) (upper bound =) $183 + 1.5 \times 15$ **OR** $205.5$ seen ***A1***

$205.5 > 204$ **OR** $204 - 183 < 22.5$ **OR** $204 - 22.5 < 183$ ) ***R1***

Laszlo's height is not an outlier ***A1***

**Note:** Do not award ***R0A1***.

***[3 marks]***

(d) $H_0$ : The heights of the students can be modelled by $N(176, 13.5^2)$

$H_1$ : The heights of the students cannot be modelled by $N(176, 13.5^2)$ ***A1A1***

**Note:** Award ***A1*** for each correct hypothesis that includes a reference to normal distribution with a mean of $176$ and a standard deviation of $13.5$ (or variance of $13.5^2$). "Correlation", "independence", "association", and "relationship" are incorrect.

Award at most ***A0A1*** for correctly worded hypotheses that include a reference to a normal distribution but omit the distribution's parameters in one or both hypotheses. Award ***A0A1*** for correct hypotheses that are reversed.

***[2 marks]***

(e) (i) $h \sim \mathrm{N}(176, 13.5^2)$

attempt to find normal probability in either correct range ***(M1)***
$\mathrm{P}(170 \leq h < 180)$ **OR** $\mathrm{P}(h \geq 190)$

recognition of multiplying either of their probabilities by $200$ ***(M1)***
$0.288137... \times 200$ **OR** $0.149859... \times 200$

$a = 57.6$ $(57.6274...)$, $b = 30.0$ $(29.9718...)$ ***A1A1***

(ii) $\mathrm{df} = 4$ ***(A1)***

$(p =)$ $0.0166$ $(= 0.0166282\ldots)$ ***A1***

comparing their $p$-value to $0.05$ ***R1***
$0.0166 < 0.05$

**Note:** Accept $p$ value of $0.0165$ $(= 0.0164693...)$ from using $a$ and $b$ to 3 sf.

(Reject $\mathrm{H}_0$ There is sufficient evidence to say that) the data has not been drawn from the ($\mathrm{N}(176, 13.5^2)$) distribution. ***A1***

**Note:** Do not award ***R0A1***.

The conclusion to part (e)(ii) **MUST** follow through from their hypotheses seen in part (d); if hypotheses are incorrect/reversed etc., the answer to part (e)(ii) must reflect this in order for the ***A1*** to be credited.

***[8 marks]***
***[Total 18 marks]***

**3.** (a) (i) attempt to find $15\%$ or $85\%$ of $285000$ ***(M1)***
$285000 \times 0.85$
$242\,250$ (USD) ***A1***

**Note:** Do not award ***A1*** if answer is not given exact.

(ii) $N = 360$
$I\% = 4$
$PV = (\pm)\,242\,250$
$FV = 0$
$P/Y = 12$
$C/Y = 12$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology with at least two entries seen, award ***A1*** for all entries correct.

$(PMT =)$ $1156.54$ (USD) ***A1***

**Note:** Do not award final ***A1*** if answer is not given to $2$ dp.

***[5 marks]***

(b) $1156.54 \times 360$ ***(M1)***

$416354$ (USD) ***A1***

**Note:** Do not award ***A1*** if answer is not given to the nearest dollar, unless already penalized in part (a)(ii).

***[2 marks]***

(c) $I\% = 4$
$PV = (\pm)\,242\,250$
$PMT = (\mp)1300$
$FV = 0$
$P/Y = 12$
$C/Y = 12$ ***(A1)***

**Note:** Award ***A1*** for $PMT = (\mp)1300$ seen.

$(N =)\ 292$ ***A1***

***[2 marks]***

(d) **METHOD 1**

$N = 291$
$I\% = 4$
$PV = (\pm)\,242\,250$
$PMT = (\mp)1300$
$P/Y = 12$
$F/Y = 12$ ***(A1)***

**Note:** Award ***A1*** for $N = 291$ seen.

$(FV =)\ 871.91\ (871.908\ldots)$ ***A1***

valid attempt to find interest in final month (e.g. $N = 1$ **OR** $PV = 871.91$) ***(M1)***

$N = 1$
$I\% = 4$
$PV = 871.91\ (871.908\ldots)$
$FV = 0$
$P/Y = 12$
$F/Y = 12$

$(PMT =)\ 874.82$ (USD) ***A1***

**Note:** Do not award ***A1*** if answer is not given correct to 2dp, unless already penalized previously.

**METHOD 2**

$N = 292$
$I\% = 4$
$PV = (\pm)\,242\,250$
$PMT = (\mp)1300$
$P/Y = 12$
$F/Y = 12$ ***(A1)***

**Note:** Award ***A1*** for $N = 292$ seen.

$(FV =)\ 425.185\ldots$ ***A1***

$1300 - 425.185...$ ***(A1)***
$(PMT =)\ 874.82$ (USD) ***A1***

**Note:** Accept $874.81$. Do not award ***A1*** if answer is not given correct to 2dp, unless already penalized previously.

***[4 marks]***

(e) $291 \times 1300 + 874.82$ ***(M1)***

$379174.82$

attempt to find difference between their value and their part (b) ***(M1)***
$(416354 - 379174.82)$

$37179$ (USD) ***A1***

**Note:** Accept $37180$ (USD) from using the 2 dp. answer from part (b). Do not penalize for not rounding to nearest dollar if this has already been penalized in part (b).

***[3 marks]***
***[Total 16 marks]***

**4.** (a) (i) $h(0) = 0.00623\ (\text{km})\ (= 0.00622517)$ ***A1***

(ii) this is the height of the nose of the plane (above the runway), when the plane is on the runway ***A1***
***[2 marks]***

(b) (i) $y = 9.94$ ***A1***

**Note:** Accept $h = 9.94$.

(ii) **EITHER**
this is the height that the (nose of the) plane approaches (but does not reach) ***A1***
**OR**
this is the maximum possible height of the (nose of the) plane ***A1***
**OR**
the (nose of the) plane does not exceed this height ***A1***
***[2 marks]***

(c) **METHOD 1 (chain rule)**

$h(x) = 10\left(1+150\mathrm{e}^{-0.07x}\right)^{-1} - 0.06$ ***(M1)***

find $h'(x) = -10\left(1+150\mathrm{e}^{-0.07x}\right)^{-2} \times 150\mathrm{e}^{-0.07x} \times -0.07$ ***A1M1A1***

$\left(= \dfrac{105\mathrm{e}^{-0.07x}}{(1+150\mathrm{e}^{-0.07x})^2}\right)$

**Note:** Award ***A1*** for correct first term $\left(-10\left(1+150\mathrm{e}^{-0.07x}\right)^{-2}\right)$, ***M1*** for attempt to use the chain rule, ***A1*** for correct use of chain rule $\left(\times 150\mathrm{e}^{-0.07x} \times -0.07\right)$. Award at most ***A1M1A0*** if additional terms are seen. The answer is not required to be simplified beyond what is shown in the markscheme.

**METHOD 2 (quotient rule)**

$$\frac{\left(1+150\,\mathrm{e}^{-0.007x}\right)(0) - 10\left(150\mathrm{e}^{-0.007x} \times -0.007\right)}{\left(1+150\,\mathrm{e}^{-0.007x}\right)^2}$$ ***M1A1***

**Note:** Award ***M1*** for attempt to use quotient rule, ***A1*** for correct use.

$$= \frac{-10\left(150\mathrm{e}^{-0.007x} \times -0.007\right)}{\left(1+150\,\mathrm{e}^{-0.007x}\right)^2} \quad \left(= \frac{105\mathrm{e}^{-0.07x}}{(1+150\mathrm{e}^{-0.07x})^2}\right)$$ ***A1A1***

**Note:** Award ***A1*** for correct numerator and ***A1*** for correct denominator.

***[4 marks]***

(d) evidence of a graph of $h'(x)$ ***(M1)***

maximum at $x = 71.6 \quad (= 71.58051...)$ ***(A1)***

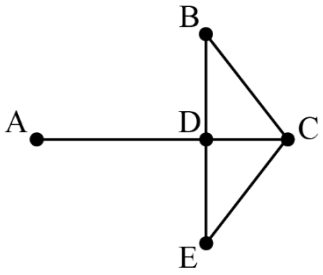
$h'(71.58051...) = 0.175$ ***A1***

maximum gradient is less than $0.2$
and hence the regulation is being followed ***A1***

***[4 marks]***

***[Total 12 marks]***

**5.** (a)




***A1***

***[1 mark]***

(b) (i) $\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$ ***(M1)***

$\boldsymbol{P}^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & 6 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & 6 & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$

$a = 6$ ***A1***

(ii) 5 (routes) ***A1***

***[3 marks]***

(c) A and C identified as start/finish points (in either order) ***(A1)***
for example : A – D – E – C – D – B – C ***A1***

***[2 marks]***

(d) cost of their Eulerian trail A to C (=180) ***(A1)***
consider edges to get from C to A ***(M1)***

235 (USD) ***A1***

***[3 marks]***

(e) (i) A to C (or C to A) ***A1***

(ii) best is CBDA
55 (USD) ***A1***
***[2 marks]***

(f) (i) A – D – C – B – E – A **OR** 50, 45, 30, 120, 60 ***(A1)***
summing their 5 edges ***(M1)***
$50 + 45 + 30 + 120 + 60$
(upper bound =) 305 (km) ***A1***

(ii) attempt to find MST without vertex A ***(M1)***
(MST =) 130 ***(A1)***
$130 + 50 + 60$ ***(M1)***
(lower bound =) 240 (km) ***A1***
***[7 marks]***
***[Total 18 marks]***

**6.** (a) $x = -1 + 2\lambda,\ y = 1 - \lambda$ ***A1***
***[1 mark]***

(b) $\begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}\begin{pmatrix} -1+2\lambda \\ 1-\lambda \end{pmatrix} = \begin{pmatrix} 6-5\lambda \\ -8+15\lambda \end{pmatrix}$ ***(M1)(A1)***

$\boldsymbol{r} = \begin{pmatrix} 6 \\ -8 \end{pmatrix} + \lambda\begin{pmatrix} -5 \\ 15 \end{pmatrix}$ (or equivalent) ***(M1)A1***

**Note:** Award ***(M1)*** for the correct format of a vector equation of a line, ***A1*** for the line being completely correct.

***[4 marks]***

(c) (i) $\begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix}$ **OR** $\begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ ***A1***

(ii) $\begin{pmatrix} 5\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix}$ ***A1***
***[2 marks]***

(d) $(\boldsymbol{R} =) \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ ***A1***
***[1 mark]***

(e) (i) attempt to multiply matrices from part (c) (in any order) ***(M1)***

e.g. $\boldsymbol{X} = \begin{pmatrix} 5\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix}$

$\boldsymbol{X} = \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}$ ***A1***

(ii) substituting $\boldsymbol{T}$, $\boldsymbol{R}$ and $\boldsymbol{X}$ ***(M1)***

$\begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}$

multiplying by inverse (in any order) ***(M1)***

$\begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$

$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ ***A1***

$\cos 2\alpha = -\frac{3}{5}$ **AND** $\sin 2\alpha = \frac{4}{5}$

$\alpha = 1.11 \ \ (= 1.107148...)$ **OR** $63.4^\circ \ \ (63.4349\ldots^\circ)$ ***A1***

***[6 marks]***

***[Total 14 marks]***

**7.** (a) (let $X$ be the random variable the weight of an individual in the city of Melba)

$X \sim \mathrm{N}(72, 10^2)$

recognizing need to find $\mathrm{P}(X > 85)$ (condone "86" for the ***M1***) ***(M1)***

e.g. correct sketch of normal curve **OR** $0.0968 \ \ (= 0.0968005...)$ seen

let $Y$ be the random variable the number of people more than $85 \text{ kg}$

attempt to use a binomial distribution ***(M1)***

$Y \sim \mathrm{B}(10, 0.0968005...)$ ***(A1)***

**Note:** This ***(A1)*** can be implied by the value $0.988580\ldots$

$\left(\mathrm{P}(Y \geq 4) =\right) \ 0.0114 \ \ (= 0.0114196...)$ ***A1***

***[4 marks]***

(b) let $W$ be the random variable the total weight of a sample of eight people

$W \sim \mathrm{N}(576, 8\times10^2)$ ***A1A1A1***

**Note:** Award ***A1*** for normal distribution; ***A1*** correct mean; ***A1*** correct variance or SD (SD = 28.2842…).

***[3 marks]***

(c) attempt to use inverse normal (or equivalent) ***(M1)***

$\mathrm{P}(W > w) = 0.01$

$(w =)\ 642\ (\mathrm{kg})\ \ (641.799\ldots)$ ***A1***

***[2 marks]***

(d) (i) *Any two correct assumptions identified,* ***A1A1***

*e.g.*

That Laetitia's clients are a random sample of the city's population

That people take only one holiday a year

That the choice of individual holidays is independent

That Laetitia is her clients' only agent

**Note:** Accept "assumes the proportion that takes a holiday abroad is 42%".

(ii) $\mathrm{H}_0 : p = 0.42$ ***A1***

$\mathrm{H}_1 : p < 0.42$ ***A1***

(iii) let $Q$ be the random variable the number who go holiday abroad

$Q \sim \mathrm{B}(200,\ 0.42)$ ***(A1)***

$\left(\mathrm{P}(Q \le 67) =\right)\ 0.00850\ \ (= 0.00849906...)$ ***A1***

$0.00850 < 0.05$ ***R1***

**EITHER**

there is evidence that Laetitia's claim is reasonable ***A1***

**OR**

there is insufficient evidence to accept the newspaper's claim ***A1***

**Note:** Follow through within this part, for correctly comparing and concluding with their **probability**, e.g. it is possible to award ***A0A0R1A1***.
The conclusion to part (e)(iii) MUST follow through from their hypotheses seen in part (e)(ii); if hypotheses are incorrect/reversed etc., the answer to part (e)(iii) must reflect this in order for the ***A1*** to be credited.

***[8 marks]***

***[Total 17 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2023

## Mathematics: applications and interpretation

## Higher level

## Paper 2

23 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | Correct answer seen | Further working seen | Any FT issues? | Action |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$ , non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

**5 Mis-read**

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

**6 Alternative methods**

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and $1.000$.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) $\frac{9.45-8.73}{1958-1708}$ ***(M1)***

$= 0.00288 \left(\frac{9}{3125}\right)$ ***A1***

***[2 marks]***

(b) (i) the (mean) yearly change in (mean annual) temperature ***A1***

**Note:** Accept equivalent statements, e.g. "rate of change of temperature".

(ii) $^{\circ}\text{C / year}$ **OR** degrees C per year ***A1***

**Note:** Do not follow through from part (b)(i) into (b)(ii).

***[2 marks]***

(c) attempt to substitute point and gradient into appropriate formula ***(M1)***

$8.73 = 0.00288 \times 1708 + c \Rightarrow c = 3.81096...$

**or**

$9.45 = 0.00288 \times 1958 + c \Rightarrow c = 3.81096$.

equation is $y = 0.00288x + 3.81$ ***A1***

***[2 marks]***

(d) attempt to substitute 2000 into their part (c) ***(M1)***

$0.00288 \times 2000 + 3.81096...$

$= 9.57\ (^{\circ}\text{C})\ \ (9.57096...)$ ***A1***

***[2 marks]***

*continued…*

*Question 1 continued*

(e) (i) $y = 0.00256x + 4.46$ $(0.00255714...x + 4.46454...)$ ***(M1)A1***

**Note:** Award ***(M1)A0*** for answers that show the correct method, but are presented incorrectly (e.g. no "$y =$" or truncated values etc.). Accept $4.465$ as the correct answer to 4 sf.

(ii) $0.861$ $(0.861333...)$ ***A1***

***[3 marks]***

(f) attempt to substitute $2000$ into their part (e)(i) ***(M1)***

$0.00255714... \times 2000 + 4.46454...$

$= 9.58(^{\circ}\text{C})$ $(9.57882...(^{\circ}\text{C}))$ ***A1***

**Note:** Award ***A1*** for $9.57$ from $0.00255714 \times 2000 + 4.46$.

***[2 marks]***

***[Total: 13 marks]***

**2.** (a) $\frac{18-4}{2}$ ***(M1)***

$(a=)\ 7$ ***A1***

***[2 marks]***

(b) $\frac{18+4}{2}$ **OR** $18-7$ **OR** $4+7$ ***(M1)***

$(d=)\ 11$ ***A1***

***[2 marks]***

(c) (time between high and low tide is) $6\text{h}15\text{m}$ **OR** $375$ minutes ***(A1)***

multiplying by $2$ ***(M1)***

$750$ minutes ***A1***

***[3 marks]***

(d) **EITHER**

$\frac{360^\circ}{b}=750$ ***(A1)***

**OR**

$7\cos(b\times 375)+11=4$ ***(A1)***

**THEN**

$(b=)\ 0.48$ ***A1***

**Note:** Award ***A1A0*** for an answer of $\frac{2\pi}{750}\left(=\frac{\pi}{375}=0.00837758...\right)$.

***[2 marks]***

(e) equating their cos function to $6$ **OR** graphing their cos function and $6$ ***(M1)***

$7\cos(0.48t)+11=6$

$\Rightarrow t=282.468...$ (minutes) ***(A1)***

$=4.70780...$ (hr) **OR** 4hr 42 mins (4hr $42.4681\ldots$ mins) ***(A1)***

so the time is $10:42$ ***A1***

***[4 marks]***

*continued…*

*Question 2 continued*

(f) next solution is $t = 467.531...$ ***(A1)***

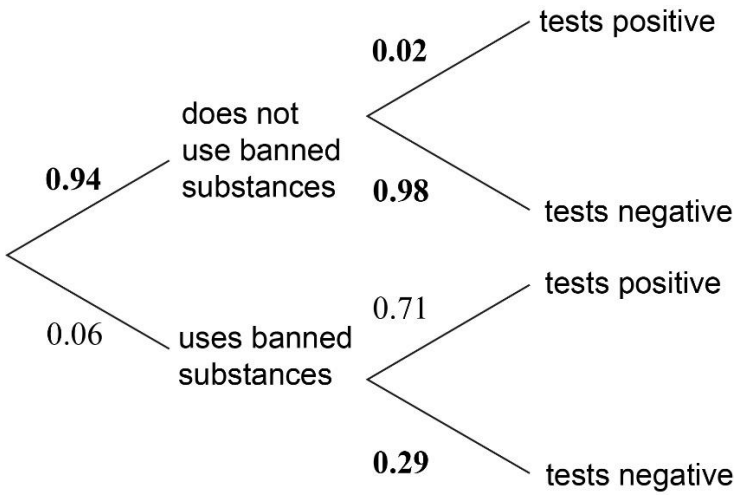
$467.531... - 282.468...$

$185\ (\text{mins})\ (185.063...)$ ***A1***

***[2 marks]***

***[Total: 15 marks]***

**3.** (a)




***A1A1***

| **Note:** | Award ***A1*** for any one value correct, ***A1*** for other three values correct. Accept percentage responses as equivalent forms on **all** branches. |
|---|---|

***[2 marks]***

(b) (i) multiplication of two probabilities along the tree diagram ***(M1)***

$0.94 \times 0.98$

$= 0.921 \ (0.9212, 92.1\%, 92.12\%)$ ***A1***

(ii) $(0.9212)^2$ ***(A1)***

$= 0.849 \ (0.848609\ldots, 84.9\%, 84.8609...\%)$ ***A1***

***[4 marks]***

*continued…*

*Question 3 continued*

(c) (i) $0.94\times0.02+0.06\times0.29$ ***(A1)(M1)***

**Note:** Award ***A1*** for two correct products from their tree diagram seen, ***M1*** for the addition of their two products.

$0.0362$ $(3.62\%)$ ***A1***

(ii) multiplying their part (c)(i) by $1300$

$0.0362\times1300$ ***(M1)***

$47.1\,(47.06)$ ***A1***

***[5 marks]***

(d) $p=0.02$ **OR** $p=0.98$ ***(A1)***

recognition of binomial probability with $n=20$ ***(M1)***

$\mathrm{P}(X=0)$ **OR** $\mathrm{P}(X=20)$ ***(M1)***

$0.668$ $(0.667607...)$ ***A1***

**Note:** Award ***(A1)(M1)(M1)A0*** for an answer of $0.667$.

$0.98^{20}=0.668$ $(0.667607\ldots)$ is awarded full marks.

***[4 marks]***

(e) $\mathrm{P}(X\geq3)$ **OR** $\mathrm{P}(X\leq17)$ ***(M1)***

$0.00707$ $(0.00706869...)$ ***A1***

**Note:** Award ***(M1)A0*** for an answer of $0.00706$. Award ***(M1)A0*** for an answer of $0.0599$ $(0.0598989\ldots)$, obtained from the use of $\mathrm{P}(X\geq2)$.

***FT*** from their value of $p$ in part (d)

***[2 marks]***

***[Total: 17 marks]***

**4.** (a) there are more than two vertices with odd degree ***R1***

so it is not possible to travel along each road exactly once ***A1***

**Note:** Do not award ***R0A1***.

Award ***R1*** for "There are 4 vertices with odd degree".

***[2 marks]***

(b) $a = 11,\ b = 18,\ c = 17,\ d = 15$ ***A2***

**Note:** Award ***A1*** for any one correct, ***A2*** for all four correct.

***[2 marks]***

(c) attempt to use nearest neighbour algorithm ***(M1)***

**Note:** Award ***M1*** for first $3$ vertices correct or $11, 4, 3$ seen.

$\mathrm{G} - \mathrm{E} - \mathrm{F} - \mathrm{B} - \mathrm{D} - \mathrm{A} - \mathrm{C}(-\mathrm{E}) - \mathrm{G}$ **OR** $11 + 4 + 3 + 5 + 5 + 8 + \text{their } b$ ***(A1)***

upper bound $= 54\ (\mathrm{km})$ ***A1***

***[3 marks]***

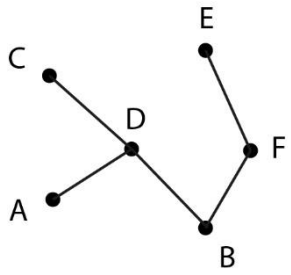
*continued…*

*Question 4 continued*

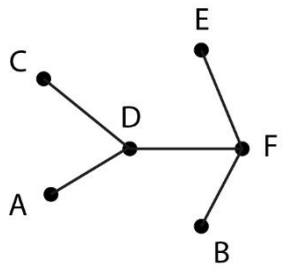
(d) (i) a diagram of **any** spanning tree of the subgraph ABCDEF ***(A1)***

attempt at Kruskal's algorithm or Prim's algorithm ***(M1)***

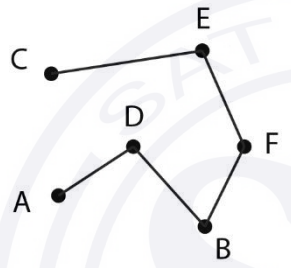
*e.g.* edges BF $(3)$, EF $(4)$ and an edge of length $5$ listed or seen in any spanning tree



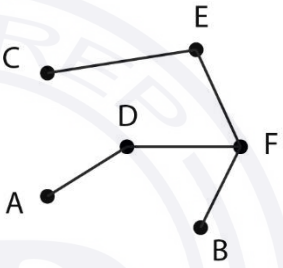

**OR**




**OR**




**OR**




***A1***

(ii) $24\ (\text{km})$ ***A1***

| |
|---|
| **Note:** ***FT*** from their sketch, only if it is a spanning tree. It is not required to see the edge lengths on the sketch, since they are given in the question. |

***[4 marks]***

*continued…*

*Question 4 continued*

(e) adding vertex G's two shortest edges to their part (d)(ii) ***(M1)***

$24+11+13$

$=48$ ***A1***

***[2 marks]***

(f) try removing a different vertex ***A1***

***[1 mark]***

(g) recognize $7$ edges in optimum route ***(M1)***

**Note:** Award ***M1*** for a total length of $52$ seen.

subtracting $0.5\times$ edges from $52$ ***(M1)***

$52-7\times 0.5$

$=48.5\ (\text{km})$ ***A1***

***[3 marks]***

***[Total: 17 marks]***

**5.** (a) $(s_{n-1} =) 1.30243...$ ***(M1)(A1)***

$1.70\ (1.69632)$ ***A1***

**Note:** Award ***(M1)A0A0*** for a value of $(s_n =)\ 1.28934...$ or $(s_n{}^2 =)\ 1.6624$ seen.

***[3 marks]***

(b) the variance and the mean are similar ***R1***

**Note:** Do not accept a general statement "the variance and the mean are equal" unless their answer in part (a) is $1.76$.

***[1 mark]***

(c) (i) attempt to find $\mathrm{P}(X = 4)$ under the null hypothesis $(= 0.0687830...)$ ***(M1)***

multiplying by $50$ ***(M1)***

$j = 3.44\ (3.43915...)$ ***A1***

(ii) **EITHER**

attempt to find $\mathrm{P}(X \geq 5)$ under the null hypothesis and multiply by $50$ ***(M1)***

**OR**

$50 - (8.60 + 15.14 + 13.32 + 7.82 + 3.44)\quad (= 5.12 - 3.44)$ ***(M1)***

**THEN**

$k = 1.68\ (1.67925...)$ ***A1***

***[5 marks]***

(d) there are expected frequencies less than $5$ ***A1***

***[1 mark]***

*continued…*

*Question 5 continued*

(e) 3 ***A1***

***[1 mark]***

(f) 0.991 (0.991187) ***(M1)A1***

**Note:** Award ***M1*** for a table of observed and expected frequencies with columns for 4 and 5 or more combined.

***[2 marks]***

(g) $99\% > 5\%$ ***R1***

**EITHER**

so there is insufficient evidence to reject $H_0$. ***A1***

**OR**

we accept that the number of sightings follows a Poisson distribution ***A1***

**Note:** Do not award ***R0A1***.

A $p$-value must be seen in part (f) to award ***FT***.

***[2 marks]***

***[Total: 15 marks]***

**6.** (a) attempt to solve $\det(A-\lambda I)=0$ ***(M1)***

$(-0.05-\lambda)^2+25=0$ ***(A1)***

$-0.05-\lambda=\pm 5\text{i}$ ***(A1)***

$\lambda=-0.05\pm 5\text{i}$ ***A1***

***[4 marks]***

(b) (i) spiral ***A1***

(ii) inwards / towards O ***A1***

***[2 marks]***

(c) (i) attempt to substitute $(20,\ 0)$ into expression for $\frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$-5(20)-0.05(0)$

$\frac{\mathrm{d}y}{\mathrm{d}t}=-100\ (\text{m s}^{-1})$ ***A1***

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t}=-1$ ***(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$ ***(M1)***

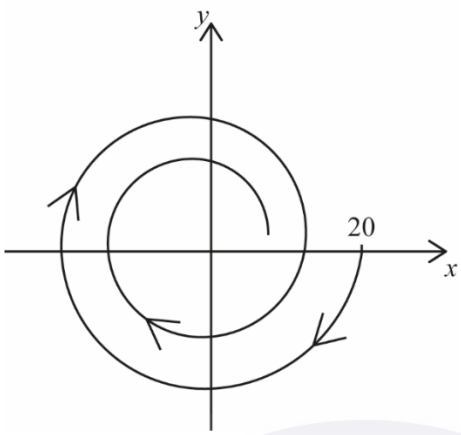
$(=-100\div -1=)\ 100$ ***A1***

***[5 marks]***

*continued…*

*Question 6 continued*

(d)




***A4***

**Note:** Award ***A1*** for starting at $(20,\ 0)$, ***A1*** for spiral inwards, ***A1*** for clockwise, ***A1*** for non-negative gradient at $(20,\ 0)$.

***[4 marks]***

***[Total: 15 marks]***

**7.** (a) using area of trapezoid formula ***M1***

$\sin(15^\circ)\times\frac{1+2}{2}$ ***A1***

$=\frac{3}{2}\sin(15^\circ)$ ***AG***

***[2 marks]***

(b) (i) $\boldsymbol{M}_6=\begin{pmatrix}\frac{1}{2}\cos 90^\circ & -\frac{1}{2}\sin 90^\circ\\ \frac{1}{2}\sin 90^\circ & \frac{1}{2}\cos 90^\circ\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$ ***A1***

(ii) multiplying their part (b)(i) and point $(0,\ -1)$ (in any order) ***M1***

$\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\times\begin{pmatrix}0\\ -1\end{pmatrix}$

$\left(\frac{1}{2},\ 0\right)$ ***A1***

***[4 marks]***

*continued…*

*Question 7 continued*

(c) (i) $\begin{pmatrix} \cos\left(k\times 15^\circ\right) & -\sin\left(k\times 15^\circ\right) \\ \sin\left(k\times 15^\circ\right) & \cos\left(k\times 15^\circ\right) \end{pmatrix}$ ***A1***

(ii) $\begin{pmatrix} 1-\frac{k}{12} & 0 \\ 0 & 1-\frac{k}{12} \end{pmatrix}$ ***A1***

(iii) $k\times 15^\circ$ ***A1***

(iv) $1-\frac{k}{12}$ ***A1***

***[4 marks]***

(d) **METHOD 1 (using part (c)(iv))**

$\left(1-\frac{k}{12}\right)^2$ ***A2***

**METHOD 2 (using full matrix $M_k$)**

$$\left|\begin{pmatrix} \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) & -\left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) \\ \left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) & \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) \end{pmatrix}\right|$$

$=\left(1-\frac{k}{12}\right)^2\cos^2\left(k\times 15^\circ\right)+\left(1-\frac{k}{12}\right)^2\sin^2\left(k\times 15^\circ\right)$ ***(M1)***

$=\left(1-\frac{k}{12}\right)^2\left(\cos^2\left(k\times 15^\circ\right)+\sin^2\left(k\times 15^\circ\right)\right)$

$=\left(1-\frac{k}{12}\right)^2$ ***A1***

***[2 marks]***

*continued…*

*Question 7 continued*

(e) recognizing to multiply by 2 and by original area ***(M1)***

attempt to sum their answer to part (d), $k = 0, 1, ..., 11$ ***(M1)***

a correct expression ***(A1)***

e.g. $0.776457...\left(1^2 + \left(\frac{11}{12}\right)^2 + ... + \left(\frac{1}{12}\right)^2\right)$ **OR** $2\sum_{k=0}^{11}\left(1-\frac{k}{12}\right)^2 \times \frac{3}{2}\sin 15^\circ$

**OR** $\sum_{k=0}^{11}\left(1-\frac{k}{12}\right)^2 \times 0.776457...$ **OR** $2\sum_{k=1}^{12}\left(\frac{k}{12}\right)^2 \times \frac{3}{2}\sin\left(15^\circ\right)$

3.50 (3.50484...) (square units)

**Note:** Award at most ***M0(M1)(A1)A0*** for an unsupported final answer of "1.75242..."

***(A1)***

***[4 marks]***

(f) $\boldsymbol{N}_k = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times \boldsymbol{M}_k$ ***A1A1***

**Note:** Award ***A1A0*** if correct matrices are written in the wrong order.

***[2 marks]***

***[Total: 18 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2023

## Mathematics: applications and interpretation

## Higher level

## Paper 2

21 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

**1 General**

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | Correct answer seen | Further working seen | Any FT issues? | Action |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to a "correct" level of accuracy (e.g 3 sf) in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) attempt to use area of triangle formula ***(M1)***

$\frac{1}{2} \times 25.9 \times 6.36 \times \sin(125^\circ)$ ***(A1)***

$67.5 \text{ m}^2$ $(67.4700... \text{ m}^2)$ ***A1***

**Note:** Units are required. The final ***A1*** is only awarded if the correct units are seen in their answer; hence award ***(M1)(A1)A0*** for an unsupported answer of $67.5$.

***[3 marks]***

(b) attempt to use cosine rule ***(M1)***

$(\text{BK} =)\sqrt{12^2 + 6.36^2 - 2 \times 12 \times 6.36 \times \cos 45^\circ}$ ***(A1)***

$8.75$ (m) $(8.74738\ldots(\text{m}))$ ***A1***

**Note:** Award ***(M1)(A1)(A0)*** for radian answer of $10.2$ (m) $(10.2109\ldots(\text{m}))$ with or without working shown.

***[3 marks]***

(c) **METHOD 1**

attempt to use sine rule with measurements from triangle $\text{OKX}$ ***(M1)***

$\frac{\text{OX}}{\sin 51.1^\circ} = \frac{22.2}{\sin 53.8^\circ}$ ***(A1)***

$(\text{OX} =)$ $21.4$ (m) $(21.4099\ldots)(\text{m})$ ***A1***

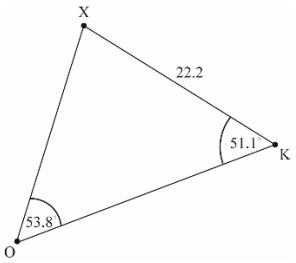
$(21.4 \text{ (m)} < 22.2 \text{ (m)})$

Odette is closer to the football / Khemil is further from the football ***A1***

**Note:** For the final ***A1*** to be awarded $21.4$ $(21.4099\ldots)$ must be seen. Follow through within question part for final ***A1*** for a consistent comparison with their $\text{OX}$.

**METHOD 2**

sketch of triangle $\text{OXK}$ with vertices, angles and lengths ***(A1)***




$51.1^\circ$ is smallest angle in triangle $\text{OXK}$ ***R1***

opposite side $(\text{OX})$ is smallest length ***R1***

therefore Odette is closest ***A1***

***[4 marks]***

*continued…*

*Question 1 continued*

(d) attempt to use length of arc formula ***(M1)***

$\frac{135}{360} \times 2\pi \times 12$ ***(A1)***

$28.3\,(\text{m})$ $(9\pi,\ 28.2743...)$ (m) ***A1***

***[3 marks]***

***Total [13 marks]***

**2.** (a) (i) $1200$ ***A1***

(ii) the initial population of the bacteria ***A1***

***[2 marks]***

(b) $1200 \times k^3 = 18750$ ***(A1)***

$(k =)\ 2.5$ ***A1***

***[2 marks]***

(c) $1200 \times 2.5^{1.5}$ ***(A1)***

$4740\ (4743.41\ldots)$ ***A1***

**Note:** Do not penalize if final answer is not given as an integer. Award ***(A1)A0*** for an answer of $3950\ (3949.14\ldots)$ from use of $1.3$ in the exponent, but only if working is shown.

***[2 marks]***

(d) equating $P(t)$ and $S(t)$ **OR** equating each function to a common variable ***(M1)***

$1200 \times 2.5^t = 5000 \times 1.65^t$; $1200 \times 2.5^t = x$ and $5000 \times 1.65^t = x$

$t = 3.43$ (hours) $(3.43456...)$ ***A1***

***[2 marks]***

*continued…*

*Question 2 continued*

(e) **METHOD 1**

$5000 \times 1.65^t = 19000$ ***(M1)***

$(t =)\ 2.66586...$ **OR** $(t - 2 =)\ 0.66586...$ (seen) ***(A1)***

multiplying by $60$ seen to convert to minutes ***(M1)***
$(m = 39.9521\ldots)$

$(m =)\ 40$ (minutes) **OR** $2$ hours and $40$ minutes ***A1***

**METHOD 2**

equating an expression for $S(t)$ to $19000$ ***(M1)***

expressing $t$ as $2 + \frac{m}{60}$ ***(A1)***

$5000 \times 1.65^{2+\frac{m}{60}} = 19000$

$2 + \frac{m}{60} = 2.66586\ldots$ ***A1***

$(m =)\ 40$ (minutes) **OR** $2$ hours and $40$ minutes ***A1***

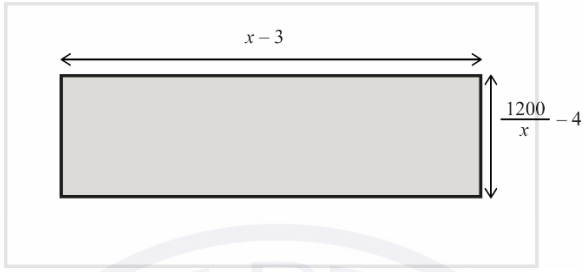
**Note:** Award ***(M1)(A1)(M1)A0*** for an answer of $39.9521...$ or $39$ with or without working.

***[4 marks]***
***Total [12 marks]***

**3.** (a)

**Note:** In methods 1 and 2, full marks are available for candidates who work with a dummy variable, e.g. $y$, that represents the width of the park and hence is equal to $\frac{1200}{x}$.
The substitution to express an answer in only $x$ may come as late as the final line.

**METHOD 1 (finding dimensions of garden)**




(width of park =) $\frac{1200}{x}$ ***(A1)***

(length of garden =) $x-3$, (width of garden =) $\frac{1200}{x}-4$ ***(A1)(A1)***

$A=(x-3)\times\left(\frac{1200}{x}-4\right)$ ***A1***

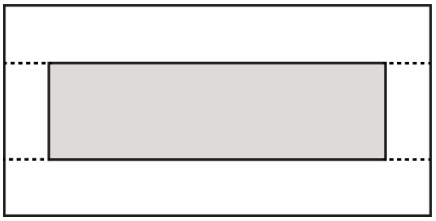
$=1200-4x-\frac{3600}{x}+12$ ***A1***

$=1212-4x-\frac{3600}{x}$ ***AG***

*continued…*

*Question 3 continued*

**METHOD 2 (subtracting the area of the path)**



width of park = $\frac{1200}{x}$ ***(A1)***

attempt to cut path into 4 (or 8) pieces ***(M1)***

four (or eight) areas of the path expressed in terms of $x$ ***(A1)***

$A = 1200 - 2x - 2x - 1.5\left(\frac{1200}{x} - 4\right) - 1.5\left(\frac{1200}{x} - 4\right)$ ***A1***

correct manipulation leading to given result ***A1***

$= 1212 - 4x - \frac{1800}{x} - \frac{1800}{x}$

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

**Note:** To award ***(M1)(A1)*** without a diagram the division of the park must be clear.

***[5 marks]***

(b) setting $1212 - 4x - \frac{3600}{x} = 800$ (accept a sketch) ***(M1)***

$x = 9.64$ $(9.64011\ldots)$ (m) **OR** $x = 93.4$ $(93.3598\ldots)$ (m) ***A1***

(width $=$) $124$ $(124.479\ldots)$ (m) ***A1***

(width $=$) $12.9$ $(12.8534\ldots)$ (m) ***A1***

**Note:** To award the final ***A1*** both values of $x$ **and** both values of the width must be seen. Accept $12.8$ for second value of width from candidate dividing $1200$ by 3 sf value of $93.4$.

***[4 marks]***

(c) $\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) -4 + \frac{3600}{x^2}$ **OR** $-4 + 3600x^{-2}$ ***A1A1A1***

**Note:** Award ***A1*** for $-4$, ***A1*** for $+3600$, and ***A1*** for $x^{-2}$ or $x^2$ in denominator.

***[3 marks]***

*continued…*

*Question 3 continued*

(d) setting *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ equal to $0$ **OR** sketch of *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ with $x$-intercept highlighted ***M1***

$(x =)\ 30\ (\mathrm{m})$ ***A1***

**Note:** To award ***A1FT*** the candidate's value of $x$ must be within the domain given in the problem $(3 < x < 300)$.

***[2 marks]***

(e) **EITHER**

evidence of using GDC to find maximum of graph of $A = 1212 - 4x - \frac{3600}{x}$ ***(M1)***

**OR**

substitution of *their* $x$ into $A$ ***(M1)***

**OR**

dividing $1200$ by *their* $x$ to find width of park **and** subtracting $3$ from *their* $x$ and $4$ from the width to find park dimensions ***(M1)***

**Note:** For the last two methods, only follow through if $3 <$ their $x < 300$.

**THEN**

$(A =)\ 972\ (\mathrm{m}^2)$ ***A1***

***[2 marks]***

***Total [16 marks]***

**4.** (a) any city can be travelled to or from any other city (so is connected) ***R1***

**EITHER**

but there is no direct flight between Los Angeles and Dallas (for example) ***R1***

**OR**

but not every vertex has degree 4 ***R1***

**Note:** Accept equivalent statements for the cities being connected and the graph not being complete.

***[2 marks]***

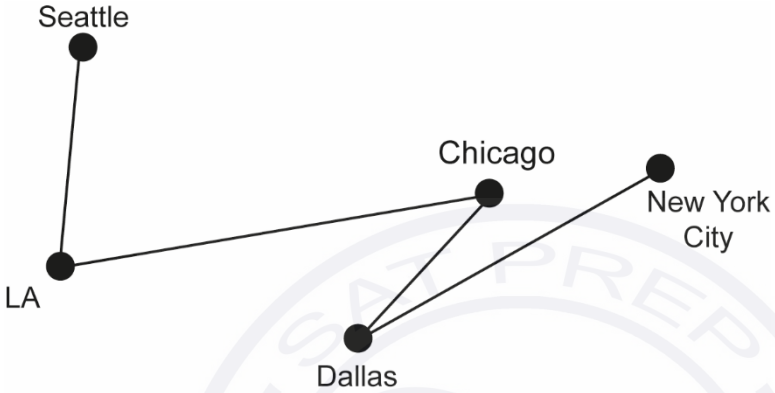
*continued…*

*Question 4 continued*

(b) edge CD selected first ***M1***

DN,
CL,
LS ***A1***

**Note:** Award marks if the answers are written as sums in the correct order.
***M1*** if $30$ is seen first, ***A1*** for $30 + 39 + 41 + 58$.




***A1***

**Note:** The final ***A1*** can be awarded independently. Award ***M0A0A1*** for a correct MST graph with no other working. Award ***M1A0A1*** if Prim's algorithm is seen to be used correctly with $\mathrm{CD}$ first.

***[3 marks]***

(c) $2 \times \mathrm{MST}$ weight ***(M1)***

$= \$336$ ***A1***

**Note:** Allow any integer multiple (>1) of MST weight for ***M1***, and if correctly calculated, award ***M1A1***.

***[2 marks]***

(d) attempt at nearest neighbour algorithm ***M1***

order is $\mathrm{LA} \to \mathrm{D} \to \mathrm{C} \to \mathrm{NYC} \to \mathrm{S} \to \mathrm{LA}$ ***A1***

**Note:** Award ***M1*** for a route that begins with $\mathrm{LA}$ and then $\mathrm{D}$, this includes seeing $26$ as the first value in a sum.
Award ***A1*** if $26+30+68+66+58$ seen in order.

**Note:** Award ***M1A0*** for an incorrect first nearest neighbour proceeding 'correctly' to the next vertex. For example, $\mathrm{LA}$ to $\mathrm{C}$ and then $\mathrm{C}$ to $\mathrm{D}$.

upper bound is $(26 + 30 + 68 + 66 + 58 =)\ \$248$ ***A1***

**Note:** Award ***M1A0*** for correct nearest neighbour algorithm starting from a vertex other than LA. Condone the correct tour written backwards i.e. $58 + 66 + 68 + 30 + 26 = 248$

***[3 marks]***

*continued…*

*Question 4 continued*

(e) (i) attempt to find MST of L, N, D and S ***(M1)***
by deleting C, Kruskal gives MST for the remainder as LD, DN, LS
weight 123 ***(A1)***
(lower bound is therefore $123 + (30 + 41) =)$ \$194 ***A1***

**Note:** Award ***(M1)*** for a graph or list of edges that does not include C.
Award ***(A1)*** if $26 + 39 + 58$ seen in any order.

(ii) by deleting S, Kruskal gives MST for the remainder as LD, DC, DN
weight 95 ***(A1)***
(lower bound is therefore $95 + (58 + 66) =)$ \$219 ***A1***

**Note:** Award ***(A1)*** if $26 + 30 + 39$ seen in any order.

***[5 marks]***

(f) $219 \leq C \leq 248$ ***A1A1***

**Note:** Award ***A1*** for $219 \leq C$ and ***A1*** for $C \leq 248$. Award at most ***A1A0*** for $219 < C < 248$.
***FT*** for their values from part (e) if higher value from (e)(i) and (e)(ii) used for the lower bound, and part (d) for the upper.

***[2 marks]***

(g) any valid tour, within their interval from part (f), from any starting point **OR**
any valid tour that starts and finishes at N ***(M1)***
valid tour starting point N **AND** within their interval ***A1***
e.g NDCLSN (weight 234)

**Note:** If part (f) not correct, **only** award ***A1FT*** if their valid tour begins and ends at N **AND** lies within **BOTH** their interval (including if one-sided) in part (f) **AND** $219 \leq C \leq 248$.

If no response in the form of an interval seen in part (f) then award ***M1A0*** for a valid tour beginning and ending at N **AND** within $219 \leq C \leq 248$.

***[2 marks]***

***Total [19 marks]***

**5.**

(a) $$(\boldsymbol{T} =) \begin{matrix} \\ \text{(B)} \\ \text{(G)} \\ \text{(N)} \end{matrix} \begin{matrix} \begin{matrix} \text{(B)} & \text{(G)} & \text{(N)} \end{matrix} \\ \begin{pmatrix} 0.945 & 0.015 & 0.02 \\ 0.05 & 0.965 & 0.03 \\ 0.005 & 0.02 & 0.95 \end{pmatrix} \end{matrix}$$ ***M1A1A1***

**Note:** Accept the columns in any order. Accept the transpose of this matrix.

Award ***M1*** for a 3x3 matrix with all values between (but not including) $0$ and $1$, and all columns (or rows if transposed) adding up to $1$, award ***A1*** for one correct row (or column if transposed) and ***A1*** for all rows (or columns if transposed ) correct.

***[3 marks]***

(b) $$(\boldsymbol{T}^6 =) \begin{pmatrix} 0.72 & 0.077 & 0.098 \\ 0.24 & 0.83 & 0.16 \\ 0.035 & 0.098 & 0.74 \end{pmatrix}$$ ***(M1)***

**Note:** Accept a transposed matrix.

multiplying their $\boldsymbol{T}^6$ by a correct matrix of the initial populations ***(M1)***

$$\begin{pmatrix} 0.72 & 0.077 & 0.098 \\ 0.24 & 0.83 & 0.16 \\ 0.035 & 0.098 & 0.74 \end{pmatrix} \begin{pmatrix} 26000 \\ 240000 \\ 50000 \end{pmatrix}$$

**Note:** Award this ***M1*** for a transposed $\boldsymbol{T}$ if used correctly in part (b) i.e. preceded by $1\times3$ matrix rather than followed by a $3\times1$ matrix.

$$= \begin{pmatrix} 42133 \\ 212205 \\ 61661 \end{pmatrix}$$ ***(A1)***

so the expected population of the German side would be $212000\ (212205)$ ***A1***

**Note:** Award ***M0M1A0A1*** for an answer of $174000\ \ (=174031)$. This is the case when $\boldsymbol{T}^{30}$ has been used.

***[4 marks]***

*continued…*

*Question 5 continued*

(c) (i) $\begin{pmatrix} 0.945 & 0.015 & 0.02 \\ 0.05 & 0.965 & 0.03 \\ 0.005 & 0.02 & 0.95 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

at least two of these three:

$0.945u_1 + 0.015u_2 + 0.02u_3 = u_1$

$0.05u_1 + 0.965u_2 + 0.03u_3 = u_2$ ***A1***

$0.005u_1 + 0.02u_2 + 0.95u_3 = u_3$

and

$u_1 + u_2 + u_3 = 1$ (may be seen in part (c)(ii)) ***A1***

(ii) $(\boldsymbol{u} =) \begin{pmatrix} 0.231 \\ 0.533 \\ 0.236 \end{pmatrix}$ $\left(\boldsymbol{u} = \begin{pmatrix} 0.231155... \\ 0.532663... \\ 0.236180... \end{pmatrix}\right)$ ***A1***

**Note:** The ***A1*** in part (c)(ii) can be awarded independently of the working in part (c)(i).

***[3 marks]***

(d) $0.532663... \times (26000 + 240000 + 50000)$ ***(M1)***

$= 168000$ $(168321...)$ ***A1***

**Note:** Award ***(M1)A1*** for answers using $\boldsymbol{T}^n$ with $n$ large that lead to a correct answer.
Award ***(M0)A0*** for answers that use $\boldsymbol{T}^n$ that lead to an incorrect answer.

***[2 marks]***

(e) *Award **R1** for each appropriate reason. For example:*

Movement unlikely to be constant

Total population for entire region likely to grow over time

Each power of the transition matrix takes five years; a relatively long time in terms of population movement.

There may be other/new external factors such as wars in other adjoining countries, leading to an influx of economic migrants.

***R1R1***

**Note:** Do not award ***R1*** for any response that shows a lack of understanding of the assumption that the total population remains constant.

***[2 marks]***

***Total [14 marks]***

**6.** (a) slugs appear discretely / independently / randomly / at a constant (average) rate / mean is (approximately) equal to variance ***R1R1***

***[2 marks]***

(b) new $(m=)\ 0.2\times12\ (=2.4)$ (so $X \sim \text{Po}(2.4)$) ***(A1)***

attempt to use a pdf (e.g $\text{P}(X=4)$) ***(M1)***

$0.125\ (0.125408...)$ ***A1***

***[3 marks]***

(c) $\text{P}(X<3)$ **OR** $\text{P}(X\leq 2)$ ***(A1)***

$0.570\ (0.569708...)$ ***A1***

***[2 marks]***

(d) $\text{P}(X\geq 1)=0.909282...$ ***(A1)***

raising a probability to a power of $3$ ***(M1)***

$0.909282...^3$

$=0.752\ (0.751788...)$ ***A1***

**Note:** Award at most ***(A1)(M1)(A0)*** for a final answer of $0.751$. Working may not be seen.

***[3 marks]***

(e) $\text{H}_0 : m = 2.4$, ***A1***

$\text{H}_1 : m > 2.4$ ***A1***

**Note:** The hypotheses may be written in words but must include reference to the mean (e.g. "number of snails" is not sufficient to award ***A1***), and state clearly for $\text{H}_1$ that the mean increases.

***[2 marks]***

(f) **EITHER**

finding either $\text{P}(X\geq 7)$ or $\text{P}(X\geq 8)$ ***(M1)***

$\left(\text{P}(X\geq 7)=\right)\ 0.01160...$ **AND** $\left(\text{P}(X\geq 8)=\right)\ 0.00334...$ ***A1***

**OR**

finding either $\text{P}(X\leq 7)$ or $\text{P}(X\leq 6)$ ***(M1)***

$\left(\text{P}(X\leq 7)=\right)\ 0.996661...$ **AND** $\left(\text{P}(X\leq 6)=\right)\ 0.988405...$ ***A1***

**THEN**

so critical region is $X\geq 8$ **OR** $X>7$ ***A1***

**Note:** ***(M1)A0A1*** can be awarded for a correct answer that is unsupported.

***[3 marks]***

(g) $(0.75\times12=)\ 9$ ***(A1)***

$\text{P}\left(X\leq 7 \mid m=9\right)$ ***(M1)***

$=0.324$ ***A1***

***[3 marks]***

***Total [18 marks]***

**7.** (a) $\begin{vmatrix} -4-\lambda & 6 \\ 9 & -1-\lambda \end{vmatrix} = 0$ ***(M1)***

**Note:** Do not accept $\det(\boldsymbol{A}-\lambda\boldsymbol{I})=0$ or similar as evidence of a correct method unless A is explicitly defined to be the given matrix.

$(-4-\lambda)(-1-\lambda)-54=0$

$\lambda=-10,\ \lambda=5$ ***A1A1***

For $\lambda=-10$

$\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -10x \\ -10y \end{pmatrix}$ ***(M1)***

$-4x+6y=-10x$

$x+y=0$

possible eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or equivalent) ***A1***

for $\lambda=5$

$\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 5x \\ 5y \end{pmatrix}$

$-4x+6y=5x$

$3x=2y$

possible eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

**Note:** If both eigenvalues are incorrect then award at most ***M1A0A0M1A0A0***.

***[6 marks]***

(b) attempt to substitute their eigenvalues and eigenvectors equation ***(M1)***

$\begin{pmatrix} x \\ y \end{pmatrix}=A\mathrm{e}^{-10t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}+B\mathrm{e}^{5t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ***A1***

**Note:** Award at most ***(M1)A0*** if $\begin{pmatrix} x \\ y \end{pmatrix}$ not seen.

***[2 marks]***

*continued…*

*Question 7 continued*

(c) At $t = 0$, $x = 500$ and $y = 125$

$x = -A + 2B$ and $y = A + 3B$

Solving simultaneously: ***(M1)***

$A = -250$ and $B = 125$ ***A1***

$$\begin{pmatrix} x \\ y \end{pmatrix} = -250\mathrm{e}^{-10t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 125\mathrm{e}^{5t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

**Note:** Follow through from their eigenvectors.
Accept equivalent values for A and B based on the direction of their eigenvectors and the order of their eigenvalues in the equation.

***[2 marks]***

(d) $2:3$ ***A1***

***[1 mark]***

(e) attempt to eliminate $\mathrm{d}t$ from the two differential equations ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x - y}{-4x + 6y}$$

substituting initial conditions ***(M1)***

$$= \frac{9(500) - 125}{-4(500) + 6(125)}$$

$= -3.5$ ***A1***

**Note:** Award ***M1*** for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4x + 6y}{9x - y}$.

***[3 marks]***

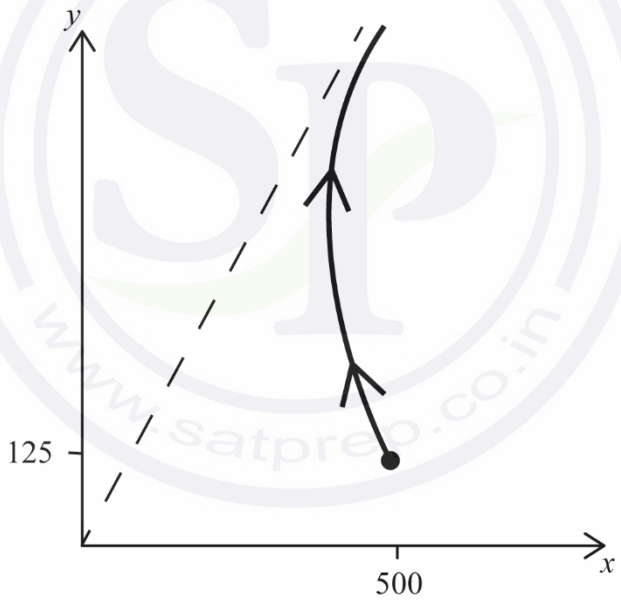
*continued…*

*Question 7 continued*

(f) trajectory or trajectories that are consistent with their eigenvalues ***A1***
a trajectory that passes through the point $(500, 125)$ with gradient that is consistent with the response to part (e) ***A1***
the diagram contains at least one of their eigenvectors ***A1***
(e.g. labelled $y = 1.5x$; $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\lambda = 5$ etc.)
the trajectory that passes through $(125, 500)$ tends towards an oblique asymptote that corresponds to their eigenvector and the direction is indicated by at least one arrow on the trajectory ***A1***

**Note:** For the second ***A1***, the point $(500, 125)$ may not be labelled but there should be a point marked on the trajectory that is consistent with these coordinates.

The final ***A1*** will depend on their eigenvalues. Follow through can be awarded as long as the direction of the trajectory is consistent with the nature of their eigenvalues and eigenvectors.




***[4 marks]***
***Total [18 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2022

## Mathematics: applications and interpretation

## Higher level

## Paper 2

16 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.
***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
***R*** Marks awarded for clear **Reasoning**.
***AG*** Answer given in the question and so no marks are awarded.
***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) (i) $(m =)\ 54(\%)$ ***A1***

(ii) $(n =)\ 14(\%)$ ***A1***

(iii) $(p =)\ 22(\%)$ ***A1***

(iv) $(q =)\ 10(\%)$ ***A1***

**Note:** Based on their $n$, follow through for parts (i) and (iii), but only if it does not contradict the given information. Follow through for part (iv) but only if the total is $100\%$.

***[4 marks]***

(b) (i) $0.54\ \left(\frac{54}{100}, \frac{27}{50}, 54\%\right)$ ***A1***

(ii) $\frac{54}{64}\ \left(0.844, \frac{27}{32}, 84.4\%, 0.84375\right)$ ***A1A1***

**Note:** Award ***A1*** for a correct denominator ($0.64$ or $64$ seen), ***A1*** for the correct final answer.

***[3 marks]***

(c) (i) recognizing Binomial distribution with correct parameters ***(M1)***

$X \sim \mathrm{B}(10, 0.68)$

$\left(\mathrm{P}(X = 5) =\right)\ 0.123\ (0.122940..., 12.3\%)$ ***A1***

(ii) $1 - \mathrm{P}(X \leq 3)$ **OR** $\mathrm{P}(X \geq 4)$ **OR** $\mathrm{P}(4 \leq X \leq 10)$ ***(M1)***

$0.984\ (0.984497..., 98.4\%)$ ***A1***

(iii) $(0.68)^9 \times 0.32$ ***(M1)***

recognition of two possible cases ***(M1)***

$2 \times \left((0.68)^9 \times 0.32\right)$

$0.0199\ (0.0198957..., 1.99\%)$ ***A1***

***[7 marks]***

(d) **EITHER**

the probability is not constant ***A1***

**OR**

the events are not independent ***A1***

**OR**

the events should be modelled by the hypergeometric distribution instead ***A1***

***[1 mark]***

***Total [15 marks]***

**2.** (a) (i) B ***A1***

(ii) F ***A1***

***[2 marks]***

(b) correct substitution into the midpoint formula ***(M1)***

$\frac{8+5}{2}$

$y = 6.5$ ***A1***

**Note:** Answer must be an equation for the ***A1*** to be awarded.

***[2 marks]***

(c) midpoint $= (5, 7)$ ***(A1)***

correct use of gradient formula ***(M1)***

$\frac{8-6}{7-3}$

gradient of $\text{BC} = 0.5$ ***(A1)***

negative reciprocal of gradient ***(M1)***

perpendicular gradient $= -2$

$y - 7 = -2(x - 5)$ (or $y = -2x + 17$) ***A1***

**Note:** Do not follow through within the part for the final ***A1***.

***[5 marks]***

(d) (i) attempt to find the intersection of two perpendicular bisectors (BC & CD) ***(M1)***

**Note:** This may be seen graphically or algebraically.

$6.5 - 7 = -2(x - 5)$ **OR** $6.5 = -2x + 17$

**Note:** Accept equivalent methods using the perpendicular bisector of BD, $y - 5.5 = 4(x - 5)$ **OR** $y = 4x - 14.5$

$x = 5.25, y = 6.5$ **OR** $(5.25, 6.5)$ ***A1***

**Note:** The $x$-coordinate must be exact or expressed to at least 3 sf.

(ii) their correct substitution into distance formula ***(M1)***

$\sqrt{(5.25-7)^2 + (6.5-5)^2}$

$= 2.30 \text{ km} \left( 2.30488..., \frac{\sqrt{85}}{4} \right)$ ***A1***

***[4 marks]***

***Total [13 marks]***

**3.** (a) (i) $f'(x) = \frac{-2x}{50} + 2 \quad \left( = \frac{-x}{25} + 2, -0.04x + 2 \right)$ ***A1A1***

**Note:** Award ***A1*** for each correct term. Award at most ***A0A1*** if extra terms are seen.

(ii) $0 = \frac{-x}{25} + 2$ **OR** sketch of $f'(x)$ with $x$-intercept indicated ***M1***

$x = 50$ ***A1***

$y = 80$ ***A1***

$(50, 80)$

**Note:** Award ***M0A0A1*** for the coordinate $(50, 80)$ seen either with no working or found from a graph of $f(x)$.

***[5 marks]***

(b) (i) $\int_0^{70} \frac{-x^2}{50} + 2x + 30 \, \mathrm{d}x$ ***A1A1***

**Note:** Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if $\mathrm{d}x$ is omitted.

(ii) (Area =) $4710 \text{ m}^2 \quad \left( 4713.33\ldots, \frac{14140}{3} \right)$ ***A2***

***[4 marks]***

(c) (i) $\frac{11.4}{4713.33...} \times 100\%$ **OR** $\left| \frac{4701.93... - 4713.33...}{4713.33...} \right| \times 100\%$ ***(M1)***

**Note:** Award ***(M1)*** for their correct substitution into the percentage error formula.

$0.242\% \quad (0.241867...\%)$ ***A1***

**Note:** Percentage sign is required. Accept $0.242038...\%$ if $4710$ is used.

(ii) **EITHER**

reduce the width of the intervals (trapezoids) ***A1***

**OR**

increase the number of intervals (trapezoids) ***A1***

**Note:** Accept equivalent statements. Award ***A0*** for the ambiguous answer "increase the intervals".

***[3 marks]***

*continued…*

*Question 3 continued*

(d) (i) width of the square is $70-x$ **OR** the length of the square is $\frac{-x^2}{50}+2x+30$

***(M1)***

**Note:** Award ***(M1)*** for $70-x$ seen anywhere. Accept $\frac{-x^2}{50}+2x+30$ but only if this expression is explicitly identified as a dimension of the square.

in term of $x$, equating the length to the width $\mathrm{ED}$ ***(M1)***

$\frac{-x^2}{50}+2x+30=70-x$

$(x=14.7920...$ or $135.21)$

$(x=)\ 14.8\ \mathrm{m}\ (14.7920...)$ ***A1***

**Note:** Award ***M0M0A0*** for an unsupported answer of $15$. Award at most ***M1M0A0*** for an approach which leads to $A'(x)=0$. This will lead to a square base which extends beyond the east boundary of the property. Similar for any solution where $\mathrm{F}$ is not on the northern boundary, or $\mathrm{GH}$ is not on the east boundary.

(ii) **EITHER**

$(70-14.7920...)^2$ ***(M1)***

**OR**

$(55.2079...)^2$ ***(M1)***

**OR**

$\left(\frac{-(14.7920...)^2}{50}+2(14.7920...)+30\right)^2$ ***(M1)***

**THEN**

$(\text{Area}=)\ 3050\ \mathrm{m}^2\ (3047.92...)$ ***A1***

**Note:** Follow through from part (d)(i), provided $x$ is between $0$ and $70$. Award at most ***M1A0*** if their answer is outside the range of their $[0,\ 4713.33\ldots]$ from part (b).

***[5 marks]***

***Total [17 marks]***

**4.** (a) any correct Hamiltonian cycle e.g. ABCDEFA ***A1***
***[1 mark]***

(b) no, since not all vertices have an even degree (or equivalent) ***R1***
***[1 mark]***

(c) (i) 49 ***A1***

(ii) 34 ***A1***

(iii) 50 ***A1***
***[3 marks]***

(d) cycle is EBCDFAE
$UB = 12 + 25 + 17 + 34 + 18 + 35$ ***(M1)(A1)***

**Note:** Award ***M1*** for $12 + 25 + 17 + ...$ **OR** EBCD.

$= 141$ ***A1***
***[3 marks]***

(e) attempt to find MST for vertices A, B, C, D and E ***M1***
$12 + 14 + 17 + 27 \ (= 70)$ ***A1***
$LB = 70 + 18 + 22$ ***(M1)***
$= 110$ ***A1***
***[4 marks]***

(f) **EITHER**
deleting a different vertex ***A1***
might give a higher value (and hence a better lower bound). ***R1***
**OR**
the edges selected in part (e) do not form a cycle. ***A1***
so a higher value is possible ***R1***
***[2 marks]***
***[14 marks]***

**5.** (a) $\frac{1}{2}x^3+1=(x-1)^4$ ***(M1)***

$(p=)\ 2.91\ \text{cm}\ (2.91082...)$ ***A1***

***[2 marks]***

(b) attempt to make $x$ (or $x^2$) the subject of $y=\frac{1}{2}x^3+1$ ***(M1)***

$x=\sqrt[3]{2(y-1)}$ (or $x^2=\left(2(y-1)\right)^{\frac{2}{3}}$) ***(A1)***

(upper limit $=$) $13.3(315\ldots)$ ***(A1)***

$V=\int_1^{13.3315...}\pi\left(2(y-1)\right)^{\frac{2}{3}}\ \mathrm{d}y$ ***(M1)***

**Note:** Award ***(M1)*** for setting up correct integral squaring their expression for $x$ with both correct lower limit and their upper limit, and $\pi$. Condone omission of $\mathrm{d}y$.

$=197\ \text{cm}^3\ (196.946...)$ ***A1***

***[5 marks]***

(c) $x=y^{\frac{1}{4}}+1$ (or $x^2=\left(y^{\frac{1}{4}}+1\right)^2$) ***(A1)***

$V_2=\int_0^{13.3315...}\pi(y^{\frac{1}{4}}+1)^2\ \mathrm{d}y$ ***(M1)(A1)***

**Note:** Award ***(M1)*** for setting up correct integral squaring their expression for $x$ with their upper limit, and $\pi$. Award ***(A1)*** for lower limit of $0$, dependent on ***M1***. Condone omission of $\mathrm{d}y$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

$=271.87668\ldots$ ***(A1)***

**Note:** Accept $271.038\ldots$ from use of 3sf in the upper limit.

subtracting their volumes ***(M1)***

$271.87668\ldots-196.946...$

$=74.9\ \text{cm}^3\ (74.93033\ldots)$ ***A1***

**Note:** Accept any answer that rounds to $75\ (\text{cm}^3)$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

***[6 marks]***

***[13 marks]***

**6.** (a) wood layer, $W \sim \mathrm{N}(7,\ 0.3^2)$; plastic, $P \sim \mathrm{N}(3,\ 0.16^2)$

door: $X = W + P$

$\mathrm{E}(X) = 10$ (mm) ***(A1)***

$\mathrm{Var}(X) = \mathrm{Var}(W) + \mathrm{Var}(P) = 0.1156$ $(\mathrm{mm}^2)$ ***(M1)(A1)***

recognizing the distribution is Normal, with their mean and variance ***(M1)***

$X \sim \mathrm{N}(10,\ 0.34^2)$

$\mathrm{P}(X < 9.5) = 0.0707$ $(0.07070125...)$ ***A1***

***[5 marks]***

(b) $\mathrm{E}(T) = 80$ ***(A1)***

$\mathrm{Var}(T)\left(= 0.1156 \times 8\right) = 0.9248$ ***(M1)(A1)***

$T \sim \mathrm{N}(80,\ 0.9248)$

$\mathrm{P}(T > 82) = 0.0188$ $(0.0187753\ldots)$ ***A1***

***[4 marks]***

(c) (i) 6.93 mm $(6.93428\ldots)$ ***A1***

(ii) $(s_{n-1} =)$ 0.404 ***(A1)***

$(s_{n-1}^2 =)$ 0.163 $\mathrm{mm}^2$ $(0.162928\ldots)$ ***A1***

***[3 marks]***

(d) $\mathrm{H}_0 : \mu_A = \mu_B$ and $\mathrm{H}_1 : \mu_A > \mu_B$ ***A1A1***

**Note:** Award ***A1*** for use of $\mu$ or in words "population mean", and ***A1*** for both correct equality in null hypothesis and correct inequality in alternative hypothesis. Accept an equivalent statement in words, must include mean and reference to "**population** mean" / "mean for **all** Machine B layers" for the first ***A1*** to be awarded.

use a two-sample *t*-test ***(M1)***

*p*-value $= 0.406975...$ ***A1***

since $0.406975... > 0.05$ **OR** $p$-value $> 0.05$ ***R1***

Do not reject $\mathrm{H}_0$ (Insufficient evidence to support the employee's claim) ***A1***

**Note:** Accept a *p*-value of $0.415861\ldots$ from use of 3sf values from part (c). Follow through within the question for the final ***R1*** and ***A1*** for their *p*-value provided $0 \le p \le 1$. Do not award ***R0A1***.

***[6 marks]***

***Total [18 marks]***

**7.** (a) (i) use of chain rule ***(M1)***

$\boldsymbol{v} = -9\sin(3t)\boldsymbol{i} + 12\cos(3t)\boldsymbol{j}$ ***A1***

**Note:** Award ***(M1)*** for at least one correct term seen but condone omission of $\boldsymbol{i}$ or $\boldsymbol{j}$.

(ii) $|\boldsymbol{v}| = \sqrt{(-9\sin(9))^2 + (12\cos(9))^2}$ ***(M1)***

$= 11.5 \text{ m s}^{-1}$ (11.5455...) ***A1***

***[4 marks]***

(b) (i) $\boldsymbol{a} = -27\cos(3t)\boldsymbol{i} - 36\sin(3t)\boldsymbol{j}$ ***A1***

(ii) $\boldsymbol{a} = -9(3\cos(3t)\boldsymbol{i} - 4\sin(3t)\boldsymbol{j})$ ***M1***

$\boldsymbol{a} = -9\boldsymbol{r}$ (where $\boldsymbol{r}$ is a position vector from the origin) ***A1***

$\boldsymbol{a}$ is in opposite direction to the position vector ***R1***

hence $\boldsymbol{a}$ is always directed towards the origin ***AG***

***[4 marks]***

(c) relative position $\boldsymbol{d} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ ***(M1)***

distance between particles $= |\boldsymbol{d}|$ $(= |\boldsymbol{r}_2 - \boldsymbol{r}_1|)$ ***(M1)***

$|\boldsymbol{d}| = \sqrt{(-4\sin(4t) - 3\cos(3t))^2 + (3\cos(4t) - 4\sin(3t))^2}$ ***(A1)***

minimum value of $|\boldsymbol{d}|$ when $t = 4.71(\text{s})$ $\left(4.71238..., \frac{3\pi}{2}\right)$ ***(M1)A1***

***[5 marks]***

(d) (i) for 2nd particle, $\boldsymbol{v} = -16\cos(4t)\boldsymbol{i} - 12\sin(4t)\boldsymbol{j}$ ***(A1)***

**EITHER**

consider the gradient of either $\boldsymbol{v}$ ***(M1)***

$m_1 = -\frac{12\cos(3t)}{9\sin(3t)}$ and $m_2 = \frac{12\sin(4t)}{16\cos(4t)}$ ***(A1)***

attempt to solve $m_1 = m_2$ ***(M1)***

**OR**

vectors are parallel therefore one is a multiple of the other, $\boldsymbol{v}_2 = l\,\boldsymbol{v}_1$ ***(M1)***

$(l =)$ $\frac{16\cos(4t)}{9\sin(3t)} = -\frac{\sin(4t)}{\cos(3t)}$ ***(A1)***

attempt to solve ***(M1)***

**THEN**

$t = 1.30$ s (1.30135...) ***A1***

*continued…*

*Question 7 continued*

(ii) **EITHER**

at $t = 1.30$, $\boldsymbol{v}_1 = 6.22\boldsymbol{i} - 8.68\boldsymbol{j}$ and $\boldsymbol{v}_2 = -7.57\boldsymbol{i} + 10.6\boldsymbol{j}$ ***A1***

**OR**

$l = -1.22$ (following second method in part (d)(i)) ***A1***

**THEN**

$\boldsymbol{v}_2$ is a negative multiple of $\boldsymbol{v}_1$ $(\boldsymbol{v}_2 = -1.22\boldsymbol{v}_1)$ ***R1***

the two particles are moving in the opposite direction ***AG***

***[7 marks]***

***Total [20 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2022

## Mathematics: applications and interpretation

## Higher level

## Paper 2

21 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.
***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
***R*** Marks awarded for clear **Reasoning**.
***AG*** Answer given in the question and so no marks are awarded.
***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) (i) **EITHER**

$115.5 = u_1 + (3-1)\times d \quad (115.5 = u_1 + 2d)$

$108 = u_1 + (8-1)\times d \quad (108 = u_1 + 7d)$ ***(M1)(A1)***

**Note:** Award ***M1*** for attempting to use the arithmetic sequence term formula, ***A1*** for both equations correct. Working for ***M1*** and ***A1*** can be found in parts (i) or (ii).

$(d = -1.5)$

$1.5$ (cups/day) ***A1***

**Note:** Answer must be written as a positive value to award ***A1***.

**OR**

$(d =) \ \dfrac{115.5-108}{5}$ ***M1A1***

**Note:** Award ***M1*** for attempting a calculation using the difference between term $3$ and term $8$; ***A1*** for a correct substitution.

$(d =) \ 1.5$ (cups/day) ***A1***

(ii) $(u_1 =) \ 118.5$ (cups) ***A1***

***[4 marks]***

(b) attempting to substitute their values into the term formula for arithmetic sequence equated to zero ***(M1)***

$0 = 118.5 + (n-1)\times(-1.5)$

$(n =) \ 80$ days ***A1***

**Note:** Follow through from part (a) only if their answer is positive.

***[2 marks]***

(c) $(t_5 =) \ 625\times 1.064^{(5-1)}$ ***(M1)(A1)***

**Note:** Award ***M1*** for attempting to use the geometric sequence term formula; ***A1*** for a correct substitution.

$\$ 801$ ***A1***

**Note:** The answer must be rounded to a whole number to award the final ***A1***.

***[3 marks]***

*continued…*

*Question 1 continued*

(d) (i) $(S_{10} =)\ (\$)\ 8390\ \ (8394.39...)$ ***A1***

(ii) **EITHER**

the total cost (of dog food) ***R1***
for 10 years beginning in 2021 **OR** 10 years before 2031 ***R1***

**OR**

the total cost (of dog food) ***R1***
from 2021 to 2030 (inclusive) **OR** from 2021 to (the start of) 2031 ***R1***

***[3 marks]***

(e) **EITHER**
According to the model, the cost of dog food per year will eventually be too high to keep a dog.
**OR**
The model does not necessarily consider changes in inflation rate.
**OR**
The model is appropriate as long as inflation increases at a similar rate.
**OR**
The model does not account for changes in the amount of food the dog eats as it ages/becomes ill/stops growing.
**OR**
The model is appropriate since dog food bags can only be bought in discrete quantities. ***R1***

**Note:** Accept reasonable answers commenting on the appropriateness of the model for the specific scenario. There should be a reference to the given context. A reference to the geometric model must be clear: either "model" is mentioned specifically, or other mathematical terms such as "increasing" or "discrete quantities" are seen. Do not accept a contextual argument in isolation, e.g. "The dog will eventually die".

***[1 mark]***
***Total [13 marks]***

**2.** (a) attempt to expand given expression **OR** attempt at product rule ***(M1)***

$$C = \frac{xk^2}{10} - \frac{3x^3}{1000}$$

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{k^2}{10} - \frac{9x^2}{1000}$$ ***M1A1***

**Note:** Award ***M1*** for power rule correctly applied to at least one term and ***A1*** for correct answer.

***[3 marks]***

(b) equating their $\frac{\mathrm{d}C}{\mathrm{d}x}$ to zero ***(M1)***

$$\frac{k^2}{10} - \frac{9x^2}{1000} = 0$$

$$x^2 = \frac{100k^2}{9}$$

$$x = \frac{10k}{3}$$ ***(A1)***

substituting their $x$ back into given expression ***(M1)***

$$C_{\max} = \frac{10k}{30}\left(k^2 - \frac{300k^2}{900}\right)$$

$$C_{\max} = \frac{2k^3}{9}\left(0.222\ldots k^3\right)$$ ***A1***

***[4 marks]***

(c) (i) substituting $20$ into given expression and equating to $426$ ***M1***

$$426 = \frac{20}{10}\left(k^2 - \frac{3}{100}(20)^2\right)$$

$$k = 15$$ ***A1***

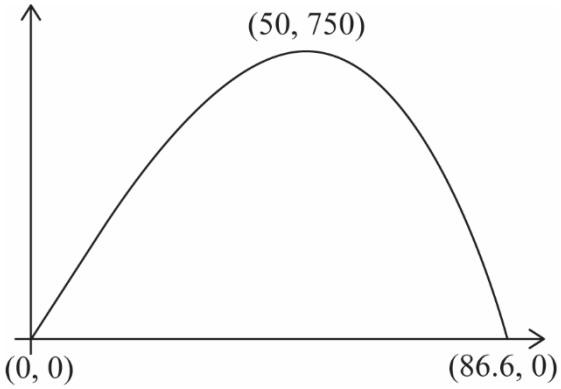
(ii) $50$ ***A1***

***[3 marks]***

*continued…*

*Question 2 continued*

(d)




***A1A1A1***

**Note:** Award ***A1*** for graph indicating an increasing and then decreasing function (drawn in first quadrant), ***A1*** for maximum labelled and ***A1*** for graph drawn for positive $x$, passing through the origin and $86.6$ which is marked on the $x$-axis or its coordinates are given.

***[3 marks]***

(e) setting their expression for $C$ to zero **OR** choosing correct $x$-intercept on their graph of $C$ ***(M1)***

$x_{\max} = 86.6 \quad (86.6025\ldots)$ litres ***A1***

***[2 marks]***

***Total [15 marks]***

**3.** (a) $\left(\frac{2+6}{2}, \frac{2+0}{2}\right)$ ***(M1)***

$(4, 1)$ ***A1***

**Note:** Award ***A0*** if parentheses are omitted in the final answer.

***[2 marks]***

(b) attempt to substitute values into gradient formula ***(M1)***

$\left(\frac{0-2}{6-2} = \right) -\frac{1}{2}$ ***(A1)***

therefore the gradient of perpendicular bisector is $2$ ***(M1)***

so $y-1=2(x-4)$ $(y=2x-7)$ ***A1***

***[4 marks]***

(c) identifying the correct equations to use: ***(M1)***

$y=2-x$ and $y=2x-7$

evidence of solving their correct equations or finding points of intersection graphically ***(M1)***

$(3, -1)$ ***A1***

**Note:** Accept an answer expressed as "$x=3$, $y=-1$".

***[3 marks]***

(d) attempt to use distance formula ***(M1)***

$\mathrm{YZ} = \sqrt{(7-(-1))^2 + (7-3)^2}$

$= \sqrt{80}$ $\left(4\sqrt{5}\right)$ ***A1***

***[2 marks]***

(e) **METHOD 1 (cosine rule)**

length of XZ is $\sqrt{80}$ $\left(4\sqrt{5}, 8.94427...\right)$ ***(A1)***

**Note:** Accept $8.94$ and $8.9$.

attempt to substitute into cosine rule ***(M1)***

$\cos \mathrm{X\hat{Y}Z} = \frac{80+32-80}{2\times\sqrt{80}\sqrt{32}}$ $(=0.316227\ldots)$ ***(A1)***

**Note:** Award ***A1*** for correct substitution of XZ, YZ, $\sqrt{32}$ values in the cos rule. Exact values do not need to be used in the substitution.

$\left(\mathrm{X\hat{Y}Z} = \right) 71.6°$ $(71.5650\ldots°)$ ***A1***

**Note:** Last ***A1*** mark may be lost if prematurely rounded values of XZ, YZ and/or XY are used.

*continued…*

*Question 3 continued*

**METHOD 2 (splitting isosceles triangle in half)**

length of XZ is $\sqrt{80}$ $\left(4\sqrt{5}, 8.94427...\right)$ ***(A1)***

**Note:** Accept $8.94$ and $8.9$.

required angle is $\cos^{-1}\left(\frac{\sqrt{32}}{2\sqrt{80}}\right)$ ***(M1)(A1)***

**Note:** Award ***A1*** for correct substitution of XZ (or YZ), $\frac{\sqrt{32}}{2}$ values in the cos rule. Exact values do not need to be used in the substitution.

$\left(\mathrm{X\hat{Y}Z} =\right) 71.6°$ $(71.5650°)$ ***A1***

**Note:** Last ***A1*** mark may be lost if prematurely rounded values of XZ, YZ and/or XY are used.

***[4 marks]***

(f) (area =) $\frac{1}{2}\sqrt{80}\sqrt{32}\sin 71.5650\ldots$ **OR** (area =) $\frac{1}{2}\sqrt{32}\sqrt{72}$ ***(M1)***

$= 24 \text{ km}^2$ ***A1***

***[2 marks]***

(g) *Any sensible answer such as:*
There might be factors other than proximity which influence shopping choices.
A larger area does not necessarily result in an increase in population.
The supermarkets might be specialized / have a particular clientele who visit even if other shops are closer.
Transport links might not be represented by Euclidean distances.
etc. ***R1***

***[1 mark]***

***Total [18 marks]***

**4.** (a) attempt to use chain rule, including the differentiation of $\frac{1}{T}$ ***(M1)***

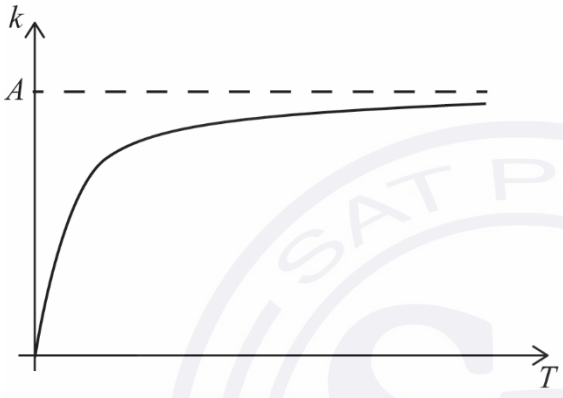
$\frac{\mathrm{d}k}{\mathrm{d}T} = A \times \frac{c}{T^2} \times \mathrm{e}^{-\frac{c}{T}}$ ***A1***

this is the product of positive quantities so must be positive ***R1***

**Note:** The ***R1*** may be awarded for correct argument from **their** derivative. ***R1*** is not possible if their derivative is not always positive.

***[3 marks]***

(b)




***A1A1A1***

**Note:** Award ***A1*** for an increasing graph, entirely in first quadrant, becoming concave down for larger values of $T$, ***A1*** for tending towards the origin and ***A1*** for asymptote labelled at $k = A$.

***[3 marks]***

(c) taking ln of both sides **OR** substituting $y = \ln x$ and $x = \frac{1}{T}$ ***(M1)***

$\ln k = \ln A - \frac{c}{T}$ **OR** $y = -cx + \ln A$ ***(A1)***

(i) so gradient is $-c$ ***A1***

(ii) $y$-intercept is $\ln A$ ***A1***

**Note:** The implied ***(M1)*** and ***(A1)*** can only be awarded if **both** correct answers are seen. Award zero if only one value is correct **and** no working is seen.

***[4 marks]***

*continued…*

*Question 4 continued*

(d) an attempt to convert data to $\frac{1}{T}$ and $\ln k$ ***(M1)***

e.g. at least one correct row in the following table

| $\frac{1}{T}$ | $\ln k$ |
|---|---|
| $1.69491\ldots\times10^{-3}$ | $-7.60090\ldots$ |
| $1.66666\ldots\times10^{-3}$ | $-7.41858\ldots$ |
| $1.63934\ldots\times10^{-3}$ | $-6.90775\ldots$ |
| $1.61290\ldots\times10^{-3}$ | $-6.57128\ldots$ |
| $1.58730\ldots\times10^{-3}$ | $-6.21460\ldots$ |
| $1.5625\times10^{-3}$ | $-5.84304\ldots$ |
| $1.53846\ldots\times10^{-3}$ | $-5.62682\ldots$ |

line is $\ln k = -13400\times\frac{1}{T}+15.0 \quad \left(=-13383.1\ldots\times\frac{1}{T}+15.0107\ldots\right)$ ***A1***

***[2 marks]***

(e) (i) $c = 13400 \quad (13383.1\ldots)$ ***A1***

(ii) attempt to rearrange or solve graphically $\ln A = 15.0107\ldots$ ***(M1)***

$A = 3300000 \quad (3304258\ldots)$ ***A1***

**Note:** Accept an $A$ value of $3269017\ldots$ from use of $3\text{sf}$ value.

***[3 marks]***

***Total [15 marks]***

**5.** (a) (i) $0.02$ ***A1***

(ii) the probability of mutating from 'not normal state' to 'normal state' ***A1***

**Note:** The ***A1*** can only be awarded if it is clear that transformation is **from** the mutated state.

***[2 marks]***

(b) $\det\begin{pmatrix} 0.94-\lambda & 0.02 \\ 0.06 & 0.98-\lambda \end{pmatrix}=0$ ***(M1)***

**Note:** Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M}-\lambda\boldsymbol{I})=0$ has been used is sufficient for the ***(M1)***.

$(0.94-\lambda)(0.98-\lambda)-0.0012=0$ **OR** $\lambda^2-1.92\lambda+0.92=0$ ***(A1)***

$\lambda=1,\ 0.92\ \left(\frac{23}{25}\right)$ ***A1***

***[3 marks]***

(c) $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$ **OR** $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0.92\begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

**Note:** This ***M1*** can be awarded for attempting to find either eigenvector.

$0.02y-0.06x=0$ **OR** $0.02y+0.02x=0$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1A1***

**Note:** Accept any multiple of the given eigenvectors.

***[3 marks]***

(d) (i) $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}^5\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.744 & 0.0852 \\ 0.256 & 0.915 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ***(M1)***

**Note:** Condone omission of the initial state vector for the ***M1***.

$0.744\quad(0.744311\ldots)$ ***A1***

(ii) $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ ***(A1)***

**Note:** Award ***A1*** for $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{pmatrix}$ seen.

$0.25$ ***A1***

***[4 marks]***

***Total [12 marks]***

**6.** (a) (i) $\sqrt{10^2+8^2}$ ***(M1)***

$=12.8\ \ (12.8062\ldots, \sqrt{164})\ (\text{m s}^{-1})$ ***A1***

(ii) $\tan^{-1}\left(\frac{10}{8}\right)$ ***(M1)***

$=0.896$ **OR** $51.3\ \ (0.896055\ldots$ **OR** $51.3401\ldots°)$ ***A1***

**Note:** Accept $0.897$ or $51.4$ from use of $\arcsin\left(\frac{10}{12.8}\right)$.

***[4 marks]***

(b) $y=t(10-5t)$ ***(M1)***

**Note:** The ***M1*** might be implied by a correct graph or use of the correct equation.

**METHOD 1 – graphical Method**

sketch graph ***(M1)***

**Note:** The ***M1*** might be implied by correct graph or correct maximum (eg $t=1$).

max occurs when $y=5\ \text{m}$ ***A1***

**METHOD 2 – calculus**

differentiating and equating to zero ***(M1)***

$\frac{\text{d}y}{\text{d}t}=10-10t=0$

$t=1$

$y\ (=1(10-5))=5\ \text{m}$ ***A1***

**METHOD 3 – symmetry**

line of symmetry is $t=1$ ***(M1)***

$y\ (=1(10-5))=5\ \text{m}$ ***A1***

***[3 marks]***

(c) attempt to solve $t(10-5t)=0$ ***(M1)***

$t=2$ (or $t=0$) ***(A1)***

$x\ (=5+8\times 2)=21\ \text{m}$ ***A1***

**Note:** Do not award the final ***A1*** if $x=5$ is also seen.

***[3 marks]***

*continued…*

*Question 6 continued*

(d) **METHOD 1**

$t = \frac{x-5}{8}$ ***M1A1***

$y = \left(\frac{x-5}{8}\right)\left(10 - 5 \times \frac{x-5}{8}\right)$ ***A1***

**METHOD 2**

$y = k(x-5)(x-21)$ ***A1***

when $x = 13,\ y = 5$ so $k = \frac{5}{(13-5)(13-21)} = -\frac{5}{64}$ ***M1A1***

$\left(y = -\frac{5}{64}(x-5)(x-21)\right)$

**METHOD 3**

if $y = ax^2 + bx + c$

$0 = 25a + 5b + c$

$5 = 169a + 13b + c$

$0 = 441a + 21b + c$ ***M1A1***

solving simultaneously, $a = -\frac{5}{64},\ b = \frac{130}{64},\ c = -\frac{525}{64}$ ***A1***

$(y = -\frac{5}{64}x^2 + \frac{130}{64}x - \frac{525}{64})$

**METHOD 4**

use quadratic regression on $(5, 0),\ (13, 5), (21, 0)$ ***M1A1***

$y = -\frac{5}{64}x^2 + \frac{130}{64}x - \frac{525}{64}$ ***A1***

**Note:** Question asks for expression; condone omission of "$y =$ ".

***[3 marks]***

(e) trajectory of arrow is $y = x\tan 10 + 2$ ***(A1)***

intersecting $y = x\tan 10 + 2$ and their answer to (d) ***(M1)***

$(8.66, 3.53)$ $((8.65705..., 3.52647\ldots))$ ***A1***

$(15.1,\ 4.66)$ $((15.0859\ldots, 4.66006\ldots))$ ***A1***

***[4 marks]***

*continued…*

*Question 6 continued*

(f) when $x_{\text{target}} = 8.65705\ldots$, $t_{\text{target}} = \dfrac{8.65705\ldots - 5}{8} = 0.457132\ldots$ s ***(A1)***

attempt to find the distance from point of release to intersection ***(M1)***

$\sqrt{8.65705\ldots^2 + (3.52647\ldots - 2)^2}$ $(= 8.79060\ldots \text{ m})$

time for arrow to get there is $\dfrac{8.79060\ldots}{60} = 0.146510\ldots \text{s}$ ***(A1)***

so the arrow should be released when

$t = 0.311$ (s) $(0.310622\ldots$ (s)) ***A1***

***[4 marks]***

***Total [21 marks]***

**7.** (a) differentiating first equation. ***M1***

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}y}{\mathrm{d}t}$$

substituting in for $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ ***M1***

$$= -2x - 3y = -2x - 3\frac{\mathrm{d}x}{\mathrm{d}t}$$

therefore $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\dfrac{\mathrm{d}x}{\mathrm{d}t} + 2x = 0$ ***AG***

**Note:** The ***AG*** line must be seen to award the final ***M1*** mark.

***[2 marks]***

(b) the relevant matrix is $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ ***(M1)***

**Note:** $\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$ is also possible.

(this has characteristic equation) $-\lambda(-3 - \lambda) + 2 = 0$ ***(A1)***

$\lambda = -1, -2$ ***A1***

***[3 marks]***

*continued…*

*Question 7 continued*

(c) **EITHER**

the general solution is $x = Ae^{-t} + Be^{-2t}$ ***M1***

**Note:** Must have constants, but condone sign error for the ***M1***.

so $\frac{dx}{dt} = -Ae^{-t} - 2Be^{-2t}$ ***M1A1***

**OR**

attempt to find eigenvectors ***(M1)***

respective eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any multiple)

$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ***(M1)A1***

**THEN**

the initial conditions become:

$0 = A + B$

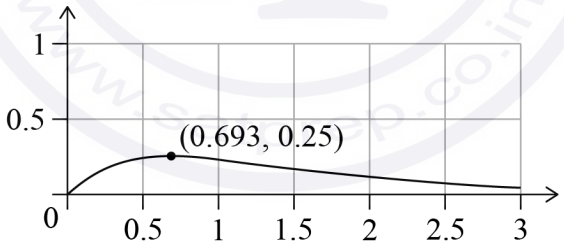
$1 = -A - 2B$ ***M1***

this is solved by $A = 1,\ B = -1$

so the solution is $x = e^{-t} - e^{-2t}$ ***A1***

***[5 marks]***

(d)




***A1A1***

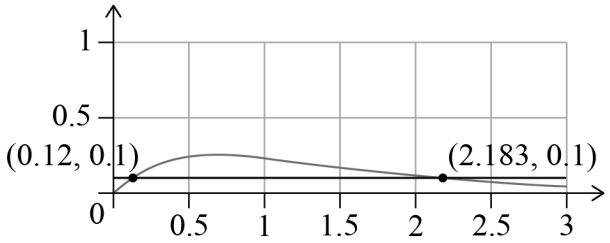
**Note:** Award ***A1*** for correct shape (needs to go through origin, have asymptote at $y = 0$ and a single maximum; condone $x < 0$ ). Award ***A1*** for correct coordinates of maximum.

***[2 marks]***

*continued…*

*Question 7 continued*

(e) intersecting graph with $y = 0.1$ ***(M1)***




so the time fishing is stopped between $2.1830\ldots$ and $0.11957\ldots$ ***(A1)***
$= 2.06(343\ldots)$ days ***A1***

***[3 marks]***

(f) *Any reasonable answer. For example:*
There are greater downsides to allowing fishing when the levels may be dangerous than preventing fishing when the levels are safe.
The concentration of mercury may not be uniform across the river due to natural variation / randomness.
The situation at the power plant might get worse.
Mercury levels are low in water but still may be high in fish. ***R1***

**Note:** Award ***R1*** for a reasonable answer that refers to this specific context (and not a generic response that could apply to *any* model).

***[1 mark]***
***Total [16 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2022

## Mathematics: applications and interpretation

## Higher level

## Paper 2

20 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.
***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
***R*** Marks awarded for clear **Reasoning**.
***AG*** Answer given in the question and so no marks are awarded.
***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) (i) $\frac{370+472}{2}$ ***(M1)***

**Note:** This ***(M1)*** can also be awarded for either a correct $Q_3$ or a correct $Q_1$ in part (a)(ii).

$Q_3 = 421$ ***A1***

(ii) their part (a)(i) – their $Q_1$ (clearly stated) ***(M1)***

$\text{IQR} = (421 - 318 =)\ 103$ ***A1***

***[4 marks]***

(b) $(Q_3 + 1.5(\text{IQR}) =)\ 421 + (1.5 \times 103)$ ***(M1)***

$= 575.5$

since $498 < 575.5$ ***R1***

Netherlands is not an outlier ***A1***

**Note:** The ***R1*** is dependent on the ***(M1)***. Do not award ***R0A1***.

***[3 marks]***

(c) not appropriate ("no" is sufficient) ***A1***

as $r$ is too close to zero / too weak a correlation ***R1***

***[2 marks]***

(d) (i) $6$ ***A1***

(ii) $4.5$ ***A1***

(iii) $4.5$ ***A1***

***[3 marks]***

(e) (i) $r_s = 0.683\ \ (0.682646\ldots)$ ***A2***

(ii) **EITHER**

there is a (positive) association between the population size and the score ***A1***

**OR**

there is a (positive) linear correlation between the ranks of the population size and the ranks of the scores (when compared with the PMCC of $0.249$) ***A1***

***[3 marks]***

(f) lowering the top score by $20$ does not change its rank so $r_s$ is unchanged ***R1***

**Note:** Accept "this would not alter the rank" or "Netherlands still top rank" or similar. Condone any statement that clearly implies the ranks have not changed, for example: "The Netherlands still has the highest score."

***[1 mark]***

***[Total 16 marks]***

**2.** (a) (i) $\left(\frac{1}{2}\text{A}\hat{\text{O}}\text{B} =\right) \arccos\left(\frac{4}{4.5}\right) = 27.266...$ ***(M1)(A1)***

$\text{A}\hat{\text{O}}\text{B} = 54.532... \approx 54.5^\circ \ (0.951764... \approx 0.952 \text{ radians})$ ***A1***

**Note:** Other methods may be seen; award ***(M1)(A1)*** for use of a correct trigonometric method to find an appropriate angle and then ***A1*** for the correct answer.

(ii) a finding area of triangle

**EITHER**

area of triangle = $\frac{1}{2} \times 4.5^2 \times \sin(54.532...)$ ***(M1)***

**Note:** Award ***M1*** for correct substitution into formula.

$= 8.24621... \approx 8.25 \text{ m}^2$ ***(A1)***

**OR**

$\text{AB} = 2 \times \sqrt{4.5^2 - 4^2} = 4.1231\ldots$ ***(M1)***

area triangle $= \frac{4.1231\ldots \times 4}{2}$

$= 8.24621... \approx 8.25 \ (\text{m}^2)$ ***(A1)***

finding area of sector

**EITHER**

area of sector = $\frac{54.532...}{360} \times \pi \times 4.5^2$ ***(M1)***

$= 9.63661... \approx 9.64 \text{ m}^2$ ***(A1)***

**OR**

area of sector = $\frac{1}{2} \times 0.9517641.. \times 4.5^2$ ***(M1)***

$= 9.63661... \approx 9.64 \text{ m}^2$ ***(A1)***

**THEN**

area of segment = $9.63661... - 8.24621...$

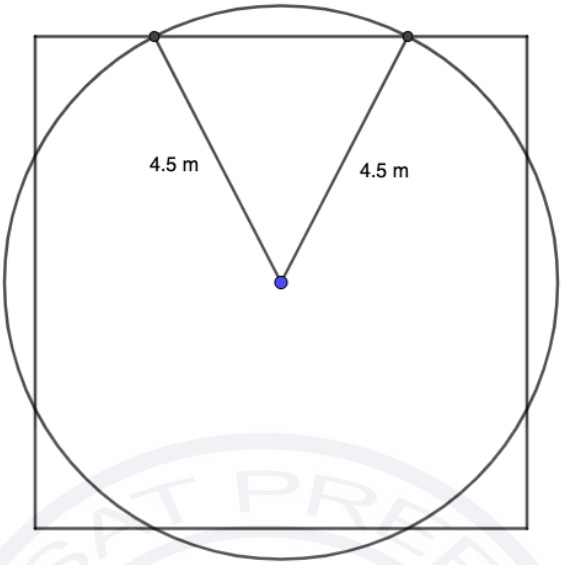
$= 1.39 \text{ m}^2 \ (1.39040...)$ ***A1***

***[8 marks]***

*continued…*

*Question 2 continued*

(b) **METHOD 1**




$\pi \times 4.5^2$ $(63.6172\ldots)$ ***(A1)***

$4 \times 1.39040...$ $(5.56160)$ ***(A1)***

subtraction of four segments from area of circle ***(M1)***

$= 58.1 \text{ m}^2$ $(58.055...)$ ***A1***

**METHOD 2**

angle of sector = $90 - 54.532... \left(\frac{\pi}{2} - 0.951764...\right)$ ***(A1)***

area of sector = $\frac{90 - 54.532...}{360} \times \pi \times 4.5^2$ $(= 6.26771...)$ ***(A1)***

area is made up of four triangles and four sectors ***(M1)***

total area = $(4 \times 8.2462...) + (4 \times 6.26771...)$

$= 58.1 \text{ m}^2$ $\left(58.055...\right)$ ***A1***

***[4 marks]***

(c) sketch of $\frac{\mathrm{d}V}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.110363...$ **OR** attempt to find where $\frac{\mathrm{d}^2V}{\mathrm{d}t^2} = 0$ ***(M1)***

$t = 1$ hour ***A1***

***[2 marks]***

(d) recognizing $V = \int \frac{\mathrm{d}V}{\mathrm{d}t}\,\mathrm{d}t$ ***(M1)***

$\int_0^8 0.3te^{-t}\mathrm{d}t$ ***(A1)***

volume eaten is $0.299... \text{ m}^3$ $(0.299094...)$ ***A1***

***[3 marks]***

***[Total 17 marks]***

**3.** (a) quota ***A1***

***[1 mark]***

(b) (i) $27.125 \approx 27.1$ ***(M1)A1***

(ii) $8.29815\ldots \approx 8.30$ ***A1***

***[3 marks]***

(c) (let $\mu$ be the national mean)

$H_0$: $\mu = 25.2$
$H_1$: $\mu > 25.2$ ***A1***

**Note:** Accept hypotheses in words if they are clearly expressed and '**population** mean' or 'school mean' is referred to. Do not accept $H_0$: $\mu = \mu_0$ unless $\mu_0$ is explicitly defined as "national standard mark" or given as $25.2$.

recognizing $t$-test ***(M1)***
$p$-value $= 0.279391\ldots$ ***A1***

$0.279391\ldots > 0.05$ ***R1***

**Note:** The ***R1*** mark is for the comparison of their $p$-value with $0.05$.

insufficient evidence to reject the null hypothesis (that the mean for the school is $25.2$) ***A1***

**Note:** Award the final ***A1*** only if the **null** hypothesis is also correct (e.g. $\mu_0 = 25.2$ or (population) mean $= 25.2$) and the conclusion is consistent with both the direction of the inequality and the alternative hypothesis.

***[5 marks]***

(d) **EITHER**
the sampling process is not random ***R1***

*For example:*
the school asked for volunteers
the students were selected from a single class

**OR**
the quota might not be representative of the student population ***R1***

*For example:*
the school may have only $4$ boys and $400$ girls.

**Note:** Do not accept 'the sample is too small'.

***[1 mark]***

(e) (i) $(28.1 \times 2 + 20 =)\ 76.2$ ***A1***

(ii) $8.4 \times 2$ ***(A1)***
$= 16.8$ ***A1***

***[3 marks]***
***[Total 13 marks]***

**4.** (a) (i) $y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} + 5\frac{dx}{dt} + 6x = 0$ **OR** $\frac{dy}{dt} + 5y + 6x = 0$ ***M1***

**Note:** Award ***M1*** for substituting $\frac{dy}{dt}$ for $\frac{d^2x}{dt^2}$.

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ ***AG***

(ii) $\det \begin{pmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{pmatrix} = 0$ ***(M1)***

**Note:** Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M} - \lambda \boldsymbol{I}) = 0$ has been used is sufficient for the ***(M1)***.

$-\lambda(-5-\lambda) + 6 = 0$ **OR** $\lambda^2 + 5\lambda + 6 = 0$ ***(A1)***

$\lambda = -2, -3$ ***A1***

(iii) (on a phase portrait the particle approaches $(0, 0)$ as $t$ increases so long term velocity ($y$) is)

$0$ ***A1***

**Note:** Only award ***A1*** for $0$ if both eigenvalues in part (a)(ii) are negative. If at least one is positive accept an answer of '*no limit*' or '*infinity*', or in the case of one positive and one negative also accept '*no limit or 0 (depending on initial conditions)*'.

***[5 marks]***

(b) (i) $y = \frac{dx}{dt}$

$\frac{d^2x}{dt^2} = \frac{dy}{dt}$ ***(A1)***

$\frac{dy}{dt} + 5y + 6x = 3t + 4$ ***A1***

(ii) recognition that $h = 0.1$ in any recurrence formula ***(M1)***

$(t_{n+1} = t_n + 0.1)$

$x_{n+1} = x_n + 0.1y_n$ ***(A1)***

$y_{n+1} = y_n + 0.1(3t_n + 4 - 5y_n - 6x_n)$ ***(A1)***

(when $t = 1$,) $x = 0.64402... \approx 0.644$ m ***A2***

(iii) recognizing that $y$ is the velocity

$0.5 \text{ m s}^{-1}$ ***A1***

***[8 marks]***

***[Total 13 marks]***

**5.** (a) (let $T$ be the number of passengers who arrive)

$\left(\mathrm{P}(T > 72) =\right)\ \mathrm{P}(T \geq 73)$ **OR** $1 - \mathrm{P}(T \leq 72)$ ***(A1)***

$T \sim \mathrm{B}(74,\ 0.9)$ **OR** $n = 74$ ***(M1)***

$= 0.00379\ \ (0.00379124...)$ ***A1***

**Note:** Using the distribution $\mathrm{B}(74,\ 0.1)$, to work with the $10\%$ that do not arrive for the flight, here and throughout this question, is a valid approach.

***[3 marks]***

(b) (i) $72 \times 0.9$ ***(M1)***

$64.8$ ***A1***

(ii) $n \times 0.9 = 72$ ***(M1)***

$80$ ***A1***

***[4 marks]***

(c) **METHOD 1**

**EITHER**

when selling $74$ tickets

| | $T \leq 72$ | $T = 73$ | $T = 74$ |
|---|---|---|---|
| Income minus compensation ($I$) | 11100 | 10800 | 10500 |
| Probability | 0.9962… | 0.003380… | 0.0004110… |

top row ***A1A1***

bottom row ***A1A1***

**Note:** Award ***A1A1*** for **each** row correct. Award ***A1*** for one correct entry and ***A1*** for the remaining entries correct.

$\mathrm{E}(I) = 11100 \times 0.9962... + 10800 \times 0.00338... + 10500 \times 0.000411 \approx 11099$ ***(M1)A1***

*continued…*

*Question 5 continued*

**OR**

income is $74 \times 150 = 11100$ ***(A1)***

expected compensation is

$0.003380... \times 300 + 0.0004110... \times 600 \ (= 1.26070...)$ ***(M1)A1A1***

**Note:** The ***(M1)*** is for an attempt to work out expected compensation by multiplying a probability for tickets sold by either $300$ or $600$.

expected income when selling $74$ tickets is $11100 - 1.26070\ldots$ ***(M1)***

**Note:** Award ***(M1)*** for subtracting their expected compensation from $11100$.

$= 11098.73..\ (= \$11099)$ ***A1***

**THEN**

income for $72$ tickets $= 72 \times 150 = 10800$ ***(A1)***

so expected gain $\approx 11099 - 10800 = \$299$ ***A1***

**METHOD 2**

for $74$ tickets sold, let $C$ be the compensation paid out

$P(T = 73) = 0.00338014\ldots,\ P(T = 74) = 0.000411098\ldots$ ***A1A1***

$E(C) = 0.003380... \times 300 + 0.0004110... \times 600 \ (= 1.26070...)$ ***(M1)A1A1***

extra expected revenue $= 300 - 1.01404... - 0.246658\ldots \ (300 - 1.26070\ldots)$

***(A1)(M1)***

**Note:** Award ***A1*** for the $300$ and ***M1*** for the subtraction.

$= \$299$ (to the nearest dollar) ***A1***

**METHOD 3**

let $D$ be the change in income when selling $74$ tickets.

| | $T \leq 72$ | $T = 73$ | $T = 74$ |
|---|---|---|---|
| Change in income | 300 | 0 | −300 |

***(A1)(A1)***

**Note:** Award ***A1*** for one error, however award ***A1A1*** if there is no explicit mention that $T = 73$ would result in $D = 0$ and the other two are correct.

$P(T \leq 73) = 0.9962...,\ P(T = 74) = 0.000411098\ldots$ ***A1A1***

$E(D) = 300 \times 0.9962... + 0 \times 0.003380... - 300 \times 0.0004110$ ***(M1)A1A1***

$= \$299$ ***A1***

***[8 marks]***

***[Total 15 marks]***

**6.** (a) (i) $y = x^{\frac{1}{2}}$ ***(M1)***

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ ***A1***

(ii) gradient at $x = 0.16$ is $\frac{1}{2} \times \frac{1}{\sqrt{0.16}}$ ***M1***

$= 1.25$

**EITHER**

$y - 0.4 = 1.25(x - 0.16)$ ***M1***

**OR**

$0.4 = 1.25(0.16) + b$ ***M1***

**Note:** Do not allow working backwards from the given answer.

**THEN**

hence $y = 1.25x + 0.2$ ***AG***

***[4 marks]***

(b) $p = 0.45,\ q = 0.4125$ (or $0.413$) (accept "$(0.45,\ 0.4125)$") ***A1A1***

***[2 marks]***

(c) (i) $(h(x) =)\ \frac{1}{2}\sqrt{2(x - 0.2)}$ ***A2***

**Note:** Award ***A1*** if only two correct transformations are seen.

(ii) $(a =)\ 0.28$ ***A1***

(iii) **EITHER**

Correct substitution of their part (b) (or $(0.28,\ 0.2)$) into the given expression ***(M1)***

**OR**

$\frac{1}{2}(1.25 \times 2(x - 0.2) + 0.2)$ ***(M1)***

**Note:** Award ***M1*** for transforming the equivalent expression for $f$ correctly.

**THEN**

$(b =)\ -0.15$ ***A1***

***[5 marks]***

*continued…*

*Question 6 continued*

(d) (i) recognizing need to add two integrals ***(M1)***

$\int_0^{0.16} \sqrt{x}\ \mathrm{d}x + \int_{0.16}^{0.5} (1.25x + 0.2)\ \mathrm{d}x$ ***(A1)***

**Note:** The second integral could be replaced by the formula for the area of a trapezoid $\frac{1}{2} \times 0.34(0.4 + 0.825)$.

$0.251\ \mathrm{m}^2\ (0.250916\ldots)$ ***A1***

(ii) **EITHER**

area of trapezoid $\frac{1}{2} \times 0.05(0.4125 + 0.825) = 0.0309375$ ***(M1)(A1)***

**OR**

$\int_{0.45}^{0.5} (8.25x - 3.3)\,\mathrm{d}x = 0.0309375$ ***(M1)(A1)***

**Note:** If the rounded answer of $0.413$ from part (b) is used, the integral is $\int_{0.45}^{0.5} (8.24x - 3.295)\,\mathrm{d}x = 0.03095$ which would be awarded ***(M1)(A1)***.

**THEN**

shaded area $= 0.250916\ldots - 0.0627292 - 0.0309375$ ***(M1)***

**Note:** Award ***(M1)*** for the subtraction of both $0.0627292\ldots$ and their area for the trapezoid from their answer to (a)(i).

$= 0.157\ \mathrm{m}^2\ (0.15725)$ ***A1***

***[7 marks]***

***[Total 18 marks]***

**7.** (a) (i) $\boldsymbol{P}\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{q}=\begin{pmatrix}0\\1\end{pmatrix}$ ***(M1)***

$\boldsymbol{q}=\begin{pmatrix}0\\1\end{pmatrix}$ ***A1***

(ii) **EITHER**

$\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\\frac{3}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\-\frac{1}{4}\end{pmatrix}$ ***A1***

$\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\1+\frac{\sqrt{3}}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{\sqrt{3}}{4}\end{pmatrix}$ ***A1***

*continued…*

*Question 7 continued*

**OR**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{4} \\ \frac{3}{4} \end{pmatrix}$$ ***M1***

hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix}$ ***A1***

$$\begin{pmatrix} a \\ c \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{1}{4} \\ 1+\frac{\sqrt{3}}{4} \end{pmatrix}$$ ***M1***

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} b \\ d \end{pmatrix}=\begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$

**THEN**

$$\Rightarrow \boldsymbol{P}=\begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***AG***

***[6 marks]***

(b) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ ***A1***

***[1 mark]***

*continued…*

*Question 7 continued*

(c) (i) **EITHER**

$$\boldsymbol{S}^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{(A1)}$$

$$\boldsymbol{R} = \boldsymbol{P}\boldsymbol{S}^{-1} \quad \textit{(M1)}$$

**Note:** The ***M1*** is for an attempt at rearranging the matrix equation. Award even if the order of the product is reversed.

$$\boldsymbol{R} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{(A1)}$$

**OR**

$$\begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

let $\boldsymbol{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

attempt to solve a system of equations ***M1***

$$\frac{\sqrt{3}}{4} = 0.5a, \quad \frac{1}{4} = 0.5b$$

$$-\frac{1}{4} = 0.5c, \quad \frac{\sqrt{3}}{4} = 0.5d \quad \textit{A2}$$

**Note:** Award ***A1*** for two correct equations, ***A2*** for all four equations correct.

**THEN**

$$\boldsymbol{R} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \textbf{ OR } \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix} \textbf{ OR } \left( \begin{pmatrix} 0.866025... & 0.5 \\ -0.5 & 0.866025... \end{pmatrix} \right) \quad \textit{A1}$$

**Note:** The correct answer can be obtained from reversing the matrices, so do not award if incorrect product seen. If the given answer is obtained from the product $\boldsymbol{R} = \boldsymbol{S}^{-1}\boldsymbol{P}$, award ***(A1)(M1)(A0)A0***.

*continued…*

*Question 7 continued*

(ii) clockwise ***A1***
arccosine or arcsine of value in matrix seen ***(M1)***
$30^\circ$ ***A1***

**Note:** Both ***A1*** marks are dependent on the answer to part (c)(i) and should only be awarded for a valid rotation matrix.

***[7 marks]***

(d) **METHOD 1**

(i) $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ ***A1***

(ii) solving $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ using simultaneous equations or $\boldsymbol{a} = (\boldsymbol{I} - \boldsymbol{P})^{-1}\boldsymbol{q}$

***(M1)***

$a = 0.651\ (0.651084...),\ b = 1.48\ \ (1.47662...)$ ***A1A1***

$\left( a = \dfrac{5 + 2\sqrt{3}}{13},\ b = \dfrac{14 + 3\sqrt{3}}{13} \right)$

**METHOD 2**

(i) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} x - a \\ y - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ ***A1***

**Note:** Accept substitution of $x$ and $y$ (and $x'$ and $y'$) with particular points given in the question.

(ii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} 0 - a \\ 0 - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ ***(M1)***

**Note:** This line, with any of the points substituted, may be seen in part (d)(i) and if so the ***M1*** can be awarded there.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{P})\begin{pmatrix} a \\ b \end{pmatrix}$

$a = 0.651084...,\ b = 1.47662...$ ***A1A1***

$\left( a = \dfrac{5 + 2\sqrt{3}}{13},\ b = \dfrac{14 + 3\sqrt{3}}{13} \right)$

***[4 marks]***
***[Total 18 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2021

## Mathematics: applications and interpretation

## Higher level

## Paper 2

19 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.
***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
***R*** Marks awarded for clear **Reasoning**.
***AG*** Answer given in the question and so no marks are awarded.
***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

**1 General**
Award marks using the annotations as noted in the markscheme *eg **M1**, **A2**.*

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$.

An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x} \times \mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x} \times \mathrm{e}^{3x} - \mathrm{e}^{4x} \times \mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

**OR**

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) $\tan(\theta) = \frac{6}{10}$ ***(M1)***

$(\theta =)\ 31.0^\circ\ (30.9637...^\circ)$ **OR** $0.540\ (0.540419\ldots)$ ***A1***

***[2 marks]***

(b) (i) $(\text{CV} =)\ 40\tan(\theta)$ **OR** $(\text{CV} =)\ 4\times 6$ ***(M1)***

**Note:** Award ***(M1)*** for an attempt at trigonometry or similar triangles (e.g. ratios).

$(\text{CV} =)\ 24\ \text{m}$ ***A1***

(ii) $(V =)\ \frac{1}{3}80^2\times 24 - \frac{1}{3}60^2\times 18$ ***M1A1A1***

**Note:** Award ***M1*** for finding the difference between the volumes of two pyramids, ***A1*** for each correct volume expression. The final ***A1*** is contingent on correct working leading to the given answer.
If the correct final answer is not seen, award at most ***M1A1A0***. Award ***M0A0A0*** for any height derived from $V = 29\,600$, including $18.875$ or $13.875$.

$(V =)\ 29\,600\ \text{m}^3$ ***AG***

***[5 marks]***

(c) **METHOD 1**

$\left(\frac{29\,600}{80} =\right)\ 370$ (days) ***A1***

$(370 > 366)$ Joshua is correct ***A1***

**Note:** Award ***A0A0*** for unsupported answer of "Joshua is correct". Accept $1.01... > 1$ for the first ***A1*** mark.

**METHOD 2**

$80\times 366 = 29280\ \text{m}^3$ **OR** $80\times 365 = 29200\ \text{m}^3$ ***A1***

$(29280 < 29600)$ Joshua is correct ***A1***

**Note:** The second ***A1*** can be awarded for an answer consistent with their result.

***[2 marks]***

*continued…*

*Question 1 continued*

(d) height of trapezium is $\sqrt{10^2+6^2}$ $(=11.6619...)$ ***(M1)***

area of trapezium is $\frac{80+60}{2}\times\sqrt{10^2+6^2}$ $(=816.333...)$ ***(M1)(A1)***

$(SA=)$ $4\times\left(\frac{80+60}{2}\times\sqrt{10^2+6^2}\right)+60^2$ ***(M1)***

**Note:** Award ***M1*** for adding 4 times their (MNOP) trapezium area to the area of the $(60\times60)$ base.

$(SA=)$ $6870\ \text{m}^2$ $(6865.33\ \text{m}^2)$ ***A1***

**Note:** No marks are awarded if the correct shape is not identified.

***[5 marks]***
***Total: [14 marks]***

**2.** (a) (i) maximum $h = 130$ metres ***A1***

(ii) minimum $h = 50$ metres ***A1***

***[2 marks]***

(b) (i) $(60 \div 12 =)\ 5$ seconds ***A1***

(ii) $360 \div 5$ ***(M1)***

**Note:** Award ***(M1)*** for $360$ divided by their time for one revolution.

$= 72^\circ$ ***A1***

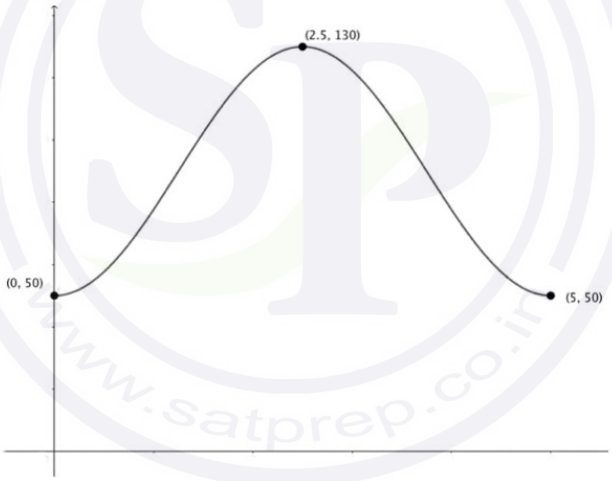
***[3 marks]***

(c) (i) (amplitude $=)\ 40$ ***A1***

(ii) (period $= \frac{360}{72} =)\ 5$ ***A1***

***[2 marks]***

(d)




Maximum point labelled with correct coordinates. ***A1***
At least one minimum point labelled. Coordinates seen for any minimum points must be correct. ***A1***
Correct shape with an attempt at symmetry and “concave up" evident as it approaches the minimum points. Graph must be drawn in the given domain. ***A1***

***[3 marks]***

*continued…*

*Question 2 continued*

(e) (i) $h = 90 - 40\cos(144^\circ)$ ***(M1)***

$(h =)\ 122\ (\text{m})\ (122.3606.....)$ ***A1***

(ii) evidence of $h = 100$ on graph **OR** $100 = 90 - 40\cos(72t)$ ***(M1)***

$t$ coordinates $3.55\ (3.54892...)$ **OR** $1.45\ (1.45107...)$ or equivalent ***(A1)***

**Note:** Award ***A1*** for either $t$-coordinate seen.

$= 2.10$ seconds $(2.09784\ldots)$ ***A1***

***[5 marks]***

(f) **METHOD 1**

$90 - 40\cos(at^\circ) = 110$ ***(M1)***

$\cos(at^\circ) = -0.5$

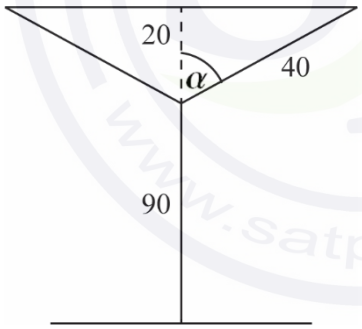
$at^\circ = 120\ ,\ 240$ ***(A1)***

$1 = \frac{240}{a} - \frac{120}{a}$ ***(M1)***

$a = 120$ ***(A1)***

period $= \frac{360}{120} = 3$ seconds ***A1***

**METHOD 2**




attempt at diagram ***(M1)***

$\cos\alpha = \frac{20}{40}$ (or recognizing special triangle) ***(M1)***

angle made by C, $2\alpha = 120^\circ$ ***(A1)***

one third of a revolution in $1$ second ***(M1)***

hence one revolution = $3$ seconds ***A1***

*continued…*

*Question 2 continued*

**METHOD 3**

considering $h(t) = 110$ on original function ***(M1)***

$t = \frac{5}{3}$ or $\frac{10}{3}$ ***(A1)***

$\frac{10}{3} - \frac{5}{3} = \frac{5}{3}$ ***(A1)***

**Note:** Accept $t = 1.67$ or equivalent.

so period is $\frac{3}{5}$ of original period ***(R1)***

so new period is $3$ seconds ***A1***

***[5 marks]***

***Total: [20 marks]***

**3.** (a) (i) Let $X$ be the random variable "distance from O".

$X \sim \mathrm{N}(10, 3^2)$

$\mathrm{P}(X < 13) = 0.841 \ (0.841344...)$ ***(M1)A1***

(ii) $\left(\mathrm{P}(X > 15) =\right) \ 0.0478 \ (0.0477903)$ ***A1***

***[3 marks]***

(b) $\mathrm{P}(X > 15) \times \mathrm{P}(X > 15)$ ***(M1)***

$= 0.00228 \ (0.00228391...)$ ***A1***

***[2 marks]***

(c) $1 - (0.8143)^3$ ***(M1)***

$= 0.460 \ (0.460050...)$ ***A1***

***[2 marks]***

(d) (i) let $Y$ be the random variable "number of points scored"

evidence of use of binomial distribution ***(M1)***

$Y \sim \mathrm{B}(10, 0.539949...)$ ***(A1)***

$\left(\mathrm{E}(Y) =\right) 10 \times 0.539949...$ ***(M1)***

$= 5.40$ ***A1***

(ii) $\left(\mathrm{P}(Y \geq 5) =\right) \ 0.717 \ (0.716650...)$ ***A1***

(iii) $\mathrm{P}(5 \leq Y < 8)$ ***(M1)***

$= 0.628 \ (0.627788...)$ ***A1***

**Note:** Award ***M1*** for a correct probability statement or indication of correct lower and upper bounds, 5 and 7.

(iv) $\dfrac{\mathrm{P}(5 \leq Y < 8)}{\mathrm{P}(Y \geq 5)} \left( = \dfrac{0.627788...}{0.716650...} \right)$ ***(M1)***

$= 0.876 \ (0.876003...)$ ***A1***

***[9 marks]***

***Total: [16 marks]***

**4.**

**Note:** For clarity, exact answers are used throughout this markscheme. However it is perfectly acceptable for candidates to write decimal values $\left(\text{e.g. } \frac{\sqrt{3}}{2} = 0.866\right)$.

(a) (i) rotation anticlockwise $\frac{\pi}{6}$ is $\begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ ***(M1)A1***

reflection in $y = \frac{x}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ ***(M1)***

$\Rightarrow 2\theta = \frac{\pi}{3}$ ***(A1)***

matrix is $\begin{pmatrix} 0.5 & 0.866 \\ 0.866 & -0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ ***A1***

rotation clockwise $\frac{\pi}{3}$ is $\begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ ***A1***

(ii) an attempt to multiply three matrices ***(M1)***

$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ ***(A1)***

$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ **OR** $\begin{pmatrix} 0.866 & -0.5 \\ -0.5 & -0.866 \end{pmatrix}$ ***A1***

*continued…*

*Question 4 continued*

(iii) $\left(\boldsymbol{P}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

**Note:** Do not award ***A1*** if final answer not resolved into the identity matrix $I$.

(iv) if the overall movement of the drone is repeated ***A1***
the drone would return to its original position ***A1***

***[12 marks]***

(b) **METHOD 1**

$\left|\det \boldsymbol{P}\right| = \left|\left(-\frac{3}{4}\right) - \left(\frac{1}{4}\right)\right| = 1$ ***A1***

area of triangle $\mathrm{ABC}$ = area of triangle $\mathrm{A'B'C'} \times \left|\det \boldsymbol{P}\right|$ ***R1***

area of triangle $\mathrm{ABC}$ = area of triangle $\mathrm{A'B'C'}$ ***AG***

**Note:** Award at most ***A1R0*** for responses that omit modulus sign.

**METHOD 2**

statement of fact that rotation leaves area unchanged ***R1***
statement of fact that reflection leaves area unchanged ***R1***
area of triangle $\mathrm{ABC}$ = area of triangle $\mathrm{A'B'C'}$ ***AG***

***[2 marks]***

(c) attempt to find angles associated with values of elements in matrix $\boldsymbol{P}$ ***(M1)***

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & \sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & -\cos\left(-\frac{\pi}{6}\right) \end{pmatrix}$$

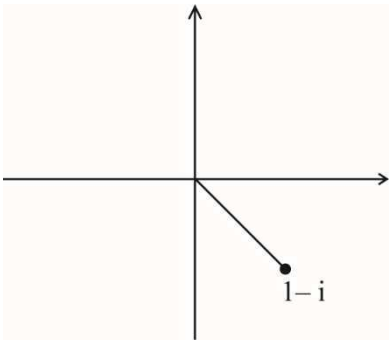
reflection (in $y = (\tan\theta)x$) ***(M1)***

where $2\theta = -\frac{\pi}{6}$ ***A1***

reflection in $y = \tan\left(-\frac{\pi}{12}\right)x \ \left(= -0.268x\right)$ ***A1***

***[4 marks]***
***Total: [18 marks]***

**5.** (a) (i)




***A1***

(ii) $z = \sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$ ***A1A1***

| **Note:** Accept an argument of $\frac{7\pi}{4}$. Do **NOT** accept answers that are not exact. |
|---|

***[3 marks]***

(b) (i) $w_1 + w_2 = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{\mathrm{i}\left(x-\frac{\pi}{2}\right)}$

$= \mathrm{e}^{\mathrm{i}x}\left(1 + \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}\right)$ ***(M1)***

$= \mathrm{e}^{\mathrm{i}x}(1 - \mathrm{i})$ ***A1***

(ii) $w_1 + w_2 = \mathrm{e}^{\mathrm{i}x} \times \sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$ ***M1***

$= \sqrt{2}\mathrm{e}^{\mathrm{i}\left(x-\frac{\pi}{4}\right)}$ ***(A1)***

attempt extract real part using cis form ***(M1)***

$\mathrm{Re}(w_1 + w_2) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ **OR** $1.4142...\cos(x - 0.785398...)$ ***A1***

***[6 marks]***

*continued…*

*Question 5 continued*

(c) (i) $I_t = 12\cos(bt) + 12\cos\left(bt - \frac{\pi}{2}\right)$ ***(M1)***

$I_t = 12\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}bt} + \mathrm{e}^{\mathrm{i}\left(bt - \frac{\pi}{2}\right)}\right)$ ***(M1)***

$I_t = 12\sqrt{2}\cos\left(bt - \frac{\pi}{4}\right)$

max $= 12\sqrt{2}$ $(= 17.0)$ ***A1***

(ii) phase shift $= \frac{\pi}{4} (= 0.785)$ ***A1***

***[4 marks]***
***Total: [13 marks]***

**6.** (a) $y = \dot{x} \Rightarrow \dot{y} = \ddot{x}$ ***A1***

$\dot{y} + 3(y) + 1.25x = 0$ ***R1***

**Note:** If no explicit reference is made to $\dot{y} = \ddot{x}$, or equivalent, award ***A0R1*** if second line is seen.
If $\frac{\mathrm{d}y}{\mathrm{d}x}$ used instead of $\frac{\mathrm{d}y}{\mathrm{d}t}$, award ***A0R0***.

$\dot{y} = -3y - 1.25x$ ***AG***

***[2 marks]***

(b) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1.25 & -3 \end{pmatrix}$ ***A1***

***[1 mark]***

(c) (i) $\begin{vmatrix} -\lambda & 1 \\ -1.25 & -3-\lambda \end{vmatrix} = 0$ ***(M1)***

$\lambda(\lambda + 3) + 1.25 = 0$ ***(A1)***

$\lambda = -2.5 \;\; ; \;\; \lambda = -0.5$ ***A1***

*continued…*

*Question 6 continued*

(ii) $\begin{pmatrix} 2.5 & 1 \\ -1.25 & -0.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***(M1)***

$2.5a + b = 0$

$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ ***A1***

$\begin{pmatrix} 0.5 & 1 \\ -1.25 & -2.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0.5a + b = 0$

$\boldsymbol{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ***A1***

**Note:** Award ***M1*** for a valid attempt to find either eigenvector. Accept equivalent forms of the eigenvectors.
Do not award ***FT*** for eigenvectors that do not satisy both rows of the matrix.

***[6 marks]***

(d) $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + B\mathrm{e}^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ***M1A1***

$t = 0 \Rightarrow x = 8,\ \dot{x} = y = 0$ ***(M1)***

$-2A - 2B = 8$
$5A + B = 0$ ***(M1)***

$A = 1;\ B = -5$ ***A1***

$x = -2\mathrm{e}^{-2.5t} + 10\mathrm{e}^{-0.5t}$ ***A1***

**Note:** Do not award the final ***A1*** if the answer is given in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + B\mathrm{e}^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

***[6 marks]***
***Total: [15 marks]***

**7.** (a) (i) let $X$ be the random variable "number of patients arriving in a minute", such that $X \sim \text{Po}(m)$.

$\text{H}_0 : m = 1.5$ ***A1***

$\text{H}_1 : m > 1.5$ ***A1***

**Note:** Allow a value of $270$ for $m$. Award at most ***A0A1*** if it is not clear that it is the population mean being referred to e.g

$\text{H}_0$ : *The number of patients is equal to 1.5 every minute*

$\text{H}_1$ : *The number of patients exceeds 1.5 every minute.*

Referring to the "expected" number of patients or the use of $\mu$ or $\lambda$ is sufficient for ***A1A1***.

(ii) under $\text{H}_0$ let $Y$ be the number of patients in $3$ hours

$Y \sim \text{Po}(270)$ ***(A1)***

$\text{P}(Y \geq 320)\ \left(= 1 - \text{P}(Y \leq 319)\right) = 0.00166\ \ (0.00165874)$ ***(M1)A1***

since $0.00166 < 0.05$ ***R1***

(reject $\text{H}_0$)

Loreto should employ more staff ***A1***

***[7 marks]***

(b) (i) $\text{H}_0$ : The probability of a patient waiting less than $20$ minutes is $0.95$ ***A1***

$\text{H}_1$ : The probability of a patient waiting less than $20$ minutes is less than $0.95$ ***A1***

(ii) under $\text{H}_0$ let $W$ be the number of patients waiting more than $20$ minutes

$W \sim \text{B}(150,\ 0.05)$ ***(A1)***

$\text{P}(W \geq 11) = 0.132\ \ (0.132215...)$ ***(M1)A1***

since $0.132 > 0.1$ ***R1***

(fail to reject $\text{H}_0$)

insufficient evidence to suggest they are not meeting their target ***A1***

**Note:** Do not accept "they are meeting target" for the ***A1***.
Accept use of $\text{B}(150,\ 0.95)$ and $\text{P}(W \leq 139)$ and any consistent use of a random variable, appropriate $p$-value and significance level.

***[7 marks]***

***Total: [14 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2021

## Mathematics: applications and interpretation

## Higher level

## Paper 2

18 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.
***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
***R*** Marks awarded for clear **Reasoning**.
***AG*** Answer given in the question and so no marks are awarded.
***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

**1.** (a) use of cosine rule ***(M1)***

$\hat{ACB} = \cos^{-1}\left(\frac{1005^2 + 1225^2 - 650^2}{2 \times 1005 \times 1225}\right)$ ***(A1)***

$= 32.0^\circ$ $(31.9980...)$ **OR** $0.558$ $(0.558471...)$ ***A1***

***[3 marks]***

(b) use of sine rule ***(M1)***

$\frac{DE}{\sin 31.9980...^\circ} = \frac{210}{\sin 100^\circ}$ ***(A1)***

$(DE =)$ 113 m $(112.9937...)$ ***A1***

***[3 marks]***

(c) **METHOD 1**

$180^\circ - (100^\circ +$ their part $(a))$ ***(M1)***

$= 48.0019\ldots^\circ$ **OR** $0.837791...$ ***(A1)***

substituted area of triangle formula ***(M1)***

$\frac{1}{2} \times 112.9937... \times 210 \times \sin 48.002^\circ$ ***(A1)***

$8820$ m$^2$ $(8817.18\ldots)$ ***A1***

**METHOD 2**

$\frac{CE}{\sin(180 - 100 - \text{their part } (a))} = \frac{210}{\sin 100}$ ***(M1)***

$(CE =)$ $158.472...$ ***(A1)***

substituted area of triangle formula ***(M1)***

**EITHER**

$\frac{1}{2} \times 112.993... \times 158.472... \times \sin 100$ ***(A1)***

**OR**

$\frac{1}{2} \times 210... \times 158.472... \times \sin(\text{their part } (a))$ ***(A1)***

**THEN**

$8820$ m$^2$ $(8817.18\ldots)$ ***A1***

*continued…*

*Question 1 continued*

**METHOD 3**

$\text{CE}^2 = 210^2 + 112.993...^2 - \left(2 \times 210 \times 112.993... \times \cos(180 - 100 - \text{their part } (a))\right)$ ***(M1)***

$(\text{CE} =)\ 158.472\ldots$ ***(A1)***

substituted area of triangle formula ***(M1)***

$\frac{1}{2} \times 112.993... \times 158.472... \times \sin 100$ ***(A1)***

$8820\ \text{m}^2\ \ (8817.18\ldots)$ ***A1***

***[5 marks]***

(d) $1005 - 210$ **OR** $795$ ***(A1)***

equating answer to part (c) to area of a triangle formula ***(M1)***

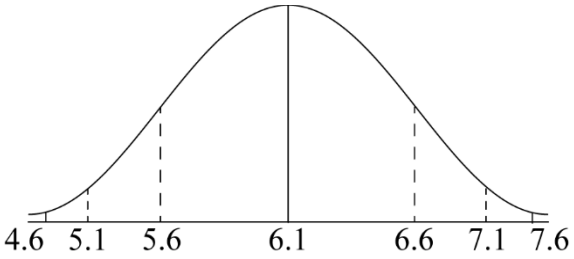
$8817.18\ldots = \frac{1}{2} \times \text{DF} \times (1005 - 210) \times \sin 48.002...^\circ$ ***(A1)***

$(\text{DF} =)\ 29.8\ \text{m}\ \ (29.8473...)$ ***A1***

***[4 marks]***

***Total [15 marks]***

**2.** (a)




***A1A1***

**Note:** Award ***A1*** for a normal curve with mean labelled $6.1$ or $\mu$, ***A1*** for indication of SD $(0.5)$: marks on horizontal axis at $5.6$ and/or $6.6$ **OR** $\mu - 0.5$ and/or $\mu + 0.5$ on the correct side and approximately correct position.

***[2 marks]***

(b) $X \sim \mathrm{N}(6.1,\ 0.5^2)$

$\mathrm{P}(5.5 < X < 6.5)$ **OR** labelled sketch of region ***(M1)***

$= 0.673\ \ (0.673074...)$ ***A1***

***[2 marks]***

(c) $(\mathrm{P}(X < 5.3) =)\ \ 0.0547992...$ ***(A1)***

$0.0547992... \times 80$ ***(M1)***

$= 4.38\ \ (4.38393...)$ ***A1***

***[3 marks]***

(d) (i) $Y \sim \mathrm{N}(4.5,\ 0.45^2)$,

$(\mathrm{P}(Y > 4.62) =)\ \ 0.394862...$ ***(A1)***

use of binomial seen or implied ***(M1)***

using $\mathrm{B}(10,\ 0.394862...)$ ***(M1)***

$0.0430\ \ (0.0429664...)$ ***A1***

(ii) $np(1-p) = 2.39\ \ (2.38946...)$ ***A1***

***[5 marks]***

(e) $\mathrm{P}\left(F \cap (W > 4.7)\right) = 0.5 \times 0.3284\ \ \left(= 0.1642\right)$ ***(A1)***

attempt use of tree diagram **OR** use of $\mathrm{P}(F \mid W > 4.7) = \dfrac{\mathrm{P}\left(F \cap (W > 4.7)\right)}{\mathrm{P}(W > 4.7)}$ ***(M1)***

$\dfrac{0.5 \times 0.3284}{0.5 \times 0.9974 + 0.5 \times 0.3284}$ ***(A1)***

$= 0.248\ \ (0.247669...)$ ***A1***

***[4 marks]***

***Total [16 marks]***

**3.** (a) evidence of splitting diagram into equilateral triangles ***M1***

area $= 6\left(\frac{1}{2}x^2 \sin 60^\circ\right)$ ***A1***

$= \frac{3\sqrt{3}x^2}{2}$ ***AG***

**Note:** The ***AG*** line must be seen for the final ***A1*** to be awarded.

***[2 marks]***

(b) total surface area of prism $1200 = 2\left(3x^2 \frac{\sqrt{3}}{2}\right) + 6xh$ ***M1A1***

**Note:** Award ***M1*** for expressing total surface areas as a sum of areas of rectangles and hexagon(s), and ***A1*** for a correctly substituted formula, equated to $1200$.

***[5 marks]***

$h = \frac{400 - \sqrt{3}x^2}{2x}$ ***A1***

volume of prism $= \frac{3\sqrt{3}}{2}x^2 h$ ***(A1)***

$= \frac{3\sqrt{3}}{2}x^2\left(\frac{400 - \sqrt{3}x^2}{2x}\right)$ ***A1***

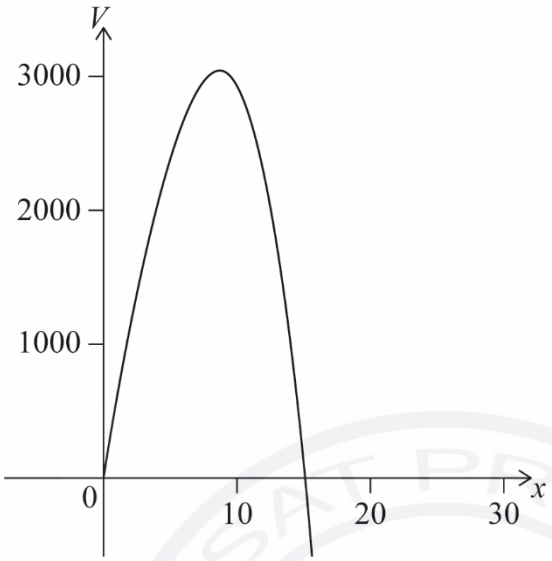
$= 300\sqrt{3}x - \frac{9}{4}x^3$ ***(AG)***

**Note:** The ***AG*** line must be seen for the final ***A1*** to be awarded.

*continued…*

*Question 3 continued*

(c)




***A1A1***

**Note:** Award ***A1*** for correct shape, ***A1*** for roots in correct place with some indication of scale (indicated by a labelled point).

***[2 marks]***

(d) $\frac{\mathrm{d}V}{\mathrm{d}x} = 300\sqrt{3} - \frac{27}{4}x^2$ ***A1A1***

**Note:** Award ***A1*** for a correct term.

***[2 marks]***

(e) from the graph of $V$ or $\frac{\mathrm{d}V}{\mathrm{d}x}$ **OR** solving $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ ***(M1)***

$x = 8.77 \ \ (8.77382...)$ ***A1***

***[2 marks]***

(f) from the graph of $V$ **OR** substituting their value for $x$ into $V$ ***(M1)***

$V_{\max} = 3040 \text{ cm}^3 \ \ (3039.34...)$ ***A1***

***[2 marks]***

***Total [15 marks]***

**4.** (a) $\boldsymbol{T} = \begin{pmatrix} 0.965 & 0.05 \\ 0.035 & 0.95 \end{pmatrix}$ ***M1A1***

**Note:** Award ***M1A1*** for $\boldsymbol{T} = \begin{pmatrix} 0.95 & 0.035 \\ 0.05 & 0.965 \end{pmatrix}$.
Award the ***A1*** for a transposed $\boldsymbol{T}$ if used correctly in part (b) i.e. preceded by $1\times2$ matrix $(2100 \quad 3500)$ rather than followed by a $2\times1$ matrix.

***[2 marks]***

(b) $\begin{pmatrix} 0.965 & 0.05 \\ 0.035 & 0.95 \end{pmatrix}^2 \begin{pmatrix} 2100 \\ 3500 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 2294 \\ 3306 \end{pmatrix}$

so ratio is $2294 : 3306 \ (= 1147 : 1653,\ 0.693889...)$ ***A1***

***[2 marks]***

(c) to solve $\boldsymbol{A}x = \lambda x$ :

$\begin{vmatrix} 0.965-\lambda & 0.05 \\ 0.035 & 0.95-\lambda \end{vmatrix} = 0$ ***(M1)***

$(0.965-\lambda)(0.95-\lambda) - 0.05\times0.035 = 0$

$\lambda = 0.915$ **OR** $\lambda = 1$ ***(A1)***

attempt to find eigenvectors for at least one eigenvalue ***(M1)***

when $\lambda = 0.915$, $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (or any real multiple) ***(A1)***

when $\lambda = 1$, $x = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$ (or any real multiple) ***(A1)***

therefore $\boldsymbol{P} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}$ (accept integer valued multiples of their eigenvectors and columns in either order) ***A1***

***[6 marks]***

*continued…*

*Question 4 continued*

(d) $\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}^{-1} = \frac{1}{17}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***(A1)***

**Note:** This mark is independent, and may be seen anywhere in part (d).

$\boldsymbol{D} = \begin{pmatrix} 0.915 & 0 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

$\boldsymbol{T}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0.915^n & 0 \\ 0 & 1^n \end{pmatrix}\frac{1}{17}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***(M1)A1***

**Note:** Award ***(M1)A0*** for finding $\boldsymbol{P}^{-1}\boldsymbol{D}^n\boldsymbol{P}$ correctly.

as $n \to \infty$ , $\boldsymbol{D}^n = \begin{pmatrix} 0.915^n & 0 \\ 0 & 1^n \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ***R1***

so $\boldsymbol{T}^n \to \frac{1}{17}\begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***A1***

$= \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$ ***AG***

**Note:** The ***AG*** line must be seen for the final ***A1*** to be awarded.

***[6 marks]***

(e) **METHOD ONE**

$\begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}\begin{pmatrix} 2100 \\ 3500 \end{pmatrix} = \begin{pmatrix} 3294 \\ 2306 \end{pmatrix}$ ***(M1)***

so ratio is $3294 : 2306$ $(1647 : 1153,\ 1.42844...,\ 0.700060...)$

***A1***

**METHOD TWO**

long term ratio is the eigenvector associated with the largest eigenvalue ***(M1)***

$10 : 7$ ***A1***

***[2 marks]***

***Total [18 marks]***

**5.** (a) $X_1 \sim \text{Po}(3.1)$

$\text{P}(X_1 = 4) = 0.173 \ \ (0.173349...)$ ***A1***

***[1 mark]***

(b) (i) $X_2 \sim \text{Po}(3 \times 3.1) = \text{Po}(9.3)$ ***(M1)***

$\text{P}(X_2 = 12) = 0.0799 \ \ (0.0798950...)$ ***A1***

(ii) $\left(\text{P}(X_1 > 0)\right)^2 \times \text{P}(X_1 = 0)$ ***(M1)***

$0.95495^2 \times 0.04505$ ***(A1)***

$= 0.0411 \ \ (0.0410817...)$ ***A1***

***[5 marks]***

(c) $\text{P}(X_1 = 0) = 0.04505$ ***(A1)***

$X_1 \sim \text{B}(12, \ 0.04505)$ ***(M1)(A1)***

**Note:** Award ***M1*** for recognizing binomial probability, and ***A1*** for correct parameters.

$= 0.0133 \ \ (0.013283....)$ ***A1***

***[4 marks]***

(d) **METHOD ONE**

| $n$ | $\lambda$ | $\text{P}(X \geq 30)$ |
|---|---|---|
| ... | … | … |
| 10 | 24.1 | 0.136705 |
| 11 | 26.2 | 0.253384 |

***(M1)(A1)(A1)***

**Note:** Award ***M1*** for evidence of a cumulative Poisson with $\lambda = 3.1 + 2.1n$, ***A1*** for $0.136705$ and ***A1*** for $0.253384$.

so require $12$ magpies (including Bill) ***A1***

**METHOD TWO**

evidence of a cumulative Poisson with $\lambda = 3.1 + 2.1n$ ***(M1)***

sketch of curve and $y = 0.2$ ***(A1)***

(intersect at) $10.5810\ldots$ ***(A1)***

rounding up gives $n = 11$

so require $12$ magpies (including Bill) ***A1***

***[4 marks]***

***Total [14 marks]***

**6.** (a) solving $v = 0$ ***M1***

$t = 2,\ t = 6$ ***A1***

***[2 marks]***

(b) use of power rule ***(M1)***

$\frac{\mathrm{d}v}{\mathrm{d}t} = -4t + 16$ ***(A1)***

$(t = 6)$

$\Rightarrow a = -8$ ***(A1)***

magnitude $= 8\ \mathrm{m\,s^{-2}}$ ***A1***

***[4 marks]***

(c) using a sketch graph of $v$ ***(M1)***

$24\ \mathrm{m\,s^{-1}}$ ***A1***

***[2 marks]***

(d) **METHOD ONE**

$x = \int v\ \mathrm{d}t$

attempt at integration of $v$ ***(M1)***

$-\frac{2t^3}{3} + 8t^2 - 24t\ (+c)$ ***A1***

attempt to find $c$ (use of $t = 0,\ x = 0$) ***(M1)***

$c = 0$ ***A1***

$\left(x = -\frac{2t^3}{3} + 8t^2 - 24t\right)$

**METHOD TWO**

$x = \int_0^t v\ \mathrm{d}t$

attempt at integration of $v$ ***(M1)***

$\left[-\frac{2t^3}{3} + 8t^2 - 24t\right]_0^t$ ***A1***

attempt to substituted limits into their integral ***(M1)***

$x = -\frac{2t^3}{3} + 8t^2 - 24t$ ***A1***

***[4 marks]***

(e) $\int_0^4 |v|\ \mathrm{d}t$ ***(M1)(A1)***

**Note:** Award ***M1*** for using the absolute value of $v$, or separating into two integrals, ***A1*** for the correct expression.

$= 32\ \mathrm{m}$ ***A1***

***[3 marks]***

***Total [15 marks]***

**7.** (a) $\begin{vmatrix} -4-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$ ***(M1)***

$(-4-\lambda)(-2-\lambda) = 0$ ***(A1)***

$\lambda = -4$ **OR** $\lambda = -2$ ***A1***

$\lambda = -4$

$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x \\ -4y \end{pmatrix}$ ***(M1)***

**Note:** This ***M1*** can be awarded for attempting to find either eigenvector.

$3x - 2y = -4y$

$3x = -2y$

possible eigenvector is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ (or any real multiple) ***A1***

$\lambda = -2$

$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$x = 0,\ y = 1$

possible eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (or any real multiple) ***A1***

***[6 marks]***

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-4t}\begin{pmatrix} -2 \\ 3 \end{pmatrix} + B\mathrm{e}^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ***(M1)A1***

**Note:** Award ***M1A1*** for $x = -2A\mathrm{e}^{-4t},\ y = 3A\mathrm{e}^{-4t} + B\mathrm{e}^{-2t}$, ***M1A0*** if LHS is missing or incorrect.

***[2 marks]***

(c) two (distinct) real negative eigenvalues ***R1***

(or equivalent (*eg* both $\mathrm{e}^{-4t} \to 0, \mathrm{e}^{-2t} \to 0$ as $t \to \infty$ ))

$\Rightarrow$ stable equilibrium point ***A1***

**Note:** Do not award ***R0A1***.

***[2 marks]***

*continued…*

*Question 7 continued*

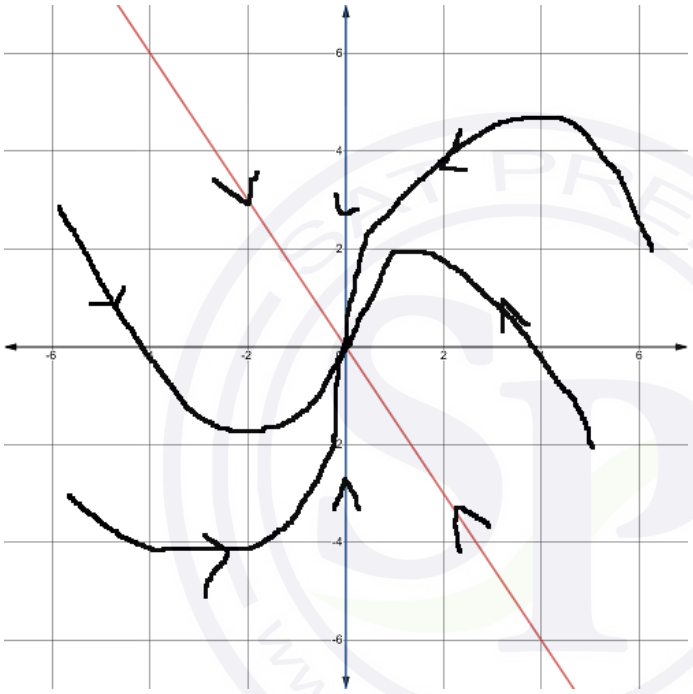
(d) $\frac{dy}{dx} = \frac{3x - 2y}{-4x}$ ***(M1)***

(i) $(4, 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ***A1***

(ii) $(-4, 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ***A1***

***[3 marks]***

(e)




***A1A1A1A1***

**Note:** Award ***A1*** for a phase plane, with correct axes (condone omission of labels) and at least three non-overlapping trajectories. Award ***A1*** for all trajectories leading to a stable node at $(0, 0)$. Award ***A1*** for showing gradient is negative at $x = 4$ and $-4$. Award ***A1*** for both eigenvectors on diagram.

***[4 marks]***

***Total [17 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## May 2021

## Mathematics: applications and interpretation

## Higher level

## Paper 2

19 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.
***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
***R*** Marks awarded for clear **Reasoning**.
***AG*** Answer given in the question and so no marks are awarded.
***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc.*, do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1{,}9$ or $1000$ and $1{,}000$ and $1.000$.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

**1.** (a) **EITHER**

$N = 2$
$PV = -37\,000$
$I\% = 6.4$
$P/Y = 1$
$C/Y = 4$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

**OR**

$N = 8$
$PV = -37\,000$
$I\% = 6.4$
$P/Y = 4$
$C/Y = 4$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

**OR**

$FV = 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{4\times 2}$ ***(M1)(A1)***

**Note:** Award ***M1*** for substitution into compound interest formula, ***(A1)*** for correct substitution.

$= 42\,010 \text{ AUD}$ ***A1***

**Note:** Award ***(M1)(A1)A0*** for unsupported $42009.87$.

***[3 marks]***

(b) **EITHER**

$PV = -37\,000$
$FV = 50\,000$
$I\% = 6.4$
$P/Y = 1$
$C/Y = 4$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. The final mark can still be awarded for the correct number of months (multiple of $3$).

*continued…*

*Question 1 continued*

**OR**

$PV = -37\,000$
$FV = 50\,000$
$I\% = 6.4$
$P/Y = 4$
$C/Y = 4$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

**OR**

$50000 < 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{4 \times n}$ **OR** $50\,000 < 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{n}$ ***(M1)(A1)***

**Note:** Award ***M1*** for the correct inequality, $50\,000$ and substituted compound interest formula. Allow an equation. Award ***A1*** for correct substitution.

**THEN**

$N = 4.74$ (years) $(4.74230...)$ **OR** $N = 18.9692...$ (quarters) ***(A1)***

$m = 57$ months ***A1***

**Note:** Award ***A1*** for rounding their $m$ to the correct number of months. The final answer must be a multiple of $3$. Follow through within this part.

***[4 marks]***

(c) $150\,000$ AUD ***A1***

***[1 mark]***

*continued…*

*Question 1 continued*

(d) (i) $120 \times 1700 - 150\,000$ ***(M1)***

$= 54\,000$ AUD ***A1***

(ii) $N = 120$

$PV = -150\,000$

$PMT = 1700$

$FV = 0$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology or an attempt to use an annuity formula or $FV = 0$ seen. If a compound interest formula is equated to zero, award ***M1***, otherwise award ***M0*** for a substituted compound interest formula.
Award ***A1*** for all entries correct in financial app or correct substitution in annuity formula, but award ***A0*** for a substituted compound interest formula.
Follow through marks in part (d)(ii) are contingent on working seen.

$r = 6.46$ (%) (6.45779...) ***A1***

***[5 marks]***

(e) $N = 60$

$I = 6.46$ (6.45779...)

$PV = -150\,000$

$PMT = 1700$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology or an attempt to use an annuity formula. Award ***(M0)*** for a substituted compound interest formula. Award ***A1*** for all entries correct. Follow through marks in part (e) are contingent on working seen.

$FV = 86\,973$ AUD ***A1***

***[3 marks]***

*continued…*

*Question 1 continued*

(f) $204\,000-(60\times1700+86\,973)$ **OR** $204\,000-188\,973$ ***(M1)(M1)***

**Note:** Award ***M1*** for $60\times1700$. Award ***M1*** for subtracting their $(60\times1700+86\,973)$ from their $(204\,000)$. Award at most ***M1M0*** for their $204\,000-(60\times1700)$ or ***M0M0*** for their $204\,000-(86\,973)$. Follow through from parts (d)(i) and (e). Follow through marks in part (f) are contingent on working seen.

$15\,027$ AUD ***A1***

***[3 marks]***

***Total [19 marks]***

**2.** (a) (i) evidence of power rule (at least one correct term seen) ***(M1)***

$\frac{dy}{dx} = -0.3x^2 + 1.6x$ ***A1***

(ii) $-0.3x^2 + 1.6x = 0$ ***M1***

$x = 5.33 \left( 5.33333..., \frac{16}{3} \right)$ ***A1***

$y = -0.1 \times 5.33333...^3 + 0.8 \times 5.33333...^2$ ***(M1)***

**Note:** Award ***M1*** for substituting their zero for $\frac{dy}{dx}$ $(5.333\ldots)$ into $y$.

7.59 m (7.58519...) ***A1***

**Note:** Award ***M0A0M0A0*** for an unsupported 7.59.
Award at most ***M0A0M1A0*** if only the last two lines in the solution are seen.
Award at most ***M1A0M1A1*** if their $x = 5.33$ is not seen.

***[6 marks]***

(b) One correct substitution seen ***(M1)***

(i) 6.4 m ***A1***

(ii) 7.2 m ***A1***

***[3 marks]***

*continued…*

*Question 2 continued*

(c) $A = \frac{1}{2} \times 2\big((2.4+0)+2(6.4+7.2)\big)$ ***(A1)(M1)***

**Note:** Award ***A1*** for $h = 2$ seen. Award ***M1*** for correct substitution into the trapezoidal rule (the zero can be omitted in working).

$= 29.6 \text{ m}^2$ ***A1***

***[3 marks]***

(d) (i) $A = \int_2^8 -0.1x^3 + 0.8x^2 \, \mathrm{d}x$ **OR** $A = \int_2^8 y \, \mathrm{d}x$ ***A1A1***

**Note:** Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if $\mathrm{d}x$ is omitted.

(ii) $A = 32.4 \text{ m}^2$ ***A2***

**Note:** As per the marking instructions, ***FT*** from their integral in part (d)(i). Award at most ***A1FTA0*** if their area is $>48$, this is outside the constraints of the question (a $6$x$8$ rectangle).

***[4 marks]***

***Total [16 marks]***

**3.** (a) (i) evidence of correct probability ***(M1)***
e.g. sketch **OR** correct probability statement $\mathrm{P}(X < 6.5)$

$0.0151$ ***A1***

(ii) $0.0228$ ***A1***

**Note:** Answers should be given to $4$ decimal place.

***[3 marks]***

(b) (i) multiplying **their** probability by $1000$ ***(M1)***
$451.7$ ***A1***

(ii) $510.5$ ***A1***

***[3 marks]***

**Note:** Answers should be given to $4$ sf.

(c) $\mathrm{H}_0$ : stopping distances can be modelled by $\mathrm{N}(6.76, 0.12^2)$
$\mathrm{H}_1$ : stopping distances cannot be modelled by $\mathrm{N}(6.76, 0.12^2)$ ***A1A1***

**Note:** Award ***A1*** for correct $\mathrm{H}_0$, including reference to the mean and standard deviation.
Award ***A1*** for the negation of their $\mathrm{H}_0$.

***[2 marks]***

(d) $15.1$ or $22.8$ seen ***(M1)***

$0.0727$ $(0.0726542\ldots, 7.27\%)$ ***A2***

***[3 marks]***

(e) $0.05 < 0.0727$ ***R1***
there is insufficient evidence to reject $\mathrm{H}_0$ (or "accept $\mathrm{H}_0$") ***A1***

**Note:** Do not award ***R0A1***.

***[2 marks]***

***Total [13 marks]***

**4.** (a) $y = \frac{5}{8}x + \frac{7}{2}$ $(y = 0.625x + 3.5)$ ***A1A1***

**Note:** Award ***A1*** for $0.625x$, ***A1*** for $3.5$.
Award a maximum of ***A0A1*** if not part of an equation.

***[2 marks]***

(b) (i) $y = -0.975x^2 + 9.56x - 16.7$ ***(M1)A1***
$(y = -0.974630x^2 + 9.55919x - 16.6569...)$

(ii) gradient of curve is positive at $x = 4$ ***R1***

**Note:** Accept a sensible rationale that refers to the gradient.

***[3 marks]***

(c) **METHOD 1**
let $y = ax^2 + bx + c$

differentiating or using $x = \frac{-b}{2a}$ ***(M1)***

$8a + b = 0$

substituting in the coordinates
$7.5^2 a + 7.5b + c = 0$ ***(A1)***
$4^2 a + 4b + c = 6$ ***(A1)***

solve to get
$y = -\frac{24}{49}x^2 + \frac{192}{49}x - \frac{90}{49}$ **OR** $y = -0.490x^2 + 3.92x - 1.84$ ***A1***

**Note:** Use of quadratic regression with points using the symmetry of the graph is a valid method.

**METHOD 2**
$y = a(x-4)^2 + 6$ ***(M1)***
$0 = a(7.5-4)^2 + 6$ ***(M1)***
$a = -\frac{24}{49}$ ***(A1)***
$y = -\frac{24}{49}(x-4)^2 + 6$ **OR** $y = -0.490(x-4)^2 + 6$ ***A1***

***[4 marks]***

*continued…*

*Question 4 continued*

(d) (i) $\pi\int_0^4\left(\frac{5}{8}x+3.5\right)^2\mathrm{d}x+\pi\int_4^{7.5}\left(-\frac{24}{49}(x-4)^2+6\right)^2\mathrm{d}x$ ***(M1)(M1)(M1)A1***

**Note:** Award ***(M1)(M1)(M1)A0*** if $\pi$ is omitted but response is otherwise correct. Award ***(M1)*** for an integral that indicates volume, ***(M1)*** for their part (a) within their volume integral, ***(M1)*** for their part (b)(i) within their volume integral, ***A1*** for their correct two integrals with all correct limits.

(ii) $501\ \mathrm{cm}^3$ (501.189...) ***A1***

***[5 marks]***

***Total [14 marks]***

**5.** (a) finding $\boldsymbol{T}^3$ **OR** use of tree diagram ***(M1)***

$\boldsymbol{T}^3 = \begin{pmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{pmatrix}$

the probability of sunny in three days' time is $0.65$ ***A1***

***[2 marks]***

(b) attempt to find eigenvalues ***(M1)***

**Note:** Any indication that $\det(\boldsymbol{T} - \lambda \boldsymbol{I}) = 0$ has been used is sufficient for the ***(M1)***.

$$\begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06 = 0$$

$(\lambda^2 - 1.5\lambda + 0.5 = 0)$

$\lambda = 1,\ \lambda = 0.5$ ***A1***

attempt to find either eigenvector ***(M1)***

$0.8x + 0.3y = x \Rightarrow -0.2x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ***A1***

$0.8x + 0.3y = 0.5x \Rightarrow 0.3x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

**Note:** Accept multiples of the stated eigenvectors.

***[5 marks]***

(c) (i) $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ **OR** $\boldsymbol{P} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ ***A1***

**Note:** Examiners should be aware that different, correct, matrices $\boldsymbol{P}$ may be seen.

(ii) $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$ **OR** $\boldsymbol{D} = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

**Note:** $\boldsymbol{P}$ and $\boldsymbol{D}$ must be consistent with each other.

***[2 marks]***

(d) $0.5^n \to 0$ ***(M1)***

$\boldsymbol{D}^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ **OR** $\boldsymbol{D}^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

**Note:** Award ***A1*** only if their $\boldsymbol{D}^n$ corresponds to their $\boldsymbol{P}$.

$\boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$ ***(M1)***

$60\ \%$ ***A1***

***[4 marks]***

***Total [13 marks]***

**6.** (a) use of product rule ***(M1)***

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}ab\mathrm{e}^{bt}\cos t-a\mathrm{e}^{bt}\sin t\\ ab\mathrm{e}^{bt}\sin t+a\mathrm{e}^{bt}\cos t\end{pmatrix}$$ ***A1A1***

***[3 marks]***

(b) $|\boldsymbol{v}|^2=\dot{x}^2+\dot{y}^2=\left[ab\mathrm{e}^{bt}\cos t-a\mathrm{e}^{bt}\sin t\right]^2+\left[ab\mathrm{e}^{bt}\sin t+a\mathrm{e}^{bt}\cos t\right]^2$ ***M1***

**Note:** It is more likely that an expression for $|\boldsymbol{v}|$ is seen.
$\sqrt{\dot{x}^2+\dot{y}^2}$ is not sufficient to award the ***M1***, their part (a) must be substituted.

$=\left[a^2\sin^2 t-2a^2b\sin t\cos t+a^2b^2\cos^2 t+a^2\cos^2 t+2a^2b\sin t\cos t+a^2b^2\sin^2 t\right]\mathrm{e}^{2bt}$ ***A1***

use of $\sin^2 t+\cos^2 t=1$ within a factorized expression that leads to the final answer ***M1***

$=a^2\left(b^2+1\right)\mathrm{e}^{2bt}$ ***A1***

magnitude of velocity is $a\,\mathrm{e}^{bt}\sqrt{\left(1+b^2\right)}$ ***AG***

***[4 marks]***

(c) when $t=0$, $a\mathrm{e}^{bt}\cos t=5$
$a=5$ ***A1***
$ab\mathrm{e}^{bt}\cos t-a\mathrm{e}^{bt}\sin t=-3.5$ ***(M1)***
$b=-0.7$ ***A1***

**Note:** Use of $a\,\mathrm{e}^{bt}\sqrt{\left(1+b^2\right)}$ result from part (b) is an alternative approach.

***[3 marks]***

(d) $5\,\mathrm{e}^{-0.7\times2}\sqrt{\left(1+(-0.7)^2\right)}$ ***(M1)***
$1.51\ \ (1.50504\ldots)$ ***A1***

***[2 marks]***

(e) $\dot{x}=0$ ***(M1)***

$a\,\mathrm{e}^{bt}(b\cos t-\sin t)=0$
$\tan t=b$
$t=2.53\ \ (2.53086...)$ ***(A1)***

correct substitution of their $t$ to find $x$ or $y$ ***(M1)***
$x=-0.697\ \ (-0.696591...)$ **and** $y=0.488\ \ (0.487614...)$ ***(A1)***

use of Pythagoras / distance formula ***(M1)***
$\mathrm{OP}=0.850\ \mathrm{m}\ \ (0.850297\ldots)$ ***A1***

***[6 marks]***
***Total [18 marks]***

**7.** (a) $\int \frac{1}{x}\mathrm{d}x = \int 2\mathrm{d}t$ ***(M1)***

$\ln x = 2t + c$

$x = A\mathrm{e}^{2t}$ ***(A1)***

$x(0) = 100 \Rightarrow A = 100$ ***(M1)***

$x = 100\mathrm{e}^{2t}$ ***(A1)***

$x(1) = 739$ ***A1***

**Note:** Accept $738$ for the final ***A1***.

***[5 marks]***

(b) $t_{n+1} = t_n + 0.25$ ***(A1)***

**Note:** This may be inferred from a correct $t$ column, where this is seen.

$x_{n+1} = x_n + 0.25x_n\left(2 - 0.01y_n\right)$ ***(A1)***

$y_{n+1} = y_n + 0.25y_n\left(0.0002x_n - 0.8\right)$ ***(A1)***

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | 1000 | 100 |
| 0.25 | 1250 | 85 |
| 0.5 | 1609 | 73 |
| 0.75 | 2119 | 65 |
| 1 | 2836 | 58 |

***(A1)***

**Note:** Award ***A1*** for whole line correct when $t = 0.5$ or $t = 0.75$. The $t$ column may be omitted and implied by the correct $x$ and $y$ values. The formulas are implied by the correct $x$ and $y$ columns.

(i) $2840$ $(2836$ **OR** $2837)$ ***A1***

(ii) $58$ **OR** $59$ ***A1***

***[6 marks]***

(c) (i) both populations are increasing ***A1***

(ii) rabbits are decreasing and foxes are increasing ***A1A1***

***[3 marks]***

(d) setting at least one DE to zero ***(M1)***

$x = 4000,\ y = 200$ ***A1A1***

***[3 marks]***

***Total [17 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## Specimen paper

## Mathematics: applications and interpretation

## Higher level

## Paper 2

14 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

### Using the markscheme

**1 General**

*Award marks using the annotations as noted in the markscheme eg* ***M1****,* ***A2****.*

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***A3***, *etc.*, do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

| | Correct answer seen | Further working seen | Action |
|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685...<br>*(incorrect decimal value)* | Award the final ***A1***<br>*(ignore the further working)* |
| **2.** | $\frac{1}{4}\sin 4x$ | $\sin x$ | Do not award the final ***A1*** |
| **3.** | $\log a - \log b$ | $\log(a-b)$ | Do not award the final ***A1*** |

## 3 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

## 4 Follow through marks (only applied after an error is made)

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) or subpart(s). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then **FT** marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

## 5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

*Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme*

- Alternative methods for complete questions are indicated by ***METHOD 1***, ***METHOD 2***, *etc*.
- Alternative solutions for part-questions are indicated by ***EITHER*** . . . ***OR***.

## 7 Alternative forms

*Unless the question specifies otherwise,* ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

## 8 Accuracy of Answers

*If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.*

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

## 9 Calculators

*A GDC is required for this examination, but calculators with symbolic manipulation features/ CAS functionality are not allowed.*

**Calculator notation**
The subject guide says:

> *Students must always use correct mathematical notation, not calculator notation.*

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

**1.** (a) $2(8\times4+3\times4+3\times8)$ ***M1***

$=136\ (\text{cm}^2)$ ***A1***

***[2 marks]***

(b) $\sqrt{8^2+4^2+3^2}$ ***M1***

(AG =) 9.43 (cm) (9.4339…, $\sqrt{89}$) ***A1***

***[2 marks]***

(c) $-2x+220=0$ ***M1***

$x=110$ ***A1***

110 000 (boxes) ***A1***

***[3 marks]***

(d) $P(x)=\int -2x+220\,\mathrm{d}x$ ***M1***

**Note:** Award ***M1*** for evidence of integration.

$P(x)=-x^2+220x+c$ ***A1A1***

**Note:** Award ***A1*** for either $-x^2$ or $220x$ award ***A1*** for both correct terms and constant of integration.

$1700=-(20)^2+220(20)+c$ ***M1***

$c=-2300$

$P(x)=-x^2+220x-2300$ ***A1***

***[5 marks]***

(e) $-x^2+220x-2300=0$ ***M1***

$x=11.005$ ***A1***

11 006 (boxes) ***A1***

**Note:** Award ***M1*** for their $P(x)=0$, award ***A1*** for their correct solution to $x$. Award the final ***A1*** for expressing their solution to the minimum number of boxes. Do not accept 11 005, the nearest integer, nor 11 000, the answer expressed to 3 significant figures, as these will not satisfy the demand of the question.

***[3 marks]***

***Total [15 marks]***

**2.** (a) (i) $\mathrm{P}(Y) = 0.8 \times 0.1 + 0.2 \times 0.3$ ***M1***

$= 0.14$ ***A1***

(ii) $\mathrm{P}(\text{Star} \mid Y) = \dfrac{0.8 \times 0.1}{0.14}$ ***M1***

$= 0.571 \left( \dfrac{4}{7}, 0.571428\ldots \right)$ ***A1***

***[4 marks]***

(b) the colours of the sweets are distributed according to manufacturer specifications ***A1***

***[1 mark]***

(c)

| **Colour** | Brown | Red | Green | Orange | Yellow | Purple |
|---|---|---|---|---|---|---|
| **Expected Frequency** | 12 | 20 | 16 | 16 | 8 | 8 |

***A2***

**Note:** Award ***A2*** for all 6 correct expected values, ***A1*** for 4 or 5 correct values, ***A0*** otherwise.

***[2 marks]***

(d) 5 ***A1***

***[1 mark]***

(e) 0.469 (0.4688117…) ***A2***

***[2 marks]***

(f) since $0.469 > 0.05$ ***R1***

fail to reject the null hypothesis. There is insufficient evidence to reject the manufacturer's specifications ***A1***

**Note:** Award ***R1*** for a correct comparison of their correct $p$-value to the test level, award ***A1*** for the correct result from that comparison. Do not award ***R0A1***.

***[2 marks]***

***Total [12 marks]***

**3.** (a) (i) $N = 24$

$I\% = 14$

$PV = -14000$

$FV = 0$

$P/Y = 4$

$C/Y = 4$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept $PV = 14000$.

$(€)871.82$ ***A1***

(ii) $4 \times 6 \times 871.82$ ***(M1)***

$(€)20923.68$ ***A1***

(iii) $20923.68 - 14000$ ***(M1)***

$(€)6923.68$ ***A1***

***[7 marks]***

(b) (i) $0.9 \times 14000$ $(= 14000 - 0.10 \times 14000)$ ***M1***

$(€)12600.00$ ***A1***

(ii) $N = 72$

$PV = 12600$

$PMT = -250$

$FV = 0$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

**Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept $PV = -12600$ provided $PMT = 250$.

$12.56(\%)$ ***A1***

***[5 marks]***

*continued…*

*Question 3 continued*

(c) ***EITHER***

Bryan should choose Option A ***A1***

no deposit is required ***R1***

**Note:** Award ***R1*** for stating that no deposit is required. Award ***A1*** for the correct choice from that fact. Do not award ***R0A1.***

***OR***

Bryan should choose Option B ***A1***

cost of Option A $(6923.69) >$ cost of Option B $(72 \times 250 - 12\,600 = 5400)$ ***R1***

**Note:** Award ***R1*** for a correct comparison of costs. Award ***A1*** for the correct choice from that comparison. Do not award ***R0A1***.

***[2 marks]***

(d) real interest rate is $0.4 - 0.1 = 0.3\%$ ***(M1)***

value of other payments $250 + 250 \times 1.003 + \ldots + 250 \times 1.003^{71}$
use of sum of geometric sequence formula or financial app on a GDC ***(M1)***

$= 20\,058.43$

value of deposit at the end of 6 years
$1400 \times (1.003)^{72} = 1736.98$ ***(A1)***

Total value is $(€)\,21\,795.41$ ***A1***

**Note:** Both ***M*** marks can awarded for a correct use of the GDC's financial app:

N = 72 $(6 \times 12)$
I% = 3.6 $(0.3 \times 12)$
PV = 0
PMT = –250
FV =
P/Y = 12
C/Y = 12

***OR***

N = 72 $(6 \times 12)$
I% = 0.3
PV = 0
PMT = –250
FV =
P/Y = 1
C/Y = 1

***[4 marks]***

***Total [18 marks]***

**4.** (a) $\boldsymbol{r} = \begin{pmatrix} 30 \\ 10 \\ 5 \end{pmatrix} + t \begin{pmatrix} -150 \\ -50 \\ -20 \end{pmatrix}$ ***A1A1***

***[2 marks]***

(b) (i) when $x = 0, t = \frac{30}{150} = 0.2$ ***M1***

***EITHER***

when $y = 0, t = \frac{10}{150} = 0.2$ ***A1***

since the two values of $t$ are equal the aircraft passes directly over the airport

***OR***

$t = 0.2,\ y = 0$ ***A1***

(ii) height $= 5 - 0.2 \times 20 = 1\text{km}$ ***A1***

(iii) time $13{:}12$ ***A1***

***[4 marks]***

(c) (i) $5 - 20t = 4 \Rightarrow t = \frac{1}{20}$ (3 minutes) ***(M1)***

time $13{:}03$ ***A1***

(ii) displacement is $\begin{pmatrix} 22.5 \\ 7.5 \\ 4 \end{pmatrix}$ ***A1***

distance is $\sqrt{22.5^2 + 7.5^2 + 4^2}$ ***(M1)***

$= 24.1\text{km}$ ***A1***

***[5 marks]***

*continued…*

*Question 4 continued*

(d) ***METHOD 1***

time until landing is $12 - 3 = 9$ minutes ***M1***

height to descend $= 4\,\text{km}$

$a = \dfrac{-4}{\dfrac{9}{60}}$ ***M1***

$= -26.7$ ***A1***

***METHOD 2***

$\begin{pmatrix} -150 \\ -50 \\ a \end{pmatrix} = s \begin{pmatrix} 22.5 \\ 7.5 \\ 4 \end{pmatrix}$ ***M1***

$-150 = 22.5s \Rightarrow s = -\dfrac{20}{3}$ ***M1***

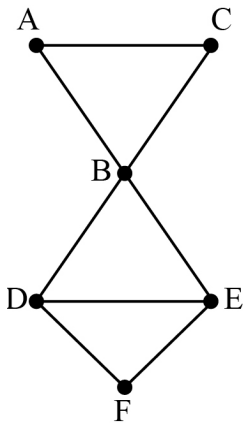
$a = -\dfrac{20}{3} \times 4$

$= -26.7$ ***A1***

***[3 marks]***

***Total [14 marks]***

**5.** (a)




***A2***

***[2 marks]***

(b) attempt to form an adjacency matrix ***M1***

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$ ***A1***

***[2 marks]***

(c) raising the matrix to the power six ***(M1)***

50 ***A1***

***[2 marks]***

(d) not possible ***A1***

because you must pass through B twice ***R1***

**Note:** Do not award ***A1R0.***

***[2 marks]***

(e) $a = 230,\ b = 340$ ***A1A1***

***[2 marks]***

(f) $\mathrm{A} \to \mathrm{B} \to \mathrm{D} \to \mathrm{E} \to \mathrm{F} \to \mathrm{C} \to \mathrm{A}$ ***(M1)***

$90 + 70 + 100 + 210 + 330 + 150$ ***(A1)***

(US$) 950 ***A1***

***[3 marks]***

*continued…*

*Question 5 continued*

(g) finding weight of minimum spanning tree ***M1***

$70 + 80 + 100 + 180 = (\text{US\$})\ 430$ ***A1***

adding in two edges of minimum weight ***M1***

$430 + 90 + 150 = (\text{US\$})\ 670$ ***A1***

***[4 marks]***

***Total [17 marks]***

**6.** (a) $\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$ ***M1A1***

***[2 marks]***

(b) $\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = 0$ ***M1***

$\lambda = 1$ and $0.7$ ***A1***

eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***(M1)A1***

**Note:** Accept any scalar multiple of the eigenvectors.

***[4 marks]***

(c) ***EITHER***

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\ \boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$ ***A1A1***

***OR***

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\ \boldsymbol{D} = \begin{pmatrix} 0.7 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1A1***

***[2 marks]***

(d) $\boldsymbol{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ ***A1***

$\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.7^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1200 \\ 1200 \end{pmatrix}$ ***M1A1***

attempt to multiply matrices ***M1***

so in company A, after $n$ years, $400\left(2 + 0.7^n\right)$ ***A1***

***[5 marks]***

(e) $400 \times 2 = 800$ ***A1***

***[1 mark]***

***Total [14 marks]***

**7.** (a) $\frac{\mathrm{d}v}{\mathrm{d}t} = 9.81 - 0.9v$ ***M1***

$\int \frac{1}{9.81 - 0.9v}\,\mathrm{d}v = \int 1\,\mathrm{d}t$ ***M1***

$-\frac{1}{0.9}\ln(9.81 - 0.9v) = t + c$ ***A1***

$9.81 - 0.9v = A\mathrm{e}^{-0.9t}$ ***A1***

$v = \frac{9.81 - A\mathrm{e}^{-0.9t}}{0.9}$ ***A1***

when $t = 0$, $v = 0$ hence $A = 9.81$ ***A1***

$v = \frac{9.81(1 - \mathrm{e}^{-0.9t})}{0.9}$

$v = 10.9(1 - \mathrm{e}^{-0.9t})$ ***A1***

***[7 marks]***

(b) **either** let $t$ tend to infinity, or $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ ***(M1)***

$v = 10.9$ ***A1***

***[2 marks]***

(c) $\frac{\mathrm{d}x}{\mathrm{d}t} = y$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}t} = 9.81 - 0.9y^2$ ***A1***

***[2 marks]***

(d) $x_{n+1} = x_n + 0.2y_n,\ y_{n+1} = y_n + 0.2(9.81 - 0.9(y_n)^2)$ ***(M1)(A1)***

$x = 1.04,\ \frac{\mathrm{d}x}{\mathrm{d}t} = 3.31$ ***(M1)A1***

***[4 marks]***

(e) 3.3015 ***A1***

***[1 mark]***

(f) $0 = 9.81 - 0.9(v)^2$ ***M1***

$\Rightarrow v = \sqrt{\frac{9.81}{0.9}} = 3.301511\ldots\ (= 3.30)$ ***A1***

***[2 marks]***

*continued…*

*Question 7 continued*

(g) the model found the terminal velocity very accurately, so good approximation ***R1***

intermediate values had object exceeding terminal velocity so not good approximation ***R1***

***[2 marks]***

***Total [20 marks]***