N20/5/MATHL/HP1/ENG/TZ0/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2020

Mathematics

Higher level

Paper 1

30 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. $\mathrm{E}(X)=(0\times p)+\left(1\times\frac{1}{4}\right)+\left(2\times\frac{1}{6}\right)+3q\left(=\frac{19}{12}\right)$ ***(M1)***

$\left(\Rightarrow\frac{1}{4}+\frac{1}{3}+3q=\frac{19}{12}\right)$

$q=\frac{1}{3}$ ***A1***

$p+\frac{1}{4}+\frac{1}{6}+q=1$ ***(M1)***

$\left(\Rightarrow p+q=\frac{7}{12}\right)$

$p=\frac{1}{4}$ ***A1***

[4 marks]

2. $(x=0\Rightarrow)\,y=1$ ***(A1)***

appreciate the need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

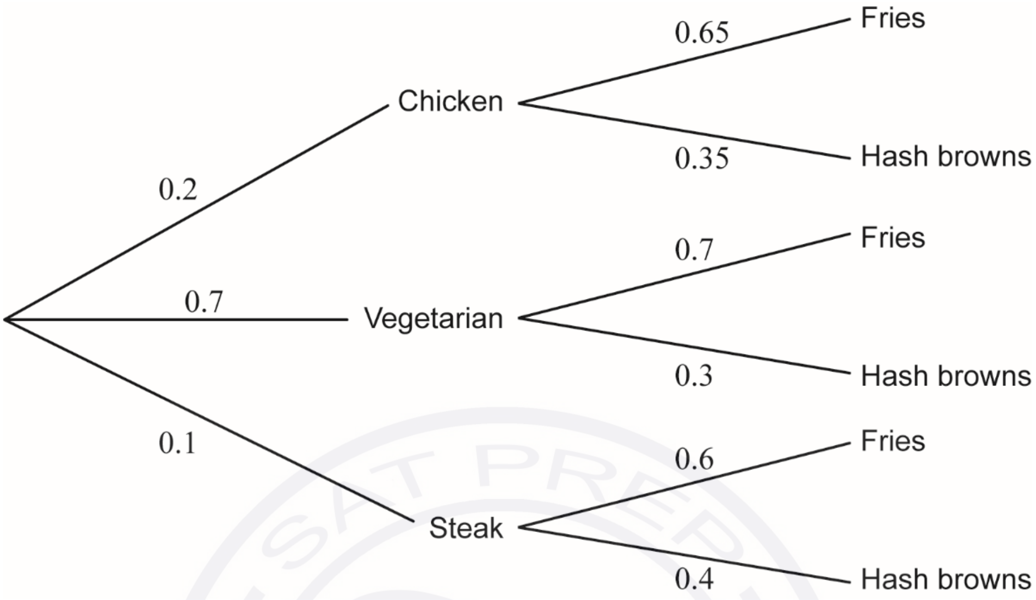
$\left(\frac{\mathrm{d}y}{\mathrm{d}x}=\right)2\mathrm{e}^{2x}-3$ ***A1***

$(x=0\Rightarrow)\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ ***A1***

$\frac{y-1}{x-0}=-1\ \ (y=1-x)$ ***A1***

[5 marks]

3. (a)



A1A1

> **Note:** Award ***A1*** for probabilities for type of omelette and ***A1*** for probabilities for fries / hash browns.

[2 marks]

(b) $(0.2 \times 0.65) + (0.7 \times 0.7) + (0.1 \times 0.6)$ ***(M1)***

$= 0.68 \left(= \dfrac{17}{25} \right)$ ***A1***

[2 marks]

(c) $\dfrac{\text{P(ordered fries and did not order chicken omelette)}}{\text{P(did not order chicken omelette)}}$ ***(M1)***

$\dfrac{0.7 \times 0.7 + 0.1 \times 0.6}{0.7 + 0.1} \left(= \dfrac{0.49 + 0.06}{0.8} = \dfrac{0.55}{0.8} \right)$ ***(A1)***

$= \dfrac{55}{80} \left(= \dfrac{11}{16} \right)$ ***A1***

[3 marks]
Total [7 marks]

4. substituting $z = x + \mathrm{i}y$ **and** $z^* = x - \mathrm{i}y$ ***M1***

$$\frac{2(x+\mathrm{i}y)}{3-(x-\mathrm{i}y)} = \mathrm{i}$$

$$2x + 2\mathrm{i}y = -y + \mathrm{i}(3-x)$$

equate real and imaginary: ***M1***

$y = -2x$ AND $2y = 3 - x$ ***A1***

Note: If they multiply top and bottom by the conjugate, the equations $6x - 2x^2 + 2y^2 = 0$ and $6y - 4xy = (3-x)^2 + y^2$ may be seen. Allow for ***A1***.

solving simultaneously:

$x = -1, y = 2$ $(z = -1 + 2\mathrm{i})$ ***A1A1***

[5 marks]

5. $u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2\left(\frac{625}{16}\right)$

$d = \frac{1}{4}\log_2\left(\frac{625}{16}\right)$

attempt to use power rule for logs ***M1***

$d = \log_2\left(\frac{625}{16}\right)^{\frac{1}{4}}$

$d = \log_2\left(\frac{5}{2}\right)$ ***A1***

[5 marks]

Note: Award method marks in any order.

6. METHOD 1

$\sin\theta\cos\theta = \frac{c}{a}$ and $\sin\theta + \cos\theta = -\frac{b}{a}$ ***A1***

attempt to square $\sin\theta + \cos\theta$ ***M1***

$\left(\frac{b^2}{a^2} =\right)\left(\sin\theta + \cos\theta\right)^2 = 1 + 2\sin\theta\cos\theta$ ***A1***

$\frac{b^2}{a^2}\left(= 1 + 2\sin\theta\cos\theta\right) = 1 + \frac{2c}{a}$ ***A1***

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

METHOD 2

$a\sin^2\theta + b\sin\theta + c = 0$ and $a\cos^2\theta + b\cos\theta + c = 0$ ***A1***

adding the two equations ***M1***

$a + b\left(\sin\theta + \cos\theta\right) + 2c = 0$ ***A1***

$a + b \times -\frac{b}{a} + 2c = 0$ ***A1***

$a^2 - b^2 + 2ac = 0$

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

7. (a) (i) $\frac{z_1}{z_2} = \cos\left(\frac{11\pi}{12} - \frac{\pi}{6}\right) + i\sin\left(\frac{11\pi}{12} - \frac{\pi}{6}\right)$ ***(M1)***

$= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ ***A1***

(ii) $\frac{z_2}{z_1} = \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}$ ***A1***

Note: Allow equivalent forms in part (a), *e.g.* $\text{cis}\left(-\frac{3\pi}{4}\right)$.

Note: Ignore subsequent work once correct answer(s) are seen.

[3 marks]

(b) valid attempt to calculate area of their triangle ***(M1)***

(angle between OA and OB is $\frac{\pi}{2}$) $\Rightarrow$ area $\left(= \frac{1}{2} \times 1 \times 1\right) = \frac{1}{2}$ ***A1***

[2 marks]

8. (a) **METHOD 1**

attempt to replace $\tan x = \dfrac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$= \frac{1-\cos^2 x}{\cos x(1-\cos x)}$$

$$= \frac{(1+\cos x)(1-\cos x)}{\cos x(1-\cos x)}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

METHOD 2

attempt to replace $\tan x = \dfrac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos x)(1+\cos x)} \equiv \frac{\sin^2 x(1+\cos x)}{\cos x\left(1-\cos^2 x\right)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x \sin^2 x}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

(b) **METHOD 1**

consider $1+\frac{1}{\cos x}=k$, leading to $\cos x=\frac{1}{k-1}$ ***(M1)***

consider graph of $y=\frac{1}{x-1}$ or range of solutions for $y=\cos x$ ***(M1)***

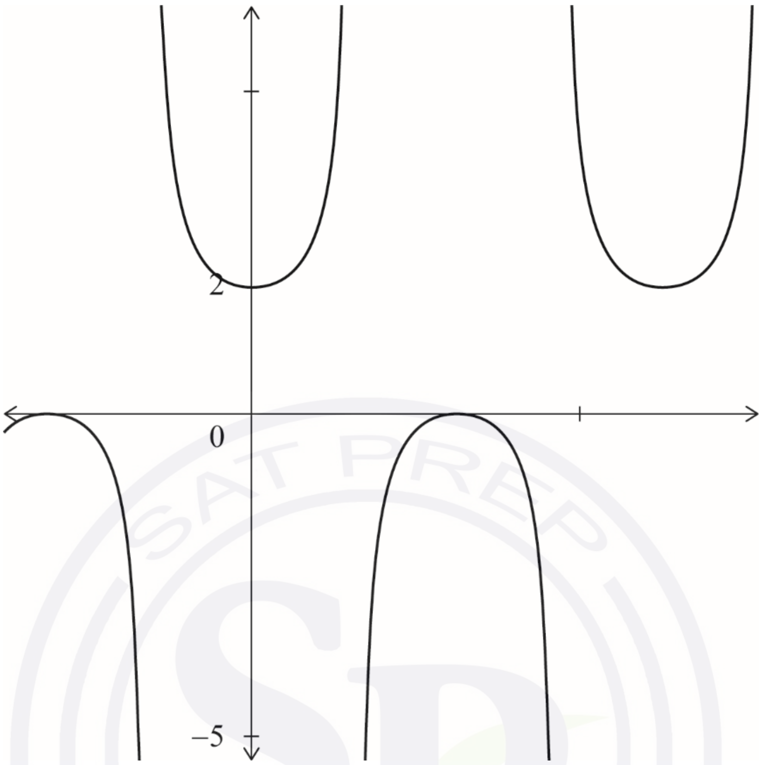
(no solutions if $y<-1$ or $y>1 \Rightarrow$) $0<k<2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k>0$ and $k<2$'.

[4 marks]

METHOD 2

consider graph of $y = 1 + \sec x$ ***M1***



A1

no real solutions if $0 < k < 2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]

METHOD 3

consider $-1 \le \cos x \le 1$, (***M1***)

$\frac{1}{\cos x} \le -1$ or $\frac{1}{\cos x} \ge 1$ (***M1***)

$1+\frac{1}{\cos x} \le 0$ or $1+\frac{1}{\cos x} \ge 2$

no solutions if $0 < k < 2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]
Total [7 marks]

9. $x = \tan u \Rightarrow \frac{dx}{du} = \sec^2 u$ OR $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$ ***A1***

attempt to write the integral in terms of u ***M1***

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u \, du}{\left(1+\tan^2 u\right)^3}$$

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u \, du}{\left(\sec^2 u\right)^3}$$ ***(A1)***

$$= \int_0^{\frac{\pi}{4}} \sin^2 u \cos^2 u \, du$$ ***A1***

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^2 2u \, du$$ ***M1***

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \left(1-\cos 4u\right) \, du$$ ***M1***

$$= \frac{1}{8}\left[u - \frac{\sin 4u}{4}\right]_0^{\frac{\pi}{4}}$$ ***A1***

$$= \frac{1}{8}\left[\frac{\pi}{4} - \frac{\sin \pi}{4} - 0 - 0\right]$$

$$= \frac{\pi}{32}$$ ***A1***

Total [8 marks]

Section B

10. (a) (i) $f'(x) = 3ax^2 + 2bx + c$ ***A1***

(ii) since f^{-1} does not exist, there must be two turning points ***R1***

($\Rightarrow f'(x) = 0$ has more than one solution)

using the discriminant $\Delta > 0$ ***M1***

$4b^2 - 12ac > 0$ ***A1***

$b^2 - 3ac > 0$ ***AG***

[4 marks]

(b) (i) **METHOD 1**

$b^2 - 3ac = (-3)^2 - 3 \times \frac{1}{2} \times 6$ ***M1***

$= 9 - 9$

$= 0$ ***A1***

hence g^{-1} exists ***AG***

METHOD 2

$g'(x) = \frac{3}{2}x^2 - 6x + 6$ ***M1***

$\Delta = (-6)^2 - 4 \times \frac{3}{2} \times 6$

$\Delta = 36 - 36 = 0 \Rightarrow$ there is (only) one point with gradient of 0
and this must be a point of inflexion (since $g(x)$ is a cubic.) ***R1***

hence g^{-1} exists ***AG***

(ii) $p = \frac{1}{2}$ ***A1***

$(x-2)^3 = x^3 - 6x^2 + 12x - 8$ ***(M1)***

$\frac{1}{2}(x^3 - 6x^2 + 12x - 8) = \frac{1}{2}x^3 - 3x^2 + 6x - 4$

$g(x) = \frac{1}{2}(x-2)^3 - 4 \Rightarrow q = -4$ ***A1***

(iii) $x = \frac{1}{2}(y-2)^3 - 4$ ***(M1)***

Note: Interchanging x and y can be done at any stage.

$2(x+4) = (y-2)^3$ ***(M1)***

$\sqrt[3]{2(x+4)} = y - 2$

$y = \sqrt[3]{2(x+4)} + 2$

$g^{-1}(x) = \sqrt[3]{2(x+4)} + 2$ ***A1***

Note: $g^{-1}(x) = \ldots$ must be seen for the final ***A*** mark.

[8 marks]

(c) translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ***A1***

Note: This can be seen anywhere.

EITHER

a stretch scale factor $\frac{1}{2}$ parallel to the y-axis then a translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ***A2***

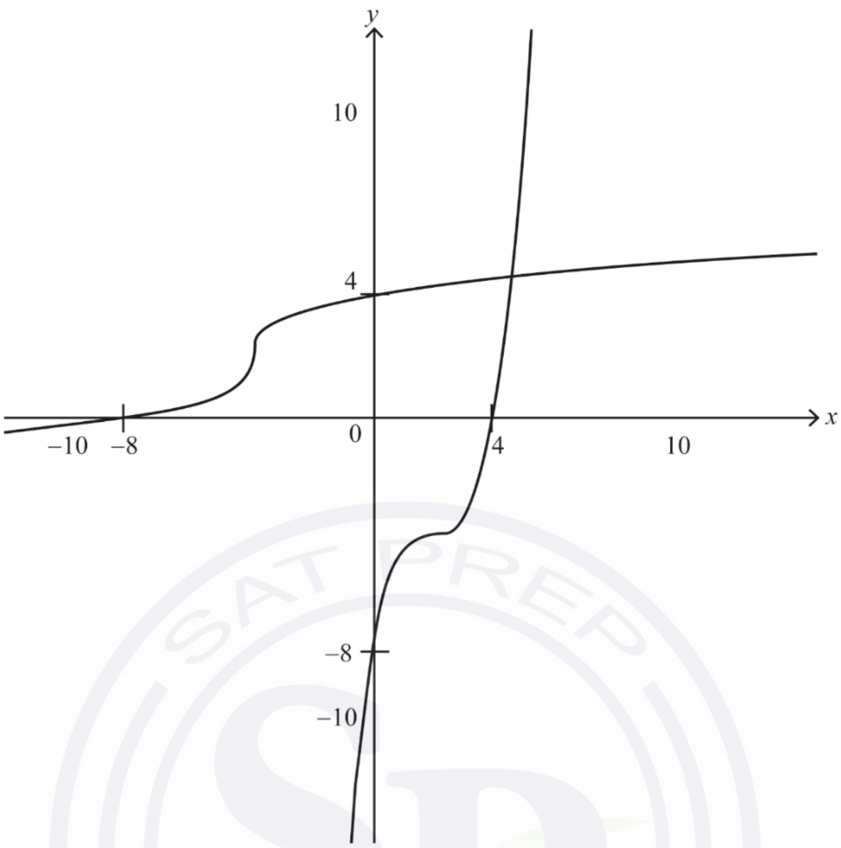
OR

a translation through $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ then a stretch scale factor $\frac{1}{2}$ parallel to the y-axis ***A2***

Note: Accept 'shift' for translation, but do not accept 'move'. Accept 'scaling' for 'stretch'.

[3 marks]

(d)



A1A1A1
M1A1

Note: Award ***A1*** for correct 'shape' of g (allow non-stationary point of inflexion)
Award ***A1*** for each correct intercept of g
Award ***M1*** for attempt to reflect their graph in $y = x$, ***A1*** for completely correct g^{-1} including intercepts

[5 marks]
Total [20 marks]

11. (a) attempt at implicit differentiation ***M1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(xy)\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right]$$ ***A1M1A1***

Note: Award ***A1*** for LHS, ***M1*** for attempt at chain rule, ***A1*** for RHS.

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) + y\cos(xy)$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) = y\cos(xy)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y - x\cos(xy)\right) = y\cos(xy)$$ ***M1***

Note: Award ***M1*** for collecting derivatives and factorising.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$$ ***AG***

[5 marks]

(b) setting $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$y\cos(xy)=0$ ***(M1)***

$(y\neq 0)\Rightarrow \cos(xy)=0$ ***A1***

$\Rightarrow \sin(xy)\left(=\pm\sqrt{1-\cos^2(xy)}=\pm\sqrt{1-0}\right)=\pm 1$ ***OR*** $xy=(2n+1)\frac{\pi}{2}\ (n\in\mathbb{Z})$

OR $xy=\frac{\pi}{2},\frac{3\pi}{2},\ldots$ ***A1***

Note: If they offer values for xy, award ***A1*** for at least two correct values in two different 'quadrants' and no incorrect values.

$y^2\left(=\sin(xy)\right)>0$ ***R1***

$\Rightarrow y^2=1$ ***A1***

$\Rightarrow y=\pm 1$ ***AG***

[5 marks]

(c) $y = \pm 1 \Rightarrow 1 = \sin(\pm x) \Rightarrow \sin x = \pm 1$ OR $y = \pm 1 \Rightarrow 0 = \cos(\pm x) \Rightarrow \cos x = 0$ ***(M1)***

$(\sin x = 1 \Rightarrow)\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$ ***A1A1***

$(\sin x = -1 \Rightarrow)\left(\frac{3\pi}{2}, -1\right), \left(\frac{7\pi}{2}, -1\right)$ ***A1A1***

Note: Allow 'coordinates' expressed as $x = \frac{\pi}{2}, y = 1$ for example.
Note: Each of the ***A*** marks may be awarded independently and are not dependent on ***(M1)*** being awarded.
Note: Mark only the candidate's first two attempts for each case of $\sin x$.

[5 marks]
Total [15 marks]

12. (a) $x = k$ ***A1***

[1 mark]

(b) $y = k$ ***A1***

[1 mark]

(c) **METHOD 1**

$$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$$ ***M1***

$$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$$ ***A1***

$$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$$

$$= \frac{k^2x-5x}{k^2-5}$$ ***A1***

$$= \frac{x(k^2-5)}{k^2-5}$$

$$= x$$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

[4 marks]

METHOD 2

$f\left(x\right)=\frac{kx-5}{x-k}$

$x=\frac{ky-5}{y-k}$ ***M1***

> **Note:** Interchanging x and y can be done at any stage.

$x\left(y-k\right)=ky-5$ ***A1***

$xy-xk=ky-5$

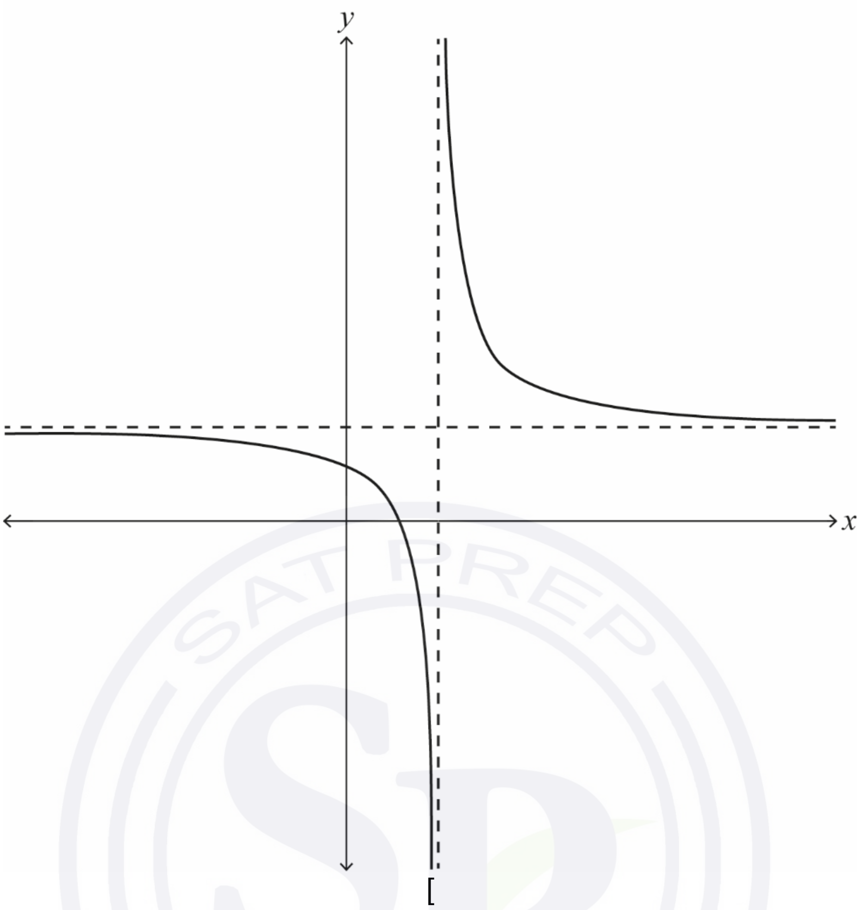
$xy-ky=xk-5$

$y\left(x-k\right)=kx-5$ ***A1***

$y=f^{-1}(x)=\frac{kx-5}{x-k}$ (hence f is self-inverse.) ***R1***

[4 marks]

(d)



attempt to draw both branches of a rectangular hyperbola **_M1_**

$x = 3$ and $y = 3$ **_A1_**

$\left(0, \frac{5}{3}\right)$ and $\left(\frac{5}{3}, 0\right)$ **_A1_**

[3 marks]

(e) **METHOD 1**

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p+\frac{q}{x-3}$ ***M1***

$$\frac{3x-5}{x-3} = 3+\frac{4}{x-3}$$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24x-56}{(x-3)^2}$$ ***A1***

THEN

$$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24}{x-3}+\frac{16}{(x-3)^2}$$ ***A1***

$$\text{volume } = \pi\int_5^7 \left(9+\frac{24}{x-3}+\frac{16}{(x-3)^2}\right)\mathrm{d}x$$

$$= \pi\left[9x+24\ln(x-3)-\frac{16}{x-3}\right]_5^7$$ ***A1***

$$= \pi\left\lfloor(63+24\ln 4-4)-(45+24\ln 2-8)\right\rfloor$$

$$= \pi(22+24\ln 2)$$ ***A1***

[6 marks]

METHOD 2

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ ***(M1)***

substituting $u = x - 3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$3x - 5 = 3(u+3) - 5 = 3u + 4$

$$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$$ ***M1***

$$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\,\mathrm{d}u$$ ***A1***

$$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$$ ***A1***

Note: Ignore absence of or incorrect limits seen up to this point.

$$= \pi\left(22 + 24\ln 2\right)$$ ***A1***

[6 marks]
Total [15 marks]

Markscheme

November 2019

Mathematics

Higher level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the “must be seen” marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N*** marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \; \left(=10\cos(5x-3)\right) \qquad \textit{A1}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $p = 1 - \frac{1}{2} - \frac{1}{5} - \frac{1}{5}$ ***(M1)***

$= \frac{1}{10}$ ***A1***

[2 marks]

(b) attempt to find $\mathrm{E}(X)$ ***(M1)***

$\frac{1}{2} + 1 + 2 + \frac{N}{10} = 10$ ***A1***

$\Rightarrow N = 65$ ***A1***

Note: Do not allow FT in part (b) if their p is outside the range $0 < p < 1$.

[3 marks]

Total [5 marks]

2. $\frac{1}{2}\mathrm{e}^{2x}$ seen ***(A1)***

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}\mathrm{e}^{2x}\right]_0^{\ln k} = \frac{1}{2}\mathrm{e}^{2\ln k} - \frac{1}{2}\mathrm{e}^{0}\right)$ ***(M1)***

$= \frac{1}{2}\mathrm{e}^{\ln k^2} - \frac{1}{2}\mathrm{e}^{0}$ ***(A1)***

Setting their equation $= 12$ ***M1***

Note: their equation must be an integrated expression with limits substituted.

$\frac{1}{2}k^2 - \frac{1}{2} = 12$ ***A1***

$\left(k^2 = 25 \Rightarrow\right) k = 5$ ***A1***

Note: Do not award final ***A1*** for $k = \pm 5$.

[6 marks]

3. attempt to eliminate a variable (or attempt to find det A) ***M1***

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 3 & -1 & 4 \\ 3 & -5 & a & b \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & -14 & a+3 & b-12 \end{array}\right)$$ (or $\det A = 14(a-3)$)

(or two correct equations in two variables) ***A1***

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & 0 & a-3 & b-6 \end{array}\right)$$ (or solving $\det A = 0$)

(or attempting to reduce to one variable, e.g. $(a-3)z = b-6$) ***M1***

$a = 3,\ b \ne 6$ ***A1A1***

[5 marks]

4. attempt to use $\cos(2A+B) = \cos 2A\cos B - \sin 2A\sin B$ (may be seen later) ***M1***

attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A\left(=\sqrt{1-\frac{4}{9}}\right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B\left(=\sqrt{1-\frac{1}{9}} = \frac{\sqrt{8}}{3}\right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A\left(=2\cos^2 A - 1\right) = -\frac{1}{9}$ ***A1***

$\sin 2A\left(=2\sin A\cos A\right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A+B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

5. (a) **METHOD 1**

$|z| = \sqrt[4]{4}\left(= \sqrt{2}\right)$ ***(A1)***

$\arg(z_1) = \dfrac{\pi}{4}$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

METHOD 2

$z^4 = -4$

$(a+\mathrm{i}b)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$\left(a^4 - 6a^4 + a^4 = -4 \Rightarrow\right) a = \pm 1$ ***and*** $\left(4a^3b - 4ab^3 = 0 \Rightarrow\right) a = \pm b$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/- their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

(b) complete method to find area of 'rectangle' ***(M1)***

$= 4$ ***A1***

[2 marks]

Total [7 marks]

6. $f'(x) = e^{2x} + 2xe^{2x}$ ***A1***

Note: This must be obtained from the candidate differentiating $f(x)$.

$= \left(2^1 x + 1 \times 2^{1-1}\right) e^{2x}$ ***A1***

(hence true for $n = 1$)

assume true for $n = k$: ***M1***

$f^{(k)}(x) = \left(2^k x + k2^{k-1}\right)e^{2x}$

Note: Award ***M1*** if truth is assumed. Do not allow "let $n = k$".

consider $n = k + 1$:

$f^{(k+1)}(x) = \frac{d}{dx}\left(\left(2^k x + k2^{k-1}\right)e^{2x}\right)$

attempt to differentiate $f^{(k)}(x)$ ***M1***

$f^{(k+1)}(x) = 2^k e^{2x} + 2\left(2^k x + k2^{k-1}\right)e^{2x}$ ***A1***

$f^{(k+1)}(x) = \left(2^k + 2^{k+1}x + k2^k\right)e^{2x}$

$f^{(k+1)}(x) = \left(2^{k+1}x + (k+1)2^k\right)e^{2x}$ ***A1***

$= \left(2^{k+1}x + (k+1)2^{(k+1)-1}\right)e^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n\left(\in \mathbb{Z}^+\right)$ ***R1***

Note: Do not award final ***R1*** if the two previous ***M1s*** are not awarded. Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

7. (a) attempt to complete the square or multiplication and equating coefficients ***(M1)***

$2x - x^2 = -(x-1)^2 + 1$ ***A1***

$a = -1,\ h = 1,\ k = 1$

[2 marks]

(b) use of their identity from part (a) $\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x\right)$ ***(M1)***

$= \left[\arcsin(x-1)\right]_{\frac{1}{2}}^{\frac{3}{2}}$ or $\left[\arcsin(u)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ ***A1***

Note: Condone lack of, or incorrect limits up to this point.

$= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$ ***(M1)***

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$ ***(A1)***

$= \frac{\pi}{3}$ ***A1***

[5 marks]

Total [7 marks]

8. a vector normal to Π_p is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ***(A1)***

Note: Allow any scalar multiple of $\begin{pmatrix}1\\0\\0\end{pmatrix}$, including $\begin{pmatrix}p\\0\\0\end{pmatrix}$

attempt to find scalar product (or vector product) of direction vector of line with any scalar multiple of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ***M1***

$\begin{pmatrix}1\\0\\0\end{pmatrix}\bullet\begin{pmatrix}5\\\sin\theta\\\cos\theta\end{pmatrix}=5$ (or $\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}5\\\sin\theta\\\cos\theta\end{pmatrix}=\begin{pmatrix}0\\-\cos\theta\\\sin\theta\end{pmatrix}$) ***A1***

(if α is the angle between the line and the normal to the plane)

$\cos\alpha=\dfrac{5}{1\times\sqrt{25+\sin^2\theta+\cos^2\theta}}$ (or $\sin\alpha=\dfrac{1}{1\times\sqrt{25+\sin^2\theta+\cos^2\theta}}$) ***A1***

$\Rightarrow\cos\alpha=\dfrac{5}{\sqrt{26}}$ or $\sin\alpha=\dfrac{1}{\sqrt{26}}$ ***A1***

this is independent of p and θ, hence the angle between the line and the plane, $(90-\alpha)$, is also independent of p and θ ***R1***

Note: The final ***R*** mark is independent, but is conditional on the candidate obtaining a value independent of p and θ.

[6 marks]

Section B

9. (a) $\cos 105^\circ = \cos\left(180^\circ - 75^\circ\right) = -\cos 75^\circ$ ***R1***

$= -q$ ***AG***

Note: Accept arguments using the unit circle or graphical/diagrammatical considerations.

[1 mark]

(b) $\text{AD} = \text{CD} \Rightarrow \text{C}\hat{\text{A}}\text{D} = 45^\circ$ ***A1***

valid method to find $\text{B}\hat{\text{A}}\text{C}$ ***(M1)***

for example: $\text{BC} = r \Rightarrow \text{B}\hat{\text{C}}\text{A} = 60^\circ$

$\Rightarrow \text{B}\hat{\text{A}}\text{C} = 30^\circ$ ***A1***

hence $\text{B}\hat{\text{A}}\text{D} = 45^\circ + 30^\circ = 75^\circ$ ***AG***

[3 marks]

(c) (i) $\text{AB} = r\sqrt{3}$, $\text{AD}(= \text{CD}) = r\sqrt{2}$ ***A1A1***

applying cosine rule ***(M1)***

$\text{BD}^2 = \left(r\sqrt{3}\right)^2 + \left(r\sqrt{2}\right)^2 - 2\left(r\sqrt{3}\right)\left(r\sqrt{2}\right)\cos 75^\circ$ ***A1***

$= 3r^2 + 2r^2 - 2r^2\sqrt{6}\cos 75^\circ$

$= 5r^2 - 2r^2 q\sqrt{6}$ ***AG***

(ii) $\text{B}\hat{\text{C}}\text{D} = 105^\circ$ ***(A1)***

attempt to use cosine rule on ΔBCD ***(M1)***

$\text{BD}^2 = r^2 + \left(r\sqrt{2}\right)^2 - 2r\left(r\sqrt{2}\right)\cos 105^\circ$

$= 3r^2 + 2r^2 q\sqrt{2}$ ***A1***

[7 marks]

(d) $5r^2 - 2r^2 q\sqrt{6} = 3r^2 + 2r^2 q\sqrt{2}$ ***(M1)(A1)***

$2r^2 = 2r^2 q\left(\sqrt{6} + \sqrt{2}\right)$ ***A1***

Note: Award ***A1*** for any correct intermediate step seen using only two terms.

$$q = \frac{1}{\sqrt{6} + \sqrt{2}}$$ ***AG***

Note: Do not award the final ***A1*** if follow through is being applied.

[3 marks]

Total [14 marks]

10. (a) (i) attempt to use quotient rule (or equivalent) ***(M1)***

$$f'(x) = \frac{(x^2-1)(2)-(2x-4)(2x)}{(x^2-1)^2}$$ ***A1***

$$= \frac{-2x^2+8x-2}{(x^2-1)^2}$$

(ii) $f'(x) = 0$

simplifying numerator (may be seen in part (i)) ***(M1)***

$\Rightarrow x^2 - 4x + 1 = 0$ or equivalent quadratic equation ***A1***

EITHER

use of quadratic formula

$$\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$$ ***A1***

OR

use of completing the square

$(x-2)^2 = 3$ ***A1***

THEN

$x = 2 - \sqrt{3}$ (since $2+\sqrt{3}$ is outside the domain) ***AG***

> **Note:** Do not condone verification that $x = 2-\sqrt{3} \Rightarrow f'(x) = 0$.
> Do not award the final ***A1*** as follow through from part (i).

[5 marks]

(b) (i) $(0, 4)$ ***A1***

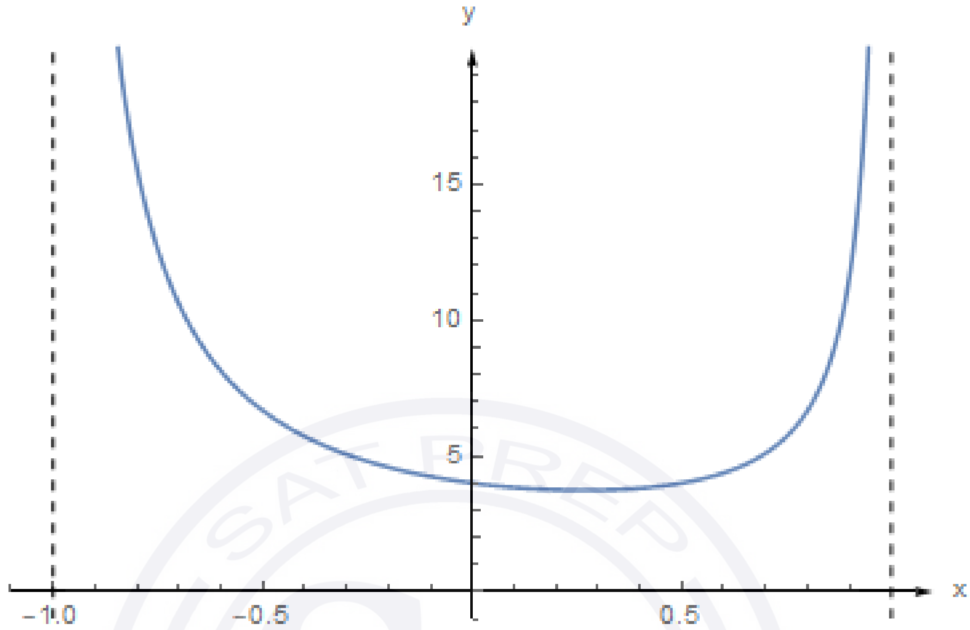
(ii) $2x - 4 = 0 \Rightarrow x = 2$ ***A1***

outside the domain ***R1***

continued…

Question 10 continued

(iii)



A1A1

award ***A1*** for concave up curve over correct domain with one minimum point in the first quadrant
award ***A1*** for approaching $x = \pm 1$ asymptotically

[5 marks]

(c) valid attempt to combine fractions (using common denominator) ***M1***

$$\frac{3(x-1)-(x+1)}{(x+1)(x-1)}$$ ***A1***

$$= \frac{3x-3-x-1}{x^2-1}$$

$$= \frac{2x-4}{x^2-1}$$ ***AG***

[2 marks]

continued…

Question 10 continued

(d)

$f(x) = 4 \Rightarrow 2x - 4 = 4x^2 - 4$ **M1**

($x = 0$ or) $x = \dfrac{1}{2}$ **A1**

area under the curve is $\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ **M1**

$= \int_0^{\frac{1}{2}} \dfrac{3}{x+1} - \dfrac{1}{x-1}\,\mathrm{d}x$

Note: Ignore absence of, or incorrect limits up to this point.

$= \left[3\ln|x+1| - \ln|x-1|\right]_0^{\frac{1}{2}}$ **A1**

$= 3\ln\dfrac{3}{2} - \ln\dfrac{1}{2}(-0)$

$= \ln\dfrac{27}{4}$ **A1**

area is $2 - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ or $\int_0^{\frac{1}{2}} 4\,\mathrm{d}x - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ **M1**

$= 2 - \ln\dfrac{27}{4}$

$= \ln\dfrac{4\mathrm{e}^2}{27}$ **A1**

$\left(\Rightarrow v = \dfrac{4\mathrm{e}^2}{27}\right)$

[7 marks]

Total [19 marks]

11. (a) (i) $\overrightarrow{AV} = \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AV} = \begin{pmatrix} 0 \\ 10 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10(p-10)+10p \\ -10p \\ -10p \end{pmatrix}$ ***A1***

$= \begin{pmatrix} 20p-100 \\ -10p \\ -10p \end{pmatrix} = -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$ ***AG***

$\overrightarrow{AC} \times \overrightarrow{AV} = \begin{pmatrix} 10 \\ 0 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10p \\ 100-20p \\ 10p \end{pmatrix}\left(=10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix}\right)$ ***A1***

(ii) attempt to find a scalar product ***M1***

$-10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 100\left(3p^2-20p\right)$

OR $-\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 3p^2-20p$ ***A1***

attempt to find magnitude of either $\overrightarrow{AB} \times \overrightarrow{AV}$ or $\overrightarrow{AC} \times \overrightarrow{AV}$ ***M1***

$\left|-10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}\right| = \left|10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix}\right| = 10\sqrt{(10-2p)^2+2p^2}$ ***A1***

$100\left(3p^2-20p\right) = 100\left(\sqrt{(10-2p)^2+2p^2}\right)^2 \cos\theta$

$\cos\theta = \dfrac{3p^2-20p}{(10-2p)^2+2p^2}$ ***A1***

Note: Award ***A1*** for any intermediate step leading to the correct answer.

$= \dfrac{p(3p-20)}{6p^2-40p+100}$ ***AG***

Note: Do not allow FT marks from part (a)(i).

[8 marks]

continued…

Question 11 continued

(b) (i) $p(3p-20)=0 \Rightarrow p=0$ or $p=\frac{20}{3}$ ***M1A1***

coordinates are $(0,0,0)$ and $\left(\frac{20}{3},\frac{20}{3},\frac{20}{3}\right)$ ***A1***

Note: Do not allow column vectors for the final ***A*** mark.

(ii) two points are mirror images in the plane
or opposite sides of the plane
or equidistant from the plane
or the line connecting the two Vs is perpendicular to the plane ***R1***

[4 marks]

(c) (i) geometrical consideration or attempt to solve $-1=\frac{p(3p-20)}{6p^2-40p+100}$ ***(M1)***

$p=\frac{10}{3}$, $\theta=\pi$ or $\theta=180^\circ$ ***A1A1***

(ii) $p\to\infty \Rightarrow \cos\theta\to\frac{1}{2}$ ***M1***

hence the asymptote has equation $\theta=\frac{\pi}{3}$ ***A1***

[5 marks]

Total [17 marks]

Markscheme

May 2019

Mathematics

Higher level

Paper 1

20 pages

Instructions to Examiners

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- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. $\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ k+2 \\ k \end{pmatrix}$

$= -6 + k(k+2) - k$ ***A1***

$\boldsymbol{a} \bullet \boldsymbol{b} = 0$ ***(M1)***

$k^2 + k - 6 = 0$

attempt at solving their quadratic equation ***(M1)***

$(k+3)(k-2) = 0$

$k = -3,\ 2$ ***A1***

Note: Attempt at solving using $|\boldsymbol{a}||\boldsymbol{b}| = |\boldsymbol{a} \times \boldsymbol{b}|$ will be ***M1A0A0A0*** if neither answer found ***M1(A1)A1A0*** for one correct answer and ***M1(A1)A1A1*** for two correct answers.

Total [4 marks]

2. attempt at binomial expansion ***M1***

$1 + \binom{11}{1}(-2x) + \binom{11}{2}(-2x)^2 + \ldots$ ***(A1)***

$\binom{11}{2} = 55$

$1 - 22x + 220x^2$ ***A1A1***

Note: ***A1*** for first two terms, ***A1*** for final term.

Note: Award ***M1(A1)A0A0*** for $(-2x)^{11} + \binom{11}{10}(-2x)^{10} + \binom{11}{9}(-2x)^9 + \ldots,$

Total [4 marks]

3. $A = P$

use of the correct formula for area and arc length ***(M1)***

perimeter is $r\theta + 2r$ ***(A1)***

Note: ***A1*** independent of previous ***M1***.

$\frac{1}{2}r^2(1) = r(1) + 2r$ ***A1***

$r^2 - 6r = 0$

$r = 6$ (as $r > 0$) ***A1***

Note: Do not award final ***A1*** if $r = 0$ is included.

Total [4 marks]

4. (a) **EITHER**

$\frac{5\sqrt{15}}{2} = \frac{1}{2} \times 4 \times 5 \sin\theta$ ***A1***

OR

height of triangle is $\frac{5\sqrt{15}}{4}$ if using 4 as the base or $\sqrt{15}$ if using 5 as the base ***A1***

THEN

$\sin\theta = \frac{\sqrt{15}}{4}$ ***AG***

[1 mark]

(b) let the third side be x

$x^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta$ ***M1***

valid attempt to find $\cos\theta$ ***(M1)***

Note: Do not accept writing $\cos\left(\arcsin\left(\frac{\sqrt{15}}{4}\right)\right)$ as a valid method.

$\cos\theta = \pm\sqrt{1 - \frac{15}{16}}$

$= \frac{1}{4}, -\frac{1}{4}$ ***A1A1***

$x^2 = 16 + 25 - 2 \times 4 \times 5 \times \pm\frac{1}{4}$

$x = \sqrt{31}$ or $\sqrt{51}$ ***A1A1***

[6 marks]

Total [7 marks]

5. let $\text{OX} = x$

METHOD 1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or -24) ***(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$3\tan\theta = x$ ***A1***

EITHER

$3\sec^2\theta = \frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{x}{3}\right)$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{3} \times \frac{1}{1 + \frac{x^2}{9}}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 24 \times \frac{1}{3\left(1 + \frac{x^2}{9}\right)}$

attempt to substitute for $x = 0$ into their differential equation ***M1***

THEN

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept $-8\,\text{rad s}^{-1}$.

continued…

Question 5 continued

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or -24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept $-8\,\text{rad}\,\text{s}^{-1}$.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d - 24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3}\left(= \frac{d}{3} - 8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{d}{3} - 8t\right)$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{8}{1+\left(\frac{d}{3} - 8t\right)^2}$ ***A1***

at O, $t = \frac{d}{24}$ ***A1***

continued…

Question 5 continued

THEN

$\frac{d\theta}{dt} = -8$ ***A1***

Total [6 marks]

6. (a) use of symmetry *eg* diagram ***(M1)***

$P(X > \mu + 5) = 0.2$ ***A1***

[2 marks]

(b) **EITHER**

$P(X < \mu + 5 \mid X > \mu - 5) = \frac{P(X > \mu - 5 \cap X < \mu + 5)}{P(X > \mu - 5)}$ ***(M1)***

$= \frac{P(\mu - 5 < X < \mu + 5)}{P(X > \mu - 5)}$ ***(A1)***

$= \frac{0.6}{0.8}$ ***A1A1***

Note: ***A1*** for denominator is independent of the previous ***A*** marks.

OR

use of diagram ***(M1)***

Note: Only award ***(M1)*** if the region $\mu - 5 < X < \mu + 5$ is indicated and used.

continued…

Question 6 continued

$\mathrm{P}\left(X > \mu - 5\right) = 0.8 \quad \mathrm{P}\left(\mu - 5 < X < \mu + 5\right) = 0.6$ ***(A1)***

Note: Probabilities can be shown on the diagram.

$= \dfrac{0.6}{0.8}$ ***M1A1***

THEN

$= \dfrac{3}{4} \ = (0.75)$ ***A1***

[5 marks]

Total [7 marks]

7. attempt at implicit differentiation ***M1***

$3y^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + 6xy\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 = 0$ ***A1A1***

Note: Award ***A1*** for the second & third terms, ***A1*** for the first term, fourth term & RHS equal to zero.

substitution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$3y^2 - 3x^2 = 0$

$\Rightarrow y = \pm x$ ***A1***

substitute either variable into original equation ***M1***

$y = x \Rightarrow x^3 = 9 \Rightarrow x = \sqrt[3]{9}$ (or $y^3 = 9 \Rightarrow y = \sqrt[3]{9}$) ***A1***

$y = -x \Rightarrow x^3 = 27 \Rightarrow x = 3$ (or $y^3 = -27 \Rightarrow y = -3$) ***A1***

$\left(\sqrt[3]{9}, \sqrt[3]{9}\right)$, $(3, -3)$ ***A1***

Total [9 marks]

8. (a) 3 ***A1***

[1 mark]

(b) attempt to use definite integral of $f'(x)$ ***(M1)***

$\int_0^1 f'(x)\mathrm{d}x = 0.5$

$f(1) - f(0) = 0.5$ ***(A1)***

$f(1) = 0.5 + 3$

$= 3.5$ ***A1***

[3 marks]

continued…

Question 8 continued

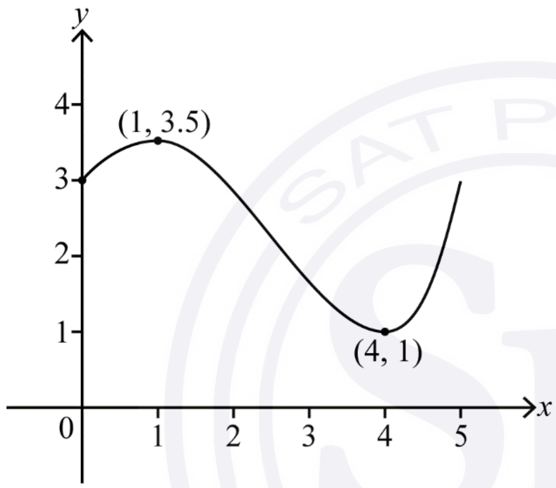
(c) $\int_1^4 f'(x)\mathrm{d}x = -2.5$ ***(A1)***

Note: *(A1)* is for -2.5.

$f(4) - f(1) = -2.5$
$f(4) = 3.5 - 2.5$
$= 1$ ***A1***

[2 marks]

(d)



A1A1A1

A1 for correct shape over approximately the correct domain

A1 for maximum and minimum (coordinates or horizontal lines from 3.5 and 1 are required),
A1 for y-intercept at 3

[3 marks]
Total [9 marks]

Section B

9. (a) $(\sin x+\cos x)^2=\sin^2 x+2\sin x\cos x+\cos^2 x$ ***M1A1***

Note: Do not award the ***M1*** for just $\sin^2 x+\cos^2 x$.

Note: Do not award ***A1*** if correct expression is followed by incorrect working.

$=1+\sin 2x$ ***AG***

[2 marks]

(b) $\sec 2x+\tan 2x=\dfrac{1}{\cos 2x}+\dfrac{\sin 2x}{\cos 2x}$ ***M1***

Note: ***M1*** is for an attempt to change both terms into sine and cosine forms (with the same argument) or both terms into functions of $\tan x$.

$=\dfrac{1+\sin 2x}{\cos 2x}$

$=\dfrac{(\sin x+\cos x)^2}{\cos^2 x-\sin^2 x}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$=\dfrac{(\sin x+\cos x)^2}{(\cos x-\sin x)(\cos x+\sin x)}$ ***M1***

$=\dfrac{\cos x+\sin x}{\cos x-\sin x}$ ***AG***

Note: Apply MS in reverse if candidates have worked from RHS to LHS.

Note: Alternative method using $\tan 2x$ and $\sec 2x$ in terms of $\tan x$.

[4 marks]

(c) **METHOD 1**

$\int_0^{\frac{\pi}{6}}\left(\dfrac{\cos x+\sin x}{\cos x-\sin x}\right)\mathrm{d}x$ ***A1***

Note: Award ***A1*** for correct expression with or without limits.

EITHER

$=\left[-\ln(\cos x-\sin x)\right]_0^{\frac{\pi}{6}}$ or $\left[\ln(\cos x-\sin x)\right]_{\frac{\pi}{6}}^{0}$ ***(M1)A1A1***

Note: Award ***M1*** for integration by inspection or substitution, ***A1*** for $\ln(\cos x-\sin x)$, ***A1*** for completely correct expression including limits.

$=-\ln\left(\cos\dfrac{\pi}{6}-\sin\dfrac{\pi}{6}\right)+\ln(\cos 0-\sin 0)$ ***M1***

Note: Award ***M1*** for substitution of limits into their integral and subtraction.

$=-\ln\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\right)$ ***(A1)***

continued…

Question 9 continued

OR

let $u = \cos x - \sin x$ ***M1***

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x - \cos x = -(\sin x + \cos x)$

$-\int_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}} \left(\frac{1}{u}\right) \mathrm{d}u$ ***A1A1***

Note: Award ***A1*** for correct limits even if seen later, ***A1*** for integral.

$= \left[-\ln u\right]_1^{\frac{\sqrt{3}}{2}-\frac{1}{2}}$ or $\left[\ln u\right]_{\frac{\sqrt{3}}{2}-\frac{1}{2}}^{1}$ ***A1***

$= -\ln\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)(+\ln 1)$ ***M1***

THEN

$= \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***M1***

Note: Award ***M1*** for both putting the expression over a common denominator and for correct use of law of logarithms.

$= \ln\left(1+\sqrt{3}\right)$ ***(M1)A1***

[9 marks]

METHOD 2

$\left[\frac{1}{2}\ln\left(\tan 2x + \sec 2x\right) - \frac{1}{2}\ln\left(\cos 2x\right)\right]_0^{\frac{\pi}{6}}$ ***A1A1***

$= \frac{1}{2}\ln\left(\sqrt{3}+2\right) - \frac{1}{2}\ln\left(\frac{1}{2}\right) - 0$ ***A1A1(A1)***

$= \frac{1}{2}\ln\left(4+2\sqrt{3}\right)$ ***M1***

$= \frac{1}{2}\ln\left(\left(1+\sqrt{3}\right)^2\right)$ ***M1A1***

$= \ln\left(1+\sqrt{3}\right)$ ***A1***

[9 marks]

Total [15 marks]

10. (a) (i) $p(2) = 8 - 12 + 16 - 24$ ***(M1)***

Note: Award ***M1*** for a valid attempt at remainder theorem or polynomial division.

$= -12$ ***A1***
remainder $= -12$

(ii) $p(3) = 27 - 27 + 24 - 24 = 0$ ***A1***
remainder $= 0$

[3 marks]

(b) $x = 3$ (is a zero) ***A1***

Note: Can be seen anywhere.

EITHER

factorise to get $(x-3)\left(x^2+8\right)$ ***(M1)A1***

$x^2 + 8 \neq 0$ (for $x \in \mathbb{R}$) (or equivalent statement) ***R1***

Note: Award ***R1*** if correct two complex roots are given.

OR

$p'(x) = 3x^2 - 6x + 8$ ***A1***
attempting to show $p'(x) \neq 0$ ***M1***
eg discriminant $= 36 - 96 < 0$, completing the square
no turning points ***R1***

THEN

only one real zero (as the curve is continuous) ***AG***
[4 marks]

(c) new graph is $y = p(2x)$ ***(M1)***
stretch parallel to the x-axis (with $x = 0$ invariant), scale factor 0.5 ***A1***
[2 marks]

Note: Accept "horizontal" instead of "parallel to the x-axis".

continued…

Question 10 continued

(d) $\frac{6\lambda^3 e^{-\lambda}}{6} = \frac{3\lambda^2 e^{-\lambda}}{2} - 2\lambda e^{-\lambda} + 3e^{-\lambda}$ ***M1A1***

Note: Allow factorials in the denominator for ***A1***.

$2\lambda^3 - 3\lambda^2 + 4\lambda - 6 = 0$ ***A1***

Note: Accept any correct cubic equation without factorials and $e^{-\lambda}$.

EITHER

$4\left(2\lambda^3 - 3\lambda^2 + 4\lambda - 6\right) = 8\lambda^3 - 12\lambda^2 + 16\lambda - 24 = 0$ ***(M1)***

$2\lambda = 3$ ***(A1)***

OR

$(2\lambda - 3)\left(\lambda^2 + 2\right) = 0$ ***(M1)(A1)***

THEN

$\lambda = 1.5$ ***A1***

[6 marks]

Total [15 marks]

11. (a) (i) appreciation that two points distinct from P need to be chosen from each line ***M1***

$^4C_2 \times {}^3C_2$

$= 18$ ***A1***

(ii) **EITHER**

consider cases for triangles including P **or** triangles not including P ***M1***

$3 \times 4 + 4 \times {}^3C_2 + 3 \times {}^4C_2$ ***(A1)(A1)***

Note: Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

OR

consider total number of ways to select 3 points and subtract those with 3 points on the same line ***M1***

$^8C_3 - {}^5C_3 - {}^4C_3$ ***(A1)(A1)***

Note: Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

$56 - 10 - 4$

THEN

$= 42$ ***A1***

[6 marks]

(b) **METHOD 1**

substitution of $(4, 6, 4)$ into both equations ***(M1)***

$\lambda = 3$ and $\mu = 1$ ***A1A1***

$(4, 6, 4)$ ***AG***

METHOD 2

attempting to solve two of the three parametric equations ***M1***

$\lambda = 3$ or $\mu = 1$ ***A1***

check both of the above give $(4, 6, 4)$ ***M1AG***

Note: If they have shown the curve intersects for all three coordinates they only need to check (4,6,4) with one of "λ" or "μ".

[3 marks]

(c) $\lambda = 2$ ***A1***

[1 mark]

(d) $\vec{PA} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, $\vec{PB} = \begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix}$ ***A1A1***

Note: Award ***A1A0*** if both are given as coordinates.

[2 marks]

continued…

Question 11 continued

(e) **METHOD 1**

area triangle $\text{ABP} = \frac{1}{2}\left|\overrightarrow{\text{PB}} \times \overrightarrow{\text{PA}}\right|$ ***M1***

$\left(= \frac{1}{2}\left|\begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right|\right) = \frac{1}{2}\left|\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right|$ ***A1***

$= \frac{\sqrt{29}}{2}$ ***A1***

EITHER

$\overrightarrow{\text{PC}} = 3\overrightarrow{\text{PA}},\ \overrightarrow{\text{PD}} = 3\overrightarrow{\text{PB}}$ ***(M1)***

area triangle $\text{PCD} = 9 \times$ area triangle ABP ***(M1)A1***

$= \frac{9\sqrt{29}}{2}$ ***A1***

OR

D has coordinates $(-11, -12, -2)$ ***A1***

area triangle $\text{PCD} = \frac{1}{2}\left|\overrightarrow{\text{PD}} \times \overrightarrow{\text{PC}}\right| = \frac{1}{2}\left|\begin{pmatrix} -15 \\ -18 \\ -6 \end{pmatrix} \times \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix}\right|$ ***M1A1***

Note: A1 is for the correct vectors in the correct formula.

$= \frac{9\sqrt{29}}{2}$ ***A1***

THEN

area of $\text{CDBA} = \frac{9\sqrt{29}}{2} - \frac{\sqrt{29}}{2}$

$= 4\sqrt{29}$ ***A1***

[8 marks]

continued…

Question 11 continued

METHOD 2

D has coordinates $(-11, -12, -2)$ ***A1***

area $= \frac{1}{2}\left|\vec{CB} \times \vec{CA}\right| + \frac{1}{2}\left|\vec{BC} \times \vec{BD}\right|$ ***M1***

Note: Award ***M1*** for use of correct formula on appropriate non-overlapping triangles.

Note: Different triangles or vectors could be used.

$\vec{CB} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \ \vec{CA} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ ***A1***

$\vec{CB} \times \vec{CA} = \begin{pmatrix} -4 \\ 6 \\ -8 \end{pmatrix}$ ***A1***

$\vec{BC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \ \vec{BD} = \begin{pmatrix} -10 \\ -12 \\ -4 \end{pmatrix}$ ***A1***

$\vec{BC} \times \vec{BD} = \begin{pmatrix} -12 \\ 18 \\ -24 \end{pmatrix}$ ***A1***

Note: Other vectors which might be used are $\vec{DA} = \begin{pmatrix} 14 \\ 16 \\ 5 \end{pmatrix}, \ \vec{BA} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \vec{DC} = \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}$.

Note: Previous ***A1A1A1A1*** are all dependent on the first ***M1***.

valid attempt to find a value of $\frac{1}{2}\left|a \times b\right|$ ***M1***

Note: ***M1*** independent of triangle chosen.

area $= \frac{1}{2} \times 2 \times \sqrt{29} + \frac{1}{2} \times 6 \times \sqrt{29}$

$= 4\sqrt{29}$ ***A1***

Note: Accept $\frac{1}{2}\sqrt{116} + \frac{1}{2}\sqrt{1044}$ or equivalent.

[8 marks]

Total [20 marks]

Markscheme

May 2019

Mathematics

Higher level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. attempting to form two equations involving u_1 and d ***M1***

$(u_1 + 2d) + (u_1 + 7d) = 1$ and $\frac{7}{2}[2u_1 + 6d] = 35$

$2u_1 + 9d = 1$

$14u_1 + 42d = 70$ $(2u_1 + 6d = 10)$ ***A1***

Note: Award ***A1*** for any two correct equations

attempting to solve their equations: ***M1***

$u_1 = 14$, $d = -3$ ***A1***

[4 marks]

2. (a) (i) $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ ***A1***

(ii) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ ***A1***

Note: Accept row vectors or equivalent.

[2 marks]

(b) **METHOD 1**

attempt at vector product using $\overrightarrow{AB}$ and $\overrightarrow{AC}$. ***(M1)***

$\pm(2\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k})$ ***A1***

attempt to use area = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ ***M1***

$= \frac{\sqrt{76}}{2} \left(= \sqrt{19}\right)$ ***A1***

[4 marks]

continued…

Question 2 continued

METHOD 2

attempt to use $\vec{AB} \bullet \vec{AC} = \left|\vec{AB}\right|\left|\vec{AC}\right| \cos\theta$ ***M1***

$$\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \sqrt{0^2 + 2^2 + (-2)^2}\sqrt{3^2 + 1^2 + (-2)^2}\cos\theta$$

$6 = \sqrt{8}\sqrt{14}\cos\theta$ ***A1***

$\cos\theta = \frac{6}{\sqrt{8}\sqrt{14}} = \frac{6}{\sqrt{112}}$

attempt to use area $= \frac{1}{2}\left|\vec{AB}\right|\left|\vec{AC}\right|\sin\theta$ ***M1***

$= \frac{1}{2}\sqrt{8}\sqrt{14}\sqrt{1 - \frac{36}{112}}\left(= \frac{1}{2}\sqrt{8}\sqrt{14}\sqrt{\frac{76}{112}}\right)$

$= \frac{\sqrt{76}}{2}\left(= \sqrt{19}\right)$ ***A1***

[4 marks]

Total [6 marks]

3. $g(x) = f(x+2)\left(= (x+2)^4 - 6(x+2)^2 - 2(x+2) + 4\right)$ ***M1***

attempt to expand $(x+2)^4$ ***M1***

$(x+2)^4 = x^4 + 4(2x^3) + 6(2^2x^2) + 4(2^3x) + 2^4$ ***(A1)***

$= x^4 + 8x^3 + 24x^2 + 32x + 16$ ***A1***

$g(x) = x^4 + 8x^3 + 24x^2 + 32x + 16 - 6(x^2 + 4x + 4) - 2x - 4 + 4$

$= x^4 + 8x^3 + 18x^2 + 6x - 8$ ***A1***

Note: For correct expansion of $f(x-2) = x^4 - 8x^3 + 18x^2 - 10x$ award max ***M0M1(A1)A0A1***.

[5 marks]

4. $u = \sin x \Rightarrow \mathrm{d}u = \cos x \mathrm{d}x$ ***(A1)***

valid attempt to write integral in terms of u and $\mathrm{d}u$ ***M1***

$\int \frac{\cos^3 x \ \mathrm{d}x}{\sqrt{\sin x}} = \int \frac{\left(1-u^2\right) \ \mathrm{d}u}{\sqrt{u}}$ ***A1***

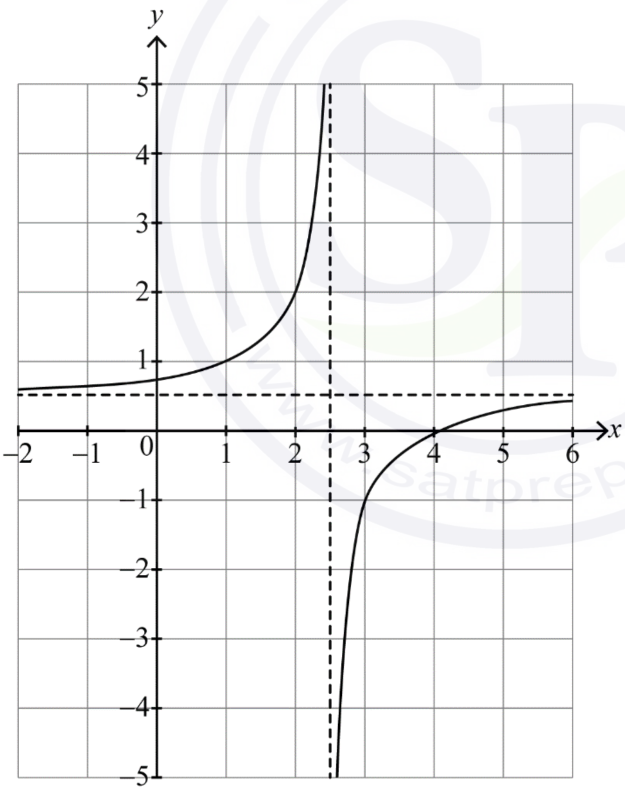
$= \int \left(u^{-\frac{1}{2}} - u^{\frac{3}{2}} \right) \mathrm{d}u$

$= 2u^{\frac{1}{2}} - \frac{2u^{\frac{5}{2}}}{5} (+c)$ ***(A1)***

$= 2\sqrt{\sin x} - \frac{2\left(\sqrt{\sin x}\right)^5}{5} (+c)$ or equivalent ***A1***

[5 marks]

5. (a)



correct shape: two branches in correct quadrants with asymptotic behaviour ***A1***

crosses at $(4, 0)$ and $\left(0, \frac{4}{5}\right)$ ***A1A1***

asymptotes at $x = \frac{5}{2}$ and $y = \frac{1}{2}$ ***A1A1***

[5 marks]

continued…

Question 5 continued

(b) (i) $x < \frac{5}{2}, x \geq 4$ ***A1A1***

(ii) $f(x) \geq 0, f(x) \neq \frac{1}{\sqrt{2}} \left(f(x) \in \mathbb{R}\right)$ ***A1***

Note: Follow through from their graph, as long as it is a rectangular hyperbola.

Note: Allow range expressed in terms of y.

[3 marks]

Total [8 marks]

6. (a) attempt to differentiate implicitly ***M1***

$$\frac{dy}{dx} = x\sec^2\left(\frac{\pi xy}{4}\right)\left[\frac{\pi}{4}x\frac{dy}{dx} + \frac{\pi}{4}y\right] + \tan\left(\frac{\pi xy}{4}\right)$$ ***A1A1***

Note: Award ***A1*** for each term.

attempt to substitute $x = 1,\ y = 1$ into their equation for $\frac{dy}{dx}$ ***M1***

$$\frac{dy}{dx} = \frac{\pi}{2}\frac{dy}{dx} + \frac{\pi}{2} + 1$$

$$\frac{dy}{dx}\left(1 - \frac{\pi}{2}\right) = \frac{\pi}{2} + 1$$ ***A1***

$$\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$$ ***AG***

[5 marks]

(b) attempt to use gradient of normal $= \frac{-1}{\frac{dy}{dx}}$ ***(M1)***

$$= \frac{\pi - 2}{\pi + 2}$$

so equation of normal is $y - 1 = \frac{\pi-2}{\pi+2}(x-1)$ or $y = \frac{\pi-2}{\pi+2}x + \frac{4}{\pi+2}$ ***A1***

[2 marks]

Total [7 marks]

7. use of at least one "log rule" applied correctly for the first equation ***M1***

$\log_2 6x = \log_2 2 + 2\log_2 y$

$= \log_2 2 + \log_2 y^2$

$= \log_2 \left(2y^2\right)$

$\Rightarrow 6x = 2y^2$ ***A1***

use of at least one "log rule" applied correctly for the second equation ***M1***

$\log_6 \left(15y - 25\right) = 1 + \log_6 x$

$= \log_6 6 + \log_6 x$

$= \log_6 6x$

$\Rightarrow 15y - 25 = 6x$ ***A1***

attempt to eliminate x (or y) from their two equations ***M1***

$2y^2 = 15y - 25$

$2y^2 - 15y + 25 = 0$

$(2y-5)(y-5) = 0$

$x = \frac{25}{12}, \; y = \frac{5}{2},$ ***A1***

or $x = \frac{25}{3}, \; y = 5$ ***A1***

Note: x, y values do not have to be "paired" to gain either of the final two ***A*** marks.

[7 marks]

8. (a) attempt to use Pythagoras in triangle OXB ***M1***

$\Rightarrow r^2 = R^2 - (h-R)^2$ ***A1***

substitution of their r^2 into formula for volume of cone $V = \frac{\pi r^2 h}{3}$ ***M1***

$= \frac{\pi h}{3}\left(R^2 - (h-R)^2\right)$

$= \frac{\pi h}{3}\left(R^2 - \left(h^2 + R^2 - 2hR\right)\right)$ ***A1***

Note: This **A** mark is independent and may be seen anywhere for the correct expansion of $(h-R)^2$.

$= \frac{\pi h}{3}\left(2hR - h^2\right)$

$= \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ ***AG***

[4 marks]

continued…

Question 8 continued

(b) at max, $\frac{dV}{dh} = 0$ ***R1***

$\frac{dV}{dh} = \frac{\pi}{3}\left(4Rh - 3h^2\right)$

$\Rightarrow 4Rh = 3h^2$

$\Rightarrow h = \frac{4R}{3}$ (since $h \neq 0$) ***A1***

EITHER

$V_{max} = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ from part (a)

$= \frac{\pi}{3}\left(2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right)$ ***A1***

$= \frac{\pi}{3}\left(2R\frac{16R^2}{9} - \left(\frac{64R^3}{27}\right)\right)$ ***A1***

OR

$r^2 = R^2 - \left(\frac{4R}{3} - R\right)^2$

$r^2 = R^2 - \frac{R^2}{9} = \frac{8R^2}{9}$ ***A1***

$\Rightarrow V_{max} = \frac{\pi r^2}{3}\left(\frac{4R}{3}\right)$

$= \frac{4\pi R}{9}\left(\frac{8R^2}{9}\right)$ ***A1***

THEN

$= \frac{32\pi R^3}{81}$ ***AG***

[4 marks]

Total [8 marks]

Section B

9. (a) $3\cos 2x = 4 - 11\cos x$

attempt to form a quadratic in $\cos x$ ***M1***

$3\left(2\cos^2 x - 1\right) = 4 - 11\cos x$ ***A1***

$\left(6\cos^2 x + 11\cos x - 7 = 0\right)$

valid attempt to solve their quadratic ***M1***

$(3\cos x + 7)(2\cos x - 1) = 0$

$\cos x = \frac{1}{2}$ ***A1***

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ ***A1A1***

Note: Ignore any "extra" solutions.

[6 marks]

(b) consider $(\pm)\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (4 - 11\cos x - 3\cos 2x)\,\mathrm{d}x$ ***M1***

$= (\pm)\left[4x - 11\sin x - \frac{3}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$ ***A1***

Note: Ignore lack of or incorrect limits at this stage.

attempt to substitute their limits into their integral ***M1***

$= \frac{20\pi}{3} - 11\sin\frac{5\pi}{3} - \frac{3}{2}\sin\frac{10\pi}{3} - \left(\frac{4\pi}{3} - 11\sin\frac{\pi}{3} - \frac{3}{2}\sin\frac{2\pi}{3}\right)$

$= \frac{16\pi}{3} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}$

$= \frac{16\pi}{3} + \frac{25\sqrt{3}}{2}$ ***A1A1***

[5 marks]

(c) attempt to differentiate both functions and equate ***M1***

$-6\sin 2x = 11\sin x$ ***A1***

attempt to solve for x ***M1***

$11\sin x + 12\sin x\cos x = 0$

$\sin x(11 + 12\cos x) = 0$

$\cos x = -\frac{11}{12}$ $(\text{or } \sin x = 0)$ ***A1***

$\Rightarrow y = 4 - 11\left(-\frac{11}{12}\right)$ ***M1***

$y = \frac{169}{12}\left(= 14\frac{1}{12}\right)$ ***A1***

[6 marks]

Total [17 marks]

10. (a) mode is 0 ***A1***

[1 mark]

(b) (i) attempt at integration by parts ***(M1)***

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}, dv = dx$$

$$= x\arcsin x - \int \frac{x dx}{\sqrt{1-x^2}}$$ ***A1***

$$= x\arcsin x + \sqrt{1-x^2} \,(+c)$$ ***A1***

(ii) $$\int_0^1 (\pi - \arcsin x)\, dx = \left[\pi x - x\arcsin x - \sqrt{1-x^2}\right]_0^1$$ ***A1***

$$= \left(\pi - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi}{2} + 1$$

$$= \frac{\pi + 2}{2}$$ ***A1***

$$\int_0^1 k(\pi - \arcsin x)\, dx = 1$$ ***(M1)***

Note: This line can be seen (or implied) anywhere.

Note: Do not allow ***FT A*** marks from bi to bii.

$$k\left(\frac{\pi+2}{2}\right) = 1$$

$$\Rightarrow k = \frac{2}{2+\pi}$$ ***AG***

[6 marks]

(c) (i) attempt to use product rule to differentiate ***M1***

$$\frac{dy}{dx} = x\arcsin x + \frac{x^2}{2\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{4}$$ ***A2***

Note: Award ***A2*** for all terms correct, ***A1*** for 4 correct terms.

$$= x\arcsin x + \frac{2x^2}{4\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{1-x^2}{4\sqrt{1-x^2}}$$ ***A1***

Note: Award ***A1*** for equivalent combination of correct terms over a common denominator.

$$= x\arcsin x$$ ***AG***

continued…

Question 10 continued

(ii) $\mathrm{E}(X) = k\int_0^1 x(\pi - \arcsin x)\,\mathrm{d}x$ ***M1***

$= k\int_0^1 (\pi x - x\arcsin x)\,\mathrm{d}x$

$= k\left[\frac{\pi x^2}{2} - \frac{x^2}{2}\arcsin x + \frac{1}{4}\arcsin x - \frac{x}{4}\sqrt{1-x^2}\right]_0^1$ ***A1A1***

Note: Award ***A1*** for first term, ***A1*** for next 3 terms.

$= k\left[\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\pi}{8}\right) - (0)\right]$ ***A1***

$= \left(\frac{2}{2+\pi}\right)\frac{3\pi}{8}$ ***A1***

$= \frac{3\pi}{4(\pi+2)}$ ***AG***

[9 marks]

Total [16 marks]

11. (a) translation k units to the left (or equivalent) ***A1***

[1 mark]

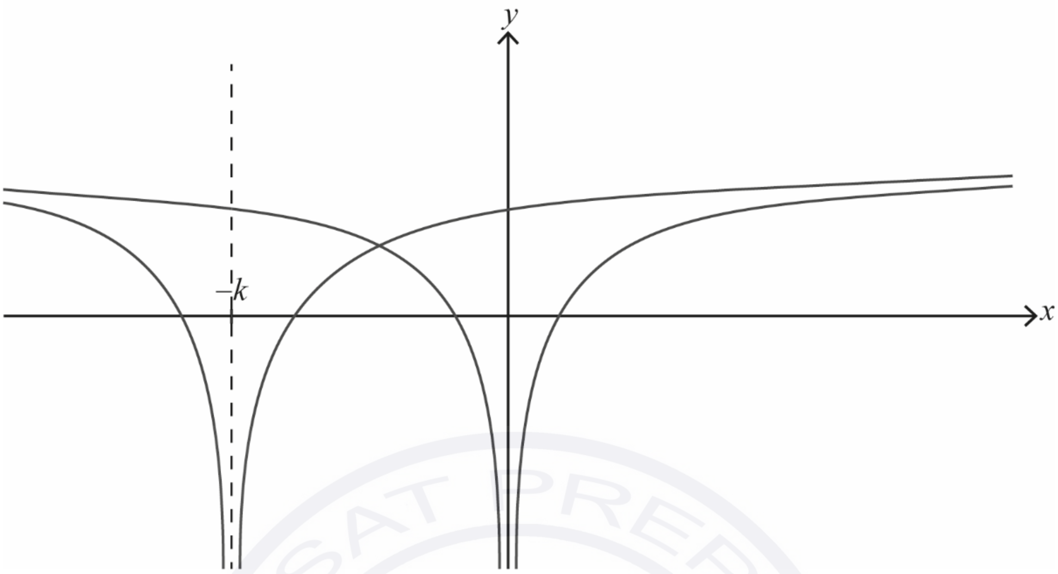
(b) range is $(g(x) \in)\mathbb{R}$ ***A1***

[1 mark]

continued…

Question 11 continued

(c)



correct shape of $y = f(x)$ ***A1***

their $f(x)$ translated k units to left (possibly shown by $x = -k$ marked on x-axis) ***A1***

asymptote included and marked as $x = -k$ ***A1***

$f(x)$ intersects x-axis at $x = -1,\ x = 1$ ***A1***

$g(x)$ intersects x-axis at $x = -k - 1,\ x = -k + 1$ ***A1***

$g(x)$ intersects y-axis at $y = \ln k$ ***A1***

Note: Do not penalise candidates if their graphs "cross" as $x \to \pm\infty$.

Note: Do not award ***FT*** marks from the candidate's part (a) to part (c).

[6 marks]

(d) at P $\ln(x + k) = \ln(-x)$

attempt to solve $x + k = -x$ (or equivalent) ***(M1)***

$x = -\frac{k}{2} \Rightarrow y = \ln\left(\frac{k}{2}\right)$ (or $y = \ln\left|\frac{k}{2}\right|$) ***A1***

$\mathrm{P}\left(-\frac{k}{2}, \ln\frac{k}{2}\right)$ (or $\mathrm{P}\left(-\frac{k}{2}, \ln\left|\frac{k}{2}\right|\right)$)

[2 marks]

continued…

Question 11 continued

(e) attempt to differentiate $\ln(-x)$ or $\ln|x|$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ ***A1***

at P, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2}{k}$ ***A1***

recognition that tangent passes through origin $\Rightarrow \frac{y}{x} = \frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\ln\left(\frac{k}{2}\right)}{-\frac{k}{2}} = \frac{-2}{k}$ ***A1***

$\ln\left(\frac{k}{2}\right) = 1$ ***(A1)***

$\Rightarrow k = 2\mathrm{e}$ ***A1***

[7 marks]

Note:	For candidates who explicitly differentiate $\ln(x)$ (rather than $\ln(-x)$ or $\ln\|x\|$, award ***M0A0A1M1A1A1A1***.

Total [17 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2018

Mathematics

Higher level

Paper 1

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2018**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
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  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
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3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

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6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).
A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

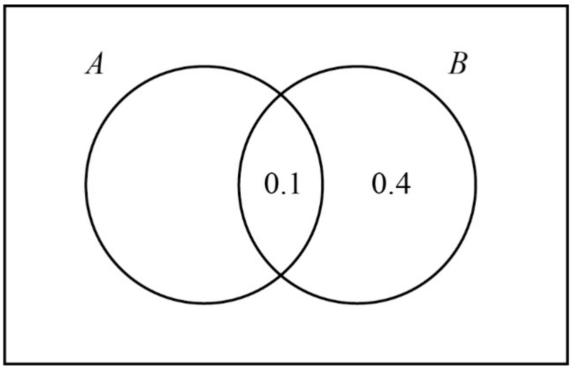
14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a)



(M1)

Note: Award ***M1*** for a Venn diagram with at least one probability in the correct region.

EITHER

$P(A \cap B') = 0.3$ ***(A1)***

$P(A \cup B) = 0.3 + 0.4 + 0.1 = 0.8$ ***A1***

OR

$P(B) = 0.5$ ***(A1)***

$P(A \cup B) = 0.5 + 0.4 - 0.1 = 0.8$ ***A1***

[3 marks]

(b) **METHOD 1**

$P(A)P(B) = 0.4 \times 0.5$ ***M1***

$= 0.2$ ***A1***

statement that their $P(A)P(B) \neq P(A \cap B)$ ***R1***

Note: Award ***R1*** for correct reasoning from their value.

$\Rightarrow A, B$ not independent ***AG***

METHOD 2

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.5}$ ***M1***

$= 0.2$ ***A1***

statement that their $P(A|B) \neq P(A)$ ***R1***

Note: Award ***R1*** for correct reasoning from their value.

$\Rightarrow A, B$ not independent ***AG***

Note: Accept equivalent argument using $P(B|A) = 0.25$.

[3 marks]

Total [6 marks]

2. (a) **METHOD 1**

$\binom{8}{4}$ ***(A1)***

$= \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5$ ***(M1)***

$= 70$ ***A1***

METHOD 2

recognition that they need to count the teams with 0 boys, 1 boy… 4 boys ***M1***

$1 + \binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} + 1$

$= 1 + (4 \times 4) + (6 \times 6) + (4 \times 4) + 1$ ***(A1)***

$= 70$ ***A1***

[3 marks]

(b) **EITHER**

recognition that the answer is the total number of teams minus the number of teams with all girls or all boys ***(M1)***

$70 - 2$

OR

recognition that the answer is the total of the number of teams with 1 boy, 2 boys, 3 boys ***(M1)***

$\binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} = (4 \times 4) + (6 \times 6) + (4 \times 4)$

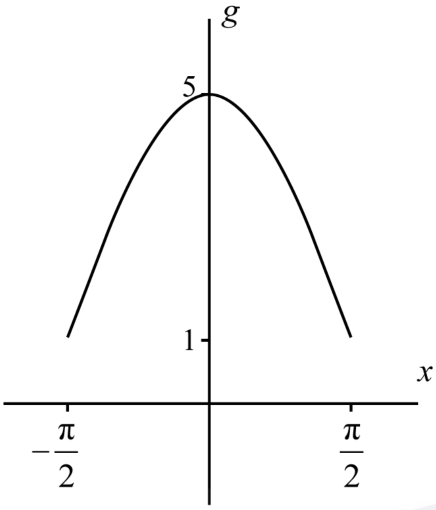
THEN

$= 68$ ***A1***

[2 marks]

Total [5 marks]

3. (a)



concave down and symmetrical over correct domain ***A1***
indication of maximum and minimum values of the function (correct range) ***A1A1***

[3 marks]

(b) $a = 0$ ***A1***

Note: Award ***A1*** for $a = 0$ only if consistent with their graph.

[1 mark]

(c) (i) $1 \le x \le 5$ ***A1***

Note: Allow FT from their graph.

(ii) $y = 4\cos x + 1$

$x = 4\cos y + 1$

$\frac{x-1}{4} = \cos y$ ***(M1)***

$\Rightarrow y = \arccos\left(\frac{x-1}{4}\right)$

$\Rightarrow g^{-1}(x) = \arccos\left(\frac{x-1}{4}\right)$ ***A1***

[3 marks]

Total [7 marks]

4. (a) an attempt at a valid method *eg* by inspection or row reduction ***(M1)***

$2 \times R_2 = R_1 \Rightarrow 2a = -1$

$\Rightarrow a = -\frac{1}{2}$ ***A1***

[2 marks]

continued…

Question 4 continued

(b) using elimination or row reduction to eliminate one variable ***(M1)***
correct pair of equations in 2 variables, such as

$$\left.\begin{aligned} 5x+10y=25 \\ 5x+12y=4 \end{aligned}\right\}$$ ***A1***

Note: Award ***A1*** for $z=0$ and one other equation in two variables.

attempting to solve for these two variables ***(M1)***
$x = 26, y = -10.5, z = 0$ ***A1A1***

Note: Award ***A1A0*** for only two correct values, and ***A0A0*** for only one.

Note: Award marks in part (b) for equivalent steps seen in part (a).

[5 marks]

Total [7 marks]

5. (a) $\boldsymbol{a}\bullet\boldsymbol{b} = (1\times 0) + (1\times -t) + (t\times 4t)$ ***(M1)***
$= -t + 4t^2$ ***A1***

[2 marks]

(b) recognition that $\boldsymbol{a}\bullet\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ ***(M1)***
$\boldsymbol{a}\bullet\boldsymbol{b} < 0$ or $-t + 4t^2 < 0$ or $\cos\theta < 0$ ***R1***

Note: Allow $\le$ for ***R1***.

attempt to solve using sketch or sign diagram ***(M1)***
$0 < t < \frac{1}{4}$ ***A1***

[4 marks]

Total [6 marks]

6. consider $n = 1$. $1(1!) = 1$ and $2! - 1 = 1$ therefore true for $n = 1$ ***R1***

Note: There must be evidence that $n = 1$ has been substituted into both expressions, or an expression such LHS=RHS=1 is used. "therefore true for $n = 1$" or an equivalent statement must be seen.

assume true for $n = k$, (so that $\sum_{r=1}^{k} r(r!) = (k+1)! - 1$) ***M1***

Note: Assumption of truth must be present.

consider $n = k + 1$

$\sum_{r=1}^{k+1} r(r!) = \sum_{r=1}^{k} r(r!) + (k+1)(k+1)!$ ***(M1)***

$= (k+1)! - 1 + (k+1)(k+1)!$ ***A1***

$= (k+2)(k+1)! - 1$ ***M1***

Note: ***M1*** is for factorising $(k+1)!$

$= (k+2)! - 1$

$= \left((k+1)+1\right)! - 1$

so if true for $n = k$, then also true for $n = k + 1$, and as true for $n = 1$ then true for all $n \left(\in \mathbb{Z}^+\right)$ ***R1***

Note: Only award final ***R1*** if all three method marks have been awarded. Award ***R0*** if the proof is developed from both LHS and RHS.

Total [6 marks]

7. (a) $C_1 : y + x\dfrac{dy}{dx} = 0$ ***(M1)***

Note: ***M1*** is for use of both product rule and implicit differentiation.

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$ ***A1***

Note: Accept $-\dfrac{4}{x^2}$.

$C_2 :\ 2y\dfrac{dy}{dx} - 2x = 0$ ***(M1)***

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$ ***A1***

Note: Accept $\pm\dfrac{x}{\sqrt{2+x^2}}$.

[4 marks]

continued…

Question 7 continued

(b) substituting a and b for x and y **M1**

product of gradients at P is $\left(-\dfrac{b}{a}\right)\left(\dfrac{a}{b}\right) = -1$ or equivalent reasoning **R1**

Note: The ***R1*** is dependent on the previous ***M1***.

so tangents are perpendicular **AG**

[2 marks]

Total [6 marks]

8. $-\mathrm{i}\sqrt{3}$ is a root ***(A1)***

$3 + \log_2 3 - \log_2 6\left(= 3 + \log_2 \dfrac{1}{2} = 3 - 1 = 2\right)$ is a root ***(A1)***

sum of roots: $-a = 3 + \log_2 3 \Rightarrow a = -3 - \log_2 3$ ***M1***

Note: Award ***M1*** for use of $-a$ is equal to the sum of the roots, do not award if minus is missing.

Note: If expanding the factored form of the equation, award ***M1*** for equating a to the coefficient of z^3.

product of roots: $(-1)^4 d \quad = 2(\log_2 6)\left(\mathrm{i}\sqrt{3}\right)\left(-\mathrm{i}\sqrt{3}\right)$ ***M1***

$= 6\log_2 6$ ***A1***

Note: Award ***M1A0*** for $d = -6\log_2 6$.

$6a + d + 12 = -18 - 6\log_2 3 + 6\log_2 6 + 12$

EITHER

$= -6 + 6\log_2 2 = 0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

OR

$= -6 - 6\log_2 3 + 6\log_2 3 + 6\log_2 2 = 0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

[7 marks]

Section B

9. (a) **METHOD 1**

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2b \\ 0 \\ b-1 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$$ ***(M1)A1***

$(0, 0, 0)$ on Π so $(b-1)x + 4by - 2bz = 0$ ***(M1)A1***

METHOD 2

using equation of the form $px + qy + rz = 0$ ***(M1)***

$(0, 1, 2)$ on $\Pi \Rightarrow q + 2r = 0$

$(2b, 0, b-1)$ on $\Pi \Rightarrow 2bp + r(b-1) = 0$ ***(M1)A1***

Note: Award ***(M1)A1*** for both equations seen.

solve for p, q, and r ***(M1)***

$(b-1)x + 4by - 2bz = 0$ ***A1***

[5 marks]

(b) M has coordinates $\left(b, 0, \dfrac{b-1}{2}\right)$ ***(A1)***

$$\boldsymbol{r} = \begin{pmatrix} b \\ 0 \\ \dfrac{b-1}{2} \end{pmatrix} + \lambda \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$$ ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ (or equivalent) is not seen.

Note: Allow equivalent forms such as $\dfrac{x-b}{b-1} = \dfrac{y}{4b} = \dfrac{2z-b+1}{-4b}$.

[3 marks]

continued…

Question 9 continued

(c) **METHOD 1**

$x = z = 0$ ***(M1)***

Note: Award ***M1*** for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ ***A1***

attempt to eliminate λ ***M1***

$\Rightarrow -\frac{b}{b-1} = \frac{b-1}{4b}$ ***(A1)***

$-4b^2 = (b-1)^2$ ***A1***

EITHER

consideration of the signs of LHS and RHS ***(M1)***
the LHS is negative and the RHS must be positive (or equivalent statement) ***R1***

OR

$-4b^2 = b^2 - 2b + 1$

$\Rightarrow 5b^2 - 2b + 1 = 0$

$\Delta = (-2)^2 - 4 \times 5 \times 1 = -16 \ (< 0)$ ***M1***

$\therefore$ no real solutions ***R1***

THEN

so no point of intersection ***AG***

METHOD 2

$x = z = 0$ ***(M1)***

Note: Award ***M1*** for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ ***A1***

attempt to eliminate b ***M1***

$\Rightarrow \frac{\lambda}{1+\lambda} = \frac{1}{1-4\lambda}$ ***(A1)***

$-4\lambda^2 = 1 \left(\Rightarrow \lambda^2 = -\frac{1}{4} \right)$ ***A1***

consideration of the signs of LHS and RHS ***(M1)***
there are no real solutions (or equivalent statement) ***R1***
so no point of intersection ***AG***

[7 marks]

Total [15 marks]

10. (a) **METHOD 1**

attempt at integration by parts with $u = e^x$, $\frac{dv}{dx} = \cos 2x$ ***M1***

$\int e^x \cos 2x \, dx = \frac{e^x}{2} \sin 2x \, dx - \int \frac{e^x}{2} \sin 2x \, dx$ ***A1***

$= \frac{e^x}{2} \sin 2x - \frac{1}{2}\left(-\frac{e^x}{2} \cos 2x + \int \frac{e^x}{2} \cos 2x\right)$ ***M1A1***

$= \frac{e^x}{2} \sin 2x + \frac{e^x}{4} \cos 2x - \frac{1}{4} \int e^x \cos 2x \, dx$

$\therefore \frac{5}{4} \int e^x \cos 2x \, dx = \frac{e^x}{2} \sin 2x + \frac{e^x}{4} \cos 2x$ ***M1***

$\int e^x \cos 2x \, dx = \frac{2e^x}{5} \sin 2x + \frac{e^x}{5} \cos 2x (+c)$ ***AG***

METHOD 2

attempt at integration by parts with $u = \cos 2x$, $\frac{dv}{dx} = e^x$ ***M1***

$\int e^x \cos 2x \, dx = e^x \cos 2x + 2\int e^x \sin 2x \, dx$ ***A1***

$= e^x \cos 2x + 2\left(e^x \sin 2x - 2\int e^x \cos 2x \, dx\right)$ ***M1A1***

$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, dx$

$\therefore 5\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$ ***M1***

$\int e^x \cos 2x \, dx = \frac{2e^x}{5} \sin 2x + \frac{e^x}{5} \cos 2x (+c)$ ***AG***

METHOD 3

attempt at use of table ***M1***

eg

$\cos 2x$	e^x
$-2\sin 2x$	e^x
$-4\cos 2x$	e^x

A1A1

Note: *A1* for first 2 lines correct, ***A1*** for third line correct.

$\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, dx$ ***M1***

$\therefore 5\int e^x \cos 2x \, dx = = e^x \cos 2x + 2e^x \sin 2x$ ***M1***

$\int e^x \cos 2x \, dx = \frac{2e^x}{5} \sin 2x + \frac{e^x}{5} \cos 2x (+c)$ ***AG***

[5 marks]

continued…

Question 10 continued

(b) $\int e^x \cos^2 x dx = \int \frac{e^x}{2}(\cos 2x + 1)\,dx$ ***M1A1***

$= \frac{1}{2}\left(\frac{2e^x}{5}\sin 2x + \frac{e^x}{5}\cos 2x\right) + \frac{e^x}{2}$ ***A1***

$= \frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2}\ (+c)$ ***AG***

Note: Do not accept solutions where the RHS is differentiated.

[3 marks]

(c) $f'(x) = e^x \cos^2 x - 2e^x \sin x \cos x$ ***M1A1***

Note: Award ***M1*** for an attempt at both the product rule and the chain rule.

$e^x \cos x(\cos x - 2\sin x) = 0$ ***(M1)***

Note: Award ***M1*** for an attempt to factorise $\cos x$ or divide by $\cos x\left(\cos x \neq 0\right)$.

discount $\cos x = 0$ (as this would also be a zero of the function)

$\Rightarrow \cos x - 2\sin x = 0$

$\Rightarrow \tan x = \frac{1}{2}$ ***(M1)***

$\Rightarrow x = \arctan\left(\frac{1}{2}\right)$ (at A) and $x = \pi + \arctan\left(\frac{1}{2}\right)$ (at C) ***A1A1***

Note: Award ***A1*** for each correct answer. If extra values are seen award ***A1A0***.

[6 marks]

(d) $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ ***A1***

Note: The ***A1*** may be awarded for work seen in part (c).

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(e^x \cos^2 x\right) dx = \left[\frac{e^x}{5}\sin 2x + \frac{e^x}{10}\cos 2x + \frac{e^x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ ***M1***

$= \left(-\frac{e^{\frac{3\pi}{2}}}{10} + \frac{e^{\frac{3\pi}{2}}}{2}\right) - \left(-\frac{e^{\frac{\pi}{2}}}{10} + \frac{e^{\frac{\pi}{2}}}{2}\right)\left(= \frac{2e^{\frac{3\pi}{2}}}{5} - \frac{2e^{\frac{\pi}{2}}}{5}\right)$ ***M1(A1)A1***

Note: Award ***M1*** for substitution of the end points and subtracting, ***(A1)*** for $\sin 3\pi = \sin \pi = 0$ and $\cos 3\pi = \cos \pi = -1$ and ***A1*** for a completely correct answer.

[5 marks]

Total [19 marks]

11. (a) $\left(r(\cos\theta + \mathrm{i}\sin\theta)\right)^{24} = 1(\cos 0 + \mathrm{i}\sin 0)$

use of De Moivre's theorem ***(M1)***

$r^{24} = 1 \Rightarrow r = 1$ ***(A1)***

$24\theta = 2\pi n \Rightarrow \theta = \frac{\pi n}{12}, (n \in \mathbb{Z})$ ***(A1)***

$0 < \arg z < \frac{\pi}{2} \Rightarrow n = 1, 2, 3, 4, 5$

$z = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{2\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{3\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{4\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{5\pi \mathrm{i}}{12}}$ ***A2***

Note: Award ***A1*** if additional roots are given or if three correct roots are given with no incorrect (or additional) roots.

[5 marks]

(b) (i) $\operatorname{Re} S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\operatorname{Im} S = \sin\frac{\pi}{12} + \sin\frac{2\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{4\pi}{12} + \sin\frac{5\pi}{12}$ ***A1***

Note: Award ***A1*** for both parts correct.

but $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}, \sin\frac{4\pi}{12} = \cos\frac{2\pi}{12}, \sin\frac{3\pi}{12} = \cos\frac{3\pi}{12},$

$\sin\frac{2\pi}{12} = \cos\frac{4\pi}{12}$ and $\sin\frac{\pi}{12} = \cos\frac{5\pi}{12}$ ***M1A1***

$\Rightarrow \operatorname{Re} S = \operatorname{Im} S$ ***AG***

Note: Accept a geometrical method.

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ ***M1A1***

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ ***A1***

continued…

Question 11 continued

(iii) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$ ***(M1)***

Note: Allow alternative methods *eg* $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)***

$\operatorname{Re} S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\operatorname{Re} S = \frac{\sqrt{2}+\sqrt{6}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}$ ***A1***

$= \frac{1}{2}\left(\sqrt{6} + 1 + \sqrt{2} + \sqrt{3}\right)$ ***A1***

$= \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)$

$S = \operatorname{Re}(S)(1+\mathrm{i})$ since $\operatorname{Re} S = \operatorname{Im} S$, ***R1***

$S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + \mathrm{i})$ ***AG***

[11 marks]

Total [16 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2018

Mathematics

Higher level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2018**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

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Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

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13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

$1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

attempt to solve their two equations ***M1***

$p = -3$, $q = 2$ ***A1***

[5 marks]

2. (a) attempt at chain rule or product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\sin\theta\cos\theta$ ***A1***

[2 marks]

(b) $2\sin\theta\cos\theta = 2\sin^2\theta$

$\sin\theta = 0$ ***(A1)***

$\theta = 0, \pi$ ***A1***

obtaining $\cos\theta = \sin\theta$ ***(M1)***

$\tan\theta = 1$ ***(M1)***

$\theta = \frac{\pi}{4}$ ***A1***

[5 marks]

Total [7 marks]

3. (a) $a = \frac{3}{16}$ and $b = \frac{5}{16}$ ***(M1)A1A1***

[3 marks]

Note: Award ***M1*** for consideration of the possible outcomes when rolling the two dice.

continued

Question 3 continued

(b) $\mathrm{E}(T) = \frac{1+6+15+28}{16} = \frac{25}{8}\left(= 3.125\right)$ ***(M1)A1***

Note: Allow follow through from part (a) even if probabilities do not add up to 1.

[2 marks]

Total [5 marks]

4. (a) $\int_{-2}^{0} f(x)\mathrm{d}x = 10 - 12 = -2$ ***(M1)(A1)***

$\int_{-2}^{0} 2\,\mathrm{d}x = \left[2x\right]_{-2}^{0} = 4$ ***A1***

$\int_{-2}^{0} \left(f(x) + 2\right)\mathrm{d}x = 2$ ***A1***

[4 marks]

(b) $\int_{-2}^{0} f(x+2)\mathrm{d}x = \int_{0}^{2} f(x)\mathrm{d}x$ ***(M1)***

$= 12$ ***A1***

[2 marks]

Total [6 marks]

5. $(\ln x)^2 - (\ln 2)(\ln x) - 2(\ln 2)^2 \left(= 0\right)$

EITHER

$\ln x = \frac{\ln 2 \pm \sqrt{(\ln 2)^2 + 8(\ln 2)^2}}{2}$ ***M1***

$= \frac{\ln 2 \pm 3\ln 2}{2}$ ***A1***

OR

$(\ln x - 2\ln 2)(\ln x + \ln 2)\left(= 0\right)$ ***M1A1***

THEN

$\ln x = 2\ln 2$ or $-\ln 2$ ***A1***

$\Rightarrow x = 4$ or $x = \frac{1}{2}$ ***(M1)A1***

Note: ***(M1)*** is for an appropriate use of a log law in either case, dependent on the previous ***M1*** being awarded, ***A1*** for both correct answers.

solution is $\frac{1}{2} < x < 4$ ***A1***

[6 marks]

6. if $n = 1$

$\text{LHS} = 1\,;\text{RHS} = 4 - \frac{3}{2^0} = 4 - 3 = 1$ ***M1***

hence true for $n = 1$

assume true for $n = k$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$

if $n = k + 1$

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$

$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$= 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k}$

$= 4 - \frac{2(k+2)-(k+1)}{2^k} = 4 - \frac{k+3}{2^k}\left(= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n = k$ then also true for $n = k + 1$, as true for $n = 1$, so true (for all $n \in \mathbb{Z}^+$) ***R1***

Note: Award the final ***R1*** only if the first four marks have been awarded.

[7 marks]

7. (a) $y = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \left(= -\frac{1}{\sqrt{4-x^2}}\right)$ ***M1A1***

Note: ***M1*** is for use of the chain rule.

[2 marks]

(b) attempt at integration by parts ***M1***

$u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{4-x^2}}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$ ***(A1)***

$\int_0^1 \arccos\left(\frac{x}{2}\right)\mathrm{d}x = \left[x\arccos\left(\frac{x}{2}\right)\right]_0^1 + \int_0^1 \frac{x}{\sqrt{4-x^2}}\mathrm{d}x$ ***A1***

using integration by substitution or inspection ***(M1)***

$\left[x\arccos\left(\frac{x}{2}\right)\right]_0^1 + \left[-\left(4-x^2\right)^{\frac{1}{2}}\right]_0^1$ ***A1***

Note: Award ***A1*** for $-\left(4-x^2\right)^{\frac{1}{2}}$ or equivalent.

Note: Condone lack of limits to this point.

attempt to substitute limits into their integral ***M1***

$= \frac{\pi}{3} - \sqrt{3} + 2$ ***A1***

[7 marks]

Total [9 marks]

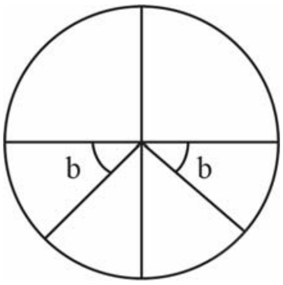
8. $\sin 2x = -\sin b$

EITHER

$\sin 2x = \sin(-b)$ or $\sin 2x = \sin(\pi + b)$ or $\sin 2x = \sin(2\pi - b) \ldots$ ***(M1)(A1)***

Note: Award ***M1*** for any one of the above, ***A1*** for having final two.

OR



(M1)(A1)

Note: Award ***M1*** for one of the angles shown with b clearly labelled, ***A1*** for both angles shown. Do not award ***A1*** if an angle is shown in the second quadrant and subsequent ***A1*** marks not awarded.

THEN

$2x = \pi + b$ or $2x = 2\pi - b$ ***(A1)(A1)***

$x = \frac{\pi}{2} + \frac{b}{2}$, $x = \pi - \frac{b}{2}$ ***A1***

[5 marks]

Section B

9. (a) attempt to differentiate ***(M1)***

$f'(x) = -3x^{-4} - 3x$ ***A1***

> **Note:** Award ***M1*** for using quotient or product rule award ***A1*** if correct derivative seen even in unsimplified form, for example $f'(x) = \frac{-15x^4 \times 2x^3 - 6x^2\left(2-3x^5\right)}{\left(2x^3\right)^2}$.

$-\frac{3}{x^4} - 3x = 0$ ***M1***

$\Rightarrow x^5 = -1 \Rightarrow x = -1$ ***A1***

$\text{A}\left(-1, -\frac{5}{2}\right)$ ***A1***

[5 marks]

(b) (i) $f''(x) = 0$ ***M1***

$f''(x) = 12x^{-5} - 3(= 0)$ ***A1***

> **Note:** Award ***A1*** for correct derivative seen even if not simplified.

$\Rightarrow x = \sqrt[5]{4}\left(= 2^{\frac{2}{5}}\right)$ ***A1***

hence (at most) one point of inflexion ***R1***

> **Note:** This mark is independent of the two ***A1*** marks above. If they have shown or stated their equation has only one solution this mark can be awarded.

$f''(x)$ changes sign at $x = \sqrt[5]{4}\left(= 2^{\frac{2}{5}}\right)$ ***R1***

so exactly one point of inflexion

continued

Question 9 continued

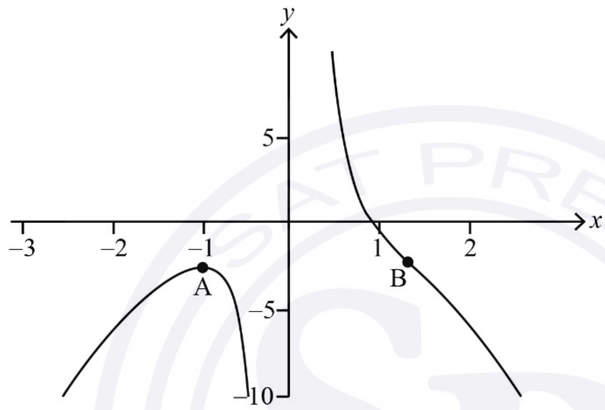
(ii) $x = \sqrt[5]{4} = 2^{\frac{2}{5}} \left(\Rightarrow a = \frac{2}{5}\right)$ ***A1***

$f\left(2^{\frac{2}{5}}\right) = \frac{2 - 3 \times 2^2}{2 \times 2^{\frac{6}{5}}} = -5 \times 2^{-\frac{6}{5}} \ (\Rightarrow b = -5)$ ***(M1)A1***

[8 marks]

Note: Award ***M1*** for the substitution of their value for x into $f(x)$.

(c)



A1A1A1A1

A1 for shape for $x < 0$
A1 for shape for $x > 0$
A1 for maximum at A
A1 for POI at B.

Note: Only award last two ***A1***s if A and B are placed in the correct quadrants, allowing for follow through.

[4 marks]

Total [17 marks]

10. (a) recognising normal to plane or attempting to find cross product of two vectors lying in the plane ***(M1)***

for example, $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_1 : x + z = 1$ ***A1***

[3 marks]

(b) **EITHER**

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 = \sqrt{2}\sqrt{2}\cos\theta$ ***M1A1***

OR

$\left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right| = \sqrt{3} = \sqrt{2}\sqrt{2}\sin\theta$ ***M1A1***

Note: ***M1*** is for an attempt to find the scalar or vector product of the two normal vectors.

$\Rightarrow \theta = 60^\circ \left(= \frac{\pi}{3} \right)$ ***A1***

angle between faces is $120^\circ \left(= \frac{2\pi}{3} \right)$ ***A1***

[4 marks]

(c) $\overrightarrow{DB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ or $\overrightarrow{BD} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_3 : x + y - z = k$ ***(M1)***

$\Pi_3 : x + y - z = 0$ ***A1***

[3 marks]

continued

Question 10 continued

(d) **METHOD 1**

line $\text{AD}: (\mathbf{r} =) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ***M1A1***

intersects Π_3 when $\lambda - (1-\lambda) = 0$ ***M1***

so $\lambda = \frac{1}{2}$ ***A1***

hence P is the midpoint of AD ***AG***

METHOD 2

midpoint of AD is $(0.5, 0, 0.5)$ ***(M1)A1***
substitute into $x + y - z = 0$ ***M1***
$0.5 + 0 - 0.5 = 0$ ***A1***
hence P is the midpoint of AD ***AG***

[4 marks]

(e) **METHOD 1**

$\text{OP} = \frac{1}{\sqrt{2}}, \ \text{O}\hat{\text{P}}\text{Q} = 90^\circ, \ \text{O}\hat{\text{Q}}\text{P} = 60^\circ$ ***A1A1A1***

$\text{PQ} = \frac{1}{\sqrt{6}}$ ***A1***

area $= \frac{1}{2\sqrt{12}} = \frac{1}{4\sqrt{3}} = \frac{\sqrt{3}}{12}$ ***A1***

continued

Question 10 continued

METHOD 2

line $\text{BD}:(\mathbf{r}=)\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\Rightarrow \lambda = \frac{2}{3}$ ***(A1)***

$\vec{\text{OQ}} = \begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$ ***A1***

area $= \frac{1}{2}\left|\vec{\text{OP}} \times \vec{\text{OQ}}\right|$ ***M1***

$\vec{\text{OP}} = \begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}$ ***A1***

Note: This ***A1*** is dependent on ***M1***.

area $= \frac{\sqrt{3}}{12}$ ***A1***

[5 marks]

Total [19 marks]

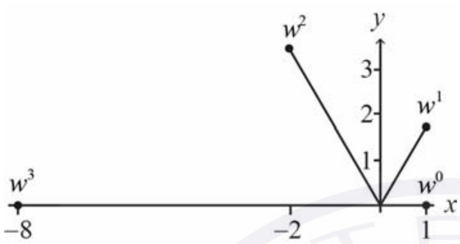
11. (a) (i) $w^2 = 4cis\left(\frac{2\pi}{3}\right)$; $w^3 = 8\text{cis}(\pi)$ ***(M1)A1A1***

Note: Accept Euler form.

Note: ***M1*** can be awarded for either both correct moduli or both correct arguments.

Note: Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

(ii)



A1A1

[5 marks]

(b) use of area $= \frac{1}{2}ab\sin C$ ***M1***

$\frac{1}{2}\times 1\times 2\times \sin\frac{\pi}{3} + \frac{1}{2}\times 2\times 4\times \sin\frac{\pi}{3} + \frac{1}{2}\times 4\times 8\times \sin\frac{\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $C = \frac{\pi}{3}$, ***A1*** for correct moduli.

$= \frac{21\sqrt{3}}{2}$ ***AG***

Note: Other methods of splitting the area may receive full marks.

[3 marks]

(c) $\frac{1}{2}\times 2^0\times 2^1\times \sin\frac{\pi}{n} + \frac{1}{2}\times 2^1\times 2^2\times \sin\frac{\pi}{n} + \frac{1}{2}\times 2^2\times 2^3\times \sin\frac{\pi}{n} + \ldots + \frac{1}{2}\times 2^{n-1}\times 2^n\times \sin\frac{\pi}{n}$

M1A1

Note: Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n}\times\left(2^0 + 2^2 + 2^4 + \ldots + 2^{2n-2}\right)$

identifying a geometric series with common ratio $2^2 (=4)$ ***(M1)A1***

$= \frac{1-2^{2n}}{1-4}\times\sin\frac{\pi}{n}$ ***M1***

Note: Award ***M1*** for use of formula for sum of geometric series.

$= \frac{1}{3}\left(4^n - 1\right)\sin\frac{\pi}{n}$ ***A1***

[6 marks]

Total [14 marks]

Markscheme

May 2018

Mathematics

Higher level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2018**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
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Section A

1. $\cos\theta = \dfrac{(3\boldsymbol{i}-4\boldsymbol{j}-5\boldsymbol{k})\bullet(5\boldsymbol{i}-4\boldsymbol{j}+3\boldsymbol{k})}{|3\boldsymbol{i}-4\boldsymbol{j}-5\boldsymbol{k}||5\boldsymbol{i}-4\boldsymbol{j}+3\boldsymbol{k}|}$ ***(M1)***

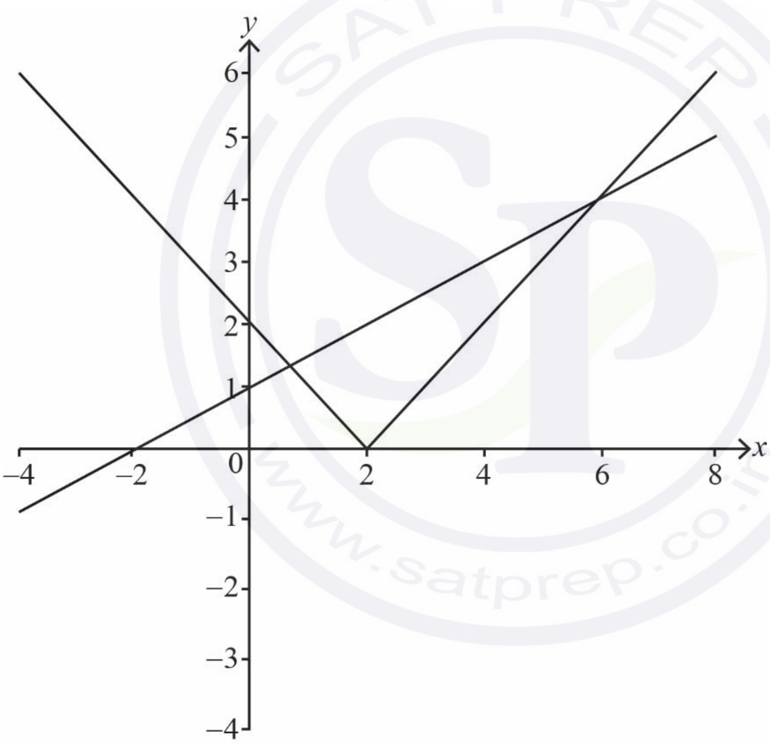
$= \dfrac{16}{\sqrt{50}\sqrt{50}}$ ***A1A1***

Note: ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \dfrac{8}{25}\left(=\dfrac{16}{50}=0.32\right)$ ***A1***

[4 marks]

2. (a)



straight line graph with correct axis intercepts ***A1***
modulus graph: V shape in upper half plane ***A1***
modulus graph having correct vertex and y-intercept ***A1***

[3 marks]

continued…

Question 2 continued

(b) **METHOD 1**

attempt to solve $\frac{x}{2} + 1 = x - 2$ ***(M1)***

$x = 6$ ***A1***

Note: Accept $x = 6$ using the graph.

attempt to solve (algebraically) $\frac{x}{2} + 1 = 2 - x$ ***M1***

$x = \frac{2}{3}$ ***A1***

[4 marks]

METHOD 2

$\left(\frac{x}{2} + 1\right)^2 = (x - 2)^2$ ***M1***

$\frac{x^2}{4} + x + 1 = x^2 - 4x + 4$

$0 = \frac{3x^2}{4} - 5x + 3$

$3x^2 - 20x + 12 = 0$

attempt to factorise (or equivalent) ***M1***

$(3x-2)(x-6) = 0$

$x = \frac{2}{3}$ ***A1***

$x = 6$ ***A1***

[4 marks]

Total [7 marks]

3. (a) equating sum of probabilities to 1 ($p + 0.5 - p + 0.25 + 0.125 + p^3 = 1$) ***M1***

$p^3 = 0.125 = \frac{1}{8}$

$p = 0.5$ ***A1***

[2 marks]

(b) (i) $\mu = 0 \times 0.5 + 1 \times 0 + 2 \times 0.25 + 3 \times 0.125 + 4 \times 0.125$ ***M1***

$= 1.375 \left(= \frac{11}{8}\right)$ ***A1***

continued…

Question 3 continued

(ii) $P(X > \mu) = P(X = 2) + P(X = 3) + P(X = 4)$ ***(M1)***
$= 0.5$ ***A1***

Note: Do not award follow through ***A*** marks in (b)(i) from an incorrect value of p.

Note: Award ***M*** marks in both (b)(i) and (b)(ii) provided no negative probabilities, and provided a numerical value for μ has been found.

[4 marks]

Total [6 marks]

4. valid attempt to find $\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = \frac{1}{(1-x)^2} - \frac{4}{(x-4)^2}$ ***A1A1***

attempt to solve $\frac{dy}{dx} = 0$ ***M1***

$x = 2,\ x = -2$ ***A1A1***

[6 marks]

5. (a) **METHOD 1**

state that $u_n = u_1 r^{n-1}$ (or equivalent) ***A1***

attempt to consider a_n and use of at least one log rule ***M1***

$\log_2 |u_n| = \log_2 |u_1| + (n-1)\log_2 |r|$ ***A1***

(which is an AP) with $d = \log_2 |r|$ (and 1st term $\log_2 |u_1|$) ***A1***

so A is an arithmetic sequence ***AG***

Note: Condone absence of modulus signs.

Note: The final ***A*** mark may be awarded independently.

Note: Consideration of the first two or three terms only will score ***M0***.

[4 marks]

continued…

Question 5 continued

METHOD 2

consideration of $(d =)a_{n+1} - a_n$ ***M1***

$(d) = \log_2 |u_{n+1}| - \log_2 |u_n|$

$(d) = \log_2 \left|\frac{u_{n+1}}{u_n}\right|$ ***M1***

$(d) = \log_2 |r|$ ***A1***

which is constant ***R1***

Note: Condone absence of modulus signs.

Note: the final ***A*** mark may be awarded independently.

Note: Consideration of the first two or three terms only will score ***M0***.

(b) attempting to solve $\frac{3}{1-r} = 4$ ***M1***

$r = \frac{1}{4}$ ***A1***

$d = -2$ ***A1***

[3 marks]

Total [7 marks]

6. (a) (i) attempt at product rule ***M1***

$f'(x) = -\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x$ ***A1***

(ii) $g'(x) = -\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x$ ***A1***

[3 marks]

(b) **METHOD 1**

Attempt to add $f'(x)$ and $g'(x)$ ***(M1)***

$f'(x) + g'(x) = -2\mathrm{e}^{-x}\sin x$ ***A1***

$\int_0^{\pi} \mathrm{e}^{-x}\sin x\,\mathrm{d}x = \left[-\frac{\mathrm{e}^{-x}}{2}(\sin x + \cos x)\right]_0^{\pi}$ (or equivalent) ***A1***

Note: Condone absence of limits.

$= \frac{1}{2}\left(1 + \mathrm{e}^{-\pi}\right)$ ***A1***

continued…

Question 6 continued

METHOD 2

$I = \int e^{-x} \sin x dx$

$= -e^{-x} \cos x - \int e^{-x} \cos x dx$ **OR** $= -e^{-x} \sin x + \int e^{-x} \cos x dx$ ***M1A1***

$= -e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \sin x dx$

$I = -\frac{1}{2} e^{-x} (\sin x + \cos x)$ ***A1***

$\int_0^{\pi} e^{-x} \sin x dx = \frac{1}{2}\left(1 + e^{-\pi}\right)$ ***A1***

[4 marks]

Total [7 marks]

7. (a) $\frac{z + w}{z - w} = \frac{(a + c) + i(b + d)}{(a - c) + i(b - d)}$

$= \frac{(a + c) + i(b + d)}{(a - c) + i(b - d)} \times \frac{(a - c) - i(b - d)}{(a - c) - i(b - d)}$ ***M1A1***

real part $= \frac{(a+c)(a-c) + (b+d)(b-d)}{(a - c)^2 + (b - d)^2} \left(= \frac{a^2 - c^2 + b^2 - d^2}{(a - c)^2 + (b - d)^2} \right)$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

[4 marks]

(b) $|z| = |w| \Rightarrow a^2 + b^2 = c^2 + d^2$ ***R1***

hence real part $= 0$ ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [6 marks]

8. (a) $\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$ (accept $du = \frac{1}{2} x^{-\frac{1}{2}} dx$ or equivalent) ***A1***

substitution, leading to an integrand in terms of u ***M1***

$\int \frac{2u du}{u^3 + u}$ or equivalent ***A1***

$= 2 \arctan(\sqrt{x})(+c)$ ***A1***

[4 marks]

continued…

Question 8 continued

(b) $\frac{1}{2}\int_1^9 \frac{dx}{x^{\frac{3}{2}} + x^{\frac{1}{2}}} = \arctan 3 - \arctan 1$ ***A1***

$\tan(\arctan 3 - \arctan 1) = \frac{3-1}{1+3\times 1}$ ***(M1)***

$\tan(\arctan 3 - \arctan 1) = \frac{1}{2}$

$\arctan 3 - \arctan 1 = \arctan \frac{1}{2}$ ***A1***

[3 marks]

Total [7 marks]

Section B

9. (a) (i) a pair of opposite sides have equal length and are parallel ***R1***
hence ABCD is a parallelogram ***AG***

(ii) attempt to rewrite the given information in vector form ***M1***
$\boldsymbol{b}-\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{d}$ ***A1***
rearranging $\boldsymbol{d}-\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{b}$ ***M1***
hence $\overrightarrow{AD}=\overrightarrow{BC}$ ***AG***

Note: Candidates may correctly answer part i) by answering part ii) correctly and then deducing there are two pairs of parallel sides.

[4 marks]

(b) **EITHER**

use of $\overrightarrow{AB}=\overrightarrow{DC}$ ***(M1)***

$\begin{pmatrix}2\\-3\\p+3\end{pmatrix}=\begin{pmatrix}q+1\\1-r\\4\end{pmatrix}$ ***A1A1***

OR

use of $\overrightarrow{AD}=\overrightarrow{BC}$ ***(M1)***

$\begin{pmatrix}-2\\r-2\\1\end{pmatrix}=\begin{pmatrix}q-3\\2\\2-p\end{pmatrix}$ ***A1A1***

THEN

attempt to compare coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ in their equation or statement to that effect ***M1***
clear demonstration that the given values satisfy their equation ***A1***
$p=1,\ q=1,\ r=4$ ***AG***

[5 marks]

(c) attempt at computing $\overrightarrow{AB}\times\overrightarrow{AD}$ (or equivalent) ***M1***

$\begin{pmatrix}-11\\-10\\-2\end{pmatrix}$ ***A1***

area $=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|\left(=\sqrt{225}\right)$ ***(M1)***

$=15$ ***A1***

[4 marks]

continued…

Question 9 continued

(d) valid attempt to find $\overrightarrow{OM}\left(=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{c})\right)$ ***(M1)***

$\begin{pmatrix}1\\ \frac{3}{2}\\ -\frac{1}{2}\end{pmatrix}$ ***A1***

the equation is

$\boldsymbol{r}=\begin{pmatrix}1\\ \frac{3}{2}\\ -\frac{1}{2}\end{pmatrix}+t\begin{pmatrix}11\\10\\2\end{pmatrix}$ or equivalent ***M1A1***

Note: Award maximum ***M1A0*** if $'\mathbf{r}=\ldots'$ (or equivalent) is not seen.

[4 marks]

(e) attempt to obtain the equation of the plane in the form $ax+by+cz=d$ ***M1***

$11x+10y+2z=25$ ***A1A1***

Note: ***A1*** for right hand side, ***A1*** for left hand side.

[3 marks]

(f) (i) putting two coordinates equal to zero ***(M1)***

$\text{X}\left(\frac{25}{11},0,0\right),\ \text{Y}\left(0,\frac{5}{2},0\right),\ \text{Z}\left(0,0,\frac{25}{2}\right)$ ***A1***

(ii) $\text{YZ}=\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{25}{2}\right)^2}$ ***M1***

$=\sqrt{\frac{325}{2}}\left(=\frac{5\sqrt{104}}{4}=\frac{5\sqrt{26}}{2}\right)$ ***A1***

[4 marks]

Total [24 marks]

10. (a) attempt to make x the subject of $y = \frac{ax+b}{cx+d}$ ***M1***

$y(cx+d) = ax + b$ ***A1***

$x = \frac{dy-b}{a-cy}$ ***A1***

$f^{-1}(x) = \frac{dx-b}{a-cx}$, ***A1***

Note: Do not allow $y =$ in place of $f^{-1}(x)$.

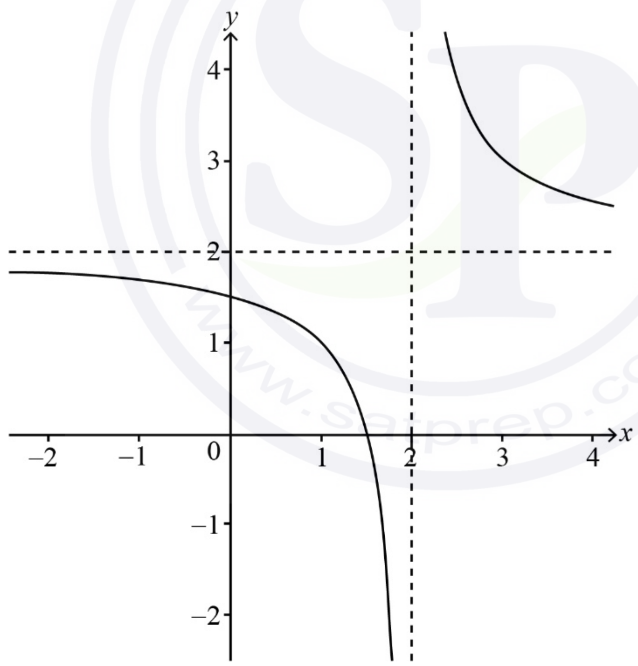
$x \neq \frac{a}{c}$, $(x \in \mathbb{R})$ ***A1***

Note: The final ***A*** mark is independent.

[5 marks]

(b) (i) $g(x) = 2 + \frac{1}{x-2}$ ***A1A1***

(ii)



hyperbola shape, with single curves in second and fourth quadrants and third quadrant blank, including vertical asymptote $x = 2$ ***A1***

horizontal asymptote $y = 2$ ***A1***

intercepts $\left(\frac{3}{2}, 0\right)$, $\left(0, \frac{3}{2}\right)$ ***A1***

[5 marks]

continued…

Question 10 continued

(c) the domain of $h \circ g$ is $x \le \frac{3}{2},\ x > 2$ ***A1A1***

the range of $h \circ g$ is $y \ge 0,\ y \ne \sqrt{2}$ ***A1A1***

[4 marks]

Total [14 marks]

11. (a) **METHOD 1**

$\log_{r^2} x = \frac{\log_r x}{\log_r r^2} \left(= \frac{\log_r x}{2\log_r r} \right)$ ***M1A1***

$= \frac{\log_r x}{2}$ ***AG***

[2 marks]

METHOD 2

$\log_{r^2} x = \frac{1}{\log_x r^2}$ ***M1***

$= \frac{1}{2\log_x r}$ ***A1***

$= \frac{\log_r x}{2}$ ***AG***

[2 marks]

(b) **METHOD 1**

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \log_4 2x^2 = 0$ ***M1***

$\log_2 y + \frac{1}{2}\log_2 2x^2 = 0$ ***M1***

$\log_2 y = -\frac{1}{2}\log_2 2x^2$

$\log_2 y = \log_2 \left(\frac{1}{\sqrt{2}x} \right)$ ***M1A1***

$y = \frac{1}{\sqrt{2}} x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

continued…

Question 11 continued

METHOD 2

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \frac{1}{2}\log_2 x + \frac{1}{2}\log_2 2x = 0$ ***M1***

$\log_2 y + \log_2 x^{\frac{1}{2}} + \log_2 (2x)^{\frac{1}{2}} = 0$ ***M1***

$\log_2\left(\sqrt{2}xy\right) = 0$ ***M1***

$\sqrt{2}xy = 1$ ***A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

(c) the area of R is $\int_1^{\alpha} \frac{1}{\sqrt{2}}x^{-1}\mathrm{d}x$ ***M1***

$= \left[\frac{1}{\sqrt{2}}\ln x\right]_1^{\alpha}$ ***A1***

$= \frac{1}{\sqrt{2}}\ln\alpha$ ***A1***

$\frac{1}{\sqrt{2}}\ln\alpha = \sqrt{2}$ ***M1***

$\alpha = \mathrm{e}^2$ ***A1***

Note: Only follow through from part (b) if y is in the form $y = px^q$.

[5 marks]

Total [12 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2017

Mathematics

Higher level

Paper 1

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2017**". It is essential that you read this document before you start marking.
In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

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Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

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Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

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Section A

1. $\log_2(x+3)+\log_2(x-3)=4$

$\log_2\left(x^2-9\right)=4$ ***(M1)***

$x^2-9=2^4\left(=16\right)$ ***M1A1***

$x^2=25$

$x=\pm 5$ ***(A1)***

$x=5$ ***A1***

[5 marks]

2. (a) $\overrightarrow{\text{AB}}=\begin{pmatrix}6\\-8\\17\end{pmatrix}$ ***(A1)***

$\boldsymbol{r}=\begin{pmatrix}0\\3\\-6\end{pmatrix}+\lambda\begin{pmatrix}6\\-8\\17\end{pmatrix}$ or $\boldsymbol{r}=\begin{pmatrix}6\\-5\\11\end{pmatrix}+\lambda\begin{pmatrix}6\\-8\\17\end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r}=$ is not seen (or equivalent).

[3 marks]

(b) substitute line L in $\Pi: 4(6\lambda)-3(3-8\lambda)+2(-6+17\lambda)=20$ ***M1***

$82\lambda=41$

$\lambda=\frac{1}{2}$ ***(A1)***

$\boldsymbol{r}=\begin{pmatrix}0\\3\\-6\end{pmatrix}+\frac{1}{2}\begin{pmatrix}6\\-8\\17\end{pmatrix}=\begin{pmatrix}3\\-1\\\frac{5}{2}\end{pmatrix}$

so coordinate is $\left(3,-1,\frac{5}{2}\right)$ ***A1***

Note: Accept coordinate expressed as position vector $\begin{pmatrix}3\\-1\\\frac{5}{2}\end{pmatrix}$.

[3 marks]

Total [6 marks]

3. (a) $q(4) = 0$ ***(M1)***

$192 - 176 + 4k + 8 = 0 \ (24 + 4k = 0)$ ***A1***

$k = -6$ ***A1***

[3 marks]

(b) $3x^3 - 11x^2 - 6x + 8 = (x - 4)\left(3x^2 + px - 2\right)$

equate coefficients of x^2: ***(M1)***

$-12 + p = -11$

$p = 1$

$(x - 4)\left(3x^2 + x - 2\right)$ ***(A1)***

$(x - 4)(3x - 2)(x + 1)$ ***A1***

Note: Allow part (b) marks if any of this work is seen in part (a).

Note: Allow equivalent methods (*eg*, synthetic division) for the ***M*** marks in each part.

[3 marks]

Total [6 marks]

4. each term is of the form $\binom{7}{r}\left(x^2\right)^{7-r}\left(\frac{-2}{x}\right)^r$ ***(M1)***

$= \binom{7}{r} x^{14-2r}(-2)^r x^{-r}$

so $14 - 3r = 8$ ***(A1)***

$r = 2$

so require $\binom{7}{2}\left(x^2\right)^5\left(\frac{-2}{x}\right)^2$ (or simply $\binom{7}{2}(-2)^2$) ***A1***

$= 21 \times 4$

$= 84$ ***A1***

Note: Candidates who attempt a full expansion, including the correct term, may only be awarded ***M1A0A0A0***.

[4 marks]

5. $s=\int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\,\mathrm{d}t$

attempt at integration by parts *M1*

$=\left[-5t\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}-5\mathrm{e}^{-2t}\,\mathrm{d}t$ *A1*

$=\left[-5t\mathrm{e}^{-2t}-\frac{5}{2}\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}}$ *(A1)*

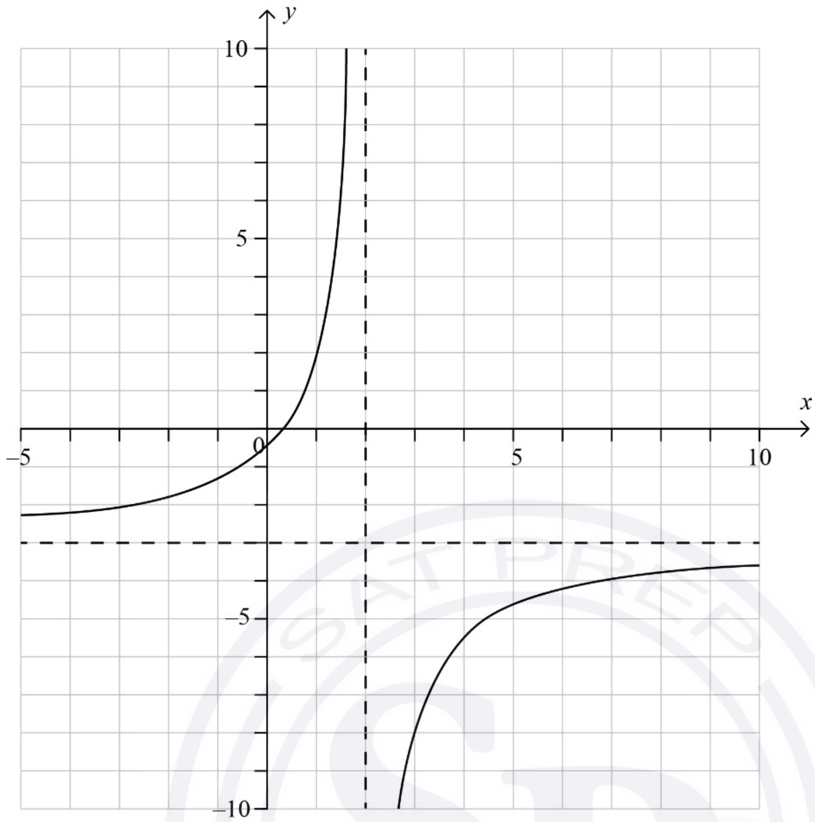
Note: Condone absence of limits (or incorrect limits) and missing factor of 10 up to this point.

$s=\int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\,\mathrm{d}t$ *(M1)*

$=-5e^{-1}+\frac{5}{2}\left(=\frac{-5}{\mathrm{e}}+\frac{5}{2}\right)\left(=\frac{5\mathrm{e}-10}{2\mathrm{e}}\right)$ *A1*

[5 marks]

6. (a)



correct vertical asymptote *A1*
shape including correct horizontal asymptote *A1*

$\left(\frac{1}{3}, 0\right)$ *A1*

$\left(0, -\frac{1}{2}\right)$ *A1*

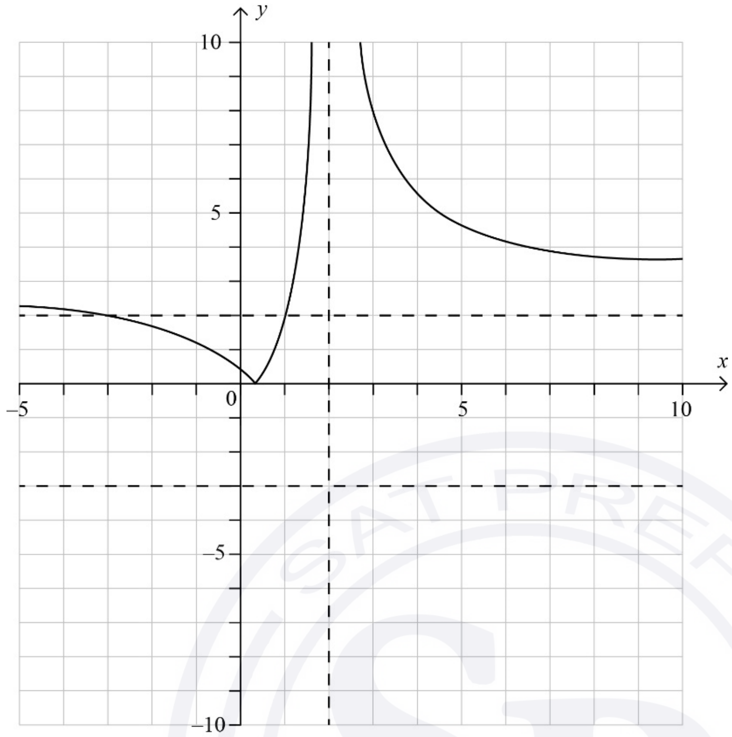
Note: Accept $x = \frac{1}{3}$ and $y = -\frac{1}{2}$ marked on the axes.

[4 marks]

continued…

Question 6 continued

(b) **METHOD 1**



$\frac{1-3x}{x-2} = 2$ ***(M1)***

$\Rightarrow x = 1$ ***A1***

$-\left(\frac{1-3x}{x-2}\right) = 2$ ***(M1)***

Note: Award this ***M1*** for the line above or a correct sketch identifying a second critical value.

$\Rightarrow x = -3$ ***A1***

solution is $-3 < x < 1$ ***A1***

[5 marks]

METHOD 2

$|1-3x| < 2|x-2|,\ x \neq 2$

$1 - 6x + 9x^2 < 4\left(x^2 - 4x + 4\right)$ ***(M1)A1***

$1 - 6x + 9x^2 < 4x^2 - 16x + 16$

$5x^2 + 10x - 15 < 0$

$x^2 + 2x - 3 < 0$ ***A1***

$(x+3)(x-1) < 0$ ***(M1)***

solution is $-3 < x < 1$ ***A1***

[5 marks]

continued…

Question 6 continued

METHOD 3

$-2 < \frac{1-3x}{x-2} < 2$

consider $\frac{1-3x}{x-2} < 2$ ***(M1)***

Note: Also allow consideration of "$>$" or "$=$" for the awarding of the ***M*** mark.

recognition of critical value at $x = 1$ ***A1***

consider $-2 < \frac{1-3x}{x-2}$ ***(M1)***

Note: Also allow consideration of "$>$" or "$=$" for the awarding of the ***M*** mark.

recognition of critical value at $x = -3$ ***A1***

solution is $-3 < x < 1$ ***A1***

[5 marks]

Total [9 marks]

7. $x^3 + y^3 - 3xy = 0$

$3x^2 + 3y^2 \frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$ ***M1A1***

Note: Differentiation wrt y is also acceptable.

$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}\left(= \frac{y - x^2}{y^2 - x}\right)$ ***(A1)***

Note: All following marks may be awarded if the denominator is correct, but the numerator incorrect.

$y^2 - x = 0$ ***M1***

EITHER

$x = y^2$

$y^6 + y^3 - 3y^3 = 0$ ***M1A1***

$y^6 - 2y^3 = 0$

$y^3\left(y^3 - 2\right) = 0$

$\left(y \neq 0\right) \quad \therefore y = \sqrt[3]{2}$ ***A1***

$x = \left(\sqrt[3]{2}\right)^2 \left(= \sqrt[3]{4}\right)$ ***A1***

continued…

Question 7 continued

OR

$x^3 + xy - 3xy = 0$ ***M1***

$x\left(x^2 - 2y\right) = 0$

$x \neq 0 \Rightarrow y = \dfrac{x^2}{2}$ ***A1***

$y^2 = \dfrac{x^4}{4}$

$x = \dfrac{x^4}{4}$

$x\left(x^3 - 4\right) = 0$

$\left(x \neq 0\right) \therefore x = \sqrt[3]{4}$ ***A1***

$y = \dfrac{\left(\sqrt[3]{4}\right)^2}{2} = \sqrt[3]{2}$ ***A1***

[8 marks]

8. METHOD 1

$216\text{i} = 216\left(\cos\dfrac{\pi}{2} + \text{i}\sin\dfrac{\pi}{2}\right)$ ***A1***

$z + 2\text{i} = \sqrt[3]{216}\left(\cos\left(\dfrac{\pi}{2} + 2\pi k\right) + \text{i}\sin\left(\dfrac{\pi}{2} + 2\pi k\right)\right)^{\frac{1}{3}}$ ***(M1)***

$z + 2\text{i} = 6\left(\cos\left(\dfrac{\pi}{6} + \dfrac{2\pi k}{3}\right) + \text{i}\sin\left(\dfrac{\pi}{6} + \dfrac{2\pi k}{3}\right)\right)$ ***A1***

$z_1 + 2\text{i} = 6\left(\cos\dfrac{\pi}{6} + \text{i}\sin\dfrac{\pi}{6}\right) = 6\left(\dfrac{\sqrt{3}}{2} + \dfrac{\text{i}}{2}\right) = 3\sqrt{3} + 3\text{i}$

$z_2 + 2\text{i} = 6\left(\cos\dfrac{5\pi}{6} + \text{i}\sin\dfrac{5\pi}{6}\right) = 6\left(\dfrac{-\sqrt{3}}{2} + \dfrac{\text{i}}{2}\right) = -3\sqrt{3} + 3\text{i}$

$z_3 + 2\text{i} = 6\left(\cos\dfrac{3\pi}{2} + \text{i}\sin\dfrac{3\pi}{2}\right) = -6\text{i}$ ***A2***

Note: Award ***A1A0*** for one correct root.

so roots are $z_1 = 3\sqrt{3} + \text{i}$, $z_2 = -3\sqrt{3} + \text{i}$ and $z_3 = -8\text{i}$ ***M1A1***

Note: Award ***M1*** for subtracting 2i from their three roots.

[7 marks]

continued…

Question 8 continued

METHOD 2

$\left(a\sqrt{3} + (b+2)\text{i}\right)^3 = 216\text{i}$

$\left(a\sqrt{3}\right)^3 + 3\left(a\sqrt{3}\right)^2(b+2)\text{i} - 3\left(a\sqrt{3}\right)(b+2)^2 - \text{i}(b+2)^3 = 216\text{i}$ ***M1A1***

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 + \text{i}\left(3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3\right) = 216\text{i}$

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 = 0$ and $3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3 = 216$ ***M1A1***

$a\left(a^2 - (b+2)^2\right) = 0$ and $9a^2(b+2) - (b+2)^3 = 216$

$a = 0$ or $a^2 = (b+2)^2$

if $a = 0, -(b+2)^3 = 216 \Rightarrow b+2 = -6$

$\therefore b = -8$ ***A1***

$(a, b) = (0, -8)$

if $a^2 = (b+2)^2,\ 9(b+2)^2(b+2) - (b+2)^3 = 216$

$8(b+2)^3 = 216$

$(b+2)^3 = 27$

$b + 2 = 3$

$b = 1$

$\therefore a^2 = 9 \Rightarrow a = \pm 3$

$\therefore (a, b) = (\pm 3, 1)$ ***A1A1***

so roots are $z_1 = 3\sqrt{3} + \text{i}$, $z_2 = -3\sqrt{3} + \text{i}$ and $z_3 = -8\text{i}$

continued…

Question 8 continued

METHOD 3

$(z+2i)^3-(-6i)^3=0$

attempt to factorise: ***M1***

$((z+2i)-(-6i))\left((z+2i)^2+(z+2i)(-6i)+(-6i)^2\right)=0$ ***A1***

$(z+8i)(z^2-2iz-28)=0$ ***A1***

$z+8i=0 \Rightarrow z=-8i$ ***A1***

$z^2-2iz-28=0 \Rightarrow z=\dfrac{2i\pm\sqrt{-4-(4\times1\times-28)}}{2}$ ***M1***

$z=\dfrac{2i\pm\sqrt{108}}{2}$

$z=\dfrac{2i\pm6\sqrt{3}}{2}$

$z=i\pm3\sqrt{3}$ ***A1A1***

Special Case:

Note: If a candidate recognises that $\sqrt[3]{216i}=-6i$ (anywhere seen), and makes no valid progress in finding three roots, award ***A1*** only.

[7 marks]

Section B

9. (a) (i) $\vec{OF} = \frac{1}{7}\boldsymbol{b}$ ***A1***

(ii) $\vec{AF} = \vec{OF} - \vec{OA}$ ***(M1)***

$= \frac{1}{7}\boldsymbol{b} - \boldsymbol{a}$ ***A1***

[3 marks]

(b) (i) $\vec{OD} = \boldsymbol{a} + \lambda\left(\frac{1}{7}\boldsymbol{b} - \boldsymbol{a}\right)\left(= (1-\lambda)\boldsymbol{a} + \frac{\lambda}{7}\boldsymbol{b}\right)$ ***M1A1***

(ii) $\vec{OD} = \frac{1}{2}\boldsymbol{a} + \mu\left(-\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}\right)\left(= \left(\frac{1}{2} - \frac{\mu}{2}\right)\boldsymbol{a} + \mu\boldsymbol{b}\right)$ ***M1A1***

[4 marks]

(c) equating coefficients: ***M1***

$\frac{\lambda}{7} = \mu,\ 1 - \lambda = \frac{1-\mu}{2}$ ***A1***

solving simultaneously: ***M1***

$\lambda = \frac{7}{13},\ \mu = \frac{1}{13}$ ***A1AG***

[4 marks]

(d) $\vec{CD} = \frac{1}{13}\vec{CB}$

$= \frac{1}{13}\left(\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right)\left(= -\frac{1}{26}\boldsymbol{a} + \frac{1}{13}\boldsymbol{b}\right)$ ***M1A1***

[2 marks]

continued…

Question 9 continued

(e) **METHOD 1**

area $\Delta\text{ACD} = \frac{1}{2}\text{CD} \times \text{AC} \times \sin \text{A}\hat{\text{C}}\text{B}$ ***(M1)***

area $\Delta\text{ACB} = \frac{1}{2}\text{CB} \times \text{AC} \times \sin\text{A}\hat{\text{C}}\text{B}$ ***(M1)***

ratio $\frac{\text{area } \Delta\text{ACD}}{\text{area } \Delta\text{ACB}} = \frac{\text{CD}}{\text{CB}} = \frac{1}{13}$ ***A1***

$k = \frac{\text{area } \Delta\text{OAB}}{\text{area } \Delta\text{CAD}} = \frac{13}{\text{area}\Delta\text{CAB}} \times \text{area}\Delta\text{OAB}$ ***(M1)***

$= 13 \times 2 = 26$ ***A1***

METHOD 2

area $\Delta\text{OAB} = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$ ***A1***

area $\Delta\text{CAD} = \frac{1}{2}\left|\vec{\text{CA}} \times \vec{\text{CD}}\right|$ or $\frac{1}{2}\left|\vec{\text{CA}} \times \vec{\text{AD}}\right|$ ***M1***

$= \frac{1}{2}\left|\frac{1}{2}\boldsymbol{a} \times \left(-\frac{1}{26}\boldsymbol{a} + \frac{1}{13}\boldsymbol{b}\right)\right|$

$= \frac{1}{2}\left|\frac{1}{2}\boldsymbol{a} \times \left(-\frac{1}{26}\boldsymbol{a}\right) + \frac{1}{2}\boldsymbol{a} \times \frac{1}{13}\boldsymbol{b}\right|$ ***(M1)***

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{13}|\boldsymbol{a} \times \boldsymbol{b}| \left(= \frac{1}{52}|\boldsymbol{a} \times \boldsymbol{b}|\right)$ ***A1***

area $\Delta\text{OAB} = k(\text{area } \Delta\text{CAD})$

$\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = k\,\frac{1}{52}|\boldsymbol{a} \times \boldsymbol{b}|$

$k = 26$ ***A1***

[5 marks]

Total [18 marks]

10. (a) **METHOD 1**

number of possible "deals" $= 4! = 24$ ***A1***

consider ways of achieving "no matches" (Chloe winning):
Selena could deal B, C, D (*ie*, 3 possibilities)
as her first card ***R1***

for each of these matches, there are only 3 possible combinations for the remaining 3 cards ***R1***

so no. ways achieving no matches $= 3 \times 3 = 9$ ***M1A1***

so probability Chloe wins $= \frac{9}{24} = \frac{3}{8}$ ***A1AG***

continued…

Question 10 continued

METHOD 2

number of possible “deals” $= 4! = 24$ ***A1***

consider ways of achieving a match (Selena winning)
Selena card A can match with Chloe card A, giving 6 possibilities for this happening ***R1***

if Selena deals B as her first card, there are only 3 possible combinations for the remaining 3 cards. Similarly for dealing C and dealing D ***R1***

so no. ways achieving one match is $= 6+3+3+3 = 15$ ***M1A1***

so probability Chloe wins $= 1 - \frac{15}{24} = \frac{3}{8}$ ***A1AG***

METHOD 3

systematic attempt to find number of outcomes where Chloe wins (no matches) (using tree diag. or otherwise) ***M1***

9 found ***A1***

each has probability $\frac{1}{4}\times\frac{1}{3}\times\frac{1}{2}\times 1$ ***M1***

$= \frac{1}{24}$ ***A1***

their 9 multiplied by their $\frac{1}{24}$ ***M1A1***

$= \frac{3}{8}$ ***AG***

[6 marks]

continued…

Question 10 continued

(b) (i) $X \sim \mathrm{B}\left(50, \frac{3}{8}\right)$ ***(M1)***

$\mu = np = 50 \times \frac{3}{8} = \frac{150}{8}\left(= \frac{75}{4}\right)(= 18.75)$ ***(M1)A1***

(ii) $\sigma^2 = np(1-p) = 50 \times \frac{3}{8} \times \frac{5}{8} = \frac{750}{64}\left(= \frac{375}{32}\right)(= 11.7)$ ***(M1)A1***

[5 marks]

Total [11 marks]

11. (a) even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x)\ldots = (\cos(-2x))(\cos(-4x))\ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

(b) consider the case $n = 1$

$\frac{\sin 4x}{2\sin 2x} = \frac{2\sin 2x \cos 2x}{2\sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x)\ldots(\cos 2^k x) = \frac{\sin 2^{k+1}x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)(\cos 2^{k+1}x)$ ***(M1)***

$= \frac{\sin 2^{k+1}x}{2^k \sin 2x}\cos 2^{k+1}x$ ***A1***

$= \frac{2\sin 2^{k+1}x \cos 2^{k+1}x}{2^{k+1}\sin 2x}$ ***A1***

$= \frac{\sin 2^{k+2}x}{2^{k+1}\sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

continued…

Question 11 continued

(c) attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) ***M1***

$$f_n'(x) = \frac{\left(2^n \sin 2x\right)\left(2^{n+1}\cos 2^{n+1}x\right) - \left(\sin 2^{n+1}x\right)\left(2^{n+1}\cos 2x\right)}{\left(2^n \sin 2x\right)^2}$$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

(d) $$f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin\frac{\pi}{2}\right)\left(2^{n+1}\cos 2^{n+1}\frac{\pi}{4}\right) - \left(\sin 2^{n+1}\frac{\pi}{4}\right)\left(2^{n+1}\cos\frac{\pi}{2}\right)}{\left(2^n \sin\frac{\pi}{2}\right)^2}$$ ***(M1)(A1)***

$$f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n\right)\left(2^{n+1}\cos 2^{n+1}\frac{\pi}{4}\right)}{\left(2^n\right)^2}$$ ***(A1)***

$$= 2\cos 2^{n+1}\frac{\pi}{4}\left(= 2\cos 2^{n-1}\pi\right)$$ ***A1***

$$f_n'\left(\frac{\pi}{4}\right) = 2$$ ***A1***

$$f_n\left(\frac{\pi}{4}\right) = 0$$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$$y = 2\left(x - \frac{\pi}{4}\right)$$ ***M1A1***

$$4x - 2y - \pi = 0$$ ***AG***

[8 marks]

Total [21 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2017

Mathematics

Higher level

Paper 1

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2017**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. $\log_2 x - \log_2 5 = 2 + \log_2 3$

collecting at least two log terms ***(M1)***

eg $\log_2 \frac{x}{5} = 2 + \log_2 3$ or $\log_2 \frac{x}{15} = 2$

obtaining a correct equation without logs ***(M1)***

eg $\frac{x}{5} = 12$ **OR** $\frac{x}{15} = 2^2$ ***(A1)***

$x = 60$ ***A1***

[4 marks]

2. (a) $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

(i) $|w| = \sqrt{2}$ ***A1***

(ii) $\arg w = \frac{\pi}{12}$ ***A1***

Notes: Allow ***FT*** from incorrect answers for z_1 and z_2 in modulus-argument form.

[4 marks]

(b) **EITHER**

$\sin\left(\frac{\pi n}{12}\right) = 0$ ***(M1)***

OR

$\arg\left(w^n\right) = \pi$ ***(M1)***

$\frac{n\pi}{12} = \pi$

THEN

$\therefore n = 12$ ***A1***

[2 marks]

Total [6 marks]

3. **METHOD 1**

use of $\sec^2 x = \tan^2 x + 1$ ***M1***

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)^2 = 0$ ***(M1)***

$\tan x = -1$ ***A1***

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ***A1A1***

METHOD 2

$\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x} = 0$ ***M1***

$1 + 2\sin x \cos x = 0$

$\sin 2x = -1$ ***M1A1***

$2x = \frac{3\pi}{2}, \frac{7\pi}{2}$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ***A1A1***

Note:	Award ***A1A0*** if extra solutions given or if solutions given in degrees (or both).

[5 marks]

4. **METHOD 1**

total number of arrangements $7!$ ***(A1)***

number of ways for girls and boys to sit together $= 3! \times 4! \times 2$ ***(M1)(A1)***

> **Note:** Award ***M1A0*** if the 2 is missing.

probability $= \dfrac{3! \times 4! \times 2}{7!}$ ***M1***

> **Note:** Award ***M1*** for attempting to write as a probability.

$$\frac{2 \times 3 \times 4! \times 2}{7 \times 6 \times 5 \times 4!}$$

$= \dfrac{2}{35}$ ***A1***

> **Note:** Award ***A0*** if not fully simplified.

METHOD 2

$\dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{1}{5} + \dfrac{4}{7} \times \dfrac{3}{6} \times \dfrac{2}{5} \times \dfrac{1}{4}$ ***(M1)A1A1***

> **Note:** Accept $\dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{1}{5} \times 2$ or $\dfrac{4}{7} \times \dfrac{3}{6} \times \dfrac{2}{5} \times \dfrac{1}{4} \times 2$.

$= \dfrac{2}{35}$ ***(M1)A1***

> **Note:** Award ***A0*** if not fully simplified.

[5 marks]

5. (a) $\overrightarrow{AB} \times \overrightarrow{AD} = -\boldsymbol{i} + 10\boldsymbol{j} - 7\boldsymbol{k}$ ***M1A1***

area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \sqrt{1^2 + 10^2 + 7^2}$

$= 5\sqrt{6}\left(\sqrt{150}\right)$ ***A1***

[3 marks]

(b) **METHOD 1**

$\overrightarrow{AB} \cdot \overrightarrow{AD} = -4 - 2 - 6$ ***M1A1***

$= -12$

considering the sign of the answer

$\overrightarrow{AB} \cdot \overrightarrow{AD} < 0$, therefore angle $D\hat{A}B$ is obtuse ***M1***

(as it is a parallelogram), $A\hat{B}C$ is acute ***A1***

[4 marks]

METHOD 2

$\overrightarrow{BA} \cdot \overrightarrow{BC} = +4 + 2 + 6$ ***M1A1***

$= 12$

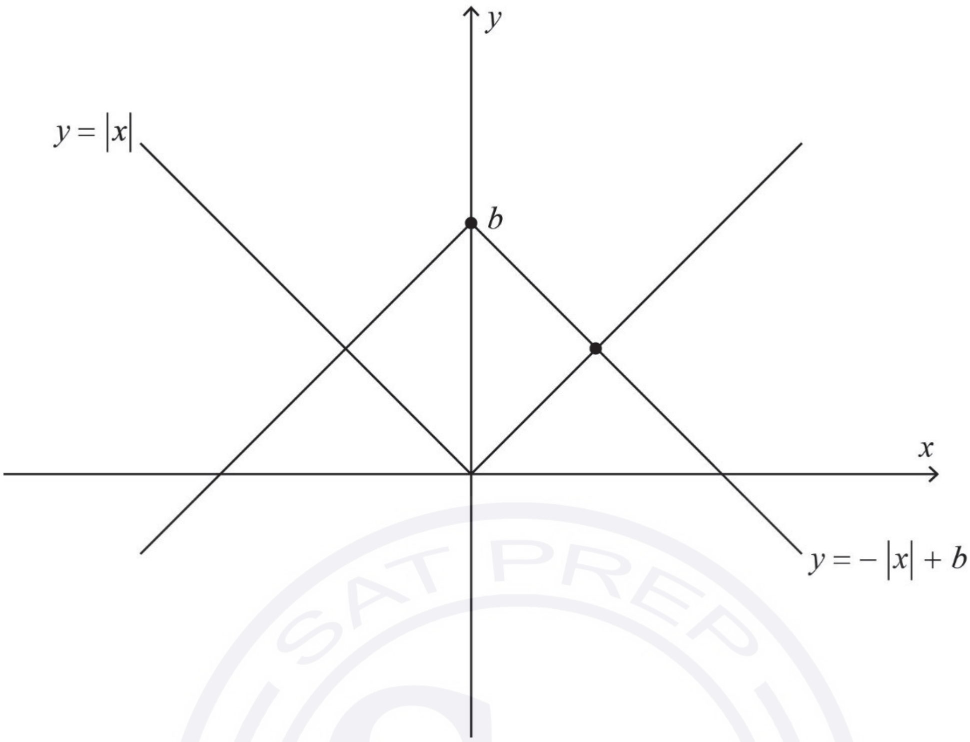
considering the sign of the answer ***M1***

$\overrightarrow{BA} \cdot \overrightarrow{BC} > 0 \Rightarrow A\hat{B}C$ is acute ***A1***

[4 marks]

Total [7 marks]

6. (a)



graphs sketched correctly (condone missing b) ***A1A1***

[2 marks]

(b) $\frac{b^2}{2} = 18$ ***(M1)A1***

$b = 6$ ***A1***

[3 marks]

Total [5 marks]

7. (a) use of $u_n = u_1 + (n-1)d$ ***M1***

$(1+2d)^2 = (1+d)(1+5d)$ (or equivalent) ***M1A1***

$d = -2$ ***A1***

[4 marks]

(b) $1 + (\mathrm{N}-1) \times -2 = -15$

$N = 9$ ***(A1)***

$\sum_{r=1}^{9} u_r = \frac{9}{2}(2 + 8 \times -2)$ ***(M1)***

$= -63$ ***A1***

[3 marks]

Total [7 marks]

8. let $P(n)$ be the proposition that $4^n + 15n - 1$ is divisible by 9

showing true for $n = 1$ ***A1***

ie for $n = 1$, $4^1 + 15 \times 1 - 1 = 18$

which is divisible by 9, therefore $P(1)$ is true

assume $P(k)$ is true so $4^k + 15k - 1 = 9A, \left(A \in \mathbb{Z}^+\right)$ ***M1***

Note: Only award ***M1*** if "truth assumed" or equivalent.

consider $4^{k+1} + 15(k+1) - 1$

$= 4 \times 4^k + 15k + 14$

$= 4(9A - 15k + 1) + 15k + 14$ ***M1***

$= 4 \times 9A - 45k + 18$ ***A1***

$= 9(4A - 5k + 2)$ which is divisible by 9 ***R1***

Note: Award ***R1*** for either the expression or the statement above.

since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, therefore (by the principle of mathematical induction) $P(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if the 2 ***M1***s have been awarded.

[6 marks]

9. attempt at integration by parts with $u = \arcsin x$ and $v' = 1$ ***M1***

$\int \arcsin x \, \mathrm{d}x = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for $x \arcsin x$ and ***A1*** for $-\int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$.

solving $\int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$ by substitution with $u = 1 - x^2$ or inspection ***(M1)***

$\int \arcsin x \, \mathrm{d}x = x \arcsin x + \sqrt{1-x^2} + c$ ***A1***

[5 marks]

Section B

10. (a) attempt to equate integral to 1 (may appear later) ***M1***

$k\int_0^6 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x = 1$

correct integral ***A1***

$k\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^6 = 1$

substituting limits ***M1***

$-\frac{6}{\pi}(-1-1) = \frac{1}{k}$

$k = \frac{\pi}{12}$ ***A1***

[4 marks]

(b) (i) mean $= 3$ ***A1***

(ii) median $= 3$ ***A1***

(iii) mode $= 3$ ***A1***

Note: Award ***A1A0A0*** for three equal answers in $(0, 6)$.

[3 marks]

(c) (i) $\frac{\pi}{12}\int_0^2 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x$ ***M1***

$= \frac{\pi}{12}\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^2$ ***A1***

Note: Accept without the $\frac{\pi}{12}$ at this stage if it is added later.

$\frac{\pi}{12}\left[-\frac{6}{\pi}\left(\cos\frac{\pi}{3} - 1\right)\right]$ ***M1***

$= \frac{1}{4}$ ***AG***

(ii) from (c)(i) $Q_1 = 2$ ***(A1)***

as the graph is symmetrical about the middle value $x = 3 \Rightarrow Q_3 = 4$ ***(A1)***

so interquartile range is

$4 - 2$ ***A1***

$= 2$

[6 marks]

continued…

Question 10 continued

(d) $P(X \le 4 | X \ge 3) = \dfrac{P(3 \le X \le 4)}{P(X \ge 3)}$

$= \dfrac{\frac{1}{4}}{\frac{1}{2}}$ ***(M1)***

$= \dfrac{1}{2}$ ***A1***

[2 marks]

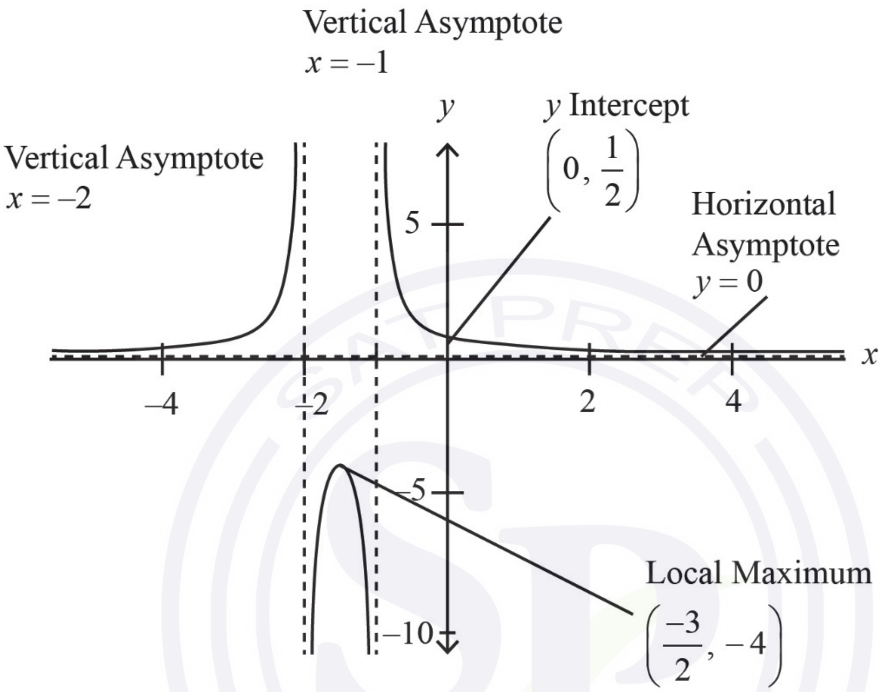
Total [15 marks]

11. (a) (i) $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

(ii) $x^2 + 3x + 2 = (x + 2)(x + 1)$ ***A1***

[2 marks]

(b)



A1 for the shape

A1 for the equation $y = 0$

A1 for asymptotes $x = -2$ and $x = -1$

A1 for coordinates $\left(-\frac{3}{2}, -4\right)$

A1 y-intercept $\left(0, \frac{1}{2}\right)$

[5 marks]

(c) $\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2) - (x+1)}{(x+1)(x+2)}$ ***M1***

$= \frac{1}{x^2 + 3x + 2}$ ***AG***

[1 mark]

continued…

Question 11 continued

(d) $\int_0^1 \frac{1}{x+1} - \frac{1}{x+2}\,dx$

$= \left[\ln(x+1) - \ln(x+2)\right]_0^1$ ***A1***

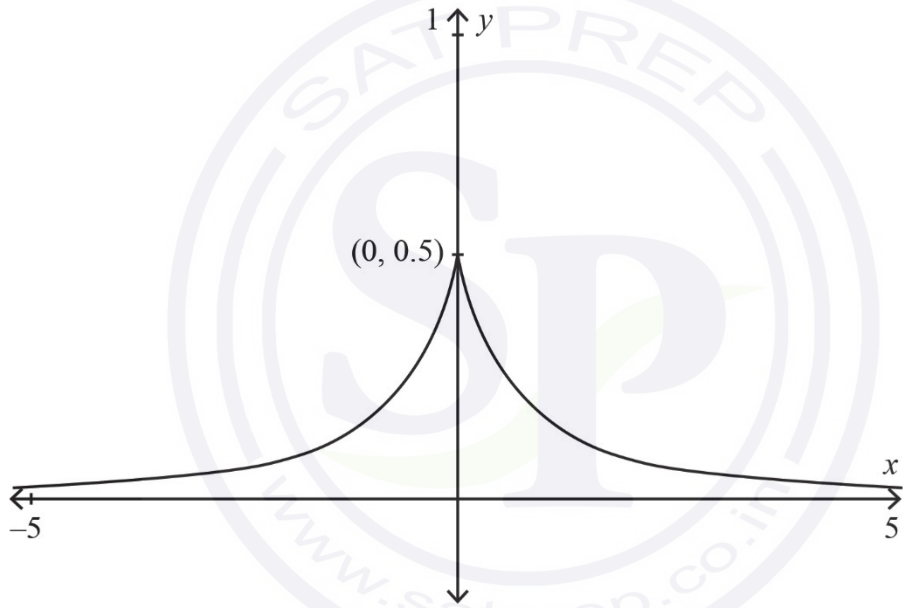
$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

$= \ln\left(\frac{4}{3}\right)$ ***M1A1***

$\therefore\ p = \frac{4}{3}$

[4 marks]

(e)



symmetry about the y-axis ***M1***
correct shape ***A1***

Note: Allow ***FT*** from part (b).

[2 marks]

(f) $2\int_0^1 f(x)\,dx$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ ***A1***

Note: Do not award ***FT*** from part (e).

[3 marks]

Total [17 marks]

12. (a) sum $= 0$ ***A1***
product $= 6$ ***A1***

[2 marks]

(b) $P(1) = 1 - 10 + 15 - 6 = 0$ ***M1A1***
$\Rightarrow (z - 1)$ is a factor of $P(z)$ ***AG***

Note: Accept use of division to show remainder is zero.

[2 marks]

(c) **METHOD 1**

$(z - 1)^3 \left(z^2 + bz + c\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)***

by inspection $c = 6$ ***A1***

$\left(z^3 - 3z^2 + 3z - 1\right)\left(z^2 + bz + 6\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)(A1)***

$b = 3$ ***A1***

METHOD 2

α, β are two roots of the quadratic

$b = -(\alpha + \beta),\ c = \alpha\beta$ ***(A1)***

from part (a) $1 + 1 + 1 + \alpha + \beta = 0$ ***(M1)***

$\Rightarrow b = 3$ ***A1***

$1 \times 1 \times 1 \times \alpha\beta = 6$ ***(M1)***

$\Rightarrow c = 6$ ***A1***

Note: Award ***FT*** if $b = -7$ following through from their sum $= 10$.

METHOD 3

$\left(z^5 - 10z^2 + 15z - 6\right) \div (z - 1) = z^4 + z^3 + z^2 - 9z + 6$ ***(M1)A1***

Note: This may have been seen in part (b).

$z^4 + z^3 + z^2 - 9z + 6 \div (z - 1) = z^3 + 2z^2 + 3z - 6$ ***(M1)***

$z^3 + 2z^2 + 3z - 6 \div (z - 1) = z^2 + 3z + 6$ ***A1A1***

[5 marks]

continued…

Question 12 continued

(d) $z^2 + 3z + 6 = 0$ **M1**

$z = \dfrac{-3 \pm \sqrt{9 - 4 \cdot 6}}{2}$ **M1**

$= \dfrac{-3 \pm \sqrt{-15}}{2}$

$z = -\dfrac{3}{2} \pm \dfrac{i\sqrt{15}}{2}$ **A1**

(or $z = 1$)

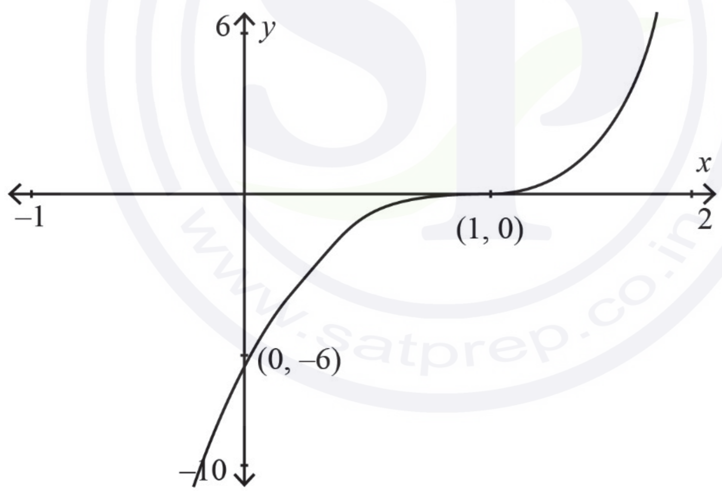
Notes:	Award the second ***M1*** for an attempt to use the quadratic formula or to complete the square. Do not award ***FT*** from (c).

[3 marks]

(e) (i) $\dfrac{d^2y}{dx^2} = 20x^3 - 20$ **M1A1**

for $x > 1$, $20x^3 - 20 > 0 \Rightarrow$ concave up **R1AG**

(ii)



x-intercept at $(1, 0)$ **A1**

y-intercept at $(0, -6)$ **A1**

stationary point of inflexion at $(1, 0)$ with correct curvature either side **A1**

[6 marks]

Total [18 marks]

Markscheme

May 2017

Mathematics

Higher level

Paper 1

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2017**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N*** marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. attempt at binomial expansion, relevant row of Pascal's triangle or use of general term with binomial coefficient must be seen ***(M1)***

term independent of x is $\binom{10}{4}(2x^2)^6\left(\frac{1}{2x^3}\right)^4$ (or equivalent) ***(A1)(A1)(A1)***

> **Notes:** x's may be omitted.
> Also accept $\binom{10}{6}$ or 210.

$= 840$ ***A1***

[5 marks]

2. (a) $-11 \le f(x) \le 21$ ***A1A1***

> **Note:** ***A1*** for correct end points, ***A1*** for correct inequalities.

[2 marks]

(b) $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$ ***(M1)A1***

[2 marks]

(c) $-11 \le x \le 21$, $-2 \le f^{-1}(x) \le 2$ ***A1A1***

[2 marks]

Total [6 marks]

3. (a) **EITHER**

the first three terms of the geometric sequence are 9, $9r$ and $9r^2$ *(M1)*

$9 + 3d = 9r$ $(\Rightarrow 3 + d = 3r)$ and $9 + 7d = 9r^2$ *(A1)*

attempt to solve simultaneously *(M1)*

$$9 + 7d = 9\left(\frac{3 + d}{3}\right)^2$$

OR

the 1st, 4th and 8th terms of the arithmetic sequence are

9, $9 + 3d$, $9 + 7d$ *(M1)*

$$\frac{9 + 7d}{9 + 3d} = \frac{9 + 3d}{9}$$ *(A1)*

attempt to solve *(M1)*

THEN

$d = 1$ ***A1***

[4 marks]

(b) $r = \dfrac{4}{3}$ ***A1***

Note:	Accept answers where a candidate obtains d by finding r first. The first two marks in either method for part (a) are awarded for the same ideas and the third mark is awarded for attempting to solve an equation in r.

[1 mark]

Total [5 marks]

4. (a) $s = t + \cos 2t$

$\frac{\mathrm{d}s}{\mathrm{d}t} = 1 - 2\sin 2t$ ***M1A1***

$= 0$ ***M1***

$\Rightarrow \sin 2t = \frac{1}{2}$

$t_1 = \frac{\pi}{12}(s),\ t_2 = \frac{5\pi}{12}(s)$ ***A1A1***

Note: Award ***A0A0*** if answers are given in degrees.

[5 marks]

(b) $s = \frac{\pi}{12} + \cos\frac{\pi}{6}\left(s = \frac{\pi}{12} + \frac{\sqrt{3}}{2}(m)\right)$ ***A1A1***

[2 marks]

Total [7 marks]

5. C represents the complex number $1 - 2\mathrm{i}$ ***A2***

D represents the complex number $3 + 2\mathrm{i}$ ***A2***

[4 marks]

6. (a) let $x = \tan\theta$

$\Rightarrow \dfrac{dx}{d\theta} = \sec^2\theta$ ***(A1)***

$\displaystyle\int \frac{1}{\left(x^2+1\right)^2}dx = \int \frac{\sec^2\theta}{\left(\tan^2\theta+1\right)^2}d\theta$ ***M1***

Note: The method mark is for an attempt to substitute for both x and dx.

$= \displaystyle\int \frac{1}{\sec^2\theta}d\theta$ (or equivalent) ***A1***

when $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \dfrac{\pi}{4}$ ***M1***

$= \displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta d\theta$ ***AG***

[4 marks]

(b) $\left(\displaystyle\int_0^1 \frac{1}{\left(x^2+1\right)^2}dx = \int_0^{\frac{\pi}{4}} \cos^2\theta d\theta\right) = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)d\theta$ ***M1***

$= \dfrac{1}{2}\left[\theta + \dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$ ***A1***

$= \dfrac{\pi}{8} + \dfrac{1}{4}$ ***A1***

[3 marks]

Total [7 marks]

7. (a) $P(X > 0) = 1 - P(X = 0)$ ***(M1)***

$\Rightarrow 1 - e^{-m} = \frac{3}{4}$ or equivalent ***A1***

$\Rightarrow m = \ln 4$ ***A1***

[3 marks]

(b) $P(Y > 1) = 1 - P(Y = 0) - P(Y = 1)$ ***(M1)***

$= 1 - e^{-2\ln 4} - e^{-2\ln 4} \times 2\ln 4$ ***A1***

recognition that $2\ln 4 = \ln 16$ ***(A1)***

$P(Y > 1) = \frac{15 - \ln 16}{16}$ ***A1***

[4 marks]

Total [7 marks]

8. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$

show true for $n=3$ ***(M1)***

LHS $=\binom{2}{2}=1$ RHS $=\binom{3}{3}=1$ ***A1***

hence true for $n=3$

assume true for $n=k$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$ ***M1***

consider for $n=k+1$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}$ ***(M1)***

$=\binom{k}{3}+\binom{k}{2}$ ***A1***

$=\frac{k!}{(k-3)!3!}+\frac{k!}{(k-2)!2!}\left(=\frac{k!}{3!}\left[\frac{1}{(k-3)!}+\frac{3}{(k-2)!}\right]\right)$ or any correct expression with a visible common factor ***(A1)***

$=\frac{k!}{3!}\left[\frac{k-2+3}{(k-2)!}\right]$ or any correct expression with a common denominator ***(A1)***

$=\frac{k!}{3!}\left[\frac{k+1}{(k-2)!}\right]$

Note: At least one of the above three lines or equivalent must be seen.

$=\frac{(k+1)!}{3!(k-2)!}$ or equivalent ***A1***

$=\binom{k+1}{3}$

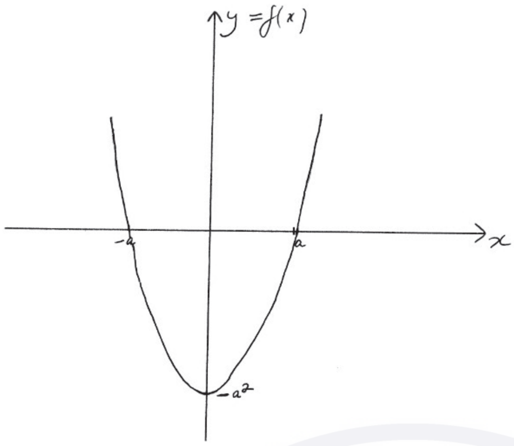
Result is true for $k=3$. If result is true for k it is true for $k+1$. Hence result is true for all $k\geq 3$. Hence proved by induction. ***R1***

Note: In order to award the ***R1*** at least ***[5 marks]*** must have been awarded.

[9 marks]

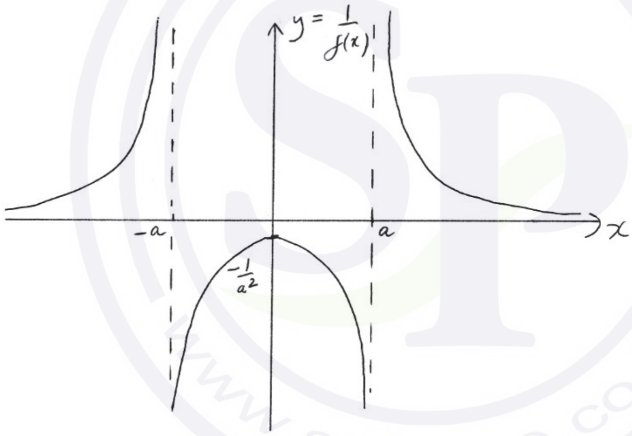
Section B

9. (a) (i)



A1 for correct shape
A1 for correct x and y intercepts and minimum point

(ii)

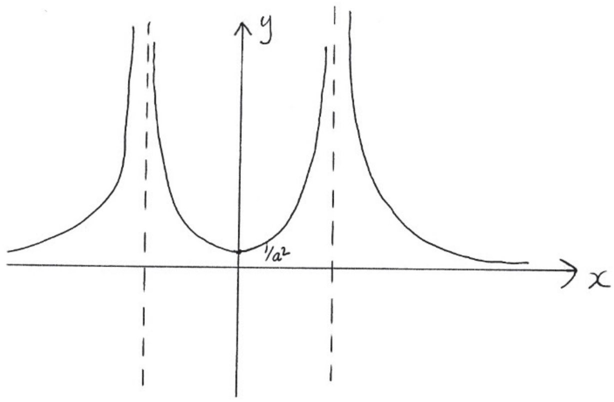


A1 for correct shape
A1 for correct vertical asymptotes
A1 for correct implied horizontal asymptote
A1 for correct maximum point

continued…

Question 9 continued

(iii)



A1 for reflecting negative branch from (ii) in the x-axis
A1 for correctly labelled minimum point

[8 marks]

(b) **EITHER**

attempt at integration by parts ***(M1)***

$\int (x^2 - a^2)\cos x\,dx = (x^2 - a^2)\sin x - \int 2x\sin x\,dx$ ***A1A1***

$= (x^2 - a^2)\sin x - 2\left[-x\cos x + \int \cos x\,dx\right]$ ***A1***

$= (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

OR

$\int (x^2 - a^2)\cos x\,dx = \int x^2\cos x\,dx - \int a^2\cos x\,dx$

attempt at integration by parts ***(M1)***

$\int x^2 \cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$ ***A1A1***

$= x^2\sin x - 2\left[-x\cos x + \int \cos x\,dx\right]$ ***A1***

$= x^2\sin x + 2x\cos x - 2\sin x$

$-\int a^2\cos x\,dx = -a^2\sin x$

$\int (x^2 - a^2)\cos x\,dx = (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

[5 marks]

continued…

Question 9 continued

(c) $g(x) = x\left(x^2 - a^2\right)^{\frac{1}{2}}$

$g'(x) = \left(x^2 - a^2\right)^{\frac{1}{2}} + \frac{1}{2}x\left(x^2 - a^2\right)^{-\frac{1}{2}}(2x)$ ***M1A1A1***

Note: Method mark is for differentiating the product. Award ***A1*** for each correct term.

$g'(x) = \left(x^2 - a^2\right)^{\frac{1}{2}} + x^2\left(x^2 - a^2\right)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive ***R1***

and therefore g is an increasing function (for $|x| > a$) ***AG***

[4 marks]

Total [17 marks]

10. (a) (i) the width of the rectangle is $2r$ and let the height of the rectangle be h

$\text{P} = 2r + 2h + \pi r$ ***(A1)***

$A = 2rh + \frac{\pi r^2}{2}$ ***(A1)***

$h = \frac{\text{P} - 2r - \pi r}{2}$

$A = 2r\left(\frac{\text{P} - 2r - \pi r}{2}\right) + \frac{\pi r^2}{2} \left(= \text{P}r - 2r^2 - \frac{\pi r^2}{2}\right)$ ***M1A1***

(ii) $\frac{\text{d}A}{\text{d}r} = \text{P} - 4r - \pi r$ ***A1***

$\frac{\text{d}A}{\text{d}r} = 0$ ***M1***

$\Rightarrow r = \frac{\text{P}}{4 + \pi}$ ***(A1)***

hence the width is $\frac{2\text{P}}{4 + \pi}$ ***A1***

$\frac{\text{d}^2 A}{\text{d}r^2} = -4 - \pi < 0$ ***R1***

hence maximum ***AG***

[9 marks]

continued…

Question 10 continued

(b) **EITHER**

$$h=\frac{P-2r-\pi r}{2}$$

$$h=\frac{P-\frac{2P}{4+\pi}-\frac{P\pi}{4+\pi}}{2}$$ ***M1***

$$h=\frac{4P+\pi P-2P-\pi P}{2(4+\pi)}$$ ***A1***

$$h=\frac{P}{(4+\pi)}=r$$ ***AG***

OR

$$h=\frac{P-2r-\pi r}{2}$$

$$P=r(4+\pi)$$ ***M1***

$$h=\frac{r(4+\pi)-2r-\pi r}{2}$$ ***A1***

$$h=\frac{4r+\pi r-2r-\pi r}{2}=r$$ ***AG***

[2 marks]

Total [11 marks]

11. (a) $2\sin(x+60^\circ)=\cos(x+30^\circ)$

$2(\sin x\cos 60^\circ+\cos x\sin 60^\circ)=\cos x\cos 30^\circ-\sin x\sin 30^\circ$ ***(M1)(A1)***

$2\sin x\times\frac{1}{2}+2\cos x\times\frac{\sqrt{3}}{2}=\cos x\times\frac{\sqrt{3}}{2}-\sin x\times\frac{1}{2}$ ***A1***

$\Rightarrow \frac{3}{2}\sin x=-\frac{\sqrt{3}}{2}\cos x$

$\Rightarrow \tan x=-\frac{1}{\sqrt{3}}$ ***M1***

$\Rightarrow x=150^\circ$ ***A1***

[5 marks]

(b) **EITHER**

choosing two appropriate angles, for example 60° and 45° ***M1***

$\sin 105^\circ=\sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ$ and

$\cos 105^\circ=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$ ***(A1)***

$\sin 105^\circ+\cos 105^\circ=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$ ***A1***

$=\frac{1}{\sqrt{2}}$ ***AG***

OR

attempt to square the expression ***M1***

$\left(\sin 105^\circ+\cos 105^\circ\right)^2=\sin^2 105^\circ+2\sin 105^\circ\cos 105^\circ+\cos^2 105^\circ$

$\left(\sin 105^\circ+\cos 105^\circ\right)^2=1+\sin 210^\circ$ ***A1***

$=\frac{1}{2}$ ***A1***

$\sin 105^\circ+\cos 105^\circ=\frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

continued…

Question 11 continued

(c) (i) **EITHER**

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1 - \cos 2\theta)}$ ***A1***

$= \sqrt{2\left(2\sin^2\theta\right)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\dfrac{\sin 2\theta}{1 - \cos 2\theta}$ ***M1***

$= \dfrac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan\left(\dfrac{\pi}{2} - \theta\right)\right)$ ***A1***

$= \theta - \dfrac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2i\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta(\sin\theta - i\cos\theta)$ ***(A1)***

$= -2i\sin\theta(\cos\theta + i\sin\theta)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + i\sin\left(\theta - \dfrac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \dfrac{\pi}{2}$ ***A1***

continued

Question 11 continued

(ii) attempt to apply De Moivre's theorem ***M1***

$$(1-\cos 2\theta - \mathrm{i}\sin 2\theta)^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)\right]$$

A1A1A1

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n=-1,\,0,\,1$.
Equivalent forms are acceptable. ***A1***

[14 marks]

Total [22 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2016

Mathematics

Higher level

Paper 1

19 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. METHOD 1

for eliminating one variable from two equations *(M1)*

eg, $\begin{cases} (x + y + z = 3) \\ 2x + 2z = 8 \\ 2x + 3z = 11 \end{cases}$ ***A1A1***

for finding correctly one coordinate

eg, $\Rightarrow \begin{cases} (x + y + z = 3) \\ (2x + 2z = 8) \\ z = 3 \end{cases}$ ***A1***

for finding correctly the other two coordinates ***A1***

$\Rightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 3 \end{cases}$

the intersection point has coordinates $(1, -1, 3)$

METHOD 2

for eliminating two variables from two equations or using row reduction *(M1)*

eg, $\begin{cases} (x + y + z = 3) \\ -2y = 2 \\ z = 3 \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ***A1A1***

for finding correctly the other coordinates ***A1A1***

$\Rightarrow \begin{cases} x = 1 \\ y = -1 \\ (z = 3) \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

the intersection point has coordinates $(1, -1, 3)$

continued…

Question 1 continued

METHOD 3

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2$ ***(A1)***

attempt to use Cramer's rule ***M1***

$x = \dfrac{\begin{vmatrix} 3 & 1 & 1 \\ 5 & -1 & 1 \\ 6 & 1 & 2 \end{vmatrix}}{-2} = \dfrac{-2}{-2} = 1$ ***A1***

$y = \dfrac{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 5 & 1 \\ 1 & 6 & 2 \end{vmatrix}}{-2} = \dfrac{2}{-2} = -1$ ***A1***

$z = \dfrac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 1 & 6 \end{vmatrix}}{-2} = \dfrac{-6}{-2} = 3$ ***A1***

Note: Award ***M1*** only if candidate attempts to determine at least one of the variables using this method.

[5 marks]

2. (a)

x	1	2	4	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

A1A1

Note: Award ***A1*** for each correct row.

[2 marks]

(b) $\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 4 \times \frac{1}{3} + 6 \times \frac{1}{6}$ ***(M1)***

$= \frac{19}{6} \left(= 3\frac{1}{6} \right)$ ***A1***

Note: If the probabilities in (a) are not values between 0 and 1 or lead to $\mathrm{E}(X) > 6$ award ***M1A0*** to correct method using the incorrect probabilities; otherwise allow ***FT*** marks.

[2 marks]

Total [4 marks]

3. (a) $a = 1$ ***A1***
$c = 3$ ***A1***

[2 marks]

(b) use the coordinates of $(1, 0)$ on the graph ***M1***

$f(1) = 0 \Rightarrow 1 + \dfrac{b}{1-3} = 0 \Rightarrow b = 2$ ***A1***

[2 marks]

Total [4 marks]

4. (a) $\boldsymbol{a} \times \boldsymbol{b} = -12\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ ***(M1)A1***

[2 marks]

(b) **METHOD 1**

$-12x - 2y - 3z = d$ ***M1***
$-12 \times 1 - 2 \times 0 - 3(-1) = d$ ***(M1)***
$\Rightarrow d = -9$ ***A1***
$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$)

METHOD 2

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix}$$ ***M1A1***

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$) ***A1***

[3 marks]

Total [5 marks]

5. $\alpha + \beta = 2k$ ***A1***
$\alpha\beta = k - 1$ ***A1***
$(\alpha + \beta)^2 = 4k^2 \Rightarrow \alpha^2 + \beta^2 + 2\underbrace{\alpha\beta}_{k-1} = 4k^2$ ***(M1)***

$\alpha^2 + \beta^2 = 4k^2 - 2k + 2$
$\alpha^2 + \beta^2 = 4 \Rightarrow 4k^2 - 2k - 2 = 0$ ***A1***
attempt to solve quadratic ***(M1)***
$k = 1, -\dfrac{1}{2}$ ***A1***

[6 marks]

6. (a) $u_1 = 1$ ***A1***

[1 mark]

(b) $u_6 = S_6 - S_5 = 31$ ***M1A1***

[2 marks]

(c) $u_n = S_n - S_{n-1}$ ***M1***

$= \left(3n^2 - 2n\right) - \left(3(n-1)^2 - 2(n-1)\right)$

$= \left(3n^2 - 2n\right) - \left(3n^2 - 6n + 3 - 2n + 2\right)$

$= 6n - 5$ ***A1***

$d = u_{n+1} - u_n$ ***R1***

$= 6n + 6 - 5 - 6n + 5$

$= \left(6(n+1) - 5\right) - (6n - 5)$

$= 6$ (constant) ***A1***

> **Notes:** Award ***R1*** only if candidate provides a clear argument that proves that the difference between **ANY** two consecutive terms of the sequence is constant. Do not accept examples involving particular terms of the sequence nor circular reasoning arguments (*eg* use of formulas of APs to prove that it is an AP). Last ***A1*** is independent of ***R1***.

[4 marks]

Total [7 marks]

7. attempt to form a quadratic in 2^x ***M1***

$\left(2^x\right)^2 + 4 \cdot 2^x - 3 = 0$ ***A1***

$2^x = \dfrac{-4 \pm \sqrt{16 + 12}}{2} \left(= -2 \pm \sqrt{7}\right)$ ***M1***

$2^x = -2 + \sqrt{7}$ (as $-2 - \sqrt{7} < 0$) ***R1***

$x = \log_2\left(-2 + \sqrt{7}\right) \quad \left(x = \dfrac{\ln\left(-2 + \sqrt{7}\right)}{\ln 2}\right)$ ***A1***

> **Note:** Award ***R0 A1*** if final answer is $x = \log_2\left(-2 \pm \sqrt{7}\right)$.

[5 marks]

8. (a) **METHOD 1**

$$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = -3+\beta \\ y = -2+4\beta \\ z = a+2\beta \end{cases}$$ ***M1***

$$\frac{6-(-3+\beta)}{3} = \frac{(-2+4\beta)-2}{4} \Rightarrow 4 = \frac{4\beta}{3} \Rightarrow \beta = 3$$ ***M1A1***

$$\frac{6-(-3+\beta)}{3} = 1-(a+2\beta) \Rightarrow 2 = -5-a \Rightarrow a = -7$$ ***A1***

METHOD 2

$$\begin{cases} -3+\beta = 6-3\lambda \\ -2+4\beta = 4\lambda+2 \\ a+2\beta = 1-\lambda \end{cases}$$ ***M1***

attempt to solve ***M1***

$\lambda = 2,\ \beta = 3$ ***A1***

$a = 1-\lambda-2\beta = -7$ ***A1***

[4 marks]

(b) $$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$$ ***A1***

$\therefore \mathrm{P}(0, 10, -1)$

[2 marks]

Total [6 marks]

9. (a) attempt to differentiate implicitly ***M1***

$$3 - \left(4y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2\right)\mathrm{e}^{x-1} = 0$$ ***A1A1A1***

Note: Award ***A1*** for correctly differentiating each term.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\cdot \mathrm{e}^{1-x} - 2y^2}{4y}$$ ***A1***

Note: This final answer may be expressed in a number of different ways.

[5 marks]

continued…

Question 9 continued

(b) $3-2y^2=2 \Rightarrow y^2=\dfrac{1}{2} \Rightarrow y=\pm\sqrt{\dfrac{1}{2}}$ ***A1***

$\dfrac{dy}{dx}=\dfrac{3-2\cdot\dfrac{1}{2}}{\pm4\sqrt{\dfrac{1}{2}}}=\pm\dfrac{\sqrt{2}}{2}$ ***M1***

at $\left(1,\sqrt{\dfrac{1}{2}}\right)$ the tangent is $y-\sqrt{\dfrac{1}{2}}=\dfrac{\sqrt{2}}{2}(x-1)$ and ***A1***

at $\left(1,-\sqrt{\dfrac{1}{2}}\right)$ the tangent is $y+\sqrt{\dfrac{1}{2}}=-\dfrac{\sqrt{2}}{2}(x-1)$ ***A1***

Note: These equations simplify to $y=\pm\dfrac{\sqrt{2}}{2}x$.

Note: Award ***A0M1A1A0*** if just the positive value of y is considered and just one tangent is found.

[4 marks]

Total [9 marks]

10 (a) **METHOD 1**

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***M1***

$= \mathrm{P}(A) + \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B) - \mathrm{P}(A \cap B)$ ***M1A1***

$= \mathrm{P}(A) + \mathrm{P}(A' \cap B)$ ***AG***

METHOD 2

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***M1***

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A|B) \times \mathrm{P}(B)$ ***M1***

$= \mathrm{P}(A) + \left(1 - \mathrm{P}(A|B)\right) \times \mathrm{P}(B)$

$= \mathrm{P}(A) + \mathrm{P}(A'|B) \times \mathrm{P}(B)$ ***A1***

$= \mathrm{P}(A) + \mathrm{P}(A' \cap B)$ ***AG***

[3 marks]

(b) (i) use $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A' \cap B)$ and $\mathrm{P}(A' \cap B) = \mathrm{P}(B \mid A')\mathrm{P}(A')$ ***(M1)***

$\frac{4}{9} = \mathrm{P}(A) + \frac{1}{6}\left(1 - \mathrm{P}(A)\right)$ ***A1***

$8 = 18\mathrm{P}(A) + 3\left(1 - \mathrm{P}(A)\right)$ ***M1***

$\mathrm{P}(A) = \frac{1}{3}$ ***AG***

(ii) **METHOD 1**

$\mathrm{P}(B) = \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)$ ***M1***

$= \mathrm{P}(B \mid A)\mathrm{P}(A) + \mathrm{P}(B \mid A')\mathrm{P}(A')$ ***M1***

$= \frac{1}{3} \times \frac{1}{3} + \frac{1}{6} \times \frac{2}{3} = \frac{2}{9}$ ***A1***

METHOD 2

$\mathrm{P}(A \cap B) = \mathrm{P}(B \mid A)\mathrm{P}(A) \Rightarrow \mathrm{P}(A \cap B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ ***M1***

$\mathrm{P}(B) = \mathrm{P}(A \cup B) + \mathrm{P}(A \cap B) - \mathrm{P}(A)$ ***M1***

$\mathrm{P}(B) = \frac{4}{9} + \frac{1}{9} - \frac{1}{3} = \frac{2}{9}$ ***A1***

[6 marks]

Total [9 marks]

Section B

11. (a) $\frac{dy}{dx} = e^x \sin x + e^x \cos x \left(= e^x (\sin x + \cos x)\right)$ ***M1A1***

[2 marks]

(b) $\frac{d^2y}{dx^2} = e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$ ***M1A1***

$= 2e^x \cos x$ ***AG***

[2 marks]

(c) $\frac{dy}{dx} = e^{\frac{3\pi}{4}}\left(\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) = 0$ ***R1***

$\frac{d^2y}{dx^2} = 2e^{\frac{3\pi}{4}} \cos\frac{3\pi}{4} < 0$ ***R1***

hence maximum at $x = \frac{3\pi}{4}$ ***AG***

[2 marks]

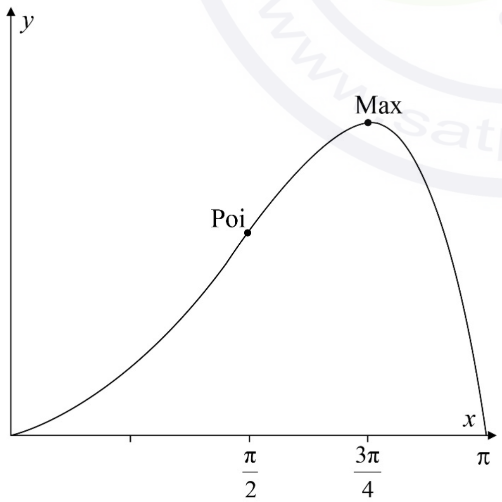
(d) $\frac{d^2y}{dx^2} = 0 \Rightarrow 2e^x \cos x = 0$ ***M1***

$\Rightarrow x = \frac{\pi}{2}$ ***A1***

Note: Award ***M1A0*** if extra zeros are seen.

[2 marks]

(e)



correct shape and correct domain ***A1***

max at $x = \frac{3\pi}{4}$, point of inflexion at $x = \frac{\pi}{2}$ ***A1***

zeros at $x = 0$ and $x = \pi$ ***A1***

Note: Penalize incorrect domain with first ***A*** mark; allow ***FT*** from (d) on extra points of inflexion.

[3 marks]

continued…

Question 11 continued

(f) **EITHER**

$$\int_0^{\pi} e^x \sin x \, dx = \left[e^x \sin x\right]_0^{\pi} - \int_0^{\pi} e^x \cos x \, dx$$ ***M1A1***

$$\int_0^{\pi} e^x \sin x \, dx = \left[e^x \sin x\right]_0^{\pi} - \left(\left[e^x \cos x\right]_0^{\pi} + \int_0^{\pi} e^x \sin x \, dx\right)$$ ***A1***

OR

$$\int_0^{\pi} e^x \sin x \, dx = \left[-e^x \cos x\right]_0^{\pi} + \int_0^{\pi} e^x \cos x \, dx$$ ***M1A1***

$$\int_0^{\pi} e^x \sin x \, dx = \left[-e^x \cos x\right]_0^{\pi} + \left(\left[e^x \sin x\right]_0^{\pi} - \int_0^{\pi} e^x \sin x \, dx\right)$$ ***A1***

THEN

$$\int_0^{\pi} e^x \sin x \, dx = \frac{1}{2}\left(\left[e^x \sin x\right]_0^{\pi} - \left[e^x \cos x\right]_0^{\pi}\right)$$ ***M1A1***

$$\int_0^{\pi} e^x \sin x \, dx = \frac{1}{2}\left(e^{\pi} + 1\right)$$ ***A1***

[6 marks]

(g) $\frac{dy}{dx} = 0$ ***(A1)***

$$\frac{d^2 y}{dx^2} = 2e^{\frac{3\pi}{4}} \cos\frac{3\pi}{4} = -\sqrt{2}e^{\frac{3\pi}{4}}$$ ***(A1)***

$$\kappa = \frac{\left|-\sqrt{2}e^{\frac{3\pi}{4}}\right|}{1} = \sqrt{2}e^{\frac{3\pi}{4}}$$ ***A1***

[3 marks]

(h) $\kappa = 0$ ***A1***

the graph is approximated by a straight line ***R1***

[2 marks]

Total [22 marks]

12. (a) (i) **METHOD 1**

$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=0$ ***A1***

as $\omega \neq 1$ ***R1***

METHOD 2

solutions of $1-\omega^3=0$ are $\omega=1,\ \omega=\frac{-1\pm\sqrt{3}\,\mathrm{i}}{2}$ ***A1***

verification that the sum of these roots is 0 ***R1***

(ii) $1+\omega^*+\left(\omega^*\right)^2=0$ ***A2***

[4 marks]

(b) $\left(\omega-3\omega^2\right)\left(\omega^2-3\omega\right)=-3\omega^4+10\omega^3-3\omega^2$ ***M1A1***

EITHER

$=-3\omega^2(\omega^2+\omega+1)+13\omega^3$ ***M1***

$=-3\omega^2\times 0+13\times 1$ ***A1***

OR

$=-3\omega+10-3\omega^2=-3(\omega^2+\omega+1)+13$ ***M1***

$=-3\times 0+13$ ***A1***

OR

substitution by $\omega=\frac{-1\pm\sqrt{3}\,\mathrm{i}}{2}$ in any form ***M1***

numerical values of each term seen ***A1***

THEN

$=13$ ***AG***

[4 marks]

(c) $|p|=|q|\Rightarrow\sqrt{1^2+3^2}=\sqrt{x^2+(2x+1)^2}$ ***(M1)(A1)***

$5x^2+4x-9=0$ ***A1***

$(5x+9)(x-1)=0$ ***(M1)***

$x=1,\ x=-\frac{9}{5}$ ***A1***

[5 marks]

continued…

Question 12 continued

(d) $pq = (1 - 3\text{i})\left(x + (2x + 1)\text{i}\right) = (7x + 3) + (1 - x)\text{i}$ ***M1A1***

$\text{Re}(pq) + 8 < \left(\text{Im}(pq)\right)^2 \Rightarrow (7x + 3) + 8 < (1 - x)^2$ ***M1***

$\Rightarrow x^2 - 9x - 10 > 0$ ***A1***

$\Rightarrow (x+1)(x-10) > 0$ ***M1***

$x < -1,\ x > 10$ ***A1***

[6 marks]

Total [19 marks]

13. (a) $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$ ***(M1)A1***

Note: Award ***M1*** for 5 equal terms with + or – signs.

[2 marks]

(b) $\frac{1 - \cos 2x}{2\sin x} \equiv \frac{1 - \left(1 - 2\sin^2 x\right)}{2\sin x}$ ***M1***

$\equiv \frac{2\sin^2 x}{2\sin x}$ ***A1***

$\equiv \sin x$ ***AG***

[2 marks]

continued…

Question 13 continued

(c) let $\mathrm{P}(n): \sin x + \sin 3x + \ldots + \sin(2n-1)x \equiv \dfrac{1-\cos 2nx}{2\sin x}$

if $n = 1$

$\mathrm{P}(1)$: $\dfrac{1-\cos 2x}{2\sin x} \equiv \sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x + \sin 3x + \ldots + \sin(2k-1)x \equiv \dfrac{1-\cos 2kx}{2\sin x}$ ***M1***

Notes: Only award ***M1*** if the words "assume" and "true" appear. Do not award ***M1*** for "let $n = k$" only. Subsequent marks are independent of this ***M1***.

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1): \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x \equiv \dfrac{1-\cos 2(k+1)x}{2\sin x}$

$LHS = \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x$ ***M1***

$\equiv \dfrac{1-\cos 2kx}{2\sin x} + \sin(2k+1)x$ ***A1***

$\equiv \dfrac{1-\cos 2kx + 2\sin x \sin(2k+1)x}{2\sin x}$

$\equiv \dfrac{1-\cos 2kx + 2\sin x\cos x\sin 2kx + 2\sin^2 x\cos 2kx}{2\sin x}$ ***M1***

$\equiv \dfrac{1-\left(\left(1-2\sin^2 x\right)\cos 2kx - \sin 2x\sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv \dfrac{1-\left(\cos 2x\cos 2kx - \sin 2x\sin 2kx\right)}{2\sin x}$ ***A1***

$\equiv \dfrac{1-\cos\left(2kx+2x\right)}{2\sin x}$ ***A1***

$\equiv \dfrac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n = k$, then also true for $n = k+1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Accept answers using transformation formula for product of sines if steps are shown clearly.

Note: Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps.

[9 marks]

continued…

Question 13 continued

(d) **EITHER**

$\sin x + \sin 3x = \cos x \Rightarrow \dfrac{1-\cos 4x}{2\sin x} = \cos x$ ***M1***

$\Rightarrow 1 - \cos 4x = 2\sin x \cos x$, $(\sin x \neq 0)$ ***A1***

$\Rightarrow 1 - \left(1 - 2\sin^2 2x\right) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2\sin 2x - 1) = 0$ ***M1***

$\Rightarrow \sin 2x = 0$ or $\sin 2x = \dfrac{1}{2}$ ***A1***

$2x = \pi$, $2x = \dfrac{\pi}{6}$ and $2x = \dfrac{5\pi}{6}$

OR

$\sin x + \sin 3x = \cos x \Rightarrow 2\sin 2x \cos x = \cos x$ ***M1A1***

$\Rightarrow (2\sin 2x - 1)\cos x = 0$, $(\sin x \neq 0)$ ***M1A1***

$\Rightarrow \sin 2x = \dfrac{1}{2}$ or $\cos x = 0$ ***A1***

$2x = \dfrac{\pi}{6}$, $2x = \dfrac{5\pi}{6}$ and $x = \dfrac{\pi}{2}$

THEN

$\therefore\ x = \dfrac{\pi}{2}$, $x = \dfrac{\pi}{12}$ and $x = \dfrac{5\pi}{12}$ ***A1***

Note: Do not award the final ***A1*** if extra solutions are seen.

[6 marks]

Total [19 marks]

Markscheme

May 2016

Mathematics

Higher level

Paper 1

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. use of either $u_n = u_1 + (n-1)d$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ ***M1***

$u_1 + 4d = 6$ ***(A1)***

$\frac{12}{2}(2u_1 + 11d) = 45$ ***(A1)***

$\Rightarrow 4u_1 + 22d = 15$

attempt to solve simultaneous equations ***M1***

$4(6-4d) + 22d = 15$

$6d = -9 \Rightarrow d = -1.5$ ***A1***

$u_1 = 12$ ***A1***

[6 marks]

2. total time of first 3 skiers $= 34.1 \times 3 = 102.3$ ***(M1)A1***

total time of first 4 skiers $= 35.0 \times 4 = 140.0$ ***A1***

time taken by fourth skier $= 140.0 - 102.3 = 37.7$ (seconds) ***A1***

[4 marks]

3. (a) $a = 1.5$ $d = 2$ ***A1A1***

[2 marks]

(b) $b = \frac{2\pi}{2} = \pi$ ***(M1)A1***

[2 marks]

(c) attempt to solve an appropriate equation or apply a horizontal translation ***(M1)***

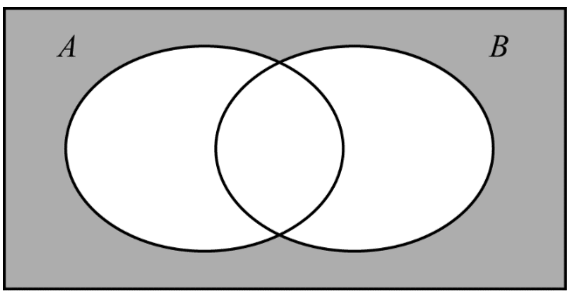
$c = 1.5$ ***A1***

Note:	Do not award a follow through mark for the final ***A1***. Award ***(M1)A0*** for $c = -0.5$.

[2 marks]

Total [6 marks]

4. (a)



A1

[1 mark]

continued…

Question 4 continued

(b) $P(A' \mid B') = \dfrac{P(A' \cap B')}{P(B')}$ ***(M1)***

$P(B') = 0.1 + 0.2 = 0.3$ ***(A1)***

$P(A' \cap B') = 0.1$ ***(A1)***

$P(A' \mid B') = \dfrac{0.1}{0.3} = \dfrac{1}{3}$ ***A1***

[4 marks]

Total [5 marks]

5. (a) $\left(1 - \sqrt{3}\right)^2 = 4 - 2\sqrt{3}$ ***A1***

Note: Award ***A0*** for $1 - 2\sqrt{3} + 3$.

[1 mark]

(b) $\cos(60^\circ - 45^\circ) = \cos(60^\circ)\cos(45^\circ) + \sin(60^\circ)\sin(45^\circ)$ ***M1***

$= \dfrac{1}{2} \times \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{2}}{2} \left(\text{or } \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}}\right)$ ***(A1)***

$= \dfrac{\sqrt{2} + \sqrt{6}}{4} \left(\text{or } \dfrac{1 + \sqrt{3}}{2\sqrt{2}}\right)$ ***A1***

[3 marks]

(c) $BC^2 = 2 + 4 - 2 \times \sqrt{2} \times 2\cos(15^\circ)$ ***M1***

$= 6 - \sqrt{2}\left(\sqrt{2} + \sqrt{6}\right)$

$= 4 - \sqrt{12} \left(= 4 - 2\sqrt{3}\right)$ ***A1***

$BC = \pm\left(1 - \sqrt{3}\right)$ ***(M1)***

$BC = -1 + \sqrt{3}$ ***A1***

Note: Accept $BC = \sqrt{3} - 1$.

Note: Award ***M1A0*** for $1 - \sqrt{3}$.

Note: Valid geometrical methods may be seen.

[4 marks]

Total [8 marks]

6. **METHOD 1**

$m - n\log_3 2 = 10\log_9 6$

$m - n\log_3 2 = 5\log_3 6$ ***M1***

$m = \log_3\left(6^5 2^n\right)$ ***(M1)***

$3^m 2^{-n} = 6^5 = 3^5 \times 2^5$ ***(M1)***

$m = 5,\ n = -5$ ***A1***

Note: First ***M1*** is for any correct change of base, second ***M1*** for writing as a single logarithm, third ***M1*** is for writing 6 as 2×3.

METHOD 2

$m - n\log_3 2 = 10\log_9 6$

$m - n\log_3 2 = 5\log_3 6$ ***M1***

$m - n\log_3 2 = 5\log_3 3 + 5\log_3 2$ ***(M1)***

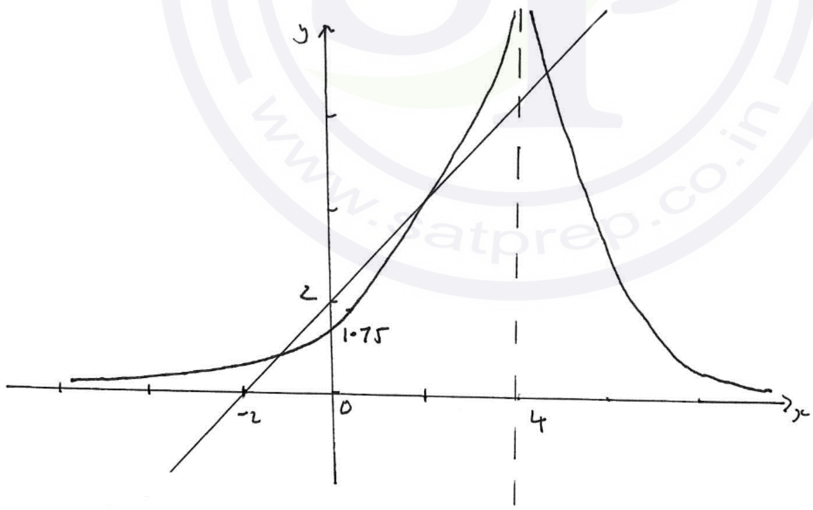
$m - n\log_3 2 = 5 + 5\log_3 2$ ***(M1)***

$m = 5,\ n = -5$ ***A1***

Note: First ***M1*** is for any correct change of base, second ***M1*** for writing 6 as 2×3 and third ***M1*** is for forming an expression without $\log_3 3$.

[4 marks]

7. (a)



A1 for vertical asymptote and for the y-intercept $\frac{7}{4}$

A1 for general shape of $y = \left|\frac{7}{x-4}\right|$ including the x-axis as asymptote

A1 for straight line with y-intercept 2 and x-intercept of -2 ***A1A1A1***

[3 marks]

continued…

Question 7 continued

(b) **METHOD 1**

for $x > 4$

$(x + 2)(x - 4) = 7$ ***(M1)***

$x^2 - 2x - 8 = 7 \Rightarrow x^2 - 2x - 15 = 0$

$(x - 5)(x + 3) = 0$

(as $x > 4$ then) $x = 5$ ***A1***

Note: Award ***A0*** if $x = -3$ is also given as a solution.

for $x < 4$

$(x + 2)(x - 4) = -7$ ***M1***

$\Rightarrow x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ***(M1)A1***

Note: Second ***M1*** is dependent on first ***M1***.

[5 marks]

METHOD 2

$(x+2)^2 = \frac{49}{(x-4)^2}$ ***M1***

$x^4 - 4x^3 - 12x^2 + 32x + 15 = 0$ ***A1***

$(x+3)(x-5)(x^2 - 2x - 1) = 0$

$x = 5$ ***A1***

Note: Award ***A0*** if $x = -3$ is also given as a solution.

$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ***(M1)A1***

[5 marks]

Total [8 marks]

8. $\boldsymbol{c} \bullet (\boldsymbol{b} - \boldsymbol{a}) = 0$ ***M1***

Note: Allow $c \bullet \overrightarrow{\mathrm{AB}} = 0$ or similar for ***M1***.

$\boldsymbol{c} \bullet \boldsymbol{b} = \boldsymbol{c} \bullet \boldsymbol{a}$ ***A1***

$\boldsymbol{b} \bullet (\boldsymbol{c} - \boldsymbol{a}) = 0$

$\boldsymbol{b} \bullet \boldsymbol{c} = \boldsymbol{b} \bullet \boldsymbol{a}$ ***A1***

$\boldsymbol{c} \bullet \boldsymbol{a} = \boldsymbol{b} \bullet \boldsymbol{a}$ ***M1***

$(\boldsymbol{c} - \boldsymbol{b}) \bullet \boldsymbol{a} = 0$ ***A1***

hence $\boldsymbol{a}$ is perpendicular to $\overrightarrow{\mathrm{BC}}$ ***AG***

Note: Only award the final ***A1*** if a dot is used throughout to indicate scalar product.
Condone any lack of specific indication that the letters represent vectors.

[5 marks]

9. $\frac{\mathrm{d}y}{\mathrm{d}x} = -\cos(\pi \cos x) \times \pi \sin x$ ***M1A1***

Note: Award follow through marks below if their answer is a multiple of the correct answer.

considering either $\sin x = 0$ or $\cos(\pi \cos x) = 0$ ***(M1)***

$x = 0, \pi$ ***A1***

$\pi \cos x = \frac{\pi}{2}, -\frac{\pi}{2} \left(\Rightarrow \cos x = \frac{1}{2}, -\frac{1}{2}\right)$ ***M1***

Note: Condone absence of $-\frac{\pi}{2}$.

$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$

$(0, 0), \left(\frac{\pi}{3}, 1\right), (\pi, 0)$ ***A1***

$\left(\frac{2\pi}{3}, -1\right)$ ***A1***

[7 marks]

10. $\frac{dy}{dx} = 8x^3 + 18x^2 + 7x - 5$ ***A1***

when $x = -1$, $\frac{dy}{dx} = -2$ ***A1***

$8x^3 + 18x^2 + 7x - 5 = -2$ ***M1***

$8x^3 + 18x^2 + 7x - 3 = 0$

$(x + 1)$ is a factor ***A1***

$8x^3 + 18x^2 + 7x - 3 = (x + 1)(8x^2 + 10x - 3)$ ***(M1)***

Note: ***M1*** is for attempting to find the quadratic factor.

$(x + 1)(4x - 1)(2x + 3) = 0$

$(x = -1)$, $x = 0.25$, $x = -1.5$ ***(M1)A1***

Note: ***M1*** is for an attempt to solve their quadratic factor.

[7 marks]

Section B

11.

Note: Throughout the question condone vectors written horizontally.

(a) angle between planes is equal to the angles between the normal to the planes *(M1)*

$$\begin{pmatrix}4\\1\\1\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-1\end{pmatrix}=18$$ *(A1)*

let θ be the angle between the normal to the planes

$$\cos\theta=\frac{18}{\sqrt{18}\sqrt{26}}=\sqrt{\frac{18}{26}}\left(\text{or equivalent,for example }\sqrt{\frac{324}{468}}\text{ or}\sqrt{\frac{9}{13}}\right)$$ ***M1A1***

[4 marks]

(b) (i) **METHOD 1**

$$\begin{pmatrix}4\\1\\1\end{pmatrix}\times\begin{pmatrix}4\\3\\-1\end{pmatrix}=\begin{pmatrix}-4\\8\\8\end{pmatrix}$$ ***M1A1***

which is a multiple of $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***R1AG***

Note: Allow any equivalent wording or $\begin{pmatrix}-4\\8\\8\end{pmatrix}=4\begin{pmatrix}-1\\2\\2\end{pmatrix}$, do not allow $\begin{pmatrix}-4\\8\\8\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$.

METHOD 2

let $z=t$ (or equivalent)

solve simultaneously to get ***M1***

$y=t-4$, $x=3-0.5t$ ***A1***

hence direction vector is $\begin{pmatrix}-0.5\\1\\1\end{pmatrix}$

which is a multiple of $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***R1AG***

continued…

Question 11 continued

METHOD 3

$$\begin{pmatrix}4\\1\\1\end{pmatrix} \bullet \begin{pmatrix}-1\\2\\2\end{pmatrix} = -4+2+2=0$$

M1A1

$$\begin{pmatrix}4\\3\\-1\end{pmatrix} \bullet \begin{pmatrix}-1\\2\\2\end{pmatrix} = -4+6-2=0$$

A1

Note: If only one scalar product is found award ***M0A0A0***.

(ii) $\Pi_1 : 4+0+4=8$ and $\Pi_2 : 4+0-4=0$ ***R1***

(iii) $\boldsymbol{r} = \begin{pmatrix}1\\0\\4\end{pmatrix} + \lambda \begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***A1A1***

Note: ***A1*** for "$\boldsymbol{r}$ =" and a correct point on the line, ***A1*** for a parameter and a correct direction vector.

[6 marks]

(c) $\vec{AB} = \begin{pmatrix}a\\b\\1\end{pmatrix} - \begin{pmatrix}1\\0\\4\end{pmatrix} = \begin{pmatrix}a-1\\b\\-3\end{pmatrix}$ ***(A1)***

$$\begin{pmatrix}a-1\\b\\-3\end{pmatrix} \bullet \begin{pmatrix}-1\\2\\2\end{pmatrix} = 0$$

M1

Note: Award ***M0*** for $\begin{pmatrix}a\\b\\1\end{pmatrix} \bullet \begin{pmatrix}-1\\2\\2\end{pmatrix} = 0$.

$-a+1+2b-6=0 \Rightarrow a-2b=-5$ ***A1***

lies on Π_1 so $4a+b+1=8 \Rightarrow 4a+b=7$ ***M1***

$a=1,\ b=3$ ***A1***

[5 marks]

(d) $\mathrm{AB} = \sqrt{0^2+3^2+(-3)^2} = 3\sqrt{2}$ ***M1AG***

[1 mark]

continued…

Question 11 continued

(e) **METHOD 1**

$\left|\vec{AB}\right| = \left|\vec{AP}\right| = 3\sqrt{2}$ ***(M1)***

$\vec{AP} = t\begin{pmatrix}-1\\2\\2\end{pmatrix}$ ***(A1)***

$\left|3t\right| = 3\sqrt{2} \Rightarrow t = \pm\sqrt{2}$ ***(M1)A1***

$P\left(1-\sqrt{2},\ 2\sqrt{2},\ 4+2\sqrt{2}\right)$ and $\left(1+\sqrt{2},\ -2\sqrt{2},\ 4-2\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 2

let P have coordinates $(1-\lambda,\ 2\lambda,\ 4+2\lambda)$ ***M1***

$\vec{BA} = \begin{pmatrix}0\\-3\\3\end{pmatrix}, \quad \vec{BP} = \begin{pmatrix}-\lambda\\2\lambda-3\\3+2\lambda\end{pmatrix}$ ***A1***

$\cos 45^0 = \dfrac{\vec{BA}\bullet\vec{BP}}{|BA||BP|}$ ***M1***

Note: Award ***M1*** even if AB rather than BA is used in the scalar product.

$\vec{BA}\bullet\vec{BP} = 18$

$\dfrac{1}{\sqrt{2}} = \dfrac{18}{\sqrt{18}\sqrt{9\lambda^2+18}}$

$\lambda = \pm\sqrt{2}$ ***A1***

$P\left(1-\sqrt{2},\ 2\sqrt{2},\ 4+2\sqrt{2}\right)$ and $\left(1+\sqrt{2},\ -2\sqrt{2},\ 4-2\sqrt{2}\right)$ ***A1***

Note: Accept answers given as position vectors.

[5 marks]

Total [21 marks]

12. (a) $\left(\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3=\cos\pi+\mathrm{i}\sin\pi$ ***M1***

$=-1$ ***A1***

[2 marks]

(b) show the expression is true for $n=1$ ***R1***

assume true for $n=k$, $(\cos\theta-\mathrm{i}\sin\theta)^k=\cos k\theta-\mathrm{i}\sin k\theta$ ***M1***

> **Note:** Do not accept "let $n=k$" or "assume $n=k$", assumption of truth must be present.

$(\cos\theta-\mathrm{i}\sin\theta)^{k+1}=(\cos\theta-\mathrm{i}\sin\theta)^k(\cos\theta-\mathrm{i}\sin\theta)$

$=(\cos k\theta-\mathrm{i}\sin k\theta)(\cos\theta-\mathrm{i}\sin\theta)$ ***M1***

$=\cos k\theta\cos\theta-\sin k\theta\sin\theta-\mathrm{i}(\cos k\theta\sin\theta+\sin k\theta\cos\theta)$ ***A1***

> **Note:** Award ***A1*** for any correct expansion.

$=\cos((k+1)\theta)-\mathrm{i}\sin((k+1)\theta)$ ***A1***

therefore if true for $n=k$ true for $n=k+1$, true for $n=1$, so true for all $n\left(\in\mathbb{Z}^+\right)$ ***R1***

> **Note:** To award the final ***R*** mark the first 4 marks must be awarded.

[6 marks]

(c) $(z)^n+\left(z^*\right)^n=(\cos\theta+\mathrm{i}\sin\theta)^n+(\cos\theta-\mathrm{i}\sin\theta)^n$

$=\cos n\theta+\mathrm{i}\sin n\theta+\cos n\theta-\mathrm{i}\sin n\theta=2\cos(n\theta)$ ***(M1)A1***

[2 marks]

(d) (i) $zz^*=(\cos\theta+\mathrm{i}\sin\theta)(\cos\theta-\mathrm{i}\sin\theta)$

$=\cos^2\theta+\sin^2\theta$ ***A1***

$=1$ ***AG***

> **Note:** Allow justification starting with $|z|=1$.

(ii) $\left(z+z^*\right)^3=z^3+3z^2z^*+3z\left(z^*\right)^2+\left(z^*\right)^3\left(=z^3+3z+3z^*+\left(z^*\right)^3\right)$ ***A1***

(iii) $\left(z+z^*\right)^3=(2\cos\theta)^3$ ***A1***

$z^3+3z+3z^*+\left(z^*\right)^3=2\cos3\theta+6\cos\theta$ ***M1A1***

$\cos3\theta=4\cos^3\theta-3\cos\theta$ ***AG***

> **Note:** ***M1*** is for using $zz^*=1$, this might be seen in d(ii).

[5 marks]

continued…

Question 12 continued

(e) $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

$\cos(3\theta) = \cos(2\theta)$ ***A1A1***

Note: ***A1*** for $\cos(3\theta)$ and ***A1*** for $\cos(2\theta)$.

$\theta = 0$ ***A1***

or $3\theta = 2\pi - 2\theta$ (or $3\theta = 4\pi - 2\theta$) ***M1***

$\theta = \frac{2\pi}{5}, \frac{4\pi}{5}$ ***A1A1***

Note: Do not accept solutions via factor theorem or other methods that do not follow "hence".

[6 marks]

Total [21 marks]

13. (a) $a = 1$ ***A1***

[1 mark]

(b) $\frac{du}{dx} = \frac{1}{x}$ ***(A1)***

$\int \frac{(\ln x)^2}{x}dx = \int u^2 du$ ***M1A1***

area $= \left[\frac{1}{3}u^3\right]_0^1$ or $\left[\frac{1}{3}(\ln x)^3\right]_1^e$ ***A1***

$= \frac{1}{3}$ ***A1***

[5 marks]

(c) (i) $I_0 = \left[-\frac{1}{x}\right]_1^e$ ***(A1)***

$= 1 - \frac{1}{e}$ ***A1***

(ii) use of integration by parts ***M1***

$I_n = \left[-\frac{1}{x}(\ln x)^n\right]_1^e + \int_1^e \frac{n(\ln x)^{n-1}}{x^2}dx$ ***A1A1***

$= -\frac{1}{e} + nI_{n-1}$ ***AG***

Note: If the substitution $u = \ln x$ is used ***A1A1*** can be awarded for $I_n = \left[-e^{-u}u^n\right]_0^1 + \int_0^1 ne^{-u}u^{n-1}\,du$.

continued…

Question 13 continued

(iii) $I_1 = -\frac{1}{\mathrm{e}} + 1 \times I_0$ ***(M1)***

$= 1 - \frac{2}{\mathrm{e}}$ ***A1***

[7 marks]

(d) volume $= \pi \int_1^{\mathrm{e}} \frac{(\ln x)^4}{x^2} \mathrm{d}x \left(= \pi I_4\right)$ ***(A1)***

EITHER

$I_4 = -\frac{1}{\mathrm{e}} + 4I_3$ ***M1A1***

$= -\frac{1}{\mathrm{e}} + 4\left(-\frac{1}{\mathrm{e}} + 3I_2\right)$ ***M1***

$= -\frac{5}{\mathrm{e}} + 12I_2 = -\frac{5}{\mathrm{e}} + 12\left(-\frac{1}{\mathrm{e}} + 2I_1\right)$

OR

using parts $\int_1^{\mathrm{e}} \frac{(\ln x)^4}{x^2} \mathrm{d}x = -\frac{1}{e} + 4\int_1^{\mathrm{e}} \frac{(\ln x)^3}{x^2} \mathrm{d}x$ ***M1A1***

$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3\int_1^{\mathrm{e}} \frac{(\ln x)^2}{x^2} \mathrm{d}x\right)$ ***M1***

THEN

$= -\frac{17}{\mathrm{e}} + 24\left(1 - \frac{2}{\mathrm{e}}\right) = 24 - \frac{65}{\mathrm{e}}$ ***A1***

volume $= \pi\left(24 - \frac{65}{\mathrm{e}}\right)$

[5 marks]

Total [18 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2016

Mathematics

Higher level

Paper 1

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. **EITHER**

eliminating a variable, x, for example to obtain $y+3z=-16$ and $-5y-3z=8$ ***M1A1***
attempting to find the value of one variable ***M1***
point of intersection is $(-1,\ 2,\ -6)$ ***A1A1A1***

OR

attempting row reduction of relevant matrix, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & 3 & 1 & -1 \\ 1 & 2 & -2 & 15 \end{array}\right)$ ***M1***

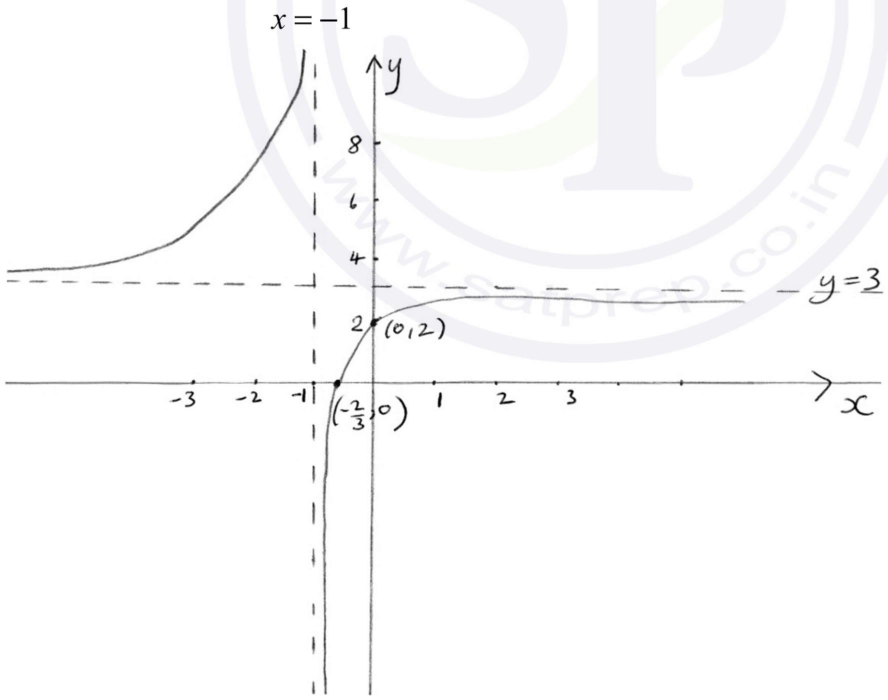
correct matrix with two zeroes in a column, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & 5 & 3 & -8 \\ 0 & 1 & 3 & -16 \end{array}\right)$ ***A1***

further attempt at reduction ***M1***
point of intersection is $(-1,\ 2,\ -6)$ ***A1A1A1***

Note: Allow solution expressed as $x=-1$, $y=2$, $z=-6$ for final ***A*** marks.

[6 marks]

2.



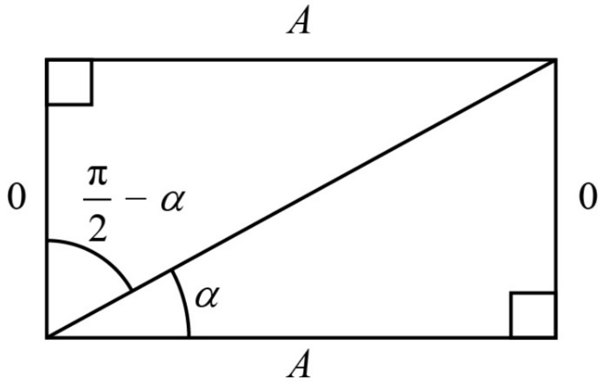
A1A1A1A1A1

Note: Award ***A1*** for correct shape, ***A1*** for $x=-1$ clearly stated and asymptote shown, ***A1*** for $y=3$ clearly stated and asymptote shown, ***A1*** for $\left(-\frac{2}{3},\ 0\right)$ and ***A1*** for $(0,\ 2)$.

[5 marks]

3. (a) **EITHER**

use of a diagram and trig ratios
eg,



$\tan\alpha = \frac{O}{A} \Rightarrow \cot\alpha = \frac{A}{O}$

from diagram, $\tan\left(\frac{\pi}{2}-\alpha\right) = \frac{A}{O}$ ***R1***

OR

use of $\tan\left(\frac{\pi}{2}-\alpha\right) = \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)} = \frac{\cos\alpha}{\sin\alpha}$ ***R1***

THEN

$\cot\alpha = \tan\left(\frac{\pi}{2}-\alpha\right)$ ***AG***

[1 mark]

(b) $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx = [\arctan x]_{\tan\alpha}^{\cot\alpha}$ ***(A1)***

Note: Limits (or absence of such) may be ignored at this stage.

$= \arctan(\cot\alpha) - \arctan(\tan\alpha)$ ***(M1)***

$= \frac{\pi}{2} - \alpha - \alpha$ ***(A1)***

$= \frac{\pi}{2} - 2\alpha$ ***A1***

[4 marks]

Total [5 marks]

4. $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=\dfrac{ax_2^{\,2}+bx_2+c-\left(ax_1^{\,2}+bx_1+c\right)}{x_2-x_1}$ ***(M1)***

$=\dfrac{a\left(x_2^{\,2}-x_1^{\,2}\right)+b\left(x_2-x_1\right)}{x_2-x_1}$ ***A1***

$=\dfrac{a\left(x_2-x_1\right)\left(x_2+x_1\right)+b\left(x_2-x_1\right)}{x_2-x_1}$ ***(A1)***

$=a\left(x_2+x_1\right)+b \qquad \left(x_1\neq x_2\right)$ ***A1***

$\dfrac{f'(x_2)+f'(x_1)}{2}=\dfrac{\left(2ax_2+b\right)+\left(2ax_1+b\right)}{2}$ ***M1***

$=\dfrac{2a\left(x_2+x_1\right)+2b}{2}$

$=a\left(x_2+x_1\right)+b$ ***A1***

so Hayley's conjecture is correct ***AG***

[6 marks]

5. (a) $X\sim \mathrm{B}(5,\ p)$ ***(M1)***

$\mathrm{P}(X=4)=\begin{pmatrix}5\\4\end{pmatrix}p^4(1-p)$ (or equivalent) ***A1***

[2 marks]

(b) (i) $\dfrac{\mathrm{d}}{\mathrm{d}p}\left(5p^4-5p^5\right)=20p^3-25p^4$ ***M1A1***

$5p^3(4-5p)=0\Rightarrow p=\dfrac{4}{5}$ ***M1A1***

Note: Do not award the final ***A1*** if $p=0$ is included in the answer.

(ii) $\mathrm{E}(X)=np=5\left(\dfrac{4}{5}\right)$ ***(M1)***

$=4$ ***A1***

[6 marks]

Total [8 marks]

6. (a) $1,\ nx,\ \frac{n(n-1)}{2}x^2,\ \frac{n(n-1)(n-2)}{6}x^3$ ***A1A1***

Note: Award ***A1*** for the first two terms and ***A1*** for the next two terms.

Note: Accept $\binom{n}{r}$ notation.

Note: Allow the terms seen in the context of an arithmetic sum.

Note: Allow unsimplified terms, *eg*, those including powers of 1 if seen.

[2 marks]

(b) (i) **EITHER**

using $u_3 - u_2 = u_4 - u_3$ ***(M1)***

$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}$ ***A1***

attempting to remove denominators and expanding (or vice versa) ***M1***

$3n^2 - 9n = n^3 - 6n^2 + 5n$ (or equivalent, *eg*, $6n^2 - 12n = n^3 - 3n^2 + 2n$) ***A1***

OR

using $u_2 + u_4 = 2u_3$ ***(M1)***

$n + \frac{n(n-1)(n-2)}{6} = n(n-1)$ ***(A1)***

attempting to remove denominators and expanding (or vice versa) ***M1***

$6n + n^3 - 3n^2 + 2n = 6n^2 - 6n$ (or equivalent) ***(A1)***

THEN

$n^3 - 9n^2 + 14n = 0$ ***AG***

(ii) $n(n-2)(n-7) = 0$ or $(n-2)(n-7) = 0$ ***(A1)***

$n = 7$ only (as $n \geq 3$) ***A1***

[6 marks]

Total [8 marks]

7. (a) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$ ***(M1)***

$= p + p - p^2$ ***A1***

$= 2p - p^2$ ***AG***

[2 marks]

(b) $\mathrm{P}(A \mid A \cup B) = \dfrac{\mathrm{P}(A \cap (A \cup B))}{\mathrm{P}(A \cup B)}$ ***(M1)***

Note: Allow $\mathrm{P}(A \cap A \cup B)$ if seen on the numerator.

$= \dfrac{\mathrm{P}(A)}{\mathrm{P}(A \cup B)}$ ***(A1)***

$= \dfrac{p}{2p - p^2}$ ***A1***

$= \dfrac{1}{2 - p}$ ***A1***

[4 marks]

Total [6 marks]

8. let $\mathrm{P}(n)$ be the proposition that $n(n^2 + 5)$ is divisible by 6 for $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n = 1$, $n(n^2 + 5) = 1 \times (1^2 + 5) = 6$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *ie*, $k(k^2 + 5) = 6m$ where $k, m \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k + 1)$:

$(k + 1)((k + 1)^2 + 5)$ ***M1***

$= (k + 1)(k^2 + 2k + 6)$

$= k^3 + 3k^2 + 8k + 6$ ***(A1)***

$= (k^3 + 5k) + (3k^2 + 3k + 6)$ ***A1***

$= k(k^2 + 5) + 3k(k + 1) + 6$ ***A1***

$k(k + 1)$ is even hence all three terms are divisible by 6 ***R1***

$\mathrm{P}(k + 1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

[8 marks]

9. (a) **EITHER**

$$\text{LHS} = \frac{\sqrt{3}-1}{\frac{\sqrt{6}-\sqrt{2}}{4}} + \frac{\sqrt{3}+1}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$ ***M1***

$$= \frac{\sqrt{3}-1}{\frac{\sqrt{3}-1}{2\sqrt{2}}} + \frac{\sqrt{3}+1}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$ ***A1***

$$= 2\sqrt{2} + 2\sqrt{2}$$ ***A1***

$\text{LHS} = 4\sqrt{2} \Rightarrow x = \frac{\pi}{12}$ is a solution ***AG***

OR

$$\text{LHS} = \frac{\sqrt{3}-1}{\frac{\sqrt{6}-\sqrt{2}}{4}} + \frac{\sqrt{3}+1}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$ ***M1***

$$= \frac{\left(\sqrt{3}-1\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) + \left(\sqrt{3}+1\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)}{\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)}$$ ***A1***

$= 2\sqrt{18} - 2\sqrt{2}$ (or equivalent) ***A1***

$\text{LHS} = 4\sqrt{2} \Rightarrow x = \frac{\pi}{12}$ is a solution ***AG***

[3 marks]

(b) $$\frac{\sqrt{2}}{4}\left(\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x}\right) = 2 \Rightarrow \frac{\sin\frac{\pi}{12}}{\sin x} + \frac{\cos\frac{\pi}{12}}{\cos x} = 2$$ ***M1***

$$\frac{\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x}{\sin x \cos x} = 2$$ ***M1***

$$\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x = 2\sin x\cos x$$

$$\sin\left(\frac{\pi}{12} + x\right) = \sin 2x$$ ***A1***

$\frac{\pi}{12} + x = \pi - 2x$ or $\pi - \left(\frac{\pi}{12} + x\right) = 2x$ ***(M1)***

$$x = \frac{11\pi}{36}$$ ***A1***

[5 marks]

Total [8 marks]

Section B

10. (a) **EITHER**

$\boldsymbol{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix}$ ***A1A1***

and $\boldsymbol{n} \neq k\boldsymbol{d}$ ***R1***

OR

$\boldsymbol{n} \times \boldsymbol{d} = \begin{pmatrix} -5 \\ 3p - 1 \\ 2 - p \end{pmatrix}$ ***M1A1***

the vector product is non-zero for $p \in \mathbb{R}$ ***R1***

THEN

L is not perpendicular to Π ***AG***

[3 marks]

(b) **METHOD 1**

$(2 + p\lambda) + (q + 2\lambda) + 3(1 + \lambda) = 9$ ***M1***

$(q + 5) + (p + 5)\lambda = 9$ ***(A1)***

$p = -5$ and $q = 4$ ***A1A1***

METHOD 2

direction vector of line is perpendicular to plane, so

$\begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$ ***M1***

$p = -5$ ***A1***

$(2, q, 1)$ is common to both L and Π

either $\begin{pmatrix} 2 \\ q \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 9$ or by substituting into $x + y + 3z = 9$ ***M1***

$q = 4$ ***A1***

[4 marks]

continued…

Question 10 continued

(c) (i) **METHOD 1**

α is the acute angle between $\boldsymbol{n}$ and L

if $\sin\theta = \frac{1}{\sqrt{11}}$ then $\cos\alpha = \frac{1}{\sqrt{11}}$ ***(M1)(A1)***

attempting to use $\cos\alpha = \frac{\boldsymbol{n}\cdot\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}$ or $\sin\theta = \frac{\boldsymbol{n}\cdot\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}$ ***M1***

$\frac{p+5}{\sqrt{11}\times\sqrt{p^2+5}} = \frac{1}{\sqrt{11}}$ ***A1A1***

$(p+5)^2 = p^2+5$ ***M1***

$10p = -20$ (or equivalent) ***A1***

$p = -2$ ***AG***

METHOD 2

α is the angle between $\boldsymbol{n}$ and L

if $\sin\theta = \frac{1}{\sqrt{11}}$ then $\sin\alpha = \frac{\sqrt{10}}{\sqrt{11}}$ ***(M1)A1***

attempting to use $\sin\alpha = \frac{|\boldsymbol{n}\times\boldsymbol{d}|}{|\boldsymbol{n}||\boldsymbol{d}|}$ ***M1***

$\frac{\sqrt{(-5)^2+(3p-1)^2+(2-p)^2}}{\sqrt{11}\times\sqrt{p^2+5}} = \frac{\sqrt{10}}{\sqrt{11}}$ ***A1A1***

$p^2 - p + 3 = p^2 + 5$ ***M1***

$-p + 3 = 5$ (or equivalent) ***A1***

$p = -2$ ***AG***

(ii) $p = -2$ and $z = -1 \Rightarrow \frac{x-2}{-2} = \frac{y-q}{2} = -2$ ***(A1)***

$x = 6$ and $y = q - 4$ ***(A1)***

this satisfies Π so $6 + q - 4 - 3 = 9$ ***M1***

$q = 10$ ***A1***

[11 marks]

Total [18 marks]

11. (a) use of $\pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

Note: Condone any or missing limits.

$V = \pi\int_0^\pi (3\cos 2y + 4)^2 \,\mathrm{d}y$ ***(A1)***

$= \pi\int_0^\pi \left(9\cos^2 2y + 24\cos 2y + 16\right)\mathrm{d}y$ ***A1***

$9\cos^2 2y = \frac{9}{2}(1 + \cos 4y)$ ***(M1)***

$= \pi\left[\frac{9y}{2} + \frac{9}{8}\sin 4y + 12\sin 2y + 16y\right]_0^\pi$ ***M1A1***

$= \pi\left(\frac{9\pi}{2} + 16\pi\right)$ ***(A1)***

$= \frac{41\pi^2}{2}\ \left(\text{cm}^3\right)$ ***A1***

Note: If the coefficient "π" is absent, or *eg*, "2π" is used, only ***M*** marks are available.

[8 marks]

(b) (i) attempting to use $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}$ with $\frac{\mathrm{d}V}{\mathrm{d}t} = 2$ ***M1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{\pi(3\cos 2h + 4)^2}$ ***A1***

(ii) substituting $h = \frac{\pi}{4}$ into $\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{8\pi}$ (cm min^{-1}) ***A1***

Note: Do not allow FT marks for (b)(ii).

[4 marks]

(c) (i) $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right) = \frac{\mathrm{d}h}{\mathrm{d}t} \times \frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)$ ***(M1)***

$= \frac{2}{\pi(3\cos 2h + 4)^2} \times \frac{24\sin 2h}{\pi(3\cos 2h + 4)^3}$ ***M1A1***

Note: Award ***M1*** for attempting to find $\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)$.

$= \frac{48\sin 2h}{\pi^2(3\cos 2h + 4)^5}$ ***A1***

continued…

Question 11 continued

(ii) $\sin 2h = 0 \Rightarrow h = 0, \frac{\pi}{2}, \pi$ ***A1***

Note: Award ***A1*** for $\sin 2h = 0 \Rightarrow h = 0, \frac{\pi}{2}, \pi$ from an incorrect $\frac{\mathrm{d}^2h}{\mathrm{d}t^2}$.

(iii) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t}$ is a minimum at $h = 0, \pi$ and the container is widest at these values

R1

$\frac{\mathrm{d}h}{\mathrm{d}t}$ is a maximum at $h = \frac{\pi}{2}$ and the container is narrowest at this value

R1

[7 marks]

Total [19 marks]

12. (a) **EITHER**

$w^7 = \left(\cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}\right)^7$ ***(M1)***

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ ***A1***

$= 1$ ***A1***

so w is a root ***AG***

OR

$z^7 = 1 = \cos(2\pi k) + i\sin(2\pi k)$ ***(M1)***

$z = \cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right)$ ***A1***

$k = 1 \Rightarrow z = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ ***A1***

so w is a root ***AG***

[3 marks]

(b) (i) $(w-1)\left(1 + w + w^2 + w^3 + w^4 + w^5 + w^6\right)$

$= w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 - 1 - w - w^2 - w^3 - w^4 - w^5 - w^6$ ***M1***

$= w^7 - 1\ (= 0)$ ***A1***

(ii) $w^7 - 1 = 0$ and $w - 1 \neq 0$ ***R1***

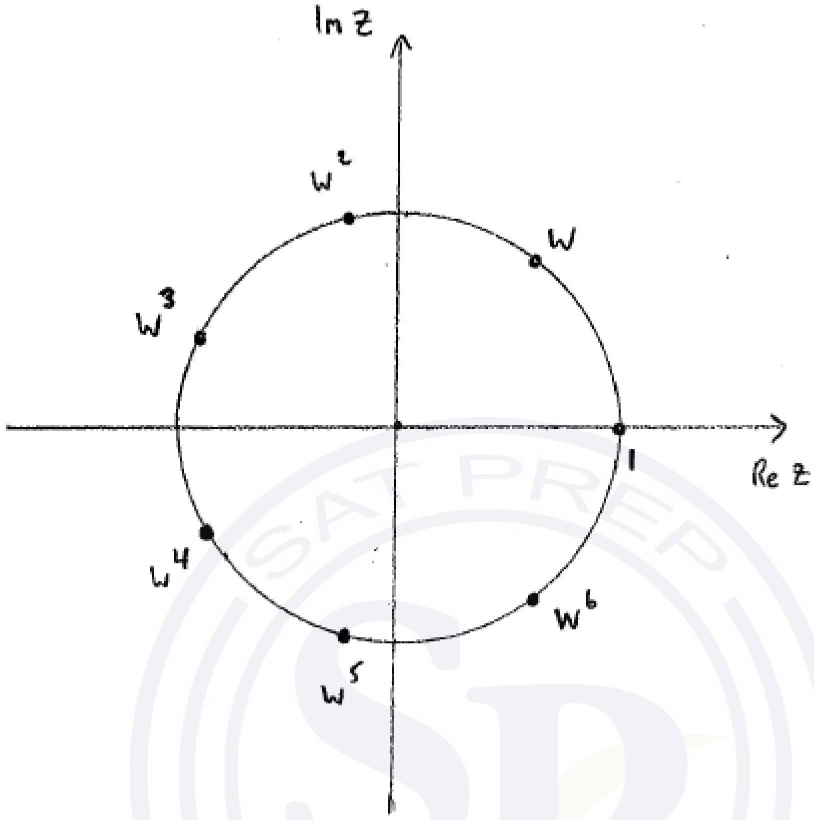
so $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$ ***AG***

[3 marks]

continued…

Question 12 continued

(c) the roots are $1,\ w,\ w^2,\ w^3,\ w^4,\ w^5$ and w^6 ***A1***



7 points equidistant from the origin ***A1***

approximately correct angular positions for $1,\ w,\ w^2,\ w^3,\ w^4,\ w^5$ and w^6 ***A1***

Note: Condone use of cis notation for the final two ***A*** marks.

Note: For the final ***A*** mark there should be one root in the first quadrant, two in the second, two in the third, one in the fourth, and one on the real axis.

[3 marks]

(d) (i) $\alpha^* = \left(w + w^2 + w^4\right)^*$

$= w^* + \left(w^2\right)^* + \left(w^4\right)^*$ ***A1***

since $w^* = w^6$, $\left(w^2\right)^* = w^5$ and $\left(w^4\right)^* = w^3$ ***R1***

$\Rightarrow \alpha^* = w^6 + w^5 + w^3$ ***AG***

continued…

Question 12 continued

(ii) $b = -\left(\alpha + \alpha^*\right)$ (using sum of roots (or otherwise)) ***(M1)***

$b = -\left(w + w^2 + w^3 + w^4 + w^5 + w^6\right)$ ***(A1)***

$= -(-1)$

$= 1$ ***A1***

$c = \alpha\alpha^*$ (using product of roots (or otherwise)) ***(M1)***

$c = \left(w + w^2 + w^4\right)\left(w^6 + w^5 + w^3\right)$

EITHER

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4$ ***A1***

$= \left(w^6 + w^5 + w^4 + w^3 + w^2 + w\right) + 3$ ***M1***

$= 3 - 1$ ***(A1)***

OR

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4 \ \left(= w^4\left(1 + w + w^3\right)\left(w^3 + w^2 + 1\right)\right)$ ***A1***

$= w^4\left(w^6 + w^5 + w^4 + w^2 + w + 1 + 3w^3\right)$ ***M1***

$= w^4\left(w^6 + w^5 + w^4 + w^3 + w^2 + w + 1 + 2w^3\right)$

$= w^4\left(2w^3\right)$ ***(A1)***

THEN

$= 2$ ***A1***

[10 marks]

(e) $z^2 + z + 2 = 0 \Rightarrow z = \dfrac{-1 \pm \mathrm{i}\sqrt{7}}{2}$ ***M1A1***

$\operatorname{Im}\left(w + w^2 + w^4\right) > 0$ ***R1***

$\operatorname{Im}\alpha = \dfrac{\sqrt{7}}{2}$ ***A1***

Note: Final ***A*** mark is independent of previous ***R*** mark.

[4 marks]

Total [23 marks]

N15/5/MATHL/HP1/ENG/TZ0/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2015

Mathematics

Higher level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. arc length $= \frac{2}{x} = rx \left(\Rightarrow r = \frac{2}{x^2} \right)$ ***M1***

$16 = \frac{1}{2}\left(\frac{2}{x^2}\right)^2 x \left(\Rightarrow \frac{2}{x^3} = 16 \right)$ ***M1***

Note: Award M1s for attempts at the use of arc-length and sector-area formulae.

$x = \frac{1}{2}$ ***A1***

arc length $= 4(\text{cm})$ ***A1***

[4 marks]

2. attempt to integrate one factor and differentiate the other, leading to a sum of two terms ***M1***

$\int x \sin x \, dx = x(-\cos x) + \int \cos x \, dx$ ***(A1)(A1)***

$= -x\cos x + \sin x + c$ ***A1***

Note: Only award final ***A1*** if $+\,c$ is seen.

[4 marks]

3. (a) $(2+x)^4 = 2^4 + 4\cdot 2^3 x + 6\cdot 2^2 x^2 + 4\cdot 2x^3 + x^4$ ***M1(A1)***

Note: Award ***M1*** for an expansion, by whatever method, giving five terms in any order.

$= 16 + 32x + 24x^2 + 8x^3 + x^4$ ***A1***

Note: Award ***M1A1A0*** for correct expansion not given in ascending powers of x.

[3 marks]

(b) let $x = 0.1$ (in the binomial expansion) ***(M1)***

$2.1^4 = 16 + 3.2 + 0.24 + 0.008 + 0.0001$ ***(A1)***

$= 19.4481$ ***A1***

Note: At most one of the marks can be implied.

[3 marks]

Total [6 marks]

4. (a) $\frac{dy}{dx} = (1-x)^{-2} \left(= \frac{1}{(1-x)^2} \right)$ ***(M1)A1***

[2 marks]

continued…

Question 4 continued

(b) gradient of Tangent $= \frac{1}{4}$ ***(A1)***

gradient of Normal $= -4$ ***(M1)***

$y + \frac{1}{2} = -4(x-3)$ or attempt to find c in $y = mx + c$ ***M1***

$8x + 2y - 23 = 0$ ***A1***

[4 marks]

Total [6 marks]

5. METHOD 1

$\int_{e}^{e^2} \frac{dx}{x \ln x} = [\ln(\ln x)]_{e}^{e^2}$ ***(M1)A1***

$= \ln(\ln e^2) - \ln(\ln e)$ $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

METHOD 2

$u = \ln x, \frac{du}{dx} = \frac{1}{x}$ ***M1***

$= \int_{1}^{2} \frac{du}{u}$ ***A1***

$= [\ln u]_1^2$ or equivalent in x $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

6. (a) probability that Darren wins $P(W) + P(RRW) + P(RRRRW)$ ***(M1)***

Note: Only award ***M1*** if three terms are seen or are implied by the following numerical equivalent.

Note: Accept equivalent tree diagram for method mark.

$= \frac{2}{6} + \frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4} + \frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot\frac{2}{2}$ $\left(= \frac{1}{3} + \frac{1}{5} + \frac{1}{15}\right)$ ***A2***

Note: *A1* for two correct.

$= \frac{3}{5}$ ***A1***

[4 marks]

continued…

Question 6 continued

(b) **METHOD 1**

the probability that Darren wins is given by

$\mathrm{P}(W) + \mathrm{P}(RRW) + \mathrm{P}(RRRRW) + \ldots$ ***(M1)***

Note: Accept equivalent tree diagram with correctly indicated path for method mark.

$\mathrm{P}(\text{Darren Win}) = \frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \ldots$

or $= \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \ldots\right)$ ***A1***

$= \frac{1}{3}\left(\frac{1}{1-\frac{4}{9}}\right)$ ***A1***

$= \frac{3}{5}$ ***AG***

[3 marks]

METHOD 2

$\mathrm{P}(\text{Darren wins}) = \mathrm{P}$

$\mathrm{P} = \frac{1}{3} + \frac{4}{9}\mathrm{P}$ ***M1A2***

$\frac{5}{9}\mathrm{P} = \frac{1}{3}$

$\mathrm{P} = \frac{3}{5}$ ***AG***

[3 marks]

Total [7 marks]

7. (a) $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1***

a horizontal tangent occurs if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so $y = 0$ ***M1***

we can see from the equation of the curve that this solution is not possible $(0 = 4)$ and so there is not a horizontal tangent ***R1***

[4 marks]

continued…

Question 7 continued

(b) $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$ or equivalent with $\dfrac{dx}{dy}$

the tangent is vertical when $2y = x$ ***M1***

substitute into the equation to give $2y^2 = y^2 + 4$ ***M1***

$y = \pm 2$ ***A1***

coordinates are $(4, 2), (-4, -2)$ ***A1***

[4 marks]

Total [8 marks]

8. (a) $\sin\left(\theta + \dfrac{\pi}{2}\right) = \sin\theta \cos\dfrac{\pi}{2} + \cos\theta \sin\dfrac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

(b) consider $n = 1$, $f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \dfrac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \dfrac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \dfrac{d\left(f^{(k)}(x)\right)}{dx} \left(= a\left(a^k \cos\left(ax + \dfrac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1}\sin\left(ax + \dfrac{k\pi}{2} + \dfrac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1}\sin\left(ax + \dfrac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior M and R marks have been awarded.

[7 marks]

Total [8 marks]

9. $(\sin 2x - \sin x) - (\cos 2x - \cos x) = 1$

attempt to use both double-angle formulae, in whatever form ***M1***

$(2\sin x \cos x - \sin x) - (2\cos^2 x - 1 - \cos x) = 1$

or $(2\sin x \cos x - \sin x) - \left(2\cos^2 x - \cos x\right) = 0$ for example ***A1***

Note: Allow any rearrangement of the above equations.

$\sin x(2\cos x - 1) - \cos x(2\cos x - 1) = 0$

$(\sin x - \cos x)(2\cos x - 1) = 0$ ***(M1)***

$\tan x = 1$ and $\cos x = \frac{1}{2}$ ***A1A1***

Note: These ***A*** marks are dependent on the ***M*** mark awarded for factorisation.

$x = -\frac{3\pi}{4}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$ ***A2***

Note: Award ***A1*** for two correct answers, which could be for both tan or both cos solutions, for example.

[7 marks]

10. (a) the sum of the roots of the polynomial $= \frac{63}{16}$ ***(A1)***

$$2\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{63}{16}$$ ***M1A1***

Note: The formula for the sum of a geometric sequence must be equated to a value for the ***M1*** to be awarded.

$$1-\left(\frac{1}{2}\right)^n = \frac{63}{64} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{1}{64}$$

$n = 6$ ***A1***

[4 marks]

(b) $\frac{a_0}{a_n} = 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}, \left(a_n = 16\right)$ ***M1***

$a_0 = 16 \times 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}$

$a_0 = 2^{-5}\left(= \frac{1}{32}\right)$ ***A1***

[2 marks]

Total [6 marks]

Section B

11. (a) $z^3 = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(A1)***

attempt the use of De Moivre's Theorem in reverse ***M1***

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right);\ 2\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right);$

$2\left(\cos\left(\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(\frac{9\pi}{6}\right)\right)$ ***A2***

Note: Accept cis form.

$z = \pm\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A1*** for two correct solutions in each of the two lines above.

[6 marks]

(b) (i) $z_1 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$ ***A1A1***

(ii) $\left(z_2 = \left(\sqrt{3} + \mathrm{i}\right)\right)$

$z_1 z_2 = (1 + \mathrm{i})\left(\sqrt{3} + \mathrm{i}\right)$ ***M1***

$= \left(\sqrt{3} - 1\right) + \mathrm{i}\left(1 + \sqrt{3}\right)$ ***A1***

(iii) $z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\right)$ ***M1A1***

$\tan\frac{5\pi}{12} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ ***A1***

$= 2 + \sqrt{3}$ ***M1A1***

Note: Award final ***M1*** for an attempt to rationalise the fraction.

(iv) $z_2{}^p = 2^p\left(\mathrm{cis}\left(\frac{p\pi}{6}\right)\right)$ ***(M1)***

$z_2{}^p$ is a positive real number when $p = 12$ ***A1***

[11 marks]

Total [17 marks]

12. (a) $f(-x) = (-x)\sqrt{1-(-x)^2}$ ***M1***

$= -x\sqrt{1-x^2}$

$= -f(x)$ ***R1***

hence f is odd ***AG***

[2 marks]

(b) $f'(x) = x \cdot \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot -2x + \left(1-x^2\right)^{\frac{1}{2}}$ ***M1A1A1***

[3 marks]

(c) $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \left(= \frac{1-2x^2}{\sqrt{1-x^2}}\right)$ ***A1***

Note: This may be seen in part (b).

$f'(x) = 0 \Rightarrow 1-2x^2 = 0$ ***M1***

$x = \pm\frac{1}{\sqrt{2}}$ ***A1***

[3 marks]

(d) y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$ ***M1A1***

so range of $f(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ ***A1***

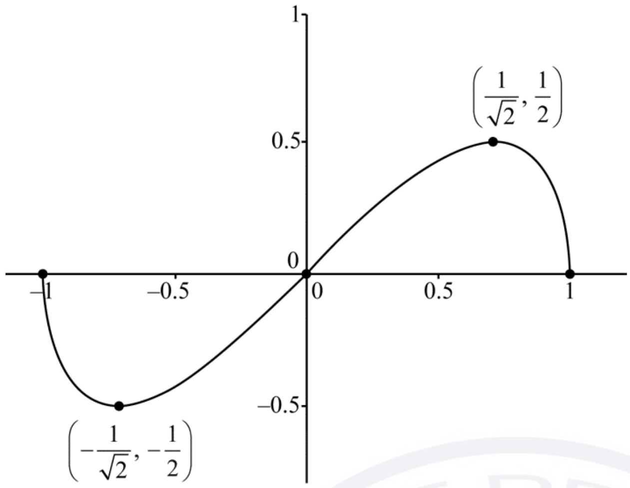
Note: Allow FT from (c) if values of x, within the domain, are used.

[3 marks]

continued…

Question 12 continued

(e)



Shape: The graph of an odd function, on the given domain, s-shaped,
where the max(min) is the right(left) of $0.5(-0.5)$ ***A1***
x-intercepts ***A1***
turning points ***A1***

[3 marks]

(f) area $= \int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$ ***(M1)***

attempt at "backwards chain rule" or substitution ***M1***

$$= -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\,\mathrm{d}x$$

$$= \left[\frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} \cdot -\frac{1}{2}\right]_0^1$$ ***A1***

$$= \left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$$ ***A1***

[4 marks]

(g) $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|\mathrm{d}x > 0$ ***R1***

$\left|\int_{-1}^{1} x\sqrt{1-x^2}\,\mathrm{d}x\right| = 0$ ***R1***

so $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|\mathrm{d}x > \left|\int_{-1}^{1} x\sqrt{1-x^2}\,\mathrm{d}x\right| = 0$ ***AG***

[2 marks]

Total [20 marks]

13. (a) $\overrightarrow{BR} = \overrightarrow{BA} + \overrightarrow{AR} \ \left(= \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}\right)$ ***(M1)***

$= (\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***A1***

[2 marks]

(b) (i) $r_{BR} = \boldsymbol{b} + \lambda\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= \frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c}\right)$ ***A1A1***

Note: Award ***A1A0*** if the r = is omitted in an otherwise correct expression/equation.

(ii) $\overrightarrow{AQ} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***(A1)***

$r_{AQ} = \boldsymbol{a} + \mu\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}\right)$ ***A1***

(iii) when $\overrightarrow{AQ}$ and $\overrightarrow{BP}$ intersect we will have $r_{BR} = r_{AQ}$ ***(M1)***

$\frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c} = (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}$

attempt to equate the coefficients of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ ***M1***

(A1)

$$\left.\begin{aligned} \frac{\lambda}{2} &= 1-\mu \\ 1-\lambda &= \frac{\mu}{2} \\ \frac{\lambda}{2} &= \frac{\mu}{2} \end{aligned}\right\}$$

$\lambda = \frac{2}{3}$ or $\mu = \frac{2}{3}$ ***A1***

substituting parameters back into one of the equations ***M1***

$\overrightarrow{OG} = \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{a} + \left(1 - \frac{2}{3}\right)\boldsymbol{b} + \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{c} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ ***AG***

[9 marks]

continued…

Question 13 continued

(c) $\vec{\mathrm{CP}} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}$ ***(M1)A1***

so we have that $\mathrm{r}_{\mathrm{CP}} = \boldsymbol{c} + \beta\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}\right)$ and when $\beta = \frac{2}{3}$ the line passes through

the point G (*ie*, with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$) ***R1***

hence $[\mathrm{AQ}]$, $[\mathrm{BR}]$ and $[\mathrm{CP}]$ all intersect in G ***AG***

[3 marks]

continued…

Question 13 continued

(d) $\vec{OG} = \frac{1}{3}\left(\begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}3\\7\\-5\end{pmatrix} + \begin{pmatrix}2\\2\\1\end{pmatrix}\right) = \begin{pmatrix}2\\4\\-1\end{pmatrix}$ ***A1***

Note: This independent mark for the vector may be awarded wherever the vector is calculated.

$\vec{AB} \times \vec{AC} = \begin{pmatrix}2\\4\\-6\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-6\\-6\\-6\end{pmatrix}$ ***M1A1***

$\vec{GX} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$ ***(M1)***

volume of Tetrahedron given by $\frac{1}{3} \times$ Area $\mathrm{ABC} \times \mathrm{GX}$

$= \frac{1}{3}\left(\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|\right) \times \mathrm{GX} = 12$ ***(M1)(A1)***

Note: Accept alternative methods, for example the use of a scalar triple product.

$= \frac{1}{6}\sqrt{(-6)^2 + (-6)^2 + (-6)^2} \times \sqrt{\alpha^2 + \alpha^2 + \alpha^2} = 12$ ***(A1)***

$= \frac{1}{6} 6\sqrt{3}\,|\alpha|\sqrt{3} = 12$

$\Rightarrow |\alpha| = 4$ ***A1***

Note: Condone absence of absolute value.

this gives us the position of X as $\begin{pmatrix}2\\4\\-1\end{pmatrix} \pm \begin{pmatrix}4\\4\\4\end{pmatrix}$

$\mathrm{X}(6, 8, 3)$ or $(-2, 0, -5)$ ***A1***

Note: Award ***A1*** for either result.

[9 marks]

Total [23 marks]

Markscheme

May 2015

Mathematics

Higher level

Paper 1

22 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) **METHOD 1**

area $= \pi 2^2 - \frac{1}{2}2^2\theta \left(= 3\pi\right)$ ***M1A1***

Note: Award ***M1*** for using area formula.

$\Rightarrow 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$ ***A1***

Note: Degrees loses final A1

METHOD 2

let $x = 2\pi - \theta$

area $= \frac{1}{2}2^2 x (= 3\pi)$ ***M1***

$\Rightarrow x = \frac{3}{2}\pi$ ***A1***

$\Rightarrow \theta = \frac{\pi}{2}$ ***A1***

METHOD 3

Area of circle is 4π ***A1***

Shaded area is $\frac{3}{4}$ of the circle ***(R1)***

$\Rightarrow \theta = \frac{\pi}{2}$ ***A1***

[3 marks]

(b) arc length $= 2\frac{3\pi}{2}$ ***A1***

perimeter $= 2\frac{3\pi}{2} + 2\times 2$

$= 3\pi + 4$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $\overline{x} = \frac{1 \times 0 + 19 \times 10}{20} = 9.5$ ***(M1)A1***

[2 marks]

(b) median is 10 ***A1***

[1 mark]

(c) (i) 19 ***A1***

(ii) 1 ***A1***

[2 marks]

Total [5 marks]

3. (a) $\int \left(1 + \tan^2 x\right) \mathrm{d}x = \int \sec^2 x \, \mathrm{d}x = \tan x (+c)$ ***M1A1***

[2 marks]

(b) $\int \sin^2 x \, \mathrm{d}x = \int \frac{1 - \cos 2x}{2} \mathrm{d}x$ ***M1A1***

$= \frac{x}{2} - \frac{\sin 2x}{4} (+c)$ ***A1***

> **Note:** Allow integration by parts followed by trig identity.
> Award ***M1*** for parts, ***A1*** for trig identity, ***A1*** final answer.

[3 marks]

Total [5 marks]

4. (a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ ***(M1)A1***

[2 marks]

(b) $f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$ ***(M1)***

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right)$ ***A1***

$= 3x^2$ ***A1***

> **Note:** Do not award final A1 on FT if $= 3x^2$ is not obtained

> **Note:** Final A1 can only be obtained if previous A1 is given

[3 marks]

Total [5 marks]

5. (a) **EITHER**

$f(-x) = f(x)$ ***M1***

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0$, $(\forall x \in \mathbb{R})$ ***A1***

OR

y-axis is eqn of symmetry ***M1***

so $\frac{-b}{2a} = 0$ ***A1***

THEN

$\Rightarrow b = 0$ ***AG***

[2 marks]

(b) $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$ ***M1***

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$ ***A1***

Note: In (a) and (b) allow substitution of a particular value of x

[2 marks]

(c) $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0$, $(\forall x)$ ***M1A1***

Note: Accept geometrical explanations.

[2 marks]

Total [6 marks]

6. (a) $f: x \to y = \frac{3x-2}{2x-1}$ $f^{-1}: y \to x$

$y = \frac{3x-2}{2x-1} \Rightarrow 3x - 2 = 2xy - y$ ***M1***

$\Rightarrow 3x - 2xy = -y + 2$ ***M1***

$x(3-2y) = 2 - y$

$x = \frac{2-y}{3-2y}$ ***A1***

$(f^{-1}(y) = \frac{2-y}{3-2y})$

$f^{-1}(x) = \frac{2-x}{3-2x}$ $\left(x \neq \frac{3}{2}\right)$ ***A1***

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

[4 marks]

(b) $\frac{3x-2}{2x-1} = A + \frac{B}{2x-1} \Rightarrow 3x - 2 = A(2x-1) + B$

equating coefficients $3 = 2A$ and $-2 = -A + B$ ***(M1)***

$A = \frac{3}{2}$ and $B = -\frac{1}{2}$ ***A1***

Note: Could also be done by division or substitution of values.

[2 marks]

(c) $\int f(x)\,dx = \frac{3}{2}x - \frac{1}{4}\ln|2x-1| + c$ ***A1***

Note: accept equivalent e.g. ln $\lvert 4x-2 \rvert$

[1 mark]

Total [7 marks]

7. (a) (i) $\left(-\frac{a_{n-1}}{a_n}=\right)-\frac{1}{2}$ ***A1***

(ii) $\left((-1)^n \frac{a_0}{a_n}=\right)-\frac{36}{2}=(-18)$ ***A1A1***

Note: First ***A1*** is for the negative sign.

[3 marks]

(b) **METHOD 1**

if λ satisfies $p(\lambda)=0$ then $q(\lambda-4)=0$

so the roots of $q(x)$ are each 4 less than the roots of $p(x)$ ***(R1)***

so sum of roots is $-\frac{1}{2}-4\times5=-20.5$ ***A1***

METHOD 2

$p(x+4)=2x^5+2\times5\times4x^4\ldots+x^4\ldots=2x^5+41x^4\ldots$ ***(M1)***

so sum of roots is $-\frac{41}{2}=-20.5$ ***A1***

[2 marks]

Total [5 marks]

8. $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ ***(A1)***

EITHER

integral is $\int \frac{\mathrm{e}^x}{\left(\mathrm{e}^x + 3\right)^2 + 2^2} \mathrm{d}x$ ***M1A1***

$= \int \frac{1}{u^2 + 2^2} \mathrm{d}u$ ***M1A1***

Note: Award ***M1*** only if the integral has completely changed to one in u.

Note: $\mathrm{d}u$ needed for final ***A1***

OR

$\mathrm{e}^x = u - 3$

integral is $\int \frac{1}{(u-3)^2 + 6(u-3) + 13} \mathrm{d}u$ ***M1A1***

Note: Award ***M1*** only if the integral has completely changed to one in u.

$= \int \frac{1}{u^2 + 2^2} \mathrm{d}u$ ***M1A1***

Note: In both solutions the two method marks are independent.

THEN

$= \frac{1}{2} \arctan\left(\frac{u}{2}\right) (+c)$ ***(A1)***

$= \frac{1}{2} \arctan\left(\frac{\mathrm{e}^x + 3}{2}\right) (+c)$ ***A1***

Total [7 marks]

9. (a) $g \circ f(x) = g(f(x))$ ***M1***

$= g\left(2x + \frac{\pi}{5}\right)$

$= 3\sin\left(2x + \frac{\pi}{5}\right) + 4$ ***AG***

[1 mark]

(b) since $-1 \le \sin\theta \le +1$, range is $[1, 7]$ ***(R1)A1***

[2 marks]

(c) $3\sin\left(2x + \frac{\pi}{5}\right) + 4 = 7 \Rightarrow 2x + \frac{\pi}{5} = \frac{\pi}{2} + 2n\pi \Rightarrow x = \frac{3\pi}{20} + n\pi$ ***(M1)***

so next biggest value is $\frac{23\pi}{20}$ ***A1***

Note: Allow use of period.

[2 marks]

(d) **Note:** Transformations can be in any order but see notes below.

stretch scale factor 3 parallel to y axis (vertically) ***A1***
vertical translation of 4 up ***A1***

Note: Vertical translation is $\frac{4}{3}$ up if it occurs before stretch parallel to y axis.

stretch scale factor $\frac{1}{2}$ parallel to x axis (horizontally) ***A1***

horizontal translation of $\frac{\pi}{10}$ to the left ***A1***

Note: Horizontal translation is $\frac{\pi}{5}$ to the left if it occurs before stretch parallel to x axis.

Note: Award ***A1*** for magnitude and direction in each case. Accept any correct terminology provided that the meaning is clear *eg* shift for translation.

[4 marks]

Total [9 marks]

10. **METHOD 1**

to have 3 consecutive losses there must be exactly 5, 4 or 3 losses

the probability of exactly 5 losses (must be 3 consecutive) is $\left(\frac{1}{3}\right)^5$ ***A1***

the probability of exactly 4 losses (with 3 consecutive) is $4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)$ ***A1A1***

Note: First ***A1*** is for the factor 4 and second ***A1*** for the other 2 factors.

the probability of exactly 3 losses (with 3 consecutive) is $3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$ ***A1A1***

Note: First ***A1*** is for the factor 3 and second ***A1*** for the other 2 factors.

(Since the events are mutually exclusive)
the total probability is $\frac{1+8+12}{3^5}=\frac{21}{243}\left(=\frac{7}{81}\right)$ ***A1***

[6 marks]

METHOD 2

Roy loses his job if
A – first 3 games are all lost (so the last 2 games can be any result)
B – first 3 games are not all lost, but middle 3 games are all lost (so the first game is not a loss and the last game can be any result)
or C – first 3 games are not all lost, middle 3 games are not all lost but last 3 games are all lost, (so the first game can be any result but the second game is not a loss)

for A 4^{th} & 5^{th} games can be anything

$\mathrm{P}(A)=\left(\frac{1}{3}\right)^3=\frac{1}{27}$ ***A1***

for B 1^{st} game not a loss & 5^{th} game can be anything ***(R1)***

$\mathrm{P}(B)=\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$ ***A1***

for C 1^{st} game anything, 2^{nd} game not a loss ***(R1)***

$\mathrm{P}(C)=1\times\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$ ***A1***

(Since the events are mutually exclusive)

total probability is $\frac{1}{27}+\frac{2}{81}+\frac{2}{81}=\frac{7}{81}$ ***A1***

continued…

Question 10 continued.

Note:	In both methods all the ***A*** marks are independent.

Note:	If the candidate misunderstands the question and thinks that it is asking for exactly 3 losses award ***A1 A1*** and ***A1*** for an answer of $\frac{12}{243}$ as in the last lines of Method 1.

[6 marks]

Total [6 marks]

Section B

11. (a) $\frac{dy}{dx} = 1 \times e^{3x} + x \times 3e^{3x} = \left(e^{3x} + 3xe^{3x}\right)$ ***M1A1***

[2 marks]

(b) let $P(n)$ be the statement $\frac{d^n y}{dx^n} = n3^{n-1}e^{3x} + x3^n e^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{dy}{dx}$ which is $1 \times e^{3x} + x \times 3e^{3x}$ and RHS is $3^0 e^{3x} + x3^1 e^{3x}$ ***R1***

as LHS = RHS, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{d^k y}{dx^k} = k3^{k-1}e^{3x} + x3^k e^{3x}$

$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$ (***M1***)

$= k3^{k-1} \times 3e^{3x} + 1 \times 3^k e^{3x} + x3^k \times 3e^{3x}$ ***A1***

$= (k+1)3^k e^{3x} + x3^{k+1}e^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true ($\forall n \in \mathbb{Z}^+$) ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

continued…

Question 11 continued

(c) $e^{3x} + x \times 3e^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1***

point is $\left(-\frac{1}{3}, -\frac{1}{3e}\right)$ ***A1***

EITHER

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$

when $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{dy}{dx}$	$-ve$ 0 $+ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

(d) $\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$ ***A1***

$2 \times 3e^{3x} + x \times 3^2 e^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3}, -\frac{2}{3e^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{d^2y}{dx^2}$	$-ve$ 0 $+ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

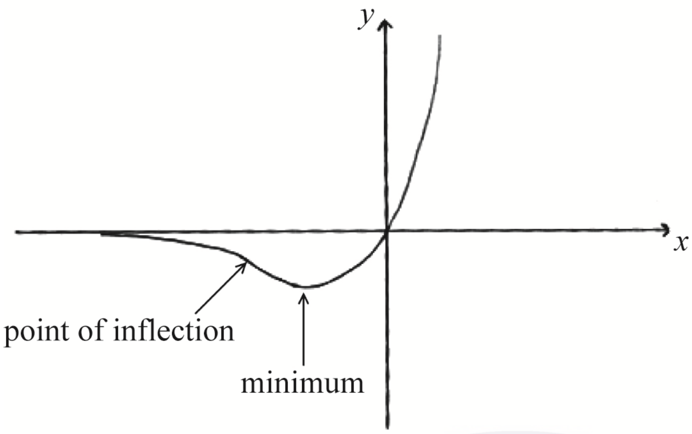
Note: Allow 3rd derivative is not zero at $-\frac{2}{3}$

[5 marks]

continued…

Question 11 continued

(e)



(general shape including asymptote and through origin) ***A1***
showing minimum and point of inflection ***A1***

Note:	Only indication of position of answers to (c) and (d) required, not coordinates.

[2 marks]

Total [21 marks]

12. (a) (i) **METHOD 1**

$\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}}$ ***M1***

$= 2^{u_{n+1}-u_n} = 2^d$ ***A1***

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}}$ ***M1***

$= 2^d$ ***A1***

(ii) 2^a ***A1***

Note: Accept 2^{u_1}.

(iii) **EITHER**

v_n is a GP with first term 2^a and common ratio 2^d

$v_n = 2^a(2^d)^{(n-1)}$

OR

$u_n = a + (n-1)d$ as it is an AP

THEN

$v_n = 2^{a+(n-1)d}$ ***A1***

[4 marks]

(b) (i) $S_n = \frac{2^a\left((2^d)^n - 1\right)}{2^d - 1} = \frac{2^a\left(2^{dn} - 1\right)}{2^d - 1}$ ***M1A1***

Note: Accept either expression.

(ii) for sum to infinity to exist need $-1 < 2^d < 1$ ***R1***

$\Rightarrow \log 2^d < 0 \Rightarrow d\log 2 < 0 \Rightarrow d < 0$ ***(M1)A1***

Note: Also allow graph of 2^d.

(iii) $S_\infty = \frac{2^a}{1-2^d}$ ***A1***

continued…

Question 12 continued

(iv) $\dfrac{2^a}{1-2^d} = 2^{a+1} \Rightarrow \dfrac{1}{1-2^d} = 2$ ***M1***

$\Rightarrow 1 = 2 - 2^{d+1} \Rightarrow 2^{d+1} = 1$

$\Rightarrow d = -1$ ***A1***

[8 marks]

(c) **METHOD 1**

$w_n = pq^{n-1}$, $z_n = \ln pq^{n-1}$ ***(A1)***

$z_n = \ln p + (n-1)\ln q$ ***M1A1***

$z_{n+1} - z_n = (\ln p + n\ln q) - (\ln p + (n-1)\ln q) = \ln q$

which is a constant so this is an AP

(with first term $\ln p$ and common difference $\ln q$)

$\sum_{i=1}^{n} z_i = \dfrac{n}{2}\left(2\ln p + (n-1)\ln q\right)$ ***M1***

$= n\left(\ln p + \ln q^{\left(\frac{n-1}{2}\right)}\right) = n\ln\left(pq^{\left(\frac{n-1}{2}\right)}\right)$ ***(M1)***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***A1***

METHOD 2

$\sum_{i=1}^{n} z_i = \ln p + \ln pq + \ln pq^2 + \ldots + \ln pq^{n-1}$ ***(M1)A1***

$= \ln\left(p^n q^{(1+2+3+\ldots+(n-1))}\right)$ ***(M1)A1***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***(M1)A1***

[6 marks]

Total [18 marks]

13. (a) $\vec{OP} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$\vec{OQ} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\vec{PQ} = \vec{OQ} - \vec{OP}$ ***(M1)***

$\vec{PQ} = \boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k} - \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$= (1 - \lambda + \mu)\boldsymbol{i} + (-1 - \lambda - \mu)\boldsymbol{j} + (-4 - \lambda + 2\mu)\boldsymbol{k}$ ***A1***

[2 marks]

(b) **METHOD 1**

use of scalar product ***M1***

perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ gives

$(1 - \lambda + \mu) + (-1 - \lambda - \mu) + (-4 - \lambda + 2\mu) = 0$

$\Rightarrow -3\lambda + 2\mu = 4$ ***A1***

perpendicular to $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ gives

$(1 - \lambda + \mu) - (-1 - \lambda - \mu) + 2(-4 - \lambda + 2\mu) = 0$

$\Rightarrow -2\lambda + 6\mu = 6$ ***A1***

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$ ***A1A1***

METHOD 2

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***M1A1***

$\vec{PQ} = a(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$1 - \lambda + \mu = 3a$

$-1 - \lambda - \mu = -a$ ***A1***

$-4 - \lambda + 2\mu = -2a$

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$ ***A1A1***

[5 marks]

(c) $\vec{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ ***A1***

shortest distance $= \left|\vec{PQ}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[3 marks]

(d) **METHOD 1**

vector perpendicular to Π is given by vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ ***(R1)***

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***(M1)A1***

so equation of Π is $3x - y - 2z + d = 0$

through $(1, 2, 3) \Rightarrow d = 5$ ***M1***

so equation is $3x - y - 2z + 5 = 0$ ***A1***

continued…

Question 13 continued

METHOD 2

from part (b) $\vec{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ is a vector perpendicular to Π ***R1A2***

so equation of Π is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + c = 0$

through $(1, 2, 3) \Rightarrow c = \frac{30}{7}$ ***M1***

so equation is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + \frac{30}{7} = 0 \quad (3x - y - 2z + 5 = 0)$ ***A1***

Note: Allow other methods *ie* via vector parametric equation.

[5 marks]

(e) $\vec{OT} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \eta(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$
$T = (2 + 3\eta, 1 - \eta, -1 - 2\eta)$ lies on Π implies
$3(2 + 3\eta) - (1 - \eta) - 2(-1 - 2\eta) + 5 = 0$ ***M1***
$\Rightarrow 12 + 14\eta = 0 \Rightarrow \eta = -\frac{6}{7}$ ***A1***

Note: If no marks awarded in (d) but correct vector product calculated in (e) award ***M1A1*** in (d).

[2 marks]

(f) $\left|\vec{BT}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[2 marks]

(g) they agree ***A1***

Note: FT is inappropriate here.

$\vec{BT}$ is perpendicular to both Π and l_2
so its length is the shortest distance between Π and l_2 which is the shortest distance between l_1 and l_2 ***R1***

[2 marks]

Total [21 marks]

Markscheme

May 2015

Mathematics

Higher level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \big(2\cos(5x-3)\big)5 \quad \big(=10\cos(5x-3)\big)$$ ***A1***

  Award ***A1*** for $\big(2\cos(5x-3)\big)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = 0.25 + 0.6 - 0.7$ ***M1***

$= 0.15$ ***A1***

[2 marks]

(b) **EITHER**

$\mathrm{P}(A)\,\mathrm{P}(B)\left(= 0.25 \times 0.6\right) = 0.15$ ***A1***

$= \mathrm{P}(A \cap B)$ so independent ***R1***

OR

$P\left(A|B\right) = \dfrac{P\left(A \cap B\right)}{P\left(B\right)} = \dfrac{0.15}{0.6} = 0.25$ ***A1***

$= P\left(A\right)$ so independent ***R1***

Note: Allow follow through for incorrect answer to (a) that will result in events being dependent in (b).

[2 marks]

Total [4 marks]

2. $(3-x)^4 = 1.3^4 + 4.3^3(-x) + 6.3^2(-x)^2 + 4.3(-x)^3 + 1(-x)^4$ or equivalent **(*M1*)(*A1*)**

$= 81 - 108x + 54x^2 - 12x^3 + x^4$ ***A1A1***

Note: ***A1*** for ascending powers, ***A1*** for correct coefficients including signs.

[4 marks]

3. $\tan x + \tan 2x = 0$

$\tan x + \dfrac{2\tan x}{1-\tan^2 x} = 0$ ***M1***

$\tan x - \tan^3 x + 2\tan x = 0$ ***A1***

$\tan x\left(3 - \tan^2 x\right) = 0$ **(*M1*)**

$\tan x = 0 \Rightarrow x = 0,\ x = 180^\circ$ ***A1***

Note: If $x = 360^\circ$ seen anywhere award ***A0***

$\tan x = \sqrt{3} \Rightarrow x = 60^\circ,\ 240^\circ$ ***A1***

$\tan x = -\sqrt{3} \Rightarrow x = 120^\circ,\ 300^\circ$ ***A1***

[6 marks]

4. (a) attempt to differentiate $f(x) = x^3 - 3x^2 + 4$ ***M1***

$f'(x) = 3x^2 - 6x$ ***A1***

$= 3x(x-2)$

(Critical values occur at) $x = 0$, $x = 2$ ***(A1)***

so f decreasing on $x \in \left]0, 2\right[$ (or $0 < x < 2$) ***A1***

[4 marks]

(b) $f''(x) = 6x - 6$ ***(A1)***

setting $f''(x) = 0$ ***M1***

$\Rightarrow x = 1$

coordinate is $(1, 2)$ ***A1***

[3 marks]

Total [7 marks]

5. any attempt at integration by parts ***M1***

$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ ***(A1)***

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^3 \Rightarrow v = \frac{x^4}{4}$ ***(A1)***

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \int_1^2 \frac{x^3}{4}\mathrm{d}x$ ***A1***

Note: Condone absence of limits at this stage.

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \left[\frac{x^4}{16}\right]_1^2$ ***A1***

Note: Condone absence of limits at this stage.

$= 4\ln 2 - \left(1 - \frac{1}{16}\right)$ ***A1***

$= 4\ln 2 - \frac{15}{16}$ ***AG***

[6 marks]

6. (a) any attempt to use sine rule ***M1***

$$\frac{\text{AB}}{\sin\frac{\pi}{3}} = \frac{\sqrt{3}}{\sin\left(\frac{2\pi}{3}-\theta\right)}$$ ***A1***

$$= \frac{\sqrt{3}}{\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta}$$ ***A1***

Note: Condone use of degrees.

$$= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$$ ***A1***

$$\frac{\text{AB}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$$

$$\therefore \text{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$$ ***AG***

[4 marks]

(b) **METHOD 1**

$$(\text{AB})' = \frac{-3\left(-\sqrt{3}\sin\theta + \cos\theta\right)}{\left(\sqrt{3}\cos\theta + \sin\theta\right)^2}$$ ***M1A1***

setting $(\text{AB})' = 0$ ***M1***

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$ ***A1***

continued…

Question 6 continued

METHOD 2

$$\mathrm{AB} = \frac{\sqrt{3}\sin\frac{\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

AB minimum when $\sin\left(\frac{2\pi}{3} - \theta\right)$ is maximum ***M1***

$\sin\left(\frac{2\pi}{3} - \theta\right) = 1$ ***(A1)***

$\frac{2\pi}{3} - \theta = \frac{\pi}{2}$ ***M1***

$\theta = \frac{\pi}{6}$ ***A1***

METHOD 3

shortest distance from B to AC is perpendicular to AC ***R1***

$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ ***M1A2***

[4 marks]

Total [8 marks]

7. (a) **METHOD 1**

$$z^3 = -\frac{27}{8} = \frac{27}{8}(\cos\pi + \mathrm{i}\sin\pi)$$ ***M1(A1)***

$$= \frac{27}{8}\left(\cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi)\right)$$ ***(A1)***

$$z = \frac{3}{2}\left(\cos\left(\frac{\pi + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2n\pi}{3}\right)\right)$$ ***M1***

$$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$$

$$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi),$$

$$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

continued…

Question 7 continued

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\frac{3}{2}$ so $(2z+3)$ is a factor

Attempt to use long division or factor theorem: ***M1***

$\Rightarrow 8z^3 + 27 \equiv (2z+3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$ ***A1***

Attempt to solve quadratic: ***M1***

$z = \frac{3 \pm 3\sqrt{3}\,\mathrm{i}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi),$

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

continued…

Question 7 continued

METHOD 3

$8z^3 + 27 = 0$

Substitute $z = x + \mathrm{i}\,y$ ***M1***

$8\left(x^3 + 3\mathrm{i}\,x^2y - 3xy^2 - \mathrm{i}\,y^3\right) + 27 = 0$

$\Rightarrow 8x^3 - 24xy^2 + 27 = 0$ and $24x^2y - 8y^3 = 0$ ***A1***

Attempt to solve simultaneously: ***M1***

$8y\left(3x^2 - y^2\right) = 0$

$y = 0, y = x\sqrt{3}, y = -x\sqrt{3}$

$\Rightarrow \left(x = -\frac{3}{2}, y = 0\right), x = \frac{3}{4}, y = \pm\frac{3\sqrt{3}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2 = \frac{3}{2}\left(\cos\pi + \mathrm{i}\sin\pi\right),$

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

continued…

Question 7 continued

(b) **EITHER**

Valid attempt to use area $=3\left(\frac{1}{2}ab\sin C\right)$ ***M1***

$=3\times\frac{1}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{\sqrt{3}}{2}$ ***A1A1***

Note: Award ***A1*** for correct sides, ***A1*** for correct $\sin C$.

OR

Valid attempt to use area $=\frac{1}{2}$ base $\times$ height ***M1***

area $=\frac{1}{2}\times\left(\frac{3}{4}+\frac{3}{2}\right)\times\frac{6\sqrt{3}}{4}$ ***A1A1***

Note: ***A1*** for correct height, ***A1*** for correct base.

THEN

$=\frac{27\sqrt{3}}{16}$ ***AG***

[3 marks]

Total [9 marks]

8. EITHER

$x = \arctan t$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+t^2}$ ***A1***

OR

$t = \tan x$

$\frac{dt}{dx} = \sec^2 x$ ***(M1)***

$= 1 + \tan^2 x$ ***A1***

$= 1 + t^2$

THEN

$\sin x = \frac{t}{\sqrt{1+t^2}}$ ***(A1)***

Note: This ***A1*** is independent of the first two marks

$\int \frac{\mathrm{d}x}{1+\sin^2 x} = \int \frac{\frac{\mathrm{d}t}{1+t^2}}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to obtain integral in terms of t and $\mathrm{d}t$

$= \int \frac{\mathrm{d}t}{\left(1+t^2\right)+t^2} = \int \frac{\mathrm{d}t}{1+2t^2}$ ***A1***

$= \frac{1}{2}\int \frac{\mathrm{d}t}{\frac{1}{2}+t^2} = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$ ***A1***

$= \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2}\tan x\right)(+c)$ ***A1***

[8 marks]

9. (a) $a > 0$ ***A1***

$a \neq 1$ ***A1***

[2 marks]

(b) **METHOD 1**

$\log_x y = \frac{\ln y}{\ln x}$ and $\log_y x = \frac{\ln x}{\ln y}$ ***M1A1***

Note: Use of any base is permissible here, not just "e".

$\left(\frac{\ln y}{\ln x}\right)^2 = 4$ ***A1***

$\ln y = \pm 2 \ln x$ ***A1***

$y = x^2$ or $\frac{1}{x^2}$ ***A1A1***

METHOD 2

$\log_y x = \frac{\log_x x}{\log_x y} = \frac{1}{\log_x y}$ ***M1A1***

$(\log_x y)^2 = 4$ ***A1***

$\log_x y = \pm 2$ ***A1***

$y = x^2$ or $y = \frac{1}{x^2}$ ***A1A1***

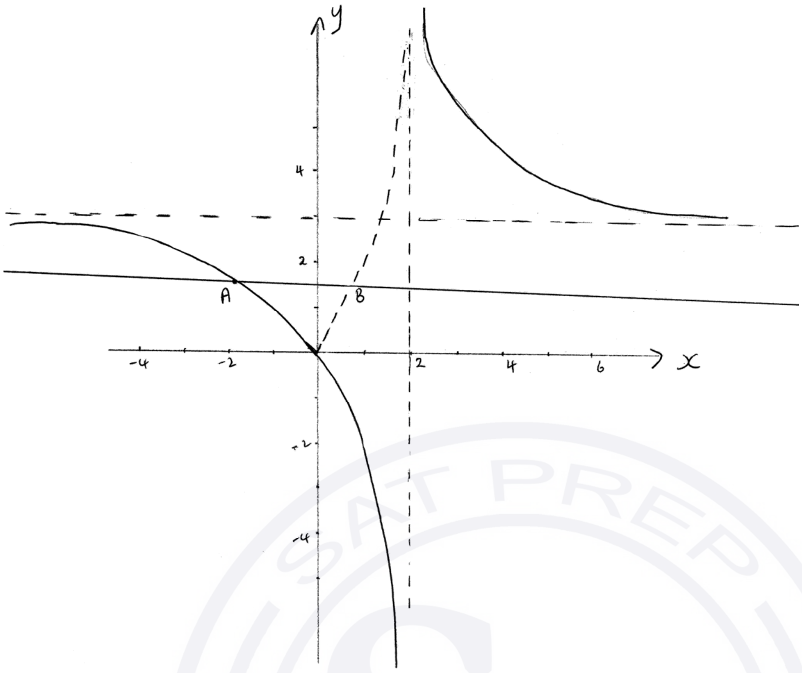
Note: The final two ***A*** marks are independent of the one coming before.

[6 marks]

Total [8 marks]

Section B

10. (a)



Note: In the diagram, points marked A and B refer to part (d) and do not need to be seen in part (a).

shape of curve *A1*

Note: This mark can only be awarded if there appear to be both horizontal and vertical asymptotes.

intersection at $(0, 0)$ *A1*
horizontal asymptote at $y = 3$ *A1*
vertical asymptote at $x = 2$ *A1*

[4 marks]

(b) $y = \frac{3x}{x-2}$

$xy - 2y = 3x$ *M1A1*
$xy - 3x = 2y$
$x = \frac{2y}{y-3}$
$\left(f^{-1}(x)\right) = \frac{2x}{x-3}$ *M1A1*

Note: Final M1 is for interchanging of x and y, which may be seen at any stage.

[4 marks]

continued…

Question 10 continued

(c) **METHOD 1**

attempt to solve $\frac{2x}{x-3}=\frac{3x}{x-2}$ ***(M1)***

$2x(x-2)=3x(x-3)$

$x[2(x-2)-3(x-3)]=0$

$x(5-x)=0$

$x=0$ or $x=5$ ***A1A1***

METHOD 2

$x=\frac{3x}{x-2}$ or $x=\frac{2x}{x-3}$ ***(M1)***

$x=0$ or $x=5$ ***A1A1***

[3 marks]

(d) **METHOD 1**

at $\mathrm{A}:\frac{3x}{x-2}=\frac{3}{2}$ AND at $\mathrm{B}:\frac{3x}{x-2}=-\frac{3}{2}$ ***M1***

$6x=3x-6$

$x=-2$ ***A1***

$6x=6-3x$

$x=\frac{2}{3}$ ***A1***

solution is $-2<x<\frac{2}{3}$ ***A1***

[4 marks]

METHOD 2

$\left(\frac{3x}{x-2}\right)^2<\left(\frac{3}{2}\right)^2$ ***M1***

$9x^2<\frac{9}{4}(x-2)^2$

$3x^2+4x-4<0$

$(3x-2)(x+2)<0$

$x=-2$ ***(A1)***

$x=\frac{2}{3}$ ***(A1)***

solution is $-2<x<\frac{2}{3}$ ***A1***

[4 marks]

continued…

Question 10 continued

(e) $-2 < x < 2$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

Note: If working is shown, then ***A*** marks may only be awarded following correct working.

[2 marks]

Total [17 marks]

11. (a) $g \circ f(x) = \dfrac{\tan x + 1}{\tan x - 1}$ ***A1***

$x \neq \dfrac{\pi}{4},\ 0 \leq x < \dfrac{\pi}{2}$ ***A1***

[2 marks]

(b) $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{\frac{\sin x}{\cos x} + 1}{\frac{\sin x}{\cos x} - 1}$ ***M1A1***

$= \dfrac{\sin x + \cos x}{\sin x - \cos x}$ ***AG***

[2 marks]

(c) **METHOD 1**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ ***M1(A1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\left(2\sin x\cos x - \cos^2 x - \sin^2 x\right) - \left(2\sin x\cos x + \cos^2 x + \sin^2 x\right)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \dfrac{-2}{1 - \sin 2x}$

Substitute $\dfrac{\pi}{6}$ into any formula for $\dfrac{dy}{dx}$ ***M1***

$\dfrac{-2}{1 - \sin\frac{\pi}{3}}$

$= \dfrac{-2}{1 - \frac{\sqrt{3}}{2}}$ ***A1***

$= \dfrac{-4}{2 - \sqrt{3}}$

$= \dfrac{-4}{2 - \sqrt{3}}\left(\dfrac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ ***M1***

$= \dfrac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ ***A1***

continued…

Question 11 continued

METHOD 2

$$\frac{dy}{dx} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$$ ***M1A1***

$$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$$ ***A1***

$$= \frac{-2\sec^2 \frac{\pi}{6}}{\left(\tan\frac{\pi}{6} - 1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}} - 1\right)^2} = \frac{-8}{\left(1 - \sqrt{3}\right)^2}$$ ***M1***

Note: Award ***M1*** for substitution of $\frac{\pi}{6}$.

$$\frac{-8}{\left(1-\sqrt{3}\right)^2} = \frac{-8}{\left(4 - 2\sqrt{3}\right)}\frac{\left(4 + 2\sqrt{3}\right)}{\left(4 + 2\sqrt{3}\right)} = -8 - 4\sqrt{3}$$ ***M1A1***

[6 marks]

continued…

Question 11 continued

(d) Area $= \left|\int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x} \mathrm{d}x\right|$ ***M1***

$= \left|\left[\ln\left|\sin x - \cos x\right|\right]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln\left|\sin 0 - \cos 0\right|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3}-1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3}-1}{2}\right) = \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$ ***M1***

$= \ln\left(\sqrt{3}+1\right)$ ***AG***

[6 marks]

Total [16 marks]

12. (a) (i)–(iii) given the three roots α, β, γ, we have

$x^3 + px^2 + qx + c = (x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$= \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)(x-\gamma)$ ***A1***

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha+\beta+\gamma)$ ***AG***

$q = (\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$c = -\alpha\beta\gamma$ ***AG***

[3 marks]

(b) **METHOD 1**

i) Given $-\alpha-\beta-\gamma=-6$

And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be α,β,γ.

So $\beta-\alpha=\gamma-\beta$ ***M1***

or $2\beta=\alpha+\gamma$

Attempt to solve simultaneous equations: ***M1***

$\beta+2\beta=6$ ***A1***

$\beta=2$ ***AG***

ii) $\alpha+\gamma=4$

$2\alpha+2\gamma+\alpha\gamma=18$

$\Rightarrow \gamma^2-4\gamma+10=0$

$\Rightarrow \gamma=\dfrac{4\pm \mathrm{i}\sqrt{24}}{2}$ ***(A1)***

Therefore $c=-\alpha\beta\gamma=-\left(\dfrac{4+\mathrm{i}\sqrt{24}}{2}\right)\left(\dfrac{4-\mathrm{i}\sqrt{24}}{2}\right)2=-20$ ***A1***

[5 marks]

continued…

Question 12 continued

METHOD 2

(i) let the three roots be α, $\alpha - d$, $\alpha + d$ ***M1***
adding roots ***M1***
to give $3\alpha = 6$ ***A1***
$\alpha = 2$ ***AG***

(ii) α is a root, so $2^3 - 6\times 2^2 + 18\times 2 + c = 0$ ***M1***

$8 - 24 + 36 + c = 0$
$c = -20$ ***A1***

[5 marks]

METHOD 3

(i) let the three roots be α, $\alpha - d$, $\alpha + d$ ***M1***
adding roots ***M1***
to give $3\alpha = 6$ ***A1***
$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2-d) + (2-d)(2+d) + 2(2+d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}\,\mathrm{i}$
$\Rightarrow c = -20$ ***A1***

[5 marks]

continued…

Question 12 continued

(c) **METHOD 1**

Given $-\alpha-\beta-\gamma=-6$
And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be α,β,γ.

So $\frac{\beta}{\alpha}=\frac{\gamma}{\beta}$ ***M1***

or $\beta^2=\alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta+\gamma\beta+\beta^2=18$
$\beta(\alpha+\beta+\gamma)=18$
$6\beta=18$
$\beta=3$ ***A1***

$\alpha+\gamma=3$, $\alpha=\frac{9}{\gamma}$
$\Rightarrow\gamma^2-3\gamma+9=0$
$\Rightarrow\gamma=\frac{3\pm i\sqrt{27}}{2}$ ***(A1)(A1)***

Therefore $c=-\alpha\beta\gamma=-\left(\frac{3+i\sqrt{27}}{2}\right)\left(\frac{3-i\sqrt{27}}{2}\right)3=-27$ ***A1***

[6 marks]

METHOD 2

let the three roots be a, ar, ar^2 ***M1***
attempt at substitution of a, ar, ar^2 and *p* and *q* into equations from (a) ***M1***
$6=a+ar+ar^2\left(=a\left(1+r+r^2\right)\right)$ ***A1***
$18=a^2r+a^2r^3+a^2r^2\left(=a^2r\left(1+r+r^2\right)\right)$ ***A1***

therefore $3=ar$ ***A1***
therefore $c=-a^3r^3=-3^3=-27$ ***A1***

[6 marks]

Total [14 marks]

13. (a) $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ ***M1***

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ ***A1***

$= \sqrt{n+1}-\sqrt{n}$ ***AG***

[2 marks]

(b) $\sqrt{2}-1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ ***A2***

$< \frac{1}{\sqrt{2}}$ ***AG***

[2 marks]

(c) consider the case $n = 2$: required to prove that $1+\frac{1}{\sqrt{2}} > \sqrt{2}$ ***M1***

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2}-1$

hence $1+\frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ ***A1***

now assume true for $n = k$: $\sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ ***M1***

$\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k+1$: $\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ (***M1***)

from assumption, we have that $\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k}+\frac{1}{\sqrt{k+1}}$ ***M1***

so attempt to show that $\sqrt{k}+\frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ (***M1***)

continued…

Question 13 continued

EITHER

$\dfrac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ ***A1***

$\dfrac{1}{\sqrt{k+1}} > \dfrac{1}{\sqrt{k} + \sqrt{k+1}}$, (from part a), which is true ***A1***

OR

$\sqrt{k} + \dfrac{1}{\sqrt{k+1}} = \dfrac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}$. ***A1***

$> \dfrac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \sqrt{k+1}$ ***A1***

THEN

so true for $n = 2$ and $n = k$ true $\Rightarrow n = k+1$ true. Hence true for all $n \geq 2$ ***R1***

Note: Award ***R1*** only if all previous ***M*** marks have been awarded.

[9 marks]

Total [13 marks]

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MARKSCHEME

November 2014

MATHEMATICS

Higher Level

Paper 1

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2014**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. (a) $g(x) = \dfrac{1}{x+3} + 1$ ***A1A1***

Note: Award ***A1*** for $x+3$ in the denominator and ***A1*** for the "+1".

[2 marks]

(b) $x = -3$ ***A1***
$y = 1$ ***A1***

[2 marks]

Total [4 marks]

2. (a) using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$ ***A1***

(ii) $\alpha\beta = \dfrac{1}{2}$ ***A1***

Note: Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

[2 marks]

(b) **METHOD 1**

required quadratic is of the form $x^2 - \left(\dfrac{2}{\alpha} + \dfrac{2}{\beta}\right)x + \left(\dfrac{2}{\alpha}\right)\left(\dfrac{2}{\beta}\right)$ ***(M1)***

$q = \dfrac{4}{\alpha\beta}$

$q = 8$ ***A1***

$p = -\left(\dfrac{2}{\alpha} + \dfrac{2}{\beta}\right)$

$= -\dfrac{2(\alpha+\beta)}{\alpha\beta}$ ***M1***

$= -\dfrac{2\times 4}{\frac{1}{2}}$

$p = -16$ ***A1***

Note: Accept the use of exact roots

continued …

Question 2 continued

METHOD 2

replacing x with $\frac{2}{x}$ ***M1***

$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$

$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$ ***(A1)***

$x^2 - 16x + 8 = 0$

$p = -16$ and $q = 8$ ***A1A1***

Note: Award ***A1A0*** for $x^2 - 16x + 8 = 0$ *ie*, if $p = -16$ and $q = 8$ are not explicitly stated.

[4 marks]

Total [6 marks]

3. METHOD 1

$\left|\overrightarrow{OP}\right| = \sqrt{(1+s)^2 + (3+2s)^2 + (1-s)^2} \quad \left(= \sqrt{6s^2 + 12s + 11}\right)$ ***A1***

Note: Award ***A1*** if the square of the distance is found.

EITHER

attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{OP}\right|^2 \left(= 12s + 12\right)$ ***M1***

attempting to solve $\frac{d}{ds}\left|\overrightarrow{OP}\right|^2 = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{OP}\right| \quad \left(= \frac{6s + 6}{\sqrt{6s^2 + 12s + 11}}\right)$ ***M1***

attempting to solve $\frac{d}{ds}\left|\overrightarrow{OP}\right| = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt at completing the square: $\left(\left|\overrightarrow{OP}\right|^2 = 6(s+1)^2 + 5\right)$ ***M1***

minimum value ***(M1)***
occurs at $s = -1$ ***(A1)***

THEN

the minimum length of $\overrightarrow{OP}$ is $\sqrt{5}$ ***A1***

METHOD 2

the length of $\overrightarrow{OP}$ is a minimum when $\overrightarrow{OP}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***(R1)***

$\begin{pmatrix} 1+s \\ 3+2s \\ 1-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$ ***A1***

attempting to solve $1 + s + 6 + 4s - 1 + s = 0$ $(6s + 6 = 0)$ for s ***(M1)***
$s = -1$ ***(A1)***
$\left|\overrightarrow{OP}\right| = \sqrt{5}$ ***A1***

Total [5 marks]

4. (a) (i) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ ***(M1)***

$\mathrm{P}(A \cup B) = 0.2 + 0.5$

$= 0.7$ ***A1***

(ii) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$ ***(M1)***

$\mathrm{P}(A \cup B) = 0.2 + 0.5 - 0.1$

$= 0.6$ ***A1***

[4 marks]

(b) $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

$\mathrm{P}(A \mid B)$ is a maximum when $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

$\mathrm{P}(A \mid B)$ is a minimum when $\mathrm{P}(A \cap B) = 0$

$0 \le \mathrm{P}(A \mid B) \le 0.4$ ***A1A1A1***

> **Note:** ***A1*** for each endpoint and ***A1*** for the correct inequalities**.**

[3 marks]

Total [7 marks]

5. use of the quotient rule or the product rule ***M1***

$$C'(t) = \frac{(3+t^2)\times 2 - 2t \times 2t}{(3+t^2)^2}\left(= \frac{6-2t^2}{(3+t^2)^2}\right) \text{ or } \frac{2}{3+t^2} - \frac{4t^2}{(3+t^2)^2}$$ ***A1A1***

> **Note:** Award ***A1*** for a correct numerator and ***A1*** for a correct denominator in the quotient rule, and ***A1*** for each correct term in the product rule.

attempting to solve $C'(t) = 0$ for t ***(M1)***

$t = \pm\sqrt{3}$ (minutes) ***A1***

$C\left(\sqrt{3}\right) = \dfrac{\sqrt{3}}{3}$ $\left(\mathrm{mg\,l^{-1}}\right)$ or equivalent. ***A1***

Total [6 marks]

6. $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ ***A1***

$dx = 2(u-1)\,du$

> **Note:** Award the ***A1*** for any correct relationship between dx and du.

$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = 2\int \frac{(u-1)^2}{u}\,du$ ***(M1)A1***

> **Note:** Award the ***M1*** for an attempt at substitution resulting in an integral only involving u.

$= 2\int u - 2 + \frac{1}{u}\,du$ ***(A1)***

$= u^2 - 4u + 2\ln u\,(+C)$ ***A1***

$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$ ***A1***

> **Note:** Award the ***A1*** for a correct expression in x, but not necessarily fully expanded/simplified.

Total [6 marks]

7. (a) $p'(3) = f'(3)\,g(3) + g'(3)\,f(3)$ ***(M1)***

> **Note:** Award ***M1*** if the derivative is in terms of x or 3.

$= 2\times4 + 3\times1$

$= 11$ ***A1***

[2 marks]

(b) $h'(x) = g'\left(f(x)\right)f'(x)$ ***(M1)(A1)***

$h'(2) = g'(1)\,f'(2)$ ***A1***

$= 4\times4$

$= 16$ ***A1***

[4 marks]

Total [6 marks]

8. let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

$2! = 2$ and $2^1 (1!)^2 = 2$ so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2$, $k \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$\left(2(k+1)\right)! = (2k+2)(2k+1)(2k)!$ ***M1***

$\left(2(k+1)\right)! \geq (2k+2)(2k+1)(k!)^2 2^k$ ***A1***

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k+1 > k+1$ ***R1***

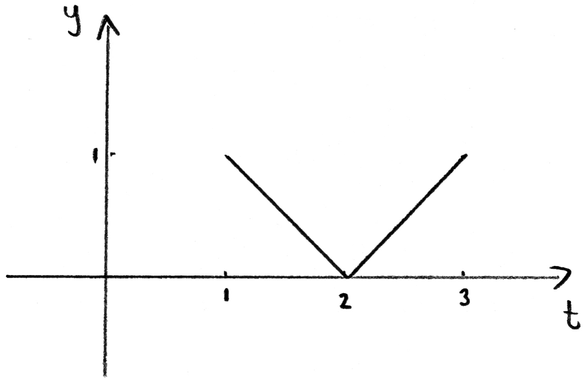
$= 2^{k+1}\left((k+1)!\right)^2$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

Total [7 marks]

9. (a)



$|2-t|$ correct for $[1, 2]$ ***A1***

$|2-t|$ correct for $[2, 3]$ ***A1***

[2 marks]

(b) **EITHER**

let q_1 be the lower quartile and let q_3 be the upper quartile

let $d = 2 - q_1 \left(= q_3 - 2\right)$ and so $\text{IQR} = 2d$ by symmetry

use of area formulae to obtain $\frac{1}{2}d^2 = \frac{1}{4}$

(or equivalent) ***M1A1***

$d = \frac{1}{\sqrt{2}}$ or the value of at least one q. ***A1***

OR

let q_1 be the lower quartile

consider $\int_1^{q_1} (2-t)\,dt = \frac{1}{4}$ ***M1A1***

obtain $q_1 = 2 - \frac{1}{\sqrt{2}}$ ***A1***

THEN

$\text{IQR} = \sqrt{2}$ ***A1***

Note: Only accept this final answer for the ***A1***.

[4 marks]

Total [6 marks]

10. (a) use of the addition principle with 3 terms *(M1)*

to obtain ${}^{4}C_{3} + {}^{5}C_{3} + {}^{6}C_{3}$ $(= 4 + 10 + 20)$ *A1*

number of possible selections is 34 *A1*

[3 marks]

(b) **EITHER**

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even) *(M1)*

$\left({}^{5}C_{2} \times {}^{4}C_{2}\right) + \left({}^{5}C_{1} \times {}^{4}C_{3}\right) + \left({}^{5}C_{0} \times {}^{4}C_{4}\right)$ $(= 60 + 20 + 1)$ *(M1)A1*

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total *(M1)*

${}^{9}C_{4} - {}^{5}C_{4} - \left({}^{5}C_{3} \times {}^{4}C_{1}\right)$ $(= 126 - 5 - 40)$ *(M1)A1*

THEN

number of possible selections is 81 *A1*

[4 marks]

Total [7 marks]

SECTION B

11. (a) (i) $x = e^{3y+1}$ ***M1***

> **Note:** The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$. ***A1***

[4 marks]

(b) $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent) ***M1A1***

$\ln x = -\frac{1}{2}$ (or equivalent) ***A1***

$x = e^{-\frac{1}{2}}$ ***A1***

coordinates of P are $\left(e^{-\frac{1}{2}}, -\frac{1}{2}\right)$ ***A1***

[5 marks]

(c) coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{dy}{dx} = \frac{1}{x}$ ***M1***

at Q, $\frac{dy}{dx} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

continued ...

Question 11 continued

(d) let the required area be A

$A = \int_1^e x - 1\mathrm{d}x - \int_1^e \ln x \mathrm{d}x$ ***M1***

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x \mathrm{d}x$ ***(M1)***

$= \left[\frac{x^2}{2} - x\right]_1^e - [x\ln x - x]_1^e$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$= \frac{e^2}{2} - e - \frac{1}{2}\left(= \frac{e^2 - 2e - 1}{2}\right)$ ***A1***

[5 marks]

(e) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \frac{1}{x}$ ***(A1)***

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$ ***R1***

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$ ***R1***

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

METHOD 2

$g''(x) = -\frac{1}{x^2}$ ***A1***

$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$ ***R1***

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$) ***R1***

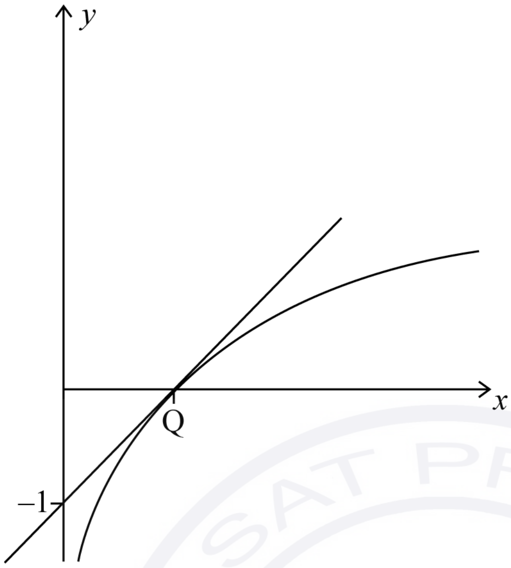
so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

continued ...

Question 11 continued

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \ge 1 - \frac{1}{x} \left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [23 marks]

12. (a) (i) $\vec{AM} = \frac{1}{2}\vec{AC}$ ***(M1)***

$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

(ii) $\vec{BM} = \vec{BA} + \vec{AM}$ ***M1***

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\vec{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***AG***

[4 marks]

(b) (i) $\vec{RA} = \frac{1}{3}\vec{BA}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ ***A1***

(ii) $\vec{RT} = \frac{2}{3}\vec{RS}$

$= \frac{2}{3}\left(\vec{RA} + \vec{AS}\right)$ ***(M1)***

$= \frac{2}{3}\left(\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})\right)$ or equivalent. ***A1A1***

$= \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\vec{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***AG***

[5 marks]

(c) $\vec{BT} = \vec{BR} + \vec{RT}$

$= \frac{2}{3}\vec{BA} + \vec{RT}$ ***(M1)***

$= \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} - \frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***A1***

$\vec{BT} = \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$ ***A1***

point B is common to $\vec{BT}$ and $\vec{BM}$ and $\vec{BT} = \frac{8}{9}\vec{BM}$ ***R1R1***

so T lies on [BM] ***AG***

[5 marks]

Total [14 marks]

13. (a) (i) METHOD 1

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = (1 + \mathrm{i}\frac{\sin\theta}{\cos\theta})^n + (1 - \mathrm{i}\frac{\sin\theta}{\cos\theta})^n$ ***M1***

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$ ***A1***

by de Moivre's theorem (***M1***)

$\left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta + i\sin n\theta}{\cos^n\theta}$ ***A1***

recognition that $\cos\theta - i\sin\theta$ is the complex conjugate of $\cos\theta + i\sin\theta$ (***R1***)

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$\left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta - i\sin n\theta}{\cos^n\theta}$ ***A1***

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ ***AG***

METHOD 2

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = (1 + \mathrm{i}\tan\theta)^n + (1 + i\tan(-\theta))^n$ ***(M1)***

$= \frac{(\cos\theta + i\sin\theta)^n}{\cos^n\theta} + \frac{(\cos(-\theta) + i\sin(-\theta))^n}{\cos^n\theta}$ ***M1A1***

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem ***(M1)***

$= \frac{1}{\cos^n\theta}(\cos n\theta + \mathrm{i}\sin n\theta + \cos(-n\theta) + \mathrm{i}\sin(-n\theta))$ ***A1***

$= \frac{2\cos n\theta}{\cos^n\theta}$ as $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$ ***R1AG***

continued …

Question 13 continued

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4+\left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4=\frac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$ ***(A1)***

$=\frac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$ ***A1***

$=0$ as $\cos\frac{3\pi}{2}=0$ ***R1***

Note: The above working could involve theta and the solution of $\cos(4\theta)=0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation ***AG***

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$ ***A1***

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.
Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

[10 marks]

(b) (i) $\tan\frac{\pi}{4}=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$ ***A1***

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t ***M1***

$t=-1\pm\sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8}>0$ ***R1***

$\tan\frac{\pi}{8}=\sqrt{2}-1$ ***AG***

continued …

Question 13 continued

(ii) $\cos 4x = 2\cos^2 2x - 1$ ***A1***

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ ***M1***

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ ***A1***

$= 8\cos^4 x - 8\cos^2 x + 1$ ***AG***

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,\mathrm{d}x$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,\mathrm{d}x$ ***M1***

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,\mathrm{d}x$ ***A1***

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [23 marks]

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MARKSCHEME

May 2014

MATHEMATICS

Higher Level

Paper 1

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2014**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. $P(2) = 24 + 2a + b = 2$, $P(-1) = -3 - a + b = 5$ ***M1A1A1***
$(2a + b = -22, -a + b = 8)$

> **Note:** Award ***M1*** for substitution of 2 or −1 and equating to remainder, ***A1*** for each correct equation.

attempt to solve simultaneously ***M1***
$a = -10$, $b = -2$ ***A1***

[5 marks]

2. using the sum divided by 4 is 13 ***M1***
two of the numbers are 15 ***A1***
(as median is 14) we need a 13 ***A1***
fourth number is 9 ***A1***
numbers are 9, 13, 15, 15 ***N4***

[4 marks]

3. $\frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} \times \ldots \times \frac{\log 32}{\log 31}$ ***M1A1***
$= \frac{\log 32}{\log 2}$ ***A1***
$= \frac{5 \log 2}{\log 2}$ ***(M1)***
$= 5$ ***A1***
hence $a = 5$

[5 marks]

> **Note:** Accept the above if done in a specific base *eg* $\log_2 x$.

4. $r_1 + r_2 + r_3 = \frac{-48}{5}$ ***(M1)(A1)***
$r_1 r_2 r_3 = \frac{a-2}{5}$ ***(M1)(A1)***
$\frac{-48}{5} + \frac{a-2}{5} = 0$ ***M1***
$a = 50$ ***A1***

> **Note:** Award ***M1A0M1A0M1A1*** if answer of 50 is found using $\frac{48}{5}$ and $\frac{2-a}{5}$.

[6 marks]

5. (a) $\cos x = 2\cos^2 \frac{1}{2}x - 1$

$\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ ***M1***

positive as $0 \le x \le \pi$ ***R1***

$\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$ ***AG***

[2 marks]

(b) $\cos 2\theta = 1 - 2\sin^2\theta$ ***(M1)***

$\sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ ***A1***

[2 marks]

(c) $\sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{1}{2}x + \sin\frac{1}{2}x \, dx$ ***A1***

$= \sqrt{2}\left[2\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \sqrt{2}(0) - \sqrt{2}(0-2)$ ***(A1)***

$= 2\sqrt{2}$ ***A1***

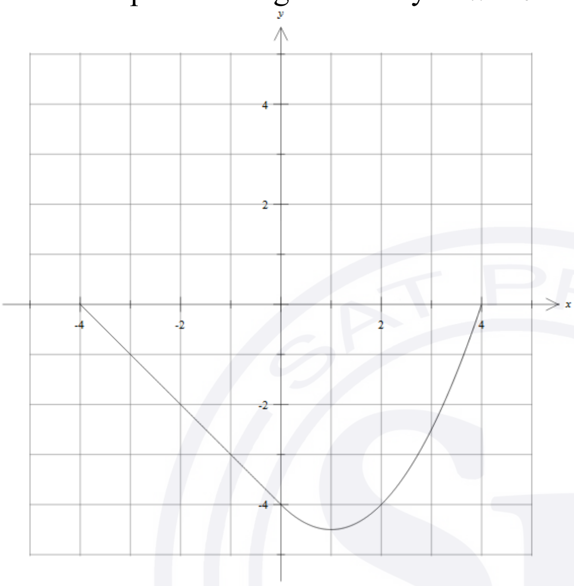
[4 marks]

Total [8 marks]

6. (a) $x = 1$ ***A1***

[1 mark]

(b) ***A1*** for point $(-4, 0)$
A1 for $(0, -4)$
A1 for min at $x = 1$ in approximately the correct place
A1 for $(4, 0)$
A1 for shape including continuity at $x = 0$



[5 marks]

Total [6 marks]

7. **METHOD 1**

$AD^2 = 2^2 + 3^2 - 2\times 2\times 3\times \cos 60^\circ$ ***M1***

$\left(\text{or } AD^2 = 1^2 + 3^2 - 2\times 1\times 3\times \cos 60^\circ\right)$

> **Note:** ***M1*** for use of cosine rule with 60° angle.

$AD^2 = 7$ ***A1***

$\cos D\hat{A}C = \dfrac{9+7-4}{2\times 3\times \sqrt{7}}$ ***M1A1***

> **Note:** ***M1*** for use of cosine rule involving $D\hat{A}C$.

$= \dfrac{2}{\sqrt{7}}$ ***A1***

METHOD 2

let point E be the foot of the perpendicular from D to AC

$EC = 1$ (by similar triangles, or triangle properties) ***M1A1***

(or $AE = 2$)

$DE = \sqrt{3}$ and $AD = \sqrt{7}$ (by Pythagoras) ***(M1)A1***

$\cos D\hat{A}C = \dfrac{2}{\sqrt{7}}$ ***A1***

[5 marks]

> **Note:** If first ***M1*** not awarded but remainder of the question is correct award ***M0A0M1A1A1***.

8. $\dfrac{dv}{ds} = 2s^{-3}$ ***M1A1***

> **Note:** Award ***M1*** for $2s^{-3}$ and ***A1*** for the whole expression.

$a = v\dfrac{dv}{ds}$ ***(M1)***

$a = -\dfrac{1}{s^2}\times\dfrac{2}{s^3}\left(= -\dfrac{2}{s^5}\right)$ ***(A1)***

when $s = \dfrac{1}{2}$, $a = -\dfrac{2}{(0.5)^5}$ $(= -64)$ (m s^{-2}) ***M1A1***

> **Note:** ***M1*** is for the substitution of 0.5 into their equation for acceleration.
> Award ***M1A0*** if $s = 50$ is substituted into the correct equation.

[6 marks]

9. (a) **METHOD 1**

$$\frac{2x}{1+x^4}+\frac{2y}{1+y^4}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$ ***M1A1A1***

> **Note:** Award ***M1*** for implicit differentiation, ***A1*** for LHS and ***A1*** for RHS.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x\left(1+y^4\right)}{y\left(1+x^4\right)}$$ ***A1***

METHOD 2

$$y^2=\tan\left(\frac{\pi}{4}-\arctan x^2\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\left(\arctan x^2\right)}{1+\left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan x^2\right)\right)}$$ ***(M1)***

$$=\frac{1-x^2}{1+x^2}$$ ***A1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2x\left(1+x^2\right)-2x\left(1-x^2\right)}{\left(1+x^2\right)^2}$$ ***M1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x}{y\left(1+x^2\right)^2}$$ ***A1***

$$\left(=\frac{2x\sqrt{1+x^2}}{\sqrt{1-x^2}\left(1+x^2\right)^2}\right)$$

[4 marks]

continued …

Question 9 continued

(b) $y^2 = \tan\left(\frac{\pi}{4} - \arctan\frac{1}{2}\right)$ ***(M1)***

$= \dfrac{\tan\frac{\pi}{4} - \tan\left(\arctan\frac{1}{2}\right)}{1 + \left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan\frac{1}{2}\right)\right)}$ ***(M1)***

Note: The two ***M1***s may be awarded for working in part (a).

$= \dfrac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \dfrac{1}{3}$ ***A1***

$y = -\dfrac{1}{\sqrt{3}}$ ***A1***

substitution into $\dfrac{dy}{dx}$

$= \dfrac{4\sqrt{6}}{9}$ ***A1***

Note: Accept $\dfrac{8\sqrt{3}}{9\sqrt{2}}$ *etc.*

[5 marks]

Total [9 marks]

10. $\sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{4}{9}$ ***(M1)(A1)***

using $\sin^2 x + \cos^2 x = 1$ ***(M1)***

$2\sin x \cos x = -\frac{5}{9}$

using $2\sin x \cos x = \sin 2x$ ***(M1)***

$\sin 2x = -\frac{5}{9}$

$\cos 4x = 1 - 2\sin^2 2x$ ***M1***

Note: Award this ***M1*** for decomposition of $\cos 4x$ using double angle formula anywhere in the solution.

$= 1 - 2 \times \frac{25}{81}$

$= \frac{31}{81}$ ***A1***

[6 marks]

SECTION B

11. (a) $f'(x) = \dfrac{x \times \frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \dfrac{1 - \ln x}{x^2}$ ***AG***

[2 marks]

(b) $\dfrac{1 - \ln x}{x^2} = 0$ has solution $x = \mathrm{e}$ ***M1A1***

$y = \dfrac{1}{\mathrm{e}}$ ***A1***

hence maximum at the point $\left(\mathrm{e}, \dfrac{1}{\mathrm{e}}\right)$

[3 marks]

(c) $f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - 2x(1 - \ln x)}{x^4}$ ***M1A1***

$= \dfrac{2\ln x - 3}{x^3}$

Note: The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where $f''(x) = 0$ ***M1***

so $x = \mathrm{e}^{\frac{3}{2}}$, $y = \dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}$ ***A1A1***

C has coordinates $\left(\mathrm{e}^{\frac{3}{2}}, \dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}\right)$

[5 marks]

(d) $f(1) = 0$ ***A1***

$f'(1) = 1$ ***(A1)***

$y = x + c$ ***(M1)***

through $(1, 0)$

equation is $y = x - 1$ ***A1***

[4 marks]

continued …

Question 11 continued

(e) **METHOD 1**

area $= \int_1^e x - 1 - \frac{\ln x}{x}\,dx$ ***M1A1A1***

Note: Award ***M1*** for integration of difference between line and curve, ***A1*** for correct limits, ***A1*** for correct expressions in either order.

$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$ ***(M1)A1***

$\int (x-1)\,dx = \frac{x^2}{2} - x(+c)$ ***A1***

$= \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^e$

$= \left(\frac{1}{2}e^2 - e - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}e^2 - e$ ***A1***

METHOD 2

area $=$ area of triangle $- \int_1^e \frac{\ln x}{x}\,dx$ ***M1A1***

Note: ***A1*** is for correct integral with limits and is dependent on the ***M1***.

$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$ ***(M1)A1***

area of triangle $= \frac{1}{2}(e-1)(e-1)$ ***M1A1***

$\frac{1}{2}(e-1)(e-1) - \left(\frac{1}{2}\right) = \frac{1}{2}e^2 - e$ ***A1***

[7 marks]

Total [21 marks]

12. (a) $|\overrightarrow{OA}| = |\overrightarrow{CB}| = |\overrightarrow{OC}| = |\overrightarrow{AB}| = 6$ (therefore a rhombus) ***A1A1***

> **Note:** Award ***A1*** for two correct lengths, ***A2*** for all four.

> **Note:** Award ***A1A0*** for $\overrightarrow{OA} = \overrightarrow{CB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ or $\overrightarrow{OC} = \overrightarrow{AB} = \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix}$ if no magnitudes are shown.

$\overrightarrow{OA} \, g \overrightarrow{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} g \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = 0$ (therefore a square) ***A1***

> **Note:** Other arguments are possible with a minimum of three conditions.

[3 marks]

(b) $M\left(3, -\frac{\sqrt{24}}{2}, \frac{\sqrt{12}}{2}\right)\left(=\left(3, -\sqrt{6}, \sqrt{3}\right)\right)$ ***A1***

[1 mark]

(c) **METHOD 1**

$\overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -6\sqrt{12} \\ -6\sqrt{24} \end{pmatrix} \left(= \begin{pmatrix} 0 \\ -12\sqrt{3} \\ -12\sqrt{6} \end{pmatrix} \right)$ ***M1A1***

> **Note:** Candidates may use other pairs of vectors.

equation of plane is $-6\sqrt{12}y - 6\sqrt{24}z = d$

any valid method showing that $d = 0$ ***M1***

$\Pi : y + \sqrt{2}z = 0$ ***AG***

METHOD 2

equation of plane is $ax + by + cz = d$

substituting O to find $d = 0$ ***(M1)***

substituting two points (A, B, C or M) ***M1***

eg

$6a = 0$, $-\sqrt{24}b + \sqrt{12}c = 0$ ***A1***

$\Pi : y + \sqrt{2}z = 0$ ***AG***

[3 marks]

continued …

Question 12 continued

(d) $\boldsymbol{r} = \begin{pmatrix} 3 \\ -\sqrt{6} \\ \sqrt{3} \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$ ***A1A1A1***

> **Note:** Award ***A1*** for $\boldsymbol{r} =$, ***A1A1*** for two correct vectors.

[3 marks]

(e) Using $y = 0$ to find λ ***M1***
Substitute their λ into their equation from part (d) ***M1***
D has coordinates $\left(3, 0, 3\sqrt{3}\right)$ ***A1***

[3 marks]

(f) λ for point E is the negative of the λ for point D ***(M1)***

> **Note:** Other possible methods may be seen.

E has coordinates $\left(3, -2\sqrt{6}, -\sqrt{3}\right)$ ***A1A1***

> **Note:** Award ***A1*** for each of the y and z coordinates.

[3 marks]

(g) (i) $\vec{DA} \bullet \vec{DO} = \begin{pmatrix} 3 \\ 0 \\ -3\sqrt{3} \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 0 \\ -3\sqrt{3} \end{pmatrix} = 18$ ***M1A1***

$\cos O\hat{D}A = \frac{18}{\sqrt{36}\sqrt{36}} = \frac{1}{2}$ ***M1***

hence $O\hat{D}A = 60^{\circ}$ ***A1***

> **Note:** Accept method showing OAD is equilateral.

(ii) OABCDE is a regular octahedron (accept equivalent description) ***A2***

> **Note:** ***A2*** for saying it is made up of 8 equilateral triangles
> Award ***A1*** for two pyramids, ***A1*** for equilateral triangles.
> (can be either stated or shown in a sketch – but there must be clear indication the triangles are equilateral)

[6 marks]

Total [22 marks]

13. (a) $r = 1 + \mathrm{i}$ *(A1)*

$u_4 = 3(1+\mathrm{i})^3$ ***M1***

$= -6 + 6\mathrm{i}$ ***A1***

[3 marks]

(b) $S_{20} = \dfrac{3\left((1+\mathrm{i})^{20} - 1\right)}{\mathrm{i}}$ *(M1)*

$= \dfrac{3\left(\left(2i\right)^{10} - 1\right)}{\mathrm{i}}$ *(M1)*

Note: Only one of the two ***M1***s can be implied. Other algebraic methods may be seen.

$= \dfrac{3\left(-2^{10} - 1\right)}{\mathrm{i}}$ *(A1)*

$= 3\mathrm{i}\left(2^{10} + 1\right)$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

$v_n = \left(3(1+\mathrm{i})^{n-1}\right)\left(3(1+\mathrm{i})^{n-1+k}\right)$ ***M1***

$9(1+\mathrm{i})^k\,(1+i)^{2n-2}$ ***A1***

$= 9(1+\mathrm{i})^k\,((1+i)^2)^{n-1}\left(= 9(1+\mathrm{i})^k\,(2\mathrm{i})^{n-1}\right)$

this is the general term of a geometrical sequence ***R1AG***

Notes: Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further. If the final expression for v_n is $9(1+\mathrm{i})^k\,(1+i)^{2n-2}$ award ***M1A1R0***.

METHOD 2

$\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1}u_{n+k+1}}{u_n u_{n+k}}$ ***M1***

$= (1+\mathrm{i})(1+\mathrm{i})$ ***A1***

this is a constant, hence sequence is geometric ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) $9(1+\mathrm{i})^k$ ***A1***

(iii) common ratio is $(1+i)^2 (= 2i)$ (which is independent of *k*) ***A1***

[5 marks]

continued …

Question 13 continued

(d) (i) **METHOD 1**

$w_n = \left|3(1+i)^{n-1} - 3(1+\mathrm{i})^n\right|$ ***M1***

$= 3\left|1+i\right|^{n-1}\left|1-(1+\mathrm{i})\right|$ ***M1***

$= 3\left|1+i\right|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

METHOD 2

$w_n = \left|u_n - (1+\mathrm{i})u_n\right|$ ***M1***

$= \left|u_n\right|\left|-\mathrm{i}\right|$

$= \left|u_n\right|$ ***A1***

$= \left|3(1+\mathrm{i})^{n-1}\right|$

$= 3\left|(1+\mathrm{i})\right|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) distance between successive points representing u_n in the complex plane forms a geometric sequence ***R1***

Note: Various possibilities but must mention distance between successive points.

[5 marks]

Total [17 marks]

M14/5/MATHL/HP1/ENG/TZ2/XX/M

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

May 2014

MATHEMATICS

Higher Level

Paper 1

24 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2014**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
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*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. (a) $P(A \cap B) = P(A | B) \times P(B)$

$P(A \cap B) = \frac{2}{11} \times \frac{11}{20}$ ***(M1)***

$= \frac{1}{10}$ ***A1***

[2 marks]

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{2}{5} + \frac{11}{20} - \frac{1}{10}$ ***(M1)***

$= \frac{17}{20}$ ***A1***

[2 marks]

(c) No – events A and B are not independent ***A1***

EITHER

$P(A | B) \neq P(A)$ ***R1***

$\left(\frac{2}{11} \neq \frac{2}{5}\right)$

OR

$P(A) \times P(B) \neq P(A \cap B)$

$\frac{2}{5} \times \frac{11}{20} = \frac{11}{50} \neq \frac{1}{10}$ ***R1***

Note: The numbers are required to gain ***R1*** in the '**OR**' method only.

Note: Do not award ***A1R0*** in either method.

[2 marks]

Total [6 marks]

2. **METHOD 1**

$2^{3(x-1)} = (2\times 3)^{3x}$ ***M1***

Note: Award ***M1*** for writing in terms of 2 and 3.

$2^{3x}\times 2^{-3} = 2^{3x}\times 3^{3x}$

$2^{-3} = 3^{3x}$ ***A1***

$\ln\left(2^{-3}\right) = \ln\left(3^{3x}\right)$ ***(M1)***

$-3\ln 2 = 3x\ln 3$ ***A1***

$x = -\frac{\ln 2}{\ln 3}$ ***A1***

METHOD 2

$\ln 8^{x-1} = \ln 6^{3x}$ ***(M1)***

$(x-1)\ln 2^3 = 3x\ln(2\times 3)$ ***M1A1***

$3x\ln 2 - 3\ln 2 = 3x\ln 2 + 3x\ln 3$ ***A1***

$x = -\frac{\ln 2}{\ln 3}$ ***A1***

METHOD 3

$\ln 8^{x-1} = \ln 6^{3x}$ ***(M1)***

$(x-1)\ln 8 = 3x\ln 6$ ***A1***

$x = \frac{\ln 8}{\ln 8 - 3\ln 6}$ ***A1***

$x = \frac{3\ln 2}{\ln\left(\frac{2^3}{6^3}\right)}$ ***M1***

$x = -\frac{\ln 2}{\ln 3}$ ***A1***

Total [5 marks]

3. (a) **EITHER**

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ ***M1***

row of zeroes implies infinite solutions, (or equivalent). ***R1***

Note: Award ***M1*** for any attempt at row reduction.

OR

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0$$ ***M1***

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \text{ with one valid point}$$ ***R1***

OR

$$x + y + 2z = -2$$
$$3x - y + 14z = 6$$
$$x + 2y = -5 \quad \Rightarrow x = -5 - 2y$$

substitute $x = -5 - 2y$ into the first two equations:

$-5 - 2y + y + 2z = -2$

$3(-5 - 2y) - y + 14z = 6$ ***M1***

$-y + 2z = 3$

$-7y + 14z = 21$

the latter two equations are equivalent (by multiplying by 7) therefore an infinite number of solutions. ***R1***

OR

for example, $7 \times \mathrm{R}_1 - \mathrm{R}_2$ gives $4x + 8y = -20$ ***M1***

this equation is a multiple of the third equation, therefore an infinite number of solutions. ***R1***

continued…

Question 3 continued

(b) let $y = t$ ***M1***

then $x = -5 - 2t$ ***A1***

$z = \frac{t+3}{2}$ ***A1***

OR

let $x = t$ ***M1***

then $y = \frac{-5-t}{2}$ ***A1***

$z = \frac{1-t}{4}$ ***A1***

OR

let $z = t$ ***M1***

then $x = 1 - 4t$ ***A1***

$y = -3 + 2t$ ***A1***

OR

attempt to find cross product of two normal vectors:

eg: $\begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4i + 2j + k$ ***M1A1***

$x = 1 - 4t$

$y = -3 + 2t$

$z = t$ ***A1***

(or equivalent)

Total [5 marks]

4. (a) using the formulae for the sum and product of roots:

$\alpha + \beta = -2$ *A1*

$\alpha\beta = -\frac{1}{2}$ *A1*

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ *M1*

$= (-2)^2 - 2\left(-\frac{1}{2}\right)$

$= 5$ *A1*

[4 marks]

Note: Award ***M0*** for attempt to solve quadratic equation.

(b) $(x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2$ *M1*

$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0$ *A1*

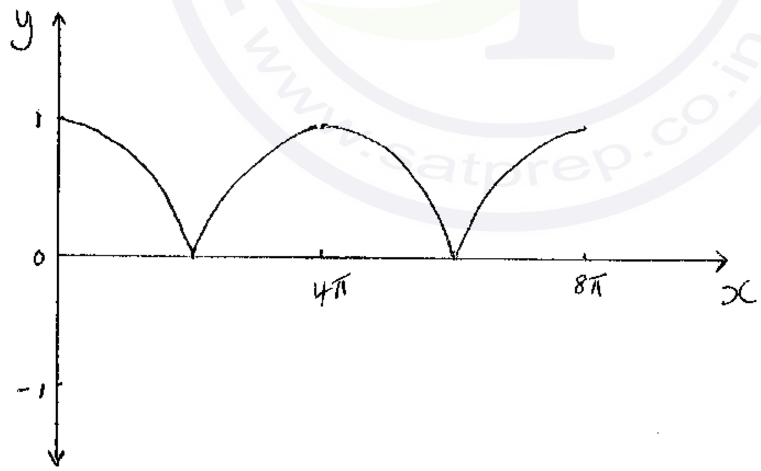
$x^2 - 5x + \frac{1}{4} = 0$

Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

5. (a)



A1A1

Note: Award ***A1*** for correct shape and ***A1*** for correct domain and range.

[2 marks]

continued…

Question 5 continued

(b) $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$ ***A1***

attempting to find any other solutions ***M1***

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$

$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$

$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$ ***A1***

Note: Award ***A1*** for all other three solutions correct and no extra solutions.

Note: If working in degrees, then max ***A0M1A0***.

[3 marks]

Total [5 marks]

6. (a) $\vec{\mathrm{PR}} = \boldsymbol{a} + \boldsymbol{b}$ ***A1***

$\vec{\mathrm{QS}} = \boldsymbol{b} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) $\vec{\mathrm{PR}} \cdot \vec{\mathrm{QS}} = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{b} - \boldsymbol{a})$ ***M1***

$= |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$ ***A1***

for a rhombus $|\boldsymbol{a}| = |\boldsymbol{b}|$ ***R1***

hence $|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 = 0$ ***A1***

Note: Do not award the final ***A1*** unless ***R1*** is awarded.

hence the diagonals intersect at right angles ***AG***

[4 marks]

Total [6 marks]

7. (a) **METHOD 1**

$\frac{1}{2+3i}+\frac{1}{3+2i}=\frac{2-3i}{4+9}+\frac{3-2i}{9+4}$ ***M1A1***

$\frac{10}{w}=\frac{5-5i}{13}$ ***A1***

$w=\frac{130}{5-5i}$

$=\frac{130\times 5\times(1+i)}{50}$

$w=13+13i$ ***A1***

[4 marks]

METHOD 2

$\frac{1}{2+3i}+\frac{1}{3+2i}=\frac{3+2i+2+3i}{(2+3i)(3+2i)}$ ***M1A1***

$\frac{10}{w}=\frac{5+5i}{13i}$ ***A1***

$\frac{w}{10}=\frac{13i}{5+5i}$

$w=\frac{130i}{(5+5i)}\times\frac{(5-5i)}{(5-5i)}$

$=\frac{650+650i}{50}$

$=13+13i$ ***A1***

[4 marks]

(b) $w^*=13-13i$ ***A1***

$z=\sqrt{338}e^{-\frac{\pi}{4}i}\left(=13\sqrt{2}\,e^{-\frac{\pi}{4}i}\right)$ ***A1A1***

> **Note:** Accept $\theta=\frac{7\pi}{4}$.
> Do not accept answers for θ given in degrees.

[3 marks]

Total [7 marks]

8. (a) $1-2(2)=-3$ and $\frac{3}{4}(2-2)^2-3=-3$ ***A1***

both answers are the same, hence f is continuous (at $x=2$) ***R1***

> **Note:** ***R1*** may be awarded for justification using a graph or referring to limits. Do not award ***A0R1***.

[2 marks]

(b) reflection in the y-axis

$$f(-x)=\begin{cases} 1+2x, & x\geq -2 \\ \frac{3}{4}(x+2)^2-3, & x<-2 \end{cases}$$ ***(M1)***

> **Note:** Award ***M1*** for evidence of reflecting a graph in y-axis.

translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$g(x)=\begin{cases} 2x-3, & x\geq 0 \\ \frac{3}{4}x^2-3, & x<0 \end{cases}$$ ***(M1)A1A1***

> **Note:** Award ***(M1)*** for attempting to substitute $(x-2)$ for x, or translating a graph along positive x-axis.
> Award ***A1*** for the correct domains (this mark can be awarded independent of the ***M1***).
> Award ***A1*** for the correct expressions.

[4 marks]

Total [6 marks]

9. (a) $\sin x$, $\sin 2x$ and $4\sin x\cos^2 x$

$r = \dfrac{2\sin x\cos x}{\sin x} = 2\cos x$ ***A1***

Note: Accept $\dfrac{\sin 2x}{\sin x}$.

[1 mark]

(b) **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1$ ***M1***

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$ ***M1***

THEN

$0 < \cos x < \dfrac{1}{2}$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$-\dfrac{\pi}{2} < x < -\dfrac{\pi}{3}$ or $\dfrac{\pi}{3} < x < \dfrac{\pi}{2}$ ***A1A1***

[3 marks]

(c) $S_\infty = \dfrac{\sin x}{1 - 2\cos x}$ ***M1***

$$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1 - 2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$$

$$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$$= \frac{\sqrt{15}}{2}$$ ***AG***

[3 marks]

Total [7marks]

10. $x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ ***(A1)***

new limits:

$x = a\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ and $x = 2a \Rightarrow \theta = \frac{\pi}{3}$ ***(A1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{a\sec\theta\tan\theta}{a^3\sec^3\theta\sqrt{a^2\sec^2\theta - a^2}}\, d\theta$ ***M1***

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2\theta}{a^3} d\theta$ ***A1***

using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ ***M1***

$\frac{1}{2a^3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or equivalent ***A1***

$= \frac{1}{4a^3}\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - 1 - \frac{\pi}{2}\right)$ or equivalent ***A1***

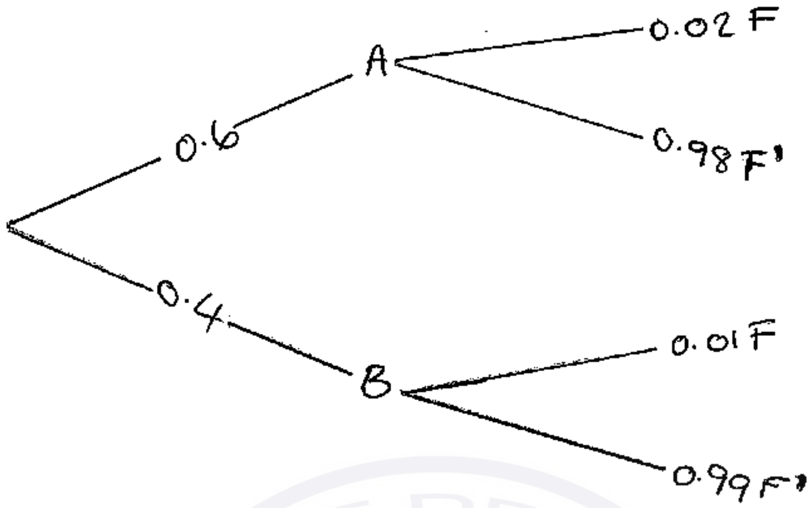
$= \frac{1}{24a^3}\left(3\sqrt{3} + \pi - 6\right)$ ***AG***

[7 marks]

Total [7 marks]

SECTION B

11. (a) (i)



A1A1

Note:	Award ***A1*** for a correctly labelled tree diagram and ***A1*** for correct probabilities.

(ii) $\mathrm{P}(F) = 0.6 \times 0.02 + 0.4 \times 0.01$ ***(M1)***

$= 0.016$ ***A1***

(iii) $\mathrm{P}(A \mid F) = \dfrac{\mathrm{P}(A \cap F)}{\mathrm{P}(F)}$

$= \dfrac{0.6 \times 0.02}{0.016} \left(= \dfrac{0.012}{0.016} \right)$ ***M1***

$= 0.75$ ***A1***

[6 marks]

continued…

Question 11 continued

(b) (i) **METHOD 1**

$\mathrm{P}(X=2)=\frac{{}^{3}C_{2}\times{}^{4}C_{1}}{{}^{7}C_{3}}$ ***(M1)***

$=\frac{12}{35}$ ***A1***

METHOD 2

$\frac{3}{7}\times\frac{2}{6}\times\frac{4}{5}\times 3$ ***(M1)***

$=\frac{12}{35}$ ***A1***

(ii)

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

A2

Note: Award ***A1*** if $\frac{4}{35}$, $\frac{18}{35}$ or $\frac{1}{35}$ is obtained.

(iii) $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$\mathrm{E}(X)=0\times\frac{4}{35}+1\times\frac{18}{35}+2\times\frac{12}{35}+3\times\frac{1}{35}$ ***M1***

$=\frac{45}{35}=\left(\frac{9}{7}\right)$ ***A1***

[6 marks]

Total [12 marks]

12. (a) direction vector $\vec{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\vec{BA} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$ ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or equivalent ***A1***

> **Note:** Do not award final ***A1*** unless ‘ $r = \ldots$ ’ (or equivalent) seen.
> Allow FT on direction vector for final ***A1***.

[2 marks]

(b) both lines expressed in parametric form:

L_1:
$x = 1 + t$
$y = 3t$
$z = 4 - 5t$

L_2:
$x = 1 + 3s$
$y = -2 + s$ ***M1A1***
$z = -2s + 1$

> **Notes:** Award ***M1*** for an attempt to convert L_2 from Cartesian to parametric form.
> Award ***A1*** for correct parametric equations for L_1 and L_2.
> Allow ***M1A1*** at this stage if same parameter is used in both lines.

attempt to solve simultaneously for x and y: ***M1***

$1 + t = 1 + 3s$
$3t = -2 + s$

$t = -\frac{3}{4}$, $s = -\frac{1}{4}$ ***A1***

substituting both values back into z values respectively gives $z = \frac{31}{4}$

and $z = \frac{3}{2}$ so a contradiction ***R1***

therefore L_1 and L_2 are skew lines ***AG***

[5 marks]

continued…

Question 12 continued

(c) finding the cross product:

$$\begin{pmatrix}1\\3\\-5\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}$$ ***(M1)***

$=-\boldsymbol{i}-13\boldsymbol{j}-8\boldsymbol{k}$ ***A1***

> **Note:** Accept $\boldsymbol{i}+13\boldsymbol{j}+8\boldsymbol{k}$

$-1(0)-13(1)-8(-2)=3$ ***(M1)***

$\Rightarrow -x-13y-8z=3$ or equivalent ***A1***

[4 marks]

(d) (i) $$\left(\cos\theta=\right)\frac{\begin{pmatrix}k\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\sqrt{k^2+1+1}\times\sqrt{1+1}}$$ ***M1***

> **Note:** Award ***M1*** for an attempt to use angle between two vectors formula.

$$\frac{\sqrt{3}}{2}=\frac{k+1}{\sqrt{2(k^2+2)}}$$ ***A1***

obtaining the quadratic equation

$4(k+1)^2=6(k^2+2)$ ***M1***

$k^2-4k+4=0$

$(k-2)^2=0$

$k=2$ ***A1***

> **Note:** Award ***M1A0M1A0*** if $\cos 60^{\circ}$ is used $(k=0$ or $k=-4)$.

continued…

Question 12 continued

(ii) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

substituting into the equation of the plane Π_2:

$3 + 2\lambda + \lambda = 12$ ***M1***

$\lambda = 3$ ***A1***

point P has the coordinates:

$(9, 3, -2)$ ***A1***

Notes: Accept $9\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix}$.

Do not allow FT if two values found for *k*.

[7 marks]

Total [18 marks]

13. (a) $f'(x) = \frac{(x^2+1) - 2x(x+1)}{(x^2+1)^2} \left(= \frac{-x^2 - 2x + 1}{(x^2+1)^2} \right)$ ***M1A1***

[2 marks]

(b) $\frac{-x^2 - 2x + 1}{(x^2+1)^2} = 0$

$x = -1 \pm \sqrt{2}$ ***A1***

[1 mark]

continued…

Question 13 continued

(c) $f''(x) = \dfrac{(-2x-2)(x^2+1)^2 - 2(2x)(x^2+1)(-x^2-2x+1)}{(x^2+1)^4}$ ***A1A1***

Note: Award ***A1*** for $(-2x-2)(x^2+1)^2$ or equivalent.

Note: Award ***A1*** for $-2(2x)(x^2+1)(-x^2-2x+1)$ or equivalent.

$= \dfrac{(-2x-2)(x^2+1) - 4x(-x^2-2x+1)}{(x^2+1)^3}$

$= \dfrac{2x^3+6x^2-6x-2}{(x^2+1)^3}$ ***A1***

$\left(= \dfrac{2(x^3+3x^2-3x-1)}{(x^2+1)^3}\right)$

[3 marks]

(d) recognition that $(x-1)$ is a factor ***(R1)***

$(x-1)(x^2+bx+c) = (x^3+3x^2-3x-1)$ ***M1***

$\Rightarrow x^2+4x+1=0$ ***A1***

$x = -2 \pm \sqrt{3}$ ***A1***

Note: Allow long division / synthetic division.

[4 marks]

(e) $\int_{-1}^{0} \dfrac{x+1}{x^2+1}\,dx$ ***M1***

$\int \dfrac{x+1}{x^2+1}\,dx = \int \dfrac{x}{x^2+1}\,dx + \int \dfrac{1}{x^2+1}\,dx$ ***M1***

$= \dfrac{1}{2}\ln(x^2+1) + \arctan(x)$ ***A1A1***

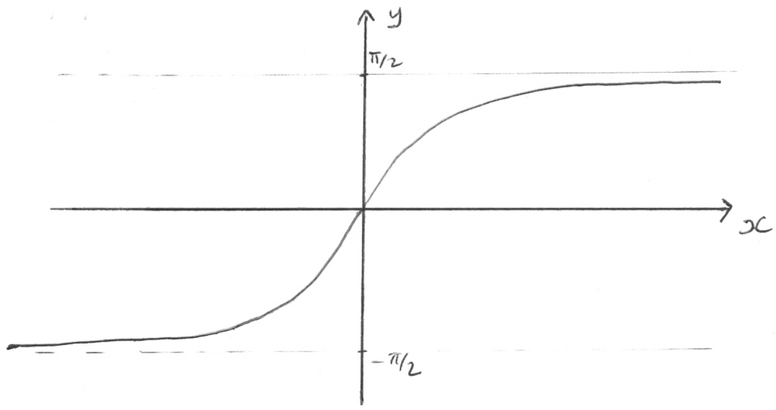
$= \left[\dfrac{1}{2}\ln(x^2+1) + \arctan(x)\right]_{-1}^{0} = \dfrac{1}{2}\ln 1 + \arctan 0 - \dfrac{1}{2}\ln 2 - \arctan(-1)$ ***M1***

$= \dfrac{\pi}{4} - \ln\sqrt{2}$ ***A1***

[6 marks]

Total [16 marks]

14. (a)



A1A1

> **Note:** ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

[2 marks]

(b) $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ *A1*

domain of $h \circ g$ is equal to the domain of g : $x \in$ ° , $x \neq 0$ *A1*

[2 marks]

(c) (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$$ ***M1A1***

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$$ ***(A1)***

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$$

$$= 0$$ ***A1***

continued…

Question 14 continued

(ii) **METHOD 1**

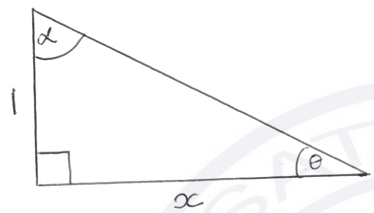
f is a constant **R1**

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ **M1A1**

$= \frac{\pi}{2}$ **AG**

METHOD 2



from diagram

$\theta = \arctan \frac{1}{x}$ **A1**

$\alpha = \arctan x$ **A1**

$\theta + \alpha = \frac{\pi}{2}$ **R1**

hence $f(x) = \frac{\pi}{2}$ **AG**

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ **M1**

$= \frac{x + \frac{1}{x}}{1 - x\left(\frac{1}{x}\right)}$ **A1**

denominator $= 0$, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) **R1**

[7 marks]

continued…

Question 14 continued

(d) (i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$

therefore f is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]

Total [14 marks]

N13/5/MATHL/HP1/ENG/TZ0/XX/M

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MARKSCHEME

November 2013

MATHEMATICS

Higher Level

Paper 1

21 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking November 2013**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. $f(-2)=0$ $(\Rightarrow -24+4p-2q-2=0)$ ***M1***

$f(-1)=4$ $(\Rightarrow -3+p-q-2=4)$ ***M1***

> **Note:** In each case award the ***M*** marks if correct substitution attempted and right-hand side correct.

attempt to solve simultaneously ($2p-q=13$, $p-q=9$) ***M1***

$p=4$ ***A1***

$q=-5$ ***A1***

Total [5 marks]

2. (a) $\frac{1}{6}+\frac{1}{2}+\frac{3}{10}+a=1 \Rightarrow a=\frac{1}{30}$ ***A1***

[1 mark]

(b) $E(X)=\frac{1}{2}+2\times\frac{3}{10}+3\times\frac{1}{30}$ ***M1***

$=\frac{6}{5}$ ***A1***

> **Note:** Do not award ***FT*** marks if a is outside $[0, 1]$.

[2 marks]

(c) $\mathrm{E}(X^2)=\frac{1}{2}+2^2\times\frac{3}{10}+3^2\times\frac{1}{30}=2$ ***(A1)***

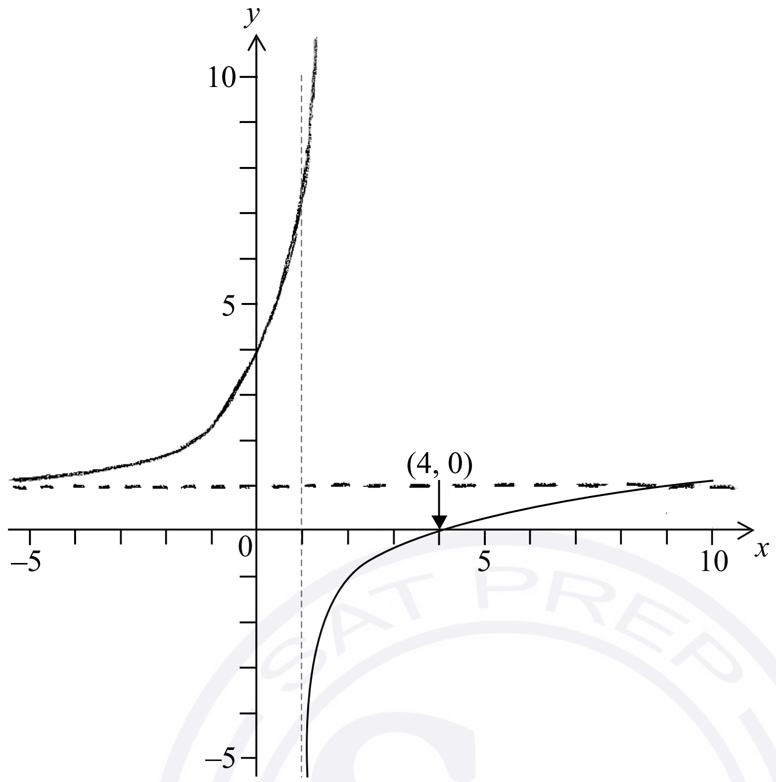
attempt to apply $\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left(\mathrm{E}(X)\right)^2$ ***M1***

$\left(=2-\frac{36}{25}\right)=\frac{14}{25}$ ***A1***

[3 marks]

Total [6 marks]

3. (a)



shape with y-axis intercept $(0, 4)$ *A1*

Note: Accept curve with an asymptote at $x = 1$ suggested.

correct asymptote $y = 1$ *A1*

[2 marks]

(b) range is $f^{-1}(x) > 1$ (or $]1, \infty[$) *A1*

Note: Also accept $]1, 10]$ or $]1, 10[$.

Note: Do not allow follow through from incorrect asymptote in (a).

[1 mark]

(c) $(4, 0) \Rightarrow \ln(4a + b) = 0$ *M1*

$\Rightarrow 4a + b = 1$ *A1*

asymptote at $x = 1 \Rightarrow a + b = 0$ *M1*

$\Rightarrow a = \frac{1}{3},\ b = -\frac{1}{3}$ *A1*

[4 marks]

Total [7 marks]

4. (if $\boldsymbol{A}$ is singular then) $\det \boldsymbol{A} = 0$ (seen anywhere) ***R1***

$\det \boldsymbol{A} = b(b+1) - a(a+1)$ ***M1***

$b + b^2 - a - a^2 = 0$

EITHER

$b - a + (b+a)(b-a) = 0$ ***(M1)***

$(b-a)(1+b+a) = 0$ ***A1***

OR

$b - a = a^2 - b^2$ ***(M1)***

$b - a = (a+b)(a-b)$ or $-(a-b) = (a+b)(a-b)$ ***A1***

THEN

$a + b = -1$ ***A1***

Total [5 marks]

5. $3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 9\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1M1A1***

Note: First ***M1*** for attempt at implicit differentiation, second ***M1*** for use of product rule.

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2y^2 + 3x^2}{3y^2 - 2x^3y - 9}\right)$

$\Rightarrow 3x^2 + 3x^2y^2 = 0$ ***(A1)***

$\Rightarrow 3x^2\left(1 + y^2\right) = 0$

$x = 0$ ***A1***

Note: Do not award ***A1*** if extra solutions given *eg* $y = \pm 1$.

substituting $x = 0$ into original equation ***(M1)***

$y^3 - 9y = 0$

$y(y+3)(y-3) = 0$

$y = 0,\ y = \pm 3$

coordinates $(0, 0), (0, 3), (0, -3)$ ***A1***

Total [7 marks]

6. $n=1:\ 1^3+11=12$
$=3\times 4$ **or** a multiple of 3 — ***A1***
assume the proposition is true for $n=k$ (*ie* $k^3+11k=3\text{m}$) — ***M1***

> **Note:** Do not award ***M1*** for statements with "Let $n=k$".

consider $n=k+1$: $(k+1)^3+11(k+1)$ — ***M1***

$=k^3+3k^2+3k+1+11k+11$ — ***A1***
$=k^3+11k+(3k^2+3k+12)$ — ***M1***
$=3(m+k^2+k+4)$ — ***A1***

> **Note:** Accept $k^3+11k+3(k^2+k+4)$ or statement that $k^3+11k+(3k^2+3k+12)$ is a multiple of 3.

true for $n=1$, and $n=k$ true $\Rightarrow n=k+1$ true
hence true for all $n\in\mathbb{Z}^+$ — ***R1***

> **Note:** Only award the final ***R1*** if at least 4 of the previous marks have been achieved.

Total [7 marks]

7. (a) **METHOD 1**

$a+ar=10$ — ***A1***
$a+ar+ar^2+ar^3=30$ — ***A1***
$a+ar=10\Rightarrow ar^2+ar^3=10r^2$ **or** $ar^2+ar^3=20$ — ***M1***
$10+10r^2=30$ **or** $r^2(a+ar)=20$ — ***A1***
$\Rightarrow r^2=2$ — ***AG***

METHOD 2

$\dfrac{a\left(1-r^2\right)}{1-r}=10$ and $\dfrac{a\left(1-r^4\right)}{1-r}=30$ — ***M1A1***

$\Rightarrow \dfrac{1-r^4}{1-r^2}=3$ — ***M1***

leading to either $1+r^2=3$ (or $r^4-3r^2+2=0$) — ***A1***
$\Rightarrow r^2=2$ — ***AG***

[4 marks]

continued

Question 7 continued

(b) (i) $a + a\sqrt{2} = 10$

$\Rightarrow a = \dfrac{10}{1+\sqrt{2}}$ **or** $a = 10\left(\sqrt{2}-1\right)$ ***A1***

(ii) $S_{10} = \dfrac{10}{1+\sqrt{2}}\left(\dfrac{\sqrt{2}^{10}-1}{\sqrt{2}-1}\right)\left(=10\times 31\right)$ ***M1***

$= 310$ ***A1***

[3 marks]

Total [7 marks]

8. (a) $\sin(x+y)\sin(x-y)$

$= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$ ***M1A1***

$= \sin^2 x\cos^2 y + \sin x\sin y\cos x\cos y - \sin x\sin y\cos x\cos y - \cos^2 x\sin^2 y$

$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$ ***A1***

$= \sin^2 x\left(1-\sin^2 y\right) - \sin^2 y\left(1-\sin^2 x\right)$ ***A1***

$= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y$

$= \sin^2 x - \sin^2 y$ ***AG***

[4 marks]

(b) $f(x) = \sin^2 x - \dfrac{1}{4}$

range is $f \in \left[-\dfrac{1}{4}, \dfrac{3}{4}\right]$ ***A1A1***

Note: Award ***A1*** for each end point. Condone incorrect brackets.

[2 marks]

(c) $g(x) = \dfrac{1}{\sin^2 x - \dfrac{1}{4}}$

range is $g \in \left]-\infty, -4\right] \cup \left[\dfrac{4}{3}, \infty\right[$ ***A1A1***

Note: Award ***A1*** for each part of range. Condone incorrect brackets.

[2 marks]

Total [8 marks]

9. (a) $\log_2(x-2) = \log_4\left(x^2-6x+12\right)$

EITHER

$\log_2(x-2) = \dfrac{\log_2\left(x^2-6x+12\right)}{\log_2 4}$ ***M1***

$2\log_2(x-2) = \log_2\left(x^2-6x+12\right)$

OR

$\dfrac{\log_4(x-2)}{\log_4 2} = \log_4\left(x^2-6x+12\right)$ ***M1***

$2\log_4(x-2) = \log_4\left(x^2-6x+12\right)$

THEN

$(x-2)^2 = x^2-6x+12$ ***A1***

$x^2-4x+4 = x^2-6x+12$

$x = 4$ ***A1*** ***N1***

[3 marks]

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$

taking ln of both sides or writing $x = \mathrm{e}^{\ln x}$ ***M1***

$(\ln x)^2 = (\ln x)^3$ ***A1***

$(\ln x)^2(\ln x - 1) = 0$ ***(A1)***

$x = 1,\ x = \mathrm{e}$ ***A1A1*** ***N2***

Note: Award second ***(A1)*** only if factorisation seen or if two correct solutions are seen.

[5 marks]

Total [8 marks]

SECTION B

10. (a) (i) $f'(x) = e^{-x} - xe^{-x}$ ***M1A1***

(ii) $f'(x) = 0 \Rightarrow x = 1$

coordinates $\left(1, e^{-1}\right)$ ***A1***

[3 marks]

(b) $f''(x) = -e^{-x} - e^{-x} + xe^{-x}\left(= -e^{-x}(2-x)\right)$ ***A1***

substituting $x = 1$ into $f''(x)$ ***M1***

$f''(1)\left(= -e^{-1}\right) < 0$ hence maximum ***R1AG***

[3 marks]

(c) $f''(x) = 0 \quad (\Rightarrow x = 2)$ ***M1***

coordinates $\left(2, 2e^{-2}\right)$ ***A1***

[2 marks]

(d) (i) $g(x) = \dfrac{x}{2}e^{-\frac{x}{2}}$ ***A1***

(ii) coordinates of maximum $\left(2, e^{-1}\right)$ ***A1***

(iii) equating $f(x) = g(x)$ and attempting to solve $xe^{-x} = \dfrac{x}{2}e^{-\frac{x}{2}}$

$\Rightarrow x\left(2e^{\frac{x}{2}} - e^{x}\right) = 0$ ***(A1)***

$\Rightarrow x = 0$ ***A1***

or $2e^{\frac{x}{2}} = e^{x}$

$\Rightarrow e^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2 \qquad (\ln 4)$ ***A1***

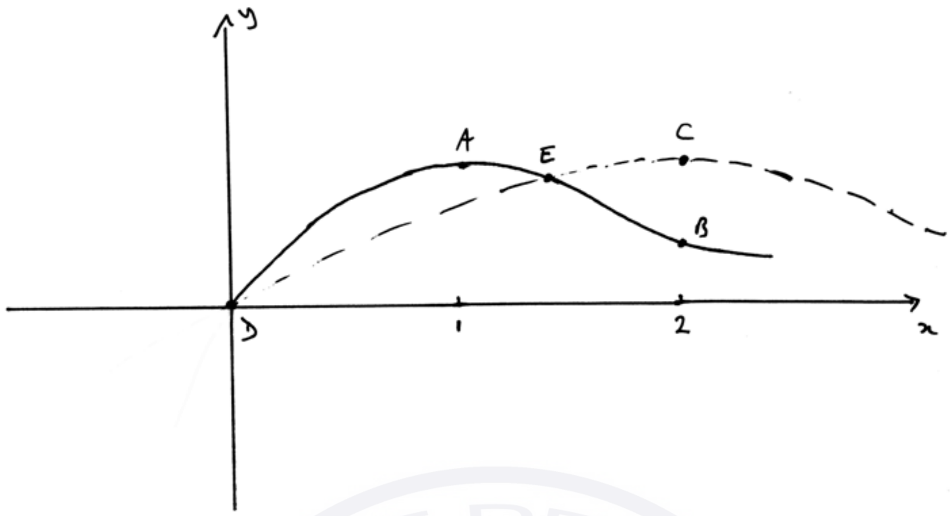
[5 marks]

> **Note:** Award first ***(A1)*** only if factorisation seen or if two correct solutions are seen.

continued …

Question 10 continued

(e)



A4

> **Note:** Award ***A1*** for shape of f, including domain extending beyond $x = 2$. Ignore any graph shown for $x < 0$.
> Award ***A1*** for A and B correctly identified.
> Award ***A1*** for shape of g, including domain extending beyond $x = 2$. Ignore any graph shown for $x < 0$. Allow follow through from f.
> Award ***A1*** for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

(f) $A = \int_0^1 \frac{x}{2} e^{-\frac{x}{2}} \, dx$ ***M1***

$= \left[-xe^{-\frac{x}{2}}\right]_0^1 - \int_0^1 -e^{-\frac{x}{2}} \, dx$ ***A1***

> **Note:** Condone absence of limits or incorrect limits.

$= -e^{-\frac{1}{2}} - \left[2e^{-\frac{x}{2}}\right]_0^1$

$= -e^{-\frac{1}{2}} - \left(2e^{-\frac{1}{2}} - 2\right) = 2 - 3e^{-\frac{1}{2}}$ ***A1***

[3 marks]

Total [20 marks]

11. (a) $\vec{CA} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ ***(A1)***

$\vec{CB} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

> **Note:** If $\vec{AC}$ and $\vec{BC}$ found correctly award ***(A1) (A0)***.

$\vec{CA} \times \vec{CB} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$ ***(M1)***

$\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$ ***A1***

[4 marks]

(b) **METHOD 1**

$\frac{1}{2}\left|\vec{CA} \times \vec{CB}\right|$ ***(M1)***

$= \frac{1}{2}\sqrt{(-2)^2 + (-3)^2 + 4^2}$ ***(A1)***

$= \frac{\sqrt{29}}{2}$ ***A1***

METHOD 2

attempt to apply $\frac{1}{2}|CA||CB|\sin C$ ***(M1)***

$CA.CB = \sqrt{5}.\sqrt{6}\cos C \Rightarrow \cos C = \frac{1}{\sqrt{30}} \Rightarrow \sin C = \frac{\sqrt{29}}{\sqrt{30}}$ ***(A1)***

$\text{area} = \frac{\sqrt{29}}{2}$ ***A1***

[3 marks]

continued …

Question 11 continued

(c) **METHOD 1**

$\mathbf{r}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}$ ***M1A1***

$\Rightarrow -2x-3y+4z=-2$ ***A1***

$\Rightarrow 2x+3y-4z=2$ ***AG***

METHOD 2

$-2x-3y+4z=\mathrm{d}$

substituting a point in the plane ***M1A1***

$\mathrm{d}=-2$ ***A1***

$\Rightarrow -2x-3y+4z=-2$

$\Rightarrow 2x+3y-4z=2$ ***AG***

Note: Accept verification that all 3 vertices of the triangle lie on the given plane.

[3 marks]

continued ...

Question 11 continued

(d) **METHOD 1**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -7 \\ -14 \\ -14 \end{pmatrix}$$ ***M1A1***

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$z = 0 \Rightarrow y = 0,\ x = 1$ ***(M1)(A1)***

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 2

eliminate 1 of the variables, *eg x* ***M1***
$-7y + 7z = 0$ ***(A1)***
introduce a parameter ***M1***
$\Rightarrow z = \lambda,$

$y = \lambda,\ x = 1 + \dfrac{\lambda}{2}$ ***(A1)***

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 3

$z = t$ ***M1***
write x and y in terms of $t \Rightarrow 4x - y = 4 + t,\ 2x + 3y = 2 + 4t$ or equivalent ***A1***
attempt to eliminate x or y ***M1***
x, y , z expressed in parameters
$\Rightarrow z = t,$

$y = t,\ x = 1 + \dfrac{t}{2}$ ***A1***

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

[5 marks]

continued ...

Question 11 continued

(e) **METHOD 1**

direction of the line is perpendicular to the normal of the plane

$$\begin{pmatrix}16\\ \alpha\\ -3\end{pmatrix}.\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=0$$ ***M1A1***

$16+2\alpha-6=0 \Rightarrow \alpha=-5$ ***A1***

METHOD 2

solving line/plane simultaneously

$16(1+\lambda)+2\alpha\lambda-6\lambda=\beta$ ***M1A1***

$16+(10+2\alpha)\lambda=\beta$

$\Rightarrow \alpha=-5$ ***A1***

METHOD 3

$$\begin{vmatrix}2 & 3 & -4\\ 4 & -1 & -1\\ 16 & \alpha & -3\end{vmatrix}=0$$ ***M1***

$2(3+\alpha)-3(-12+16)-4(4\alpha+16)=0$ ***A1***

$\Rightarrow \alpha=-5$ ***A1***

METHOD 4

attempt to use row reduction on augmented matrix ***M1***

to obtain $\left(\begin{array}{ccc|c}2 & 3 & -4 & 2\\ 0 & -1 & 1 & 0\\ 0 & 0 & \alpha+5 & \beta-16\end{array}\right)$ ***A1***

$\Rightarrow \alpha=-5$ ***A1***

[3 marks]

(f) $\alpha=-5$ ***A1***

$\beta\neq 16$ ***A1***

[2 marks]

Total [20 marks]

12. (a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ ***M1***

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$ ***A1***

$= 2\cos n\theta$ ***AG***

[2 marks]

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

> **Note:** Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

(c) **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***M1***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***A1A1***

> **Note:** Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$ ***M1***

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$ ***A1***

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$ ***A1***

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

[4 marks]

continued …

Question 12 continued

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$ ***M1***

$\left(z+z^{-1}\right)^6 = \left(z^6+\frac{1}{z^6}\right) + 6\left(z^4+\frac{1}{z^4}\right) + 15\left(z^2+\frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ ***AG***

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$ ***M1A1***

$= \frac{5\pi}{32}$ ***A1***

[3 marks]

(f) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\cos^4 x\,\mathrm{d}x$ ***M1***

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x - \pi\int_0^{\frac{\pi}{2}} \cos^6 x\,\mathrm{d}x$ ***M1***

$\int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x = \frac{3\pi}{16}$ ***A1***

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$ ***A1***

Note: Follow through from an incorrect r in (c) provided the final answer is positive.

[4 marks]

continued ...

Question 12 continued

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!}{(k!)^2}\frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$ ***A1***

[3 marks]

Total [20 marks]

M13/5/MATHL/HP1/ENG/TZ1/XX/M

International Baccalaureate®
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MARKSCHEME

May 2013

MATHEMATICS

Higher Level

Paper 1

19 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2013**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. (a) modulus $=\sqrt{8}$ ***A1***

argument $=\frac{\pi}{4}$ (accept 45°) ***A1***

Note: ***A0*** if extra values given.

[2 marks]

(b) **METHOD 1**

$w^4 z^6 = 64e^{\pi i} \times e^{5\pi i}$ ***(A1)(A1)***

Note: Allow alternative notation.

$= 64e^{6\pi i}$ ***(M1)***

$= 64$ ***A1***

METHOD 2

$w^4 = -64$ ***(M1)(A1)***

$z^6 = -1$ ***(A1)***

$w^4 z^6 = 64$ ***A1***

[4 marks]

Total [6 marks]

2. (a) $\vec{AB} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ ***A1A1***

Note: Award the above marks if the components are seen in the line below.

$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & -2 & 2 \\ 1 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ ***(M1)A1***

[4 marks]

(b) area $= \frac{1}{2}\left|\left(\vec{AB} \times \vec{AC}\right)\right|$ ***(M1)***

$= \frac{1}{2}\sqrt{4^2 + 2^2 + 2^2} = \frac{1}{2}\sqrt{24}\left(= \sqrt{6}\right)$ ***A1***

Note: Award ***M0A0*** for attempts that do not involve the answer to (a).

[2 marks]

Total [6 marks]

3. **METHOD 1**

$A^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \left(= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \right)$ ***(M1)A1***

$X = A^{-1}BA = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ ***M1***

$\begin{pmatrix} -2 & 5 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \text{or} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ ***(M1)A1***

> **Note**: Accept the answer $\frac{1}{-1}\begin{pmatrix} -3 & -1 \\ 1 & -2 \end{pmatrix}$.

METHOD 2

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$AX = BA$ ***M1***

$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} a+2c & b+2d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}$ ***(M1)(A1)***

$a+2c=1$

$b+2d=5$

$a+c=2$

$b+d=3$

$a=3, b=1, c=-1, d=2$ ***(M1)A1***

[5 marks]

METHOD 3

$A^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \left(= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \right)$ ***(M1)A1***

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$

$a=3, b=1, c=-1, d=2$ ***(M1)(M1)A1***

[5 marks]

4. $\int_0^{\frac{\pi}{2}} x \sin x \mathrm{d}x$ ***M1***

$= \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x$ ***M1(A1)***

Note: Condone the absence of limits or wrong limits to this point.

$= \left[-x\cos x + \sin x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= 1$ ***A1***

[5 marks]

5. $V = 0.5\pi r^2$ ***(A1)***

EITHER

$\frac{dV}{dr} = \pi r$ ***A1***

$\frac{dV}{dt} = 4$ ***(A1)***

applying chain rule ***M1***

for example $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

OR

$\frac{dV}{dt} = \pi r \frac{dr}{dt}$ ***M1A1***

$\frac{dV}{dt} = 4$ ***(A1)***

THEN

$\frac{dr}{dt} = 4 \times \frac{1}{\pi r}$ ***A1***

when $r = 20$, $\frac{dr}{dt} = \frac{4}{20\pi}$ or $\frac{1}{5\pi}$ (cm s^{-1}) ***A1***

Note: Allow h instead of 0.5 up until the final ***A1***.

[6 marks]

6. when $n=1$, $(\boldsymbol{A}+\boldsymbol{I})^1 = 2^0(\boldsymbol{A}+\boldsymbol{I})$ (or $1\times(\boldsymbol{A}+\boldsymbol{I})$) ***A1***
so true for $n=1$
assume the statement is true for $n=k$ ***M1***
$(\boldsymbol{A}+\boldsymbol{I})^{\boldsymbol{k}} = 2^{k-1}(\boldsymbol{A}+\boldsymbol{I})$

> **Note**: Award ***M1*** only if assumption of truth is clear.

$(\boldsymbol{A}+\boldsymbol{I})^{k+1} = (\boldsymbol{A}+\boldsymbol{I})^k(\boldsymbol{A}+\boldsymbol{I})$
$= 2^{k-1}(\boldsymbol{A}+\boldsymbol{I})(\boldsymbol{A}+\boldsymbol{I})$ ***M1***
$= 2^{k-1}(\boldsymbol{A}^2+\boldsymbol{IA}+\boldsymbol{AI}+\boldsymbol{I}^2)$ ***M1***
$= 2^{k-1}(2\boldsymbol{A}+2\boldsymbol{I})$ ***A1***
$= 2^k(\boldsymbol{A}+\boldsymbol{I})$
therefore if true for $n=k$ then true for $n=k+1$; as true for $n=1$ so true for all $n(\in\mathbb{Z}^+)$ ***R1***

> **Note**: Only award ***R1*** if all three ***M*** marks have been awarded.

[6 marks]

7. $8y\times\frac{1}{x}+8\frac{dy}{dx}\ln x-4x+8y\frac{dy}{dx}=0$ ***M1A1A1***

> **Note**: ***M1*** for attempt at implicit differentiation. ***A1*** for differentiating $8y\ln x$, ***A1*** for differentiating the rest.

when $x=1$, $8y\times0-2\times1+4y^2=7$ ***(M1)***
$y^2=\frac{9}{4}\Rightarrow y=\frac{3}{2}$ (as $y>0$) ***A1***

at $\left(1,\frac{3}{2}\right)\frac{dy}{dx}=-\frac{2}{3}$ ***A1***
$y-\frac{3}{2}=-\frac{2}{3}(x-1)$ or $y=-\frac{2}{3}x+\frac{13}{6}$ ***A1***

[7 marks]

8. METHOD 1

$$d = \frac{1}{\log_8 x} - \frac{1}{\log_2 x}$$ ***(M1)***

$$= \frac{\log_2 8}{\log_2 x} - \frac{1}{\log_2 x}$$ ***(M1)***

Note: Award this ***M1*** for a correct change of base anywhere in the question.

$$= \frac{2}{\log_2 x}$$ ***(A1)***

$$\frac{20}{2}\left(2 \times \frac{1}{\log_2 x} + 19 \times \frac{2}{\log_2 x}\right)$$ ***M1***

$$= \frac{400}{\log_2 x}$$ ***(A1)***

$$100 = \frac{400}{\log_2 x}$$

$$\log_2 x = 4 \Rightarrow x = 2^4 = 16$$ ***A1***

METHOD 2

$$20^{\text{th}} \text{ term } = \frac{1}{\log_{2^{39}} x}$$ ***A1***

$$100 = \frac{20}{2}\left(\frac{1}{\log_2 x} + \frac{1}{\log_{2^{39}} x}\right)$$ ***M1***

$$100 = \frac{20}{2}\left(\frac{1}{\log_2 x} + \frac{\log_2 2^{39}}{\log_2 x}\right)$$ ***M1(A1)***

Note: Award this ***M1*** for a correct change of base anywhere in the question.

$$100 = \frac{400}{\log_2 x}$$ ***(A1)***

$$\log_2 x = 4 \Rightarrow x = 2^4 = 16$$ ***A1***

continued …

Question 8 continued

METHOD 3

$$\frac{1}{\log_2 x}+\frac{1}{\log_8 x}+\frac{1}{\log_{32} x}+\frac{1}{\log_{128} x}+\ldots$$

$$\frac{1}{\log_2 x}+\frac{\log_2 8}{\log_2 x}+\frac{\log_2 32}{\log_2 x}+\frac{\log_2 128}{\log_2 x}+\ldots$$ ***(M1)(A1)***

Note: Award this ***M1*** for a correct change of base anywhere in the question.

$$=\frac{1}{\log_2 x}(1+3+5+\ldots)$$ ***A1***

$$=\frac{1}{\log_2 x}\left(\frac{20}{2}(2+38)\right)$$ ***(M1)(A1)***

$$100=\frac{400}{\log_2 x}$$

$$\log_2 x=4 \Rightarrow x=2^4=16$$ ***A1***

[6 marks]

9. **Note: Be aware that an unjustified assumption of independence will also lead to $\mathrm{P}(\mathrm{B}) = 0.25$, but is an invalid method.**

METHOD 1

$\mathrm{P}(A'|B') = 1 - \mathrm{P}(A|B') = 1 - 0.6 = 0.4$ ***M1A1***

$\mathrm{P}(A'|B') = \dfrac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')}$

$\mathrm{P}(A' \cap B') = \mathrm{P}\left((A \cup B)'\right) = 1 - 0.7 = 0.3$ ***A1***

$0.4 = \dfrac{0.3}{\mathrm{P}(B')} \Rightarrow \mathrm{P}(B') = 0.75$ ***(M1)A1***

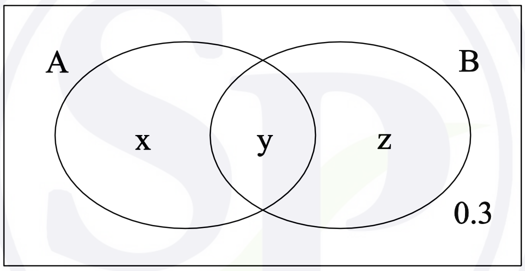
$\mathrm{P}(B) = 0.25$ ***A1***

(this method can be illustrated using a tree diagram)

[6 marks]

METHOD 2

$\mathrm{P}\left((A \cup B)'\right) = 1 - 0.7 = 0.3$ ***A1***



$\mathrm{P}(A|B') = \dfrac{x}{x + 0.3} = 0.6$ ***M1A1***

$x = 0.6x + 0.18$

$0.4x = 0.18$

$x = 0.45$ ***A1***

$\mathrm{P}(A \cup B) = x + y + z$

$\mathrm{P}(B) = y + z = 0.7 - 0.45$ ***(M1)***

$= 0.25$ ***A1***

[6 marks]

METHOD 3

$\dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = 0.6$ (or $\mathrm{P}(A \cap B') = 0.6\,\mathrm{P}(B')$) ***M1***

$\mathrm{P}(A \cap B') = \mathrm{P}(A \cup B) - \mathrm{P}(B)$ ***M1A1***

$\mathrm{P}(B') = 1 - \mathrm{P}(B)$

$0.7 - \mathrm{P}(B) = 0.6 - 0.6\,\mathrm{P}(B)$ ***M1(A1)***

$0.1 = 0.4\,\mathrm{P}(B)$

$\mathrm{P}(B) = \dfrac{1}{4}$ ***A1***

[6 marks]

10. (a) $\sin\left(\pi x^{-1}\right)=0 \quad \frac{\pi}{x}=\pi, 2\pi\ (\ldots)$ ***(A1)***

$x=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ ***A1***

[2 marks]

(b) $\left[\cos(\pi x^{-1})\right]_{\frac{1}{n+1}}^{\frac{1}{n}}$ ***M1***

$=\cos\left(\pi n\right)-\cos\left(\pi(n+1)\right)$ ***A1***

$=2$ when n is even and $=-2$ when n is odd ***A1***

[3 marks]

(c) $\int_{0.1}^{1}\left|\pi x^{-2}\sin(\pi x^{-1})\right|\mathrm{d}x=2+2+\ldots+2=18$ ***(M1)A1***

[2 marks]

Total [7 marks]

SECTION B

11. (a) (i) $\cos\left(\frac{\pi}{6}\right)\cos x - \sin\left(\frac{\pi}{6}\right)\sin x$ ***M1***

$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$ ***A1***

(ii) $\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$ ***(M1)***

$\cos\left(\frac{\pi}{6} + x\right) = \frac{1}{2}$ ***M1***

$\frac{\pi}{6} + x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ ***(A1)(A1)***

$x = \frac{\pi}{6}$ or $\frac{3\pi}{2}$ ***A1***

[7 marks]

(b) (i) **METHOD 1**

substitute $x = 1$ ***M1***

$p(1) = 0$ ***A1***

hence $x = 1$ is a zero ***AG***

METHOD 2

Correct result when dividing by $x - 1$ ***A1***

Statement that remainder is zero ***R1***

hence $x = 1$ is a zero ***AG***

continued …

Question 11 continued

(ii) $x = 1$ is a root ***A1***

valid method, for example, dividing or comparing coefficients ***(M1)***

(may be seen in (b)(i))

other roots are -1 and $\frac{1}{2}$ ***A1A1***

(iii) $2\sin\theta\cos\theta\cos\theta + \sin^2\theta$ ***M1***

$2\sin\theta\left(1 - \sin^2\theta\right) + \sin^2\theta$ ***(A1)***

$\left(2\sin\theta - 2\sin^3\theta + \sin^2\theta\right)$

(iv) $2\sin\theta - 2\sin^3\theta + \sin^2\theta = 1$ ***(A1)***

$\sin\theta = -1, \frac{1}{2}, 1$ ***A1***

$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$ ***A1A1***

Note: Award ***A1*** for two correct solutions, ***A2*** for four correct solutions.

[12 marks]

Total [19 marks]

12. (a) $4(x-0.5)^2+4$ ***A1A1***

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

[2 marks]

(b) translation $\begin{pmatrix}0.5\\0\end{pmatrix}$ (allow "0.5 to the right") ***A1***

stretch parallel to y-axis, scale factor 4 (allow vertical stretch or similar) ***A1***

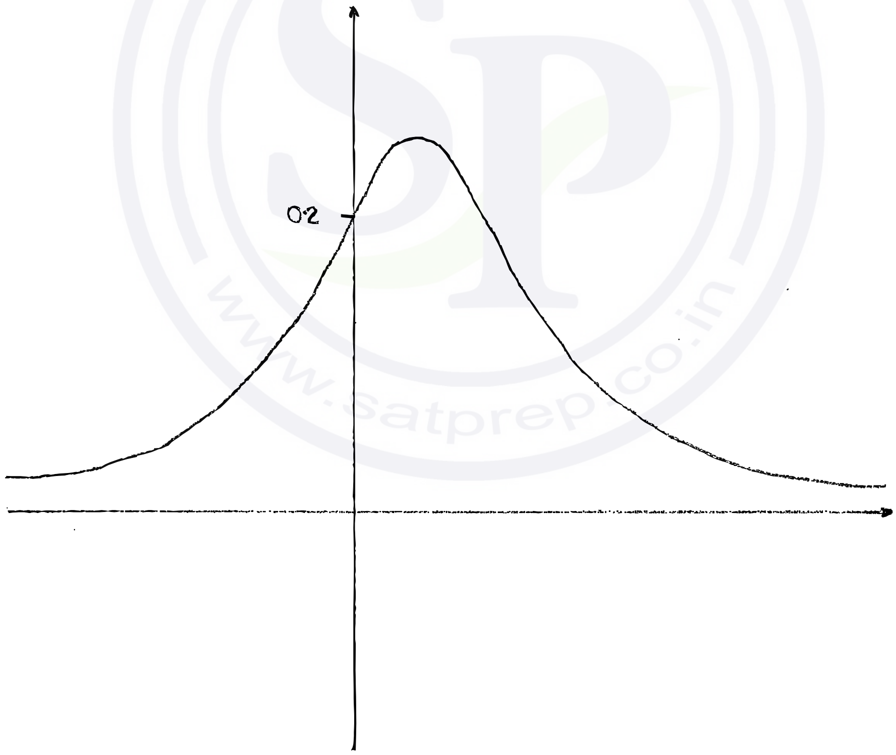
translation $\begin{pmatrix}0\\4\end{pmatrix}$ (allow "4 up") ***A1***

Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.
It could be a stretch followed by a single translation of $\begin{pmatrix}0.5\\4\end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix}0\\1\end{pmatrix}$.

[3 marks]

(c)



general shape (including asymptote and single maximum in first quadrant), ***A1***

intercept $\left(0, \frac{1}{5}\right)$ **or** maximum $\left(\frac{1}{2}, \frac{1}{4}\right)$ shown ***A1***

[2 marks]

continued …

Question 12 continued

(d) $0 < f(x) \le \frac{1}{4}$ ***A1A1***

> **Note**: ***A1*** for $\le \frac{1}{4}$, ***A1*** for $0 <$.

[2 marks]

(e) let $u = x - \frac{1}{2}$ ***A1***

$\frac{du}{dx} = 1$ (or $du = dx$) ***A1***

$\int \frac{1}{4x^2 - 4x + 5} dx = \int \frac{1}{4\left(x - \frac{1}{2}\right)^2 + 4} dx$ ***A1***

$\int \frac{1}{4u^2 + 4} du = \frac{1}{4}\int \frac{1}{u^2 + 1} du$ ***AG***

> **Note:** If following through an incorrect answer to part (a), do not award final ***A1*** mark.

[3 marks]

(f) $\int_1^{3.5} \frac{1}{4x^2 - 4x + 5} dx = \frac{1}{4}\int_{0.5}^{3} \frac{1}{u^2 + 1} du$ ***A1***

> **Note:** ***A1*** for correct change of limits. Award also if they do not change limits but go back to x values when substituting the limit (even if there is an error in the integral).

$\frac{1}{4}\left[\arctan(u)\right]_{0.5}^{3}$ ***(M1)***

$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***A1***

let the integral $= I$

$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***M1***

$\frac{3 - 0.5}{1 + 3 \times 0.5} = \frac{2.5}{2.5} = 1$ ***(M1)A1***

$4I = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{16}$ ***A1AG***

[7 marks]

Total [19 marks]

13. (a) $B\left(6, \frac{2}{3}\right)$ ***(M1)***

$p(4) = \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2$ ***A1***

$\binom{6}{4} = 15$ ***A1***

$= 15 \times \frac{2^4}{3^6} = \frac{80}{243}$ ***AG***

[3 marks]

(b) (i) 2 outcomes for each of the 6 games or $2^6 = 64$ ***R1***

(ii) $(1+x)^6 = \binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$ ***A1***

Note: Accept ${}^nC_r$ notation or $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$

setting $x=1$ in both sides of the expression ***R1***

Note: Do not award ***R1*** if the right hand side is not in the correct form.

$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$ ***AG***

(iii) the total number of outcomes $=$ number of ways Alfred can win no games, plus the number of ways he can win one game *etc*. ***R1***

[4 marks]

(c) (i) Let $\mathrm{P}(x, y)$ be the probability that Alfred wins x games on the first day and y on the second.

$\mathrm{P}(4, 2) = \binom{6}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^2 \times \binom{6}{2} \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^4$ ***M1A1***

$\binom{6}{2}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$ or $\binom{6}{4}^2\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^6$ ***A1***

$r = 2$ or 4, $s = t = 6$

continued …

Question 13 continued

(ii) $\mathrm{P}(\text{Total} = 6) =$
$\mathrm{P}(0, 6) + \mathrm{P}(1, 5) + \mathrm{P}(2, 4) + \mathrm{P}(3, 3) + \mathrm{P}(4, 2) + \mathrm{P}(5, 1) + \mathrm{P}(6, 0)$ ***(M1)***

$= \binom{6}{0}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 + \binom{6}{1}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 + \ldots + \binom{6}{6}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6$ ***A2***

$= \frac{2^6}{3^{12}} \left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 \right)$

Note: Accept any valid sum of 7 probabilities.

(iii) use of $\binom{6}{i} = \binom{6}{6-i}$ ***(M1)***

(can be used either here or in (c)(ii))

$\mathrm{P}(\text{wins 6 out of 12}) = \binom{12}{6} \times \left(\frac{2}{3}\right)^6 \times \left(\frac{1}{3}\right)^6 = \frac{2^6}{3^{12}} \binom{12}{6}$ ***A1***

$\frac{2^6}{3^{12}} \left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 \right) = \frac{2^6}{3^{12}} \binom{12}{6}$ ***A1***

therefore $\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = \binom{12}{6}$ ***AG***

[9 marks]

(d) (i) $\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{n-r} = \sum_{r=0}^{n} r \binom{n}{r} \frac{2^r}{3^n}$

$(a = 2,\ b = 3)$ ***M1A1***

Note: ***M0A0*** for $a = 2$, $b = 3$ without any method.

(ii) $n(1+x)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r x^{r-1}$ ***A1A1***

(sigma notation not necessary)
(if sigma notation used also allow lower limit to be $r = 0$)
let $x = 2$ ***M1***

$n 3^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r 2^{r-1}$

multiply by 2 and divide by 3^n ***(M1)***

$\frac{2n}{3} = \sum_{r=1}^{n} \binom{n}{r} r \frac{2^r}{3^n} \quad \left(= \sum_{r=0}^{n} \binom{n}{r} \frac{2^r}{3^n} \right)$ ***AG***

[6 marks]

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MARKSCHEME

May 2013

MATHEMATICS

Higher Level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2013**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. $\left[\frac{1}{3}(x-2)^3+\ln x-\frac{1}{\pi}\cos \pi x\right]_{(1)}^{(2)}$ ***A1A1A1***

Note: Accept $\frac{1}{3}x^3-2x^2+4x$ in place of $\frac{1}{3}(x-2)^3$.

$=\left(0+\ln 2-\frac{1}{\pi}\cos 2\pi\right)-\left(-\frac{1}{3}+\ln 1-\frac{1}{\pi}\cos \pi\right)$ ***(M1)***

$=\frac{1}{3}+\ln 2-\frac{2}{\pi}$ ***A1A1***

Note: Award ***A1*** for any two terms correct, ***A1*** for the third correct.

Total [6 marks]

2. (a) $\det A=(5\times 2-3\times 3)=1$ ***(A1)***

$A^{-1}=\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}$ ***A1***

[2 marks]

(b) $\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix}$

$=\begin{pmatrix}2\times 3+(-3)\times 2 & 2\times(-5)+(-3)\times(-3)\\ (-3)\times 3+5\times 2 & (-3)\times(-5)+5\times(-3)\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ ***A2***

[3 marks]

Note: Award ***A2*** for all correct, ***A1*** for one error, ***A0*** otherwise.

Total [5 marks]

3. clear attempt at binomial expansion for exponent 5 ***M1***

$$2^5 + 5\times 2^4 \times (-3x) + \frac{5\times 4}{2}\times 2^3 \times (-3x)^2 + \frac{5\times 4\times 3}{6}\times 2^2 \times (-3x)^3$$
$$+\frac{5\times 4\times 3\times 2}{24}\times 2\times (-3x)^4 + (-3x)^5$$ ***(A1)***

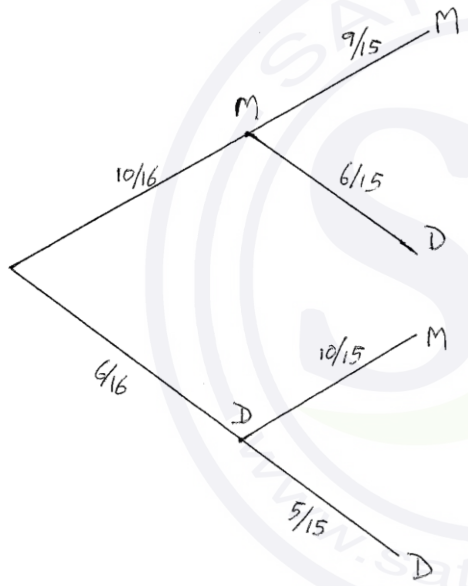
Note: Only award ***M1*** if binomial coefficients are seen.

$$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$$ ***A2***

Note: Award ***A1*** for correct moduli of coefficients and powers. ***A1*** for correct signs.

Total [4 marks]

4. (a)



A1A1A1

[3 marks]

Note: Award ***A1*** for the initial level probabilities, ***A1*** for each of the second level branch probabilities.

(b) $\frac{10}{16}\times\frac{9}{15}+\frac{6}{16}\times\frac{5}{15}$ ***(M1)***

$= \frac{120}{240} \left(= \frac{1}{2}\right)$ ***A1***

[2 marks]

Total [5 marks]

5. (a) $\frac{dy}{dx} = \frac{(x+\cos x)(\cos x - x\sin x) - x\cos x(1-\sin x)}{(x+\cos x)^2}$ ***M1A1A1***

> **Note:** Award ***M1*** for attempt at differentiation of a quotient and a product condoning sign errors in the quotient formula and the trig differentiations, ***A1*** for correct derivative of "*u*", ***A1*** for correct derivative of "*v*".

$= \frac{x\cos x + \cos^2 x - x^2 \sin x - x\cos x \sin x - x\cos x + x\cos x \sin x}{(x+\cos x)^2}$ ***A1***

$= \frac{\cos^2 x - x^2 \sin x}{(x+\cos x)^2}$ ***AG***

[4 marks]

(b) the derivative has value -1 ***(A1)***

the equation of the tangent line is $(y-0) = (-1)\left(x - \frac{\pi}{2}\right) \left(y = \frac{\pi}{2} - x\right)$ ***M1A1***

[3 marks]

Total [7 marks]

6. for the first series $\frac{a}{1-r} = 76$ ***A1***

for the second series $\frac{a}{1-r^3} = 36$ ***A1***

attempt to eliminate *a e.g.* $\frac{76(1-r)}{1-r^3} = 36$ ***M1***

simplify and obtain $9r^2 + 9r - 10 = 0$ ***(M1)A1***

> **Note:** Only award the ***M1*** if a quadratic is seen.

obtain $r = \frac{12}{18}$ and $-\frac{30}{18}$ ***(A1)***

$r = \frac{12}{18}\left(= \frac{2}{3} = 0.666\ldots\right)$ ***A1***

> **Note:** Award ***A0*** if the extra value of *r* is given in the final answer.

Total [7 marks]

7. (a) $|z_1| = \sqrt{10}$; $\arg(z_2) = -\frac{3\pi}{4}$ $\left(\text{accept } \frac{5\pi}{4}\right)$ ***A1A1***

[2 marks]

(b) $|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus ***(M1)(A1)***

attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root ***M1***

obtain $\alpha = 2$ at minimum ***(A1)***

state $\sqrt{2}$ as final answer ***A1***

[5 marks]

Total [7 marks]

8. (a) attempt at implicit differentiation ***M1***

EITHER

$$\frac{2x}{y} - \frac{x^2}{y^2}\frac{dy}{dx} - 2 = \frac{1}{y}\frac{dy}{dx}$$ ***A1A1***

Note: Award ***A1*** for each side.

$$\frac{dy}{dx} = \frac{\frac{2x}{y} - 2}{\frac{1}{y} + \frac{x^2}{y^2}} \left(= \frac{2xy - 2y^2}{x^2 + y}\right)$$ ***A1***

OR

after multiplication by y

$$2x - 2y - 2x\frac{dy}{dx} = \frac{dy}{dx}\ln y + y\frac{1}{y}\frac{dy}{dx}$$ ***A1A1***

Note: Award ***A1*** for each side.

$$\frac{dy}{dx} = \frac{2(x-y)}{1 + 2x + \ln y}$$ ***A1***

[4 marks]

(b) for $y = 1$, $x^2 - 2x = 0$

$x = (0 \text{ or})\ 2$ ***A1***

for $x = 2$, $\frac{dy}{dx} = \frac{2}{5}$ ***A1***

[2 marks]

Total [6 marks]

9. (a) **EITHER**

$$f(x)-1=\frac{1+3^{-x}}{3^{x}-3^{-x}}$$ *M1A1*

> 0 as both numerator and denominator are positive *R1*

OR

$$3^{x}+1>3^{x}>3^{x}-3^{-x}$$ *M1A1*

Note:	Accept a convincing valid argument the numerator is greater than the denominator.

numerator and denominator are positive *R1*

hence $f(x)>1$ *AG*

[3 marks]

(b) one line equation to solve, for example, $4(3^{x}-3^{-x})=3^{x}+1$, or equivalent *A1*

$(3y^{2}-y-4=0)$

attempt to solve a three-term equation *M1*

obtain $y=\frac{4}{3}$ *A1*

$x=\log_{3}\left(\frac{4}{3}\right)$ or equivalent *A1*

Note:	Award ***A0*** if an extra solution for x is given.

[4 marks]

Total [7 marks]

10. (a) attempt at use of $\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ *M1*

$$\frac{1}{p}=\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\ \left(=\frac{1}{3}\right)$$ *A1*

$p=3$ *A1*

Note:	the value of p needs to be stated for the final mark.

[3 marks]

(b) $$\tan\left(\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)\right)=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=1$$ *M1A1*

$$\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)=\frac{\pi}{4}$$ *A1*

[3 marks]

Total [6 marks]

SECTION B

11. (a) (i) $\vec{AB} = \vec{OB} - \vec{OA} = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ (or in column vector form) ***(A1)***

Note: Award ***A1*** if any one of the vectors, or its negative, representing the sides of the triangle is seen.

$|\vec{AB}| = |5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}| = \sqrt{30}$

$|\vec{BC}| = |-\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}| = \sqrt{11}$

$|\vec{CA}| = |-4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}| = \sqrt{33}$ ***A2***

Note: Award ***A1*** for two correct and ***A0*** for one correct.

(ii) **METHOD 1**

$\cos \text{BAC} = \dfrac{20+4+2}{\sqrt{30}\sqrt{33}}$ ***M1A1***

Note: Award ***M1*** for an attempt at the use of the scalar product for two vectors representing the sides AB and AC, or their negatives, ***A1*** for the correct computation using their vectors.

$= \dfrac{26}{\sqrt{990}} \quad \left(= \dfrac{26}{3\sqrt{110}}\right)$ ***A1***

Note: Candidates who use the modulus need to justify it – the angle is not stated in the question to be acute.

METHOD 2

using the cosine rule

$\cos \text{BAC} = \dfrac{30+33-11}{2\sqrt{30}\sqrt{33}}$ ***M1A1***

$= \dfrac{26}{\sqrt{990}} \quad \left(= \dfrac{26}{3\sqrt{110}}\right)$ ***A1***

[6 marks]

(b) (i) $\vec{BC} \times \vec{CA} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ -4 & 4 & 1 \end{vmatrix}$ ***A1***

$= \big((-3)\times 1 - 1\times 4\big)\boldsymbol{i} + \big(1\times(-4) - (-1)\times 1\big)\boldsymbol{j} + \big((-1)\times 4 - (-3)\times(-4)\big)\boldsymbol{k}$ ***M1A1***

$= -7\boldsymbol{i} - 3\boldsymbol{j} - 16\boldsymbol{k}$ ***AG***

(ii) the area of $\Delta\text{ABC} = \dfrac{1}{2}\left|\vec{BC} \times \vec{CA}\right|$ ***(M1)***

$\dfrac{1}{2}\sqrt{(-7)^2 + (-3)^2 + (-16)^2}$ ***A1***

$= \dfrac{1}{2}\sqrt{314}$ ***AG***

[5 marks]

(c) attempt at the use of "$(\boldsymbol{r}-\boldsymbol{a})\bullet\boldsymbol{n}=0$" *(M1)*

using $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{a}=\overrightarrow{\text{OA}}$ and $\boldsymbol{n}=-7\boldsymbol{i}-3\boldsymbol{j}-16\boldsymbol{k}$ *(A1)*

$7x+3y+16z=47$ ***A1***

Note:	Candidates who adopt a 2-parameter approach should be awarded, ***A1*** for correct 2-parameter equations for x, y and z; ***M1*** for a serious attempt at elimination of the parameters; ***A1*** for the final Cartesian equation.

[3 marks]

(d) $\boldsymbol{r}=\overrightarrow{\text{OA}}+t\overrightarrow{\text{AB}}$ (or equivalent) ***M1***

$\boldsymbol{r}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$ ***A1***

Note:	Award ***M1A0*** if "$\boldsymbol{r}=$" is missing.

Note:	Accept forms of the equation starting with B or with the direction reversed.

[2 marks]

(e) (i) $\overrightarrow{\text{OD}}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$

statement that $\overrightarrow{\text{OD}}\bullet\overrightarrow{\text{BC}}=0$ *(M1)*

$$\begin{pmatrix}-1+5t\\2-t\\3-2t\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\1\end{pmatrix}=0$$ ***A1***

$-2-4t=0$ or $t=-\frac{1}{2}$ ***A1***

coordinates of D are $\left(-\frac{7}{2},\frac{5}{2},4\right)$ ***A1***

Note:	Different forms of $\overrightarrow{\text{OD}}$ give different values of t, but the same final answer.

(ii) $t<0 \Rightarrow$ D is not between A and B ***R1***

[5 marks]

Total [21 marks]

12. (a) by division or otherwise

$f(x) = 2 - \frac{5}{x+2}$ ***A1A1***

[2 marks]

(b) $f'(x) = \frac{5}{(x+2)^2}$ ***A1***

> 0 as $(x+2)^2 > 0$ (on D) ***R1AG***

Note:	Do not penalise candidates who use the original form of the function to compute its derivative.

[2 marks]

(c) $S = \left[-3, \frac{3}{2}\right]$ ***A2***

Note:	Award ***A1A0*** for the correct endpoints and an open interval.

[2 marks]

continued …

Question 12 continued

(d) (i) **EITHER**

rearrange $y = f(x)$ to make x the subject ***M1***

obtain one-line equation, *e.g.* $2x - 1 = xy + 2y$ ***A1***

$x = \dfrac{2y+1}{2-y}$ ***A1***

OR

interchange x and y ***M1***

obtain one-line equation, *e.g.* $2y - 1 = xy + 2x$ ***A1***

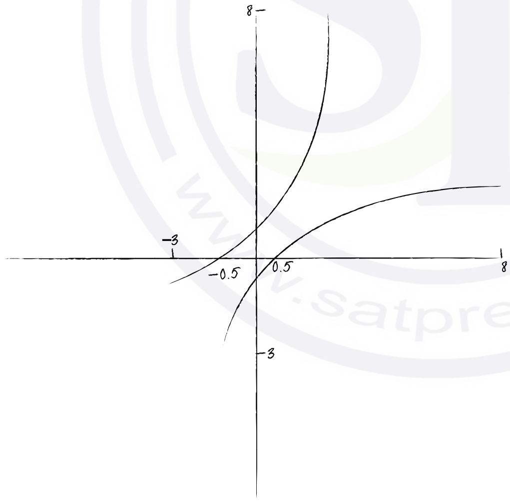
$y = \dfrac{2x+1}{2-x}$ ***A1***

THEN

$f^{-1}(x) = \dfrac{2x+1}{2-x}$ ***A1***

Note:	Accept $\dfrac{5}{2-x} - 2$

(ii), (iii)



A1A1A1A1

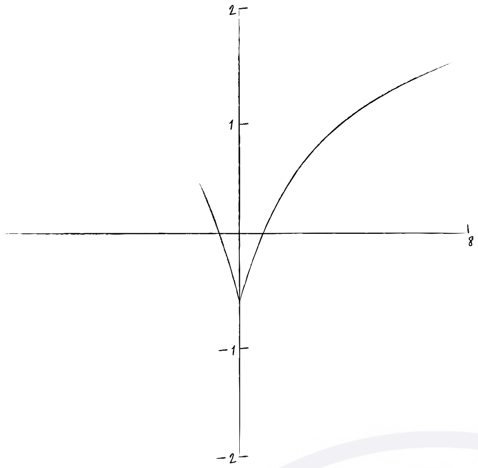
[8 marks]

Note: Award ***A1*** for correct shape of $y = f(x)$.
Award ***A1*** for x intercept $\dfrac{1}{2}$ seen. Award ***A1*** for y intercept $-\dfrac{1}{2}$ seen.
Award ***A1*** for the graph of $y = f^{-1}(x)$ being the reflection of $y = f(x)$ in the line $y = x$. Candidates are not required to indicate the full domain, but $y = f(x)$ should not be shown approaching $x = -2$. Candidates, in answering (iii), can **FT** on their sketch in (ii).

continued …

Question 12 continued

(e) (i)



A1A1A1

Note: ***A1*** for correct sketch $x > 0$, ***A1*** for symmetry, ***A1*** for correct domain (from -1 to $+8$).

Note: Candidates can **FT** on their sketch in (d)(ii).

(ii) attempt to solve $f(x) = -\frac{1}{4}$ ***(M1)***

obtain $x = \frac{2}{9}$ ***A1***

use of symmetry or valid algebraic approach ***(M1)***

obtain $x = -\frac{2}{9}$ ***A1***

[7 marks]

Total [21 marks]

13. (a) (i) $z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right)$, $z_3 = 2\text{cis}\left(-\frac{\pi}{2}\right)$ or $2\text{cis}\left(\frac{3\pi}{2}\right)$ ***A1A1A1***

Note:	Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note:	Accept arguments given in rational degrees, except where exponential form is used.

(ii) the points lie on a circle of radius 2 centre the origin ***A1***

differences are all $\frac{2\pi}{3}$ (mod 2π) ***A1***

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral ***R1AG***

Note:	Accept an approach based on a clearly marked diagram.

(iii) $z_1^{3n} + z_2^{3n} = 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\text{cis}\left(\frac{5n\pi}{2}\right)$ ***M1***

$= 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ ***A1***

$2z_3^{3n} = 2 \times 2^{3n}\text{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ ***A1AG***

[9 marks]

(b) (i) attempt to obtain **seven** solutions in modulus argument form ***M1***

$z = \text{cis}\left(\frac{2k\pi}{7}\right), k = 0, 1\ldots 6$ ***A1***

(ii) w has argument $\frac{2\pi}{7}$ and $1+w$ has argument ϕ,

then $\tan(\phi) = \dfrac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}$ ***M1***

$= \dfrac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}$ ***A1***

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$ ***A1***

Note:	Accept alternative approaches.

continued …

Question 13 continued

(iii) since roots occur in conjugate pairs, ***(R1)***

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$ ***A1***

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ ***AG***

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$ ***A1***

and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1$ ***A1***

[9 marks]

Total [18 marks]

N12/5/MATHL/HP1/ENG/TZ0/XX/M

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MARKSCHEME

November 2012

MATHEMATICS

Higher Level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking Nov 2012**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. $\sin\alpha = \sqrt{1-\left(-\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$ ***(M1)A1***

attempt to use double angle formula ***M1***

$\sin 2\alpha = 2\frac{\sqrt{7}}{4}\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$ ***A1***

Note: $\frac{\sqrt{7}}{4}$ seen would normally be awarded ***M1A1***.

[4 marks]

2. $\left(\frac{x}{y}-\frac{y}{x}\right)^4 = \left(\frac{x}{y}\right)^4 + 4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right) + 6\left(\frac{x}{y}\right)^2\left(-\frac{y}{x}\right)^2 + 4\left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3 + \left(-\frac{y}{x}\right)^4$ ***(M1)(A1)***

Note: Award ***M1*** for attempt to expand and ***A1*** for correct unsimplified expansion.

$= \frac{x^4}{y^4} - 4\frac{x^2}{y^2} + 6 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \quad \left(= \frac{x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8}{x^4y^4}\right)$ ***A1A1***

Note: Award ***A1*** for powers, ***A1*** for coefficients and signs.

Note: Final two ***A*** marks are independent of first ***A*** mark.

[4 marks]

3. (a) **METHOD 1**

$f(x) = (x+1)(x-1)(x-2)$ ***M1***

$= x^3 - 2x^2 - x + 2$ ***A1A1A1***

$a = -2$, $b = -1$ and $c = 2$

METHOD 2

from the graph or using $f(0) = 2$

$c = 2$ ***A1***

setting up linear equations using $f(1) = 0$ and $f(-1) = 0$ (or $f(2) = 0$) ***M1***

obtain $a = -2$, $b = -1$ ***A1A1***

[4 marks]

(b) (i) $(1, 0)$, $(3, 0)$ and $(4, 0)$ ***A1***

(ii) $g(0)$ occurs at $3f(-2)$ ***(M1)***

$= -36$ ***A1***

[3 marks]

Total [7 marks]

4. (a) $f'(x) = (\ln x)^2 + \dfrac{2x\ln x}{x}\left(= (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)\right)$ ***M1A1***

$f'(x) = 0$ ($\Rightarrow x = 1,\ x = e^{-2}$) ***M1***

Note: Award ***M1*** for an attempt to solve $f'(x) = 0$.

$A(e^{-2}, 4e^{-2})$ **and** $B(1, 0)$ ***A1A1***

Note: The final ***A1*** is independent of prior working.

[5 marks]

(b) $f''(x) = \dfrac{2}{x}(\ln x + 1)$ ***A1***

$f''(x) = 0\ \left(\Rightarrow x = e^{-1}\right)$ ***(M1)***

inflexion point (e^{-1}, e^{-1}) ***A1***

Note: ***M1*** for attempt to solve $f''(x) = 0$.

[3 marks]

Total [8 marks]

5. (a) 0 ***A1***

[1 mark]

(b) $\int_0^1 f(x)\,dx = 1$ ***(M1)***

$\Rightarrow a = \dfrac{1}{\int_0^1 e^{-x}\,dx}$

$\Rightarrow a = \dfrac{1}{\left[-e^{-x}\right]_0^1}$ ***A1***

$\Rightarrow a = \dfrac{e}{e-1}$ (or equivalent) ***A1***

Note: Award first ***A1*** for correct integration of $\int e^{-x}dx$. This ***A1*** is independent of previous ***M*** mark.

[3 marks]

(c) $E(X) = \int_0^1 xf(x)\,dx\left(= a\int_0^1 xe^{-x}\,dx\right)$ ***M1***

attempt to integrate by parts ***M1***

$= a\left[-xe^{-x} - e^{-x}\right]_0^1$ ***(A1)***

$= a\left(\dfrac{e-2}{e}\right)$

$= \dfrac{e-2}{e-1}$ (or equivalent) ***A1***

[4 marks]

Total [8 marks]

6. **METHOD 1**

(a) $\det\begin{pmatrix}1 & 3 & a-1\\ 2 & 2 & a-2\\ 3 & 1 & a-3\end{pmatrix}$ ***M1***

$=1\big(2(a-3)-(a-2)\big)-3\big(2(a-3)-3(a-2)\big)+(a-1)(2-6)$

(or equivalent) ***A1***

$=0$ (therefore there is no unique solution) ***A1***

[3 marks]

(b) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right):\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ ***M1A1***

$:\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ ***A1***

$b=1$ ***A1*** ***N2***

Note: Award ***M1*** for an attempt to use row operations.

[4 marks]

METHOD 2

(a) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right):\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ ***M1A1***

$:\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ (and 3 zeros imply no unique solution) ***A1***

[3 marks]

(b) $b=1$ ***A4***

Note: Award ***A4*** only if "$b-1$" seen in (a).

[4 marks]

Total [7 marks]

7. (a) attempt to apply cosine rule *M1*

$4^2 = 6^2 + QR^2 - 2 \cdot QR \cdot 6\cos 30^\circ$ $\left(\textbf{or } QR^2 - 6\sqrt{3}\ QR + 20 = 0\right)$ *A1*

$QR = 3\sqrt{3} + \sqrt{7}$ or $QR = 3\sqrt{3} - \sqrt{7}$ *A1A1*

[4 marks]

(b) **METHOD 1**

$k \geq 6$ *A1*

$k = 6\sin 30^\circ = 3$ *M1A1*

Note: The ***M1*** in (b) is for recognizing the right-angled triangle case.

METHOD 2

$k \geq 6$ *A1*

use of discriminant: $108 - 4\left(36 - k^2\right) = 0$ *M1*

$k = 3$ *A1*

Note: $k = \pm 3$ is ***M1A0.***

[3 marks]

Total [7 marks]

8. (a) attempt to differentiate implicitly *M1*

$2x + \cos y \frac{dy}{dx} - y - x\frac{dy}{dx} = 0$ *A1A1*

Note: ***A1*** for differentiating x^2 and $\sin y$; ***A1*** for differentiating xy.

substitute x and y by π *M1*

$2\pi - \frac{dy}{dx} - \pi - \pi\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{\pi}{1+\pi}$ *M1A1*

Note: ***M1*** for attempt to make dy/dx the subject. This could be seen earlier.

[6 marks]

(b) $\theta = \frac{\pi}{4} - \arctan\frac{\pi}{1+\pi}$ (or seen the other way) *M1*

$\tan\theta = \tan\left(\frac{\pi}{4} - \arctan\frac{\pi}{1+\pi}\right) = \frac{1 - \frac{\pi}{1+\pi}}{1 + \frac{\pi}{1+\pi}}$ *M1A1*

$\tan\theta = \frac{1}{1+2\pi}$ *AG*

[3 marks]

Total [9 marks]

9. **METHOD 1**

(a) $9t_A = 7 - 4t_B$ and
$3 - 6t_A = -6 + 7t_B$ ***M1A1***

solve simultaneously

$t_A = \frac{1}{3}, t_B = 1$ ***A1***

Note: Only need to see one time for the ***A1***.

therefore meet at $(3, 1)$ ***A1***

[4 marks]

(b) boats do not collide because the two times $\left(t_A = \frac{1}{3}, t_B = 1\right)$ ***(A1)***

are different ***R1***

[2 marks]

Total [6 marks]

METHOD 2

(a) path of A is a straight line: $y = -\frac{2}{3}x + 3$ ***M1A1***

Note: Award ***M1*** for an attempt at simultaneous equations.

path of B is a straight line: $y = -\frac{7}{4}x + \frac{25}{4}$ ***A1***

$-\frac{2}{3}x + 3 = -\frac{7}{4}x + \frac{25}{4}$ $(\Rightarrow x = 3)$
so the common point is $(3, 1)$ ***A1***

[4 marks]

(b) for boat A, $9t = 3 \Rightarrow t = \frac{1}{3}$ and for boat B, $7 - 4t = 3 \Rightarrow t = 1$ ***A1***

times are different so boats do not collide ***R1AG***

[2 marks]

Total [6 marks]

SECTION B

10. (a) (i) $z_1 = 2\sqrt{3}\,\text{cis}\,\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\text{i}$ ***A1***

(ii) $z_1 + z_2 = -2\sqrt{3}\text{i} - 1 + \sqrt{3}\text{i} = -1 - \sqrt{3}\text{i}$ ***A1***

$(z_1 + z_2)^* = -1 + \sqrt{3}\text{i}$ ***A1***

[3 marks]

(b) (i) $|z_2| = 2$

$\tan\theta = -\sqrt{3}$ ***(M1)***

z_2 lies on the second quadrant

$\theta = \arg z_2 = \frac{2\pi}{3}$

$z_2 = 2\text{cis}\frac{2\pi}{3}$ ***A1A1***

(ii) attempt to use De Moivre's theorem ***M1***

$z = \sqrt[3]{2}\,\text{cis}\,\dfrac{\frac{2\pi}{3} + 2k\pi}{3},\ k = 0, 1 \text{ and } 2$

$z = \sqrt[3]{2}\,\text{cis}\frac{2\pi}{9},\ \sqrt[3]{2}\,\text{cis}\frac{8\pi}{9},\ \sqrt[3]{2}\,\text{cis}\frac{14\pi}{9}\left(= \sqrt[3]{2}\,\text{cis}\left(\frac{-4\pi}{9}\right)\right)$ ***A1A1***

Note: Award ***A1*** for modulus, ***A1*** for arguments.

Note: Allow equivalent forms for z.

[6 marks]

(c) (i) **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3\left(\Rightarrow z = \pm\sqrt{3}\text{i}\right)$ ***M1***

$z = \sqrt{3}\,\text{cis}\frac{\pi}{2}$ or $z_1 = \sqrt{3}\,\text{cis}\frac{3\pi}{2}\left(= \sqrt{3}\,\text{cis}\left(\frac{-\pi}{2}\right)\right)$ ***A1A1***

so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}\left(= \frac{-\pi}{2}\right)$

Note: Accept $r\text{cis}(\theta)$ form.

METHOD 2

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3 \Rightarrow z^2 = 3\text{cis}\left((2n+1)\pi\right)$ ***M1***

$r^2 = 3 \Rightarrow r = \sqrt{3}$ ***A1***

$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (as $0 \le \theta < 2\pi$) ***A1***

Note: Accept $r\text{cis}(\theta)$ form.

continued …

Question 10 continued

(ii) **METHOD 1**

$z = -\dfrac{1}{2cis\frac{2\pi}{3}} \Rightarrow z = \dfrac{\text{cis}\pi}{2\text{cis}\frac{2\pi}{3}}$ ***M1***

$\Rightarrow z = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$

so $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ ***A1A1***

METHOD 2

$z_1 = -\dfrac{1}{-1+\sqrt{3}\text{i}} \Rightarrow z_1 = -\dfrac{-1-\sqrt{3}\text{i}}{\left(-1+\sqrt{3}\text{i}\right)\left(-1-\sqrt{3}\text{i}\right)}$ ***M1***

$z = \dfrac{1+\sqrt{3}\text{i}}{4} \Rightarrow z = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$

so $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ ***A1A1***

[6 marks]

(d) $\dfrac{z_1}{z_2} = \sqrt{3}\,\text{cis}\dfrac{5\pi}{6}$ ***(A1)***

$\left(\dfrac{z_1}{z_2}\right)^n = \sqrt{3}^n\text{cis}\dfrac{5n\pi}{6}$ ***A1***

equating imaginary part to zero and attempting to solve ***M1***

obtain $n = 12$ ***A1***

Note: Working which only includes the argument is valid.

[4 marks]

Total [19 marks]

11. (a) $\boldsymbol{A}=\begin{pmatrix}2&1\\1&1\end{pmatrix}\Rightarrow \boldsymbol{A}^2=\begin{pmatrix}5&3\\3&2\end{pmatrix}$ ***A1***

$\boldsymbol{A}=\begin{pmatrix}2&1\\1&1\end{pmatrix}\Rightarrow 3\boldsymbol{A}=\begin{pmatrix}6&3\\3&3\end{pmatrix}$ ***(A1)***

$(\boldsymbol{A}^2-3\boldsymbol{A})^2=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}^2$ ***A1***

$=\boldsymbol{I}$ ***AG***

[3 marks]

(b) (i) $\begin{pmatrix}a&a-1\\b&b\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}0\\-2\end{pmatrix}\Rightarrow\begin{cases}a+3a-3=0\\b+3b=-2\end{cases}$ ***M1(A1)***

$a=\frac{3}{4}$ and $b=-\frac{1}{2}$ ***A1***

(ii) **METHOD 1**

$\boldsymbol{A}^{-1}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}0\\-2\end{pmatrix}\Rightarrow \boldsymbol{A}\begin{pmatrix}0\\-2\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$ ***M1***

$\begin{pmatrix}a&a-1\\b&b\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}\Rightarrow\begin{cases}-2a+2=1\\-2b=3\end{cases}$ ***A1***

$a=\frac{1}{2}$ and $b=-\frac{3}{2}$ ***A1***

METHOD 2

serious attempt at finding $\boldsymbol{A}^{-1}$ ***M1***

$\left(\boldsymbol{A}^{-1}=\frac{1}{b}\begin{pmatrix}b&1-a\\-b&a\end{pmatrix}\right)$

obtain $b+3-3a=0$ and $-b+3a=-2b$ or equivalent linear equations ***A1***

$a=\frac{1}{2}$ and $b=-\frac{3}{2}$ ***A1***

[6 marks]

continued …

Question 11 continued

(c) (i) **METHOD 1**

$$\underbrace{\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ ***M1***

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{A}^{-1} = \frac{1}{b}\begin{pmatrix} b & -a+1 \\ -b & a \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-a+1}{b} \\ \frac{a}{b} \end{pmatrix}$$ ***A1***

METHOD 2

$ax+(a-1)y=0$ and $bx+by=1$ ***A1***

attempt to solve ***M1***

obtain $\left(\frac{1-a}{b}, \frac{a}{b}\right)$ ***A1A1***

(ii) gradient of l_1 is $\frac{-a}{a-1}$ and gradient of l_2 is -1 ***A1A1***

the lines are perpendicular $\Rightarrow \frac{-a}{a-1} = 1 \Rightarrow a = \frac{1}{2}$ ***M1A1***

so they intersect at $\left(\frac{1}{2b}, \frac{1}{2b}\right)$ ***A1***

[9 marks]

Total [18 marks]

12. (a) $(f \circ f)(x) = f\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$ ***M1A1***

$(f \circ f)(x) = \frac{x}{4-3x}$ ***A1***

[3 marks]

(b) $P(n): \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$

$P(1): \ f(x) = F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1 - (2^1 - 1)x} = \frac{x}{2-x}$ ***A1A1***

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$ ***M1***

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right)$ ***(M1)***

$= f\left(\frac{x}{2^k - (2^k - 1)x}\right) = \frac{\frac{x}{2^k - (2^k - 1)x}}{2 - \frac{x}{2^k - (2^k - 1)x}}$ ***A1***

$= \frac{x}{2\left(2^k - (2^k - 1)x\right) - x} = \frac{x}{2^{k+1} - (2^{k+1} - 2)x - x}$ ***A1***

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$ ***(M1)***

$= F_k\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2^k - (2^k - 1)\frac{x}{2-x}}$ ***A1***

$= \frac{x}{2^{k+1} - 2^k x - 2^k x + x}$ ***A1***

THEN

$= \frac{x}{2^{k+1} - (2^{k+1} - 1)x} = F_{k+1}(x)$ ***A1***

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ ***R1***

[8 marks]

continued ...

Question 12 continued

(c) **METHOD 1**

$x = \dfrac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$ ***M1A1***

$\Rightarrow 2^n x = \left((2^n - 1)x + 1\right)y \Rightarrow y = \dfrac{2^n x}{(2^n - 1)x + 1}$ ***A1***

$F_n^{-1}(x) = \dfrac{2^n x}{(2^n - 1)x + 1}$ ***A1***

$F_n^{-1}(x) = \dfrac{x}{\dfrac{2^n - 1}{2^n}x + \dfrac{1}{2^n}}$ ***M1***

$F_n^{-1}(x) = \dfrac{x}{(1 - 2^{-n})x + 2^{-n}}$ ***A1***

$F_n^{-1}(x) = \dfrac{x}{2^{-n} - (2^{-n} - 1)x}$ ***AG***

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$ ***M1***

$= F_{-n}\left(\dfrac{x}{2^n - (2^n - 1)x}\right) = \dfrac{\dfrac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\dfrac{x}{2^n - (2^n - 1)x}}$ ***A1A1***

$= \dfrac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \dfrac{x}{1} = x$ ***A1AG***

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$ ***M1***

$= F_n\left(\dfrac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \dfrac{\dfrac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\dfrac{x}{2^{-n} - (2^{-n} - 1)x}}$ ***A1A1***

$= \dfrac{x}{2^n\left(2^{-n} - (2^{-n} - 1)x\right) - (2^n - 1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \dfrac{x}{1} = x$ ***A1AG***

[6 marks]

continued …

Question 12 continued

(d) (i) $F_n(0) = 0,\ F_n(1) = 1$ ***A1***

(ii) **METHOD 1**

$2^n - (2^n - 1)x - 1 = (2^n - 1)(1 - x)$ ***(M1)***

> 0 if $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***A1***

so $2^n - (2^n - 1)x > 1$ and $F_n(x) = \dfrac{x}{2^n - (2^n - 1)x} < \dfrac{x}{1}\ (< x)$ ***R1***

$F_n(x) = \dfrac{x}{2^n - (2^n - 1)x} < x$ for $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***AG***

METHOD 2

$\dfrac{x}{2^n - (2^n - 1)x} < x \Leftrightarrow 2^n - (2^n - 1)x > 1$ ***(M1)***

$\Leftrightarrow (2^n - 1)x < 2^n - 1$ ***A1***

$\Leftrightarrow x < \dfrac{2^n - 1}{2^n - 1} = 1$ true in the interval $]0, 1[$ ***R1***

(iii) $B_n = 2\left(A_n - \dfrac{1}{2}\right)\ (= 2A_n - 1)$ ***(M1)A1***

[6 marks]

Total [23 marks]

M12/5/MATHL/HP1/ENG/TZ1/XX/M

International Baccalaureate®
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Bachillerato Internacional

MARKSCHEME

May 2012

MATHEMATICS

Higher Level

Paper 1

14 pages

Instructions to Examiners

Abbreviations

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3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

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*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

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7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

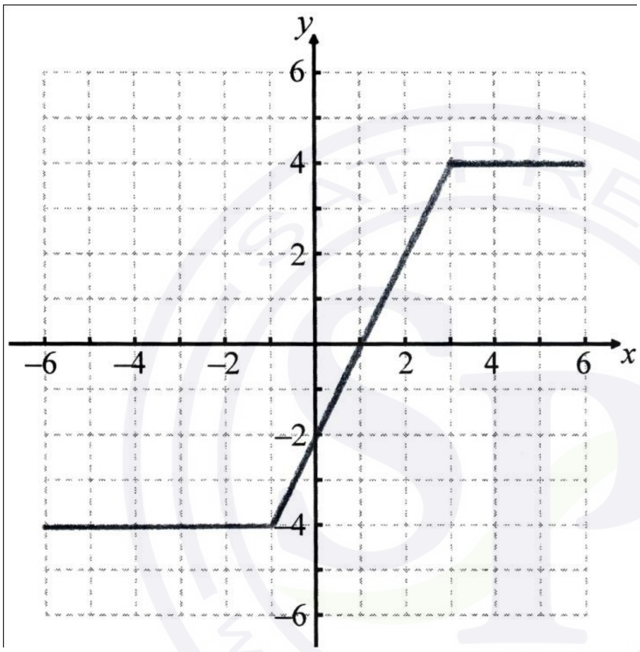
1. $u_1 = \frac{1}{3}k$, $r = \frac{1}{3}$ ***(A1)(A1)***

$7 = \dfrac{\frac{1}{3}k}{1-\frac{1}{3}}$ ***M1***

$k = 14$ ***A1***

[4 marks]

2. (a)



M1A1A1A1

Note: Award ***M1*** for any of the three sections completely correct, ***A1*** for each correct segment of the graph.

[4 marks]

(b) (i) 0 ***A1***

(ii) 2 ***A1***

(iii) finding area of rectangle ***(M1)***

–4 ***A1***

Note: Award ***M1A0*** for the answer 4.

[4 marks]

Total [8 marks]

3. $z_1 = 2a\,\text{cis}\left(\frac{\pi}{3}\right),\ z_2 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ ***M1A1A1***

EITHER

$\left(\frac{z_1}{z_2}\right)^6 = \frac{2^6 a^6 \text{cis}(0)}{\sqrt{2}^6 \text{cis}\left(\frac{\pi}{2}\right)}\ \left(= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)\right)$ ***M1A1A1***

OR

$\left(\frac{z_1}{z_2}\right)^6 = \left(\frac{2a}{\sqrt{2}}\text{cis}\left(\frac{7\pi}{12}\right)\right)^6$ ***M1A1***

$= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)$ ***A1***

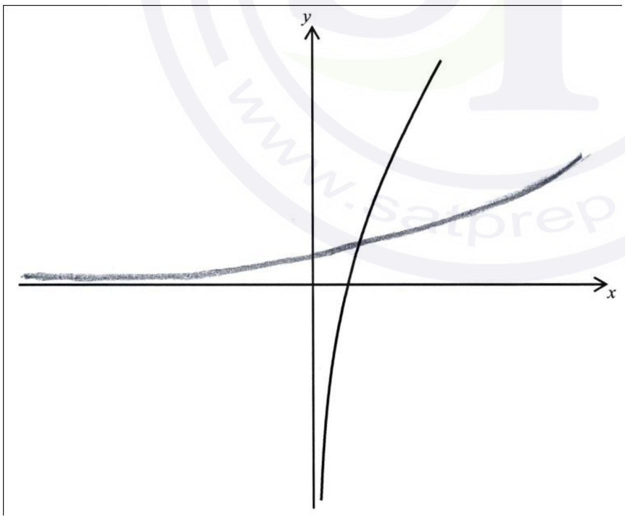
THEN

$= -8a^6 \text{i}$ ***A1***

Note: Accept equivalent angles, in radians or degrees.
Accept alternate answers without cis e.g. $= \frac{8a^6}{\text{i}}$

[7 marks]

4. (a)



A1A1

Note: Award ***A1*** for correct asymptote with correct behaviour and ***A1*** for shape.

[2 marks]

(b) intersect on $y = x$ ***(M1)***

$x + \ln x = x \Rightarrow \ln x = 0$ ***(A1)***

intersect at $(1, 1)$ ***A1A1***

[4 marks]

Total [6 marks]

5. (a) $\cos x = 0,\ \sin x = 0$ ***(M1)***

$x = \frac{n\pi}{2}, n \in \mathbb{Z}$ ***A1***

[2 marks]

(b) **EITHER**

$\frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}$ ***M1A1***

$= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x}$ ***A1A1***

$= 2$ ***A1***

OR

$\frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x} - \frac{\cos 2x \cos x - \sin 2x \sin x}{\cos x}$ ***M1***

$= \frac{2\sin x \cos^2 x + 2\cos^2 x \sin x - \sin x}{\sin x} - \frac{2\cos^3 x - \cos x - 2\sin^2 x \cos x}{\cos x}$ ***A1A1***

$= 4\cos^2 x - 1 - 2\cos^2 x + 1 + 2\sin^2 x$ ***A1***

$= 2\cos^2 x + 2\sin^2 x$

$= 2$ ***A1***

[5 marks]

Total [7 marks]

6. (a) $\int_{\frac{1}{6}}^{1} \frac{k}{x} - \frac{1}{x}\,\mathrm{d}x = (k-1)[\ln x]_{\frac{1}{6}}^{1}$ ***M1A1***

Note: Award ***M1*** for $\int \frac{k}{x} - \frac{1}{x}\,\mathrm{d}x$ or $\int \frac{1}{x} - \frac{k}{x}\,\mathrm{d}x$ and ***A1*** for $(k-1)\ln x$ seen in part (a) or later in part (b).

$= (1-k)\ln\frac{1}{6}$ ***A1***

[3 marks]

(b) $\int_{1}^{\sqrt{6}} \frac{k}{x} - \frac{1}{x}\,\mathrm{d}x = (k-1)[\ln x]_{1}^{\sqrt{6}}$ ***(A1)***

Note: Award ***A1*** for correct change of limits.

$= (k-1)\ln\sqrt{6}$ ***A1***

[2 marks]

continued …

Question 6 continued

(c) $(1-k)\ln\frac{1}{6}=(k-1)\ln 6$ ***A1***

$(k-1)\ln\sqrt{6}=\frac{1}{2}(k-1)\ln 6$ ***A1***

Note: This simplification could have occurred earlier, and marks should still be awarded.

ratio is 2 (or 2:1) ***A1***

[3 marks]

Total [8 marks]

7. $\sqrt{x^2+y^2}+x+y\mathrm{i}=6-2\mathrm{i}$ ***(A1)***

equating real and imaginary parts ***M1***

$y=-2$ ***A1***

$\sqrt{x^2+4}+x=6$ ***A1***

$x^2+4=(6-x)^2$ ***M1***

$-32=-12x\Rightarrow x=\frac{8}{3}$ ***A1***

[6 marks]

8. $\log_3\left(\frac{9}{x+7}\right)=\log_3\frac{1}{2x}$ ***M1M1A1***

Note: Award ***M1*** for changing to single base, ***M1*** for incorporating the 2 into a log and ***A1*** for a correct equation with maximum one log expression each side.

$x+7=18x$ ***M1***

$x=\frac{7}{17}$ ***A1***

[5 marks]

9. $4x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x}{y}$ ***M1A1***

Note: Allow follow through on incorrect $\frac{\mathrm{d}y}{\mathrm{d}x}$ from this point.

gradient of normal at (a, b) is $\frac{b}{2a}$

Note: No further A marks are available if a general point is not used

equation of normal at (a, b) is $y-b=\frac{b}{2a}(x-a)$ $\left(\Rightarrow y=\frac{b}{2a}x+\frac{b}{2}\right)$ ***M1A1***

substituting $(1, 0)$ ***M1***

$b=0$ or $a=-1$ ***A1A1***

four points are $(3, 0), (-3, 0), (-1, 4), (-1, -4)$ ***A1A1***

Note: Award ***A1A0*** for any two points correct.

[9 marks]

SECTION B

10. (a) $\cos\hat{A} = \frac{\text{BA}}{\sqrt{2}}$ ***A1***

$\sin\hat{A} = \frac{\text{BC}}{\sqrt{2}}$ ***A1***

$\cos\hat{A} - \sin\hat{A} = \frac{\text{BA} - \text{BC}}{\sqrt{2}}$ ***R1***

$= \frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

(b) $\cos^2\hat{A} - 2\cos\hat{A}\sin\hat{A} + \sin^2\hat{A} = \frac{1}{2}$ ***M1A1***

$1 - 2\sin\hat{A}\cos\hat{A} = \frac{1}{2}$ ***M1A1***

$\sin 2\hat{A} = \frac{1}{2}$ ***M1***

$2\hat{A} = 30^\circ$ ***A1***

angles in the triangle are 15° and 75° ***A1A1***

Note: Accept answers in radians.

[8 marks]

(c) $\text{BC}^2 + (\text{BC}+1)^2 = 2$ ***M1A1***

$2\text{BC}^2 + 2\text{BC} - 1 = 0$ ***A1***

$\text{BC} = \frac{-2+\sqrt{12}}{4}\left(= \frac{\sqrt{3}-1}{2}\right)$ ***M1A1***

$\sin\hat{A} = \frac{\text{BC}}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ ***A1***

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ ***AG***

[6 marks]

continued …

Question 10 continued

(d) **EITHER**

$h = \text{AB}\sin \hat{A}$ ***M1***

$= (\text{BC}+1)\sin \hat{A}$ ***A1***

$= \frac{\sqrt{3}+1}{2} \times \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}}{4}$ ***M1A1***

OR

$\frac{1}{2} AB.BC = \frac{1}{2} AC.h$ ***M1***

$\frac{\sqrt{3}-1}{2} \cdot \frac{\sqrt{3}+1}{2} = \sqrt{2}h$ ***A1***

$\frac{2}{4} = \sqrt{2}h$ ***M1***

$h = \frac{1}{2\sqrt{2}}$ ***A1***

[4 marks]

Total [21 marks]

11. (a) if $n = 1$

$\boldsymbol{X}^1 = \boldsymbol{U}^{-1}\boldsymbol{A}^1\boldsymbol{U}$ which is given, so true for $n = 1$ ***A1***

Assume true for $n = k$

$\boldsymbol{X}^k = \boldsymbol{U}^{-1}\boldsymbol{A}^k\boldsymbol{U}$ ***M1***

> **Note:** Only award ***M1*** if the word "true" or equivalent appears.

if $n = k+1$

$\boldsymbol{X}^{k+1} = \boldsymbol{X}\boldsymbol{X}^k$ ***M1***

$= \boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{U}\boldsymbol{U}^{-1}\boldsymbol{A}^k\boldsymbol{U}$ ***A1***

$= \boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^k\boldsymbol{U} = \boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{A}^k\boldsymbol{U}$ ***(A1)***

$= \boldsymbol{U}^{-1}\boldsymbol{A}^{k+1}\boldsymbol{U}$ ***A1***

As true for $n = 1$, and true for $n = k \Rightarrow$ true for $n = k+1$, then by the principle of mathematical induction the statement is true for all $n (\in \mathbb{Z}^+)$ ***R1***

> **Note:** Do not award ***R1*** if both ***M*** marks have not been awarded in this part.
> For ***R1*** to be awarded evidence of implication should be seen in the statement.

[7 marks]

continued …

Question 11 continued

(b) (i) **METHOD 1**

$\boldsymbol{AU} = \boldsymbol{UD} \Rightarrow \boldsymbol{D} = \boldsymbol{U}^{-1}\boldsymbol{AU}$ ***M1***

$\boldsymbol{U}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$ ***(A1)***

$\boldsymbol{D} = \boldsymbol{U}^{-1}\boldsymbol{AU} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ***M1A1***

METHOD 2

$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ***M1***

$\begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3a+c & 3b+d \\ a+c & b+d \end{pmatrix}$ ***A1***

solving simultaneously ***M1***

$a = 1,\ c = 0,\ b = 0,\ d = -1$

$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ***A1***

(ii) $\boldsymbol{D}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

(iii) $\boldsymbol{D}^{2n} = \boldsymbol{U}^{-1}\boldsymbol{A}^{2n}\boldsymbol{U} \Rightarrow \boldsymbol{A}^{2n} = \boldsymbol{UD}^{2n}\boldsymbol{U}^{-1}$ ***M1A1***

$= \boldsymbol{UIU}^{-1}$ ***A1***

$= \boldsymbol{UU}^{-1} = \boldsymbol{I}$ ***AG***

(iv) $\boldsymbol{A}^{2n} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^n\boldsymbol{A}^n = \boldsymbol{I}$ ***M1***

$\Rightarrow (\boldsymbol{A}^n)^{-1} = \boldsymbol{A}^n$ ***R1AG***

[10 marks]

Total [17 marks]

12. (a) **EITHER**

derivative of $\frac{x}{1-x}$ is $\frac{(1-x)-x(-1)}{(1-x)^2}$ ***M1A1***

$f'(x)=\frac{1}{2}\left(\frac{x}{1-x}\right)^{-\frac{1}{2}}\frac{1}{(1-x)^2}$ ***M1A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x)>0$ (for all $0<x<1$) so the function is increasing ***R1***

OR

$f(x)=\frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$

$f'(x)=\frac{(1-x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}(-1)}{1-x}$ ***M1A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}[1-x+x]$ ***M1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x)>0$ (for all $0<x<1$) so the function is increasing ***R1***

[5 marks]

(b) $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

$\Rightarrow f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}}+\frac{3}{4}x^{-\frac{1}{2}}(1-x)^{-\frac{5}{2}}$ ***M1A1***

$=-\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{5}{2}}[1-4x]$

$f''(x)=0\Rightarrow x=\frac{1}{4}$ ***M1A1***

$f''(x)$ changes sign at $x=\frac{1}{4}$ hence there is a point of inflexion ***R1***

$x=\frac{1}{4}\Rightarrow y=\frac{1}{\sqrt{3}}$ ***A1***

the coordinates are $\left(\frac{1}{4},\frac{1}{\sqrt{3}}\right)$

[6 marks]

continued …

Question 12 continued

(c) $x = \sin^2\theta \Rightarrow \dfrac{dx}{d\theta} = 2\sin\theta\cos\theta$ ***M1A1***

$\displaystyle\int \sqrt{\frac{x}{1-x}}\,dx = \int \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\ 2\sin\theta\cos\theta\,d\theta$ ***M1A1***

$= \displaystyle\int 2\sin^2\theta\,d\theta$ ***A1***

$= \displaystyle\int 1-\cos 2\theta\,d\theta$ ***M1A1***

$= \theta - \dfrac{1}{2}\sin 2\theta + c$ ***A1***

$\theta = \arcsin\sqrt{x}$ ***A1***

$\dfrac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sqrt{x}\sqrt{1-x} = \sqrt{x-x^2}$ ***M1A1***

hence $\displaystyle\int \sqrt{\frac{x}{1-x}}\,dx = \arcsin\sqrt{x} - \sqrt{x-x^2} + c$ ***AG***

[11 marks]

Total [22 marks]

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MARKSCHEME

May 2012

MATHEMATICS

Higher Level

Paper 1

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6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. let $f(x) = 2x^3 + kx^2 + 6x + 32$

let $g(x) = x^4 - 6x^2 - k^2x + 9$

$f(-1) = -2 + k - 6 + 32 (= 24 + k)$ ***A1***

$g(-1) = 1 - 6 + k^2 + 9 (= 4 + k^2)$ ***A1***

$\Rightarrow 24 + k = 4 + k^2$ ***M1***

$\Rightarrow k^2 - k - 20 = 0$

$\Rightarrow (k-5)(k+4) = 0$ ***(M1)***

$\Rightarrow k = 5, -4$ ***A1A1***

[6 marks]

2. perpendicular when $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix} = 0$ ***(M1)***

$\Rightarrow -1 + 4\sin x \cos x = 0$ ***A1***

$\Rightarrow \sin 2x = \frac{1}{2}$ ***M1***

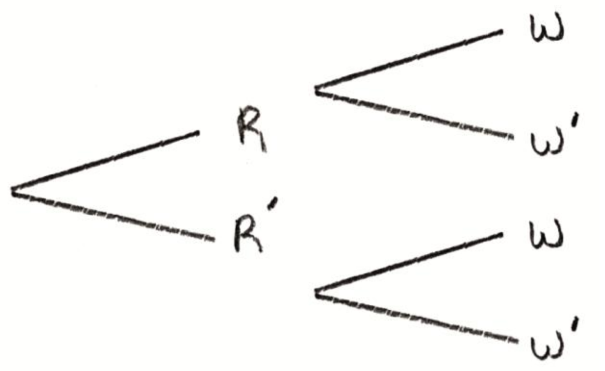
$\Rightarrow 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$ ***A1A1***

Note: Accept answers in degrees.

[5 marks]

3. (a) let R be "it rains" and W be "the 'Tigers' soccer team win"



A1

[1 mark]

(b) $\mathrm{P}(W) = \frac{2}{5} \times \frac{2}{7} + \frac{3}{5} \times \frac{4}{7}$ ***(M1)***

$= \frac{16}{35}$ ***A1***

[2 marks]

(c) $\mathrm{P}(R|W) = \dfrac{\frac{2}{5} \times \frac{2}{7}}{\frac{16}{35}}$ ***(M1)***

$= \frac{1}{4}$ ***A1***

[2 marks]

Total [5 marks]

4. (a) $\left(x - \frac{2}{x}\right)^4 = x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$ ***(A2)***

Note: Award ***(A1)*** for 3 or 4 correct terms.

Note: Accept combinatorial expressions, *e.g.* $\binom{4}{2}$ for 6.

$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ ***A1***

[3 marks]

(b) constant term from expansion of $(2x^2 + 1)\left(x - \frac{2}{x}\right)^4 = -64 + 24 = -40$ ***A2***

Note: Award ***A1*** for – 64 or 24 seen.

[2 marks]

Total [5 marks]

5. (a) $4\boldsymbol{A} - 5\boldsymbol{BX} = \boldsymbol{B}$

$\Rightarrow \boldsymbol{BX} = \frac{4}{5}\boldsymbol{A} - \frac{1}{5}\boldsymbol{B}$ ***M1***

$\Rightarrow \boldsymbol{X} = \frac{1}{5}\boldsymbol{B}^{-1}(4\boldsymbol{A} - \boldsymbol{B}) \ \left(= \frac{4}{5}\boldsymbol{B}^{-1}\boldsymbol{A} - \frac{1}{5}\boldsymbol{I}\right)$ ***A1***

[2 marks]

(b) if $\boldsymbol{A} = 2\boldsymbol{B}$ then $\boldsymbol{B}^{-1}\boldsymbol{A} = 2\boldsymbol{I}$ ***M1***

$\Rightarrow \boldsymbol{X} = \frac{8}{5}\boldsymbol{I} - \frac{1}{5}\boldsymbol{I}$

$\Rightarrow \boldsymbol{X} = \frac{7}{5}\boldsymbol{I} \left(= \begin{pmatrix} \frac{7}{5} & 0 \\ 0 & \frac{7}{5} \end{pmatrix} \right)$ ***A2***

[3 marks]

Total [5 marks]

6. (a) attempt to equate real and imaginary parts ***M1***

equate real parts: $4m + 4n = 16$; equate imaginary parts: $-5m = 15$ ***A1***

$\Rightarrow m = -3,\ n = 7$ ***A1***

[3 marks]

(b) let $m = x + \mathrm{i}y,\ n = x - \mathrm{i}y$ ***M1***

$\Rightarrow (4 - 5\mathrm{i})(x + \mathrm{i}y) + 4(x - \mathrm{i}y) = 16 + 15\mathrm{i}$

$\Rightarrow 4x - 5\mathrm{i}x + 4\mathrm{i}y + 5y + 4x - 4\mathrm{i}y = 16 + 15\mathrm{i}$

attempt to equate real and imaginary parts ***M1***

$8x + 5y = 16,\ -5x = 15$ ***A1***

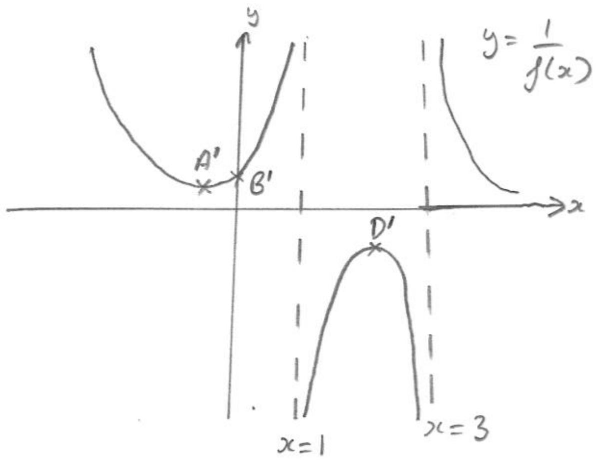
$\Rightarrow x = -3,\ y = 8$ ***A1***

$(\Rightarrow m = -3 + 8\mathrm{i},\ n = -3 - 8\mathrm{i})$

[4 marks]

Total [7 marks]

7. (a)

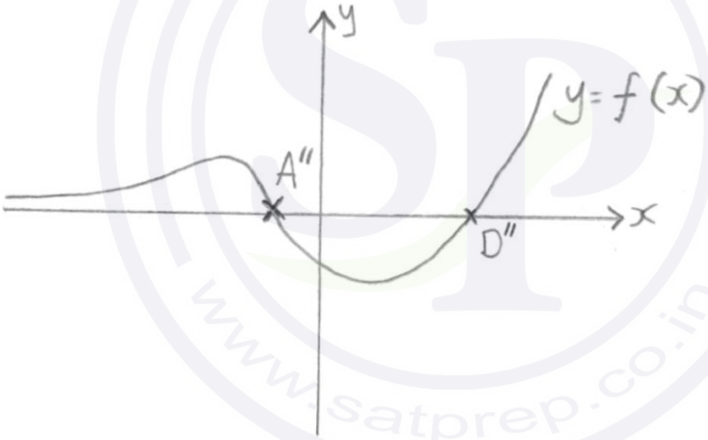


A1A1A1

Note: Award ***A1*** for correct shape.
Award ***A1*** for two correct asymptotes, $x = 1$ and $x = 3$.
Award ***A1*** for correct coordinates, $A'\left(-1, \frac{1}{4}\right)$, $B'\left(0, \frac{1}{3}\right)$ and $D'\left(2, -\frac{1}{3}\right)$.

[3 marks]

(b)



A1A1A1

Note: Award ***A1*** for correct general shape including the horizontal asymptote.
Award ***A1*** for recognition of 1 maximum point and 1 minimum point.
Award ***A1*** for correct coordinates, $A''(-1, 0)$ and $D''(2, 0)$.

[3 marks]

Total [6 marks]

8. $x^3 y = a\sin nx$

attempt to differentiate implicitly ***M1***

$\Rightarrow 3x^2 y + x^3 \frac{\mathrm{d}y}{\mathrm{d}x} = an\cos nx$ ***A2***

Note: Award ***A1*** for two out of three correct, ***A0*** otherwise.

$\Rightarrow 6xy + 3x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + x^3 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -an^2 \sin nx$ ***A2***

Note: Award ***A1*** for three or four out of five correct, ***A0*** otherwise.

$\Rightarrow 6xy + 6x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + x^3 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -an^2 \sin nx$

$\Rightarrow x^3 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 6xy + n^2 x^3 y = 0$ ***A1***

$\Rightarrow x^3 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + (n^2 x^2 + 6)xy = 0$ ***AG***

[6 marks]

9. **METHOD 1**

$\frac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$

consider right hand side

$\sec 2A + \tan 2A = \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$ ***M1A1***

$= \frac{\cos^2 A + 2\sin A \cos A + \sin^2 A}{\cos^2 A - \sin^2 A}$ ***A1A1***

Note: Award ***A1*** for recognizing the need for single angles and ***A1*** for recognizing $\cos^2 A + \sin^2 A = 1$.

$= \frac{(\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}$ ***M1A1***

$= \frac{\cos A + \sin A}{\cos A - \sin A}$ ***AG***

METHOD 2

$\frac{\cos A + \sin A}{\cos A - \sin A} = \frac{(\cos A + \sin A)^2}{(\cos A - \sin A)(\cos A + \sin A)}$ ***M1A1***

$= \frac{\cos^2 A + 2\sin A \cos A + \sin^2 A}{\cos^2 A - \sin^2 A}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{1 + \sin 2A}{\cos 2A}$ ***M1A1***

$= \sec 2A + \tan 2A$ ***AG***

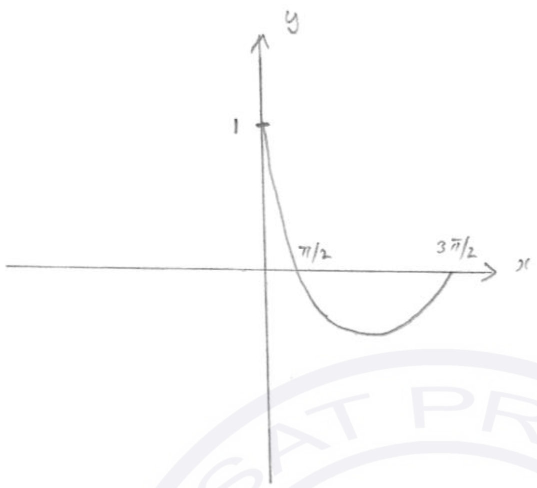
[6 marks]

10. (a) $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ *A1*

[1 mark]

(b)



A1

Note:	Accept any form of concavity for $x \in [0, \frac{\pi}{2}]$.

Note:	Do not penalize unmarked zeros if given in part (a).

Note:	Zeros written on diagram can be used to allow the mark in part (a) to be awarded retrospectively.

[1 mark]

continued …

Question 10 continued

(c) attempt at integration by parts ***M1***

EITHER

$I = \int e^{-x}\cos x dx = -e^{-x}\cos x dx - \int e^{-x}\sin x dx$ ***A1***

$\Rightarrow I = -e^{-x}\cos x dx - \left[-e^{-x}\sin x + \int e^{-x}\cos x dx\right]$ ***A1***

$\Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

OR

$I = \int e^{-x}\cos x dx = e^{-x}\sin x + \int e^{-x}\sin x dx$ ***A1***

$\Rightarrow I = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x dx$ ***A1***

$\Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

THEN

$\int_0^{\frac{\pi}{2}} e^{-x}\cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} = \frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2}$ ***A1***

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x}\cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{e^{-\frac{3\pi}{2}}}{2} - \frac{e^{-\frac{\pi}{2}}}{2}$ ***A1***

ratio of $A{:}B$ is $\dfrac{\frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2}}{\frac{e^{-\frac{3\pi}{2}}}{2} + \frac{e^{-\frac{\pi}{2}}}{2}}$

$= \dfrac{e^{\frac{3\pi}{2}}\left(e^{-\frac{\pi}{2}} + 1\right)}{e^{\frac{3\pi}{2}}\left(e^{-\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}\right)}$ ***M1***

$= \dfrac{e^{\pi}\left(e^{\frac{\pi}{2}} + 1\right)}{e^{\pi} + 1}$ ***AG***

[7 marks]

Total [9 marks]

SECTION B

11. (a) $f(x) \geq \frac{1}{25}$ ***A1***

$g(x) \in \mathbb{R}, g(x) \geq 0$ ***A1***

[2 marks]

(b) $f \circ g(x) = \dfrac{2\left(\frac{3x-4}{10}\right)^2 + 3}{75}$ ***M1A1***

$= \dfrac{\frac{2(9x^2 - 24x + 16)}{100} + 3}{75}$ ***(A1)***

$= \dfrac{9x^2 - 24x + 166}{3750}$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

$y = \dfrac{2x^2 + 3}{75}$

$x^2 = \dfrac{75y - 3}{2}$ ***M1***

$x = \sqrt{\dfrac{75y - 3}{2}}$ ***(A1)***

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x - 3}{2}}$ ***A1***

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

METHOD 2

$y = \dfrac{2x^2 + 3}{75}$

$x = \dfrac{2y^2 + 3}{75}$ ***M1***

$y = \sqrt{\dfrac{75x - 3}{2}}$ ***(A1)***

$\Rightarrow f^{-1}(x) = \sqrt{\dfrac{75x - 3}{2}}$ ***A1***

Note:	Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***. Award the ***A1*** only if written in the form $f^{-1}(x) =$.

(ii) domain: $x \geq \frac{1}{25}$; range: $f^{-1}(x) \geq 0$ ***A1***

[4 marks]

continued …

Question 11 continued

(d) probabilities from $f(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{3}{75}$	$\frac{5}{75}$	$\frac{11}{75}$	$\frac{21}{75}$	$\frac{35}{75}$

A2

Note: Award ***A1*** for one error, ***A0*** otherwise.

probabilities from $g(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{8}{10}$

A2

Note: Award ***A1*** for one error, ***A0*** otherwise.

only in the case of $f(x)$ does $\sum P(X=x)=1$, hence only $f(x)$ can be used as a probability mass function ***A2***

[6 marks]

(e)

$$E(x)=\sum x\cdot \mathrm{P}(X=x)$$ ***M1***

$$=\frac{5}{75}+\frac{22}{75}+\frac{63}{75}+\frac{140}{75}=\frac{230}{75}\left(=\frac{46}{15}\right)$$ ***A1***

[2 marks]

Total [18 marks]

12. Part A

(a) (i) $(x+\mathrm{i}y)^2 = -5+12\mathrm{i}$

$x^2 + 2\mathrm{i}xy + \mathrm{i}^2 y^2 = -5+12\mathrm{i}$ ***A1***

(ii) equating real and imaginary parts ***M1***

$x^2 - y^2 = -5$ ***AG***

$xy = 6$ ***AG***

[2 marks]

(b) substituting ***M1***

EITHER

$x^2 - \frac{36}{x^2} = -5$

$x^4 + 5x^2 - 36 = 0$ ***A1***

$x^2 = 4, -9$ ***A1***

$x = \pm 2$ and $y = \pm 3$ ***(A1)***

OR

$\frac{36}{y^2} - y^2 = -5$

$y^4 - 5y^2 - 36 = 0$ ***A1***

$y^2 = 9, -4$ ***A1***

$y = \pm 3$ **and** $x = \pm 2$ ***(A1)***

Note: Accept solution by inspection if completely correct.

THEN

the square roots are $(2+3\mathrm{i})$ and $(-2-3\mathrm{i})$ ***A1***

[5 marks]

(c) **EITHER**

consider $z = x + \mathrm{i}y$

$z^* = x - \mathrm{i}y$

$(z^*)^2 = x^2 - y^2 - 2\mathrm{i}xy$ ***A1***

$(z^2) = x^2 - y^2 + 2\mathrm{i}xy$ ***A1***

$(z^2)^* = x^2 - y^2 - 2\mathrm{i}xy$ ***A1***

$(z^*)^2 = (z^2)^*$ ***AG***

OR

$z^* = r\mathrm{e}^{-\mathrm{i}\theta}$

$(z^*)^2 = r^2\mathrm{e}^{-2\mathrm{i}\theta}$ ***A1***

$z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$ ***A1***

continued …

Question 12 continued

$(z^2)^* = r^2 e^{-2i\theta}$ ***A1***

$(z^*)^2 = (z^2)^*$ ***AG***

[3 marks]

(d) $(2-3i)$ and $(-2+3i)$ ***A1A1***

[2 marks]

Part B

(a) the graph crosses the x-axis twice, indicating two real roots ***R1***

since the quartic equation has four roots and only two are real, the other two roots must be complex ***R1***

[2 marks]

(b) $f(x) = (x+4)(x-2)(x^2+cx+d)$ ***A1A1***

$f(0) = -32 \Rightarrow d = 4$ ***A1***

Since the curve passes through (– 1, – 18),

$-18 = 3\times(-3)(5-c)$ ***M1***

$c = 3$ ***A1***

Hence $f(x) = (x+4)(x-2)(x^2+3x+4)$

[5 marks]

(c) $x = \dfrac{-3 \pm \sqrt{9-16}}{2}$ ***(M1)***

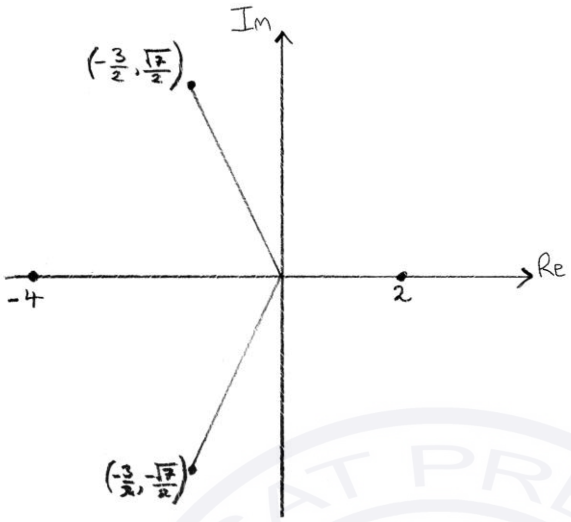
$\Rightarrow x = -\dfrac{3}{2} \pm i\dfrac{\sqrt{7}}{2}$ ***A1***

[2 marks]

continued …

Question 12 continued

(d)



A1A1

Note:	Accept points or vectors on complex plane. Award ***A1*** for two real roots and ***A1*** for two complex roots.

[2 marks]

(e) real roots are $4e^{i\pi}$ and $2e^{i0}$ ***A1A1***

considering $-\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

$r = \sqrt{\frac{9}{4} + \frac{7}{4}} = 2$ ***A1***

finding θ using $\arctan\left(\frac{\sqrt{7}}{3}\right)$ ***M1***

$\theta = \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi$ or $\theta = \arctan\left(-\frac{\sqrt{7}}{3}\right) + \pi$ ***A1***

$\Rightarrow z = 2e^{i\left(\arctan\left(\frac{\sqrt{7}}{3}\right)+\pi\right)}$ or $\Rightarrow z = 2e^{i\left(\arctan\left(\frac{-\sqrt{7}}{3}\right)+\pi\right)}$ ***A1***

Note:	Accept arguments in the range $-\pi$ to π or 0 to 2π. Accept answers in degrees.

[6 marks]

Total [29 marks]

13. (a) let $f(x)=\frac{1}{2x+1}$ and using the result $f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$$f'(x)=\lim_{h\to 0}\left(\frac{\frac{1}{2(x+h)+1}-\frac{1}{2x+1}}{h}\right)$$ ***M1A1***

$$\Rightarrow f'(x)=\lim_{h\to 0}\left(\frac{[2x+1]-[2(x+h)+1]}{h[2(x+h)+1][2x+1]}\right)$$ ***A1***

$$\Rightarrow f'(x)=\lim_{h\to 0}\left(\frac{-2}{[2(x+h)+1][2x+1]}\right)$$ ***A1***

$$\Rightarrow f'(x)=\frac{-2}{(2x+1)^2}$$ ***AG***

[4 marks]

(b) let $y=\frac{1}{2x+1}$

we want to prove that $\frac{d^n y}{dx^n}=(-1)^n\frac{2^n n!}{(2x+1)^{n+1}}$

let $n=1\Rightarrow\frac{dy}{dx}=(-1)^1\frac{2^1 1!}{(2x+1)^{1+1}}$ ***M1***

$\Rightarrow\frac{dy}{dx}=\frac{-2}{(2x+1)^2}$ which is the same result as part (a)

hence the result is true for $n=1$ ***R1***

assume the result is true for $n=k$: $\frac{d^k y}{dx^k}=(-1)^k\frac{2^k k!}{(2x+1)^{k+1}}$ ***M1***

$$\frac{d^{k+1}y}{dx^{k+1}}=\frac{d}{dx}\left[(-1)^k\frac{2^k k!}{(2x+1)^{k+1}}\right]$$ ***M1***

$$\Rightarrow\frac{d^{k+1}y}{dx^{k+1}}=\frac{d}{dx}\left[(-1)^k 2^k k!(2x+1)^{-k-1}\right]$$ ***(A1)***

$$\Rightarrow\frac{d^{k+1}y}{dx^{k+1}}=(-1)^k 2^k k!(-k-1)(2x+1)^{-k-2}\times 2$$ ***A1***

$$\Rightarrow\frac{d^{k+1}y}{dx^{k+1}}=(-1)^{k+1}2^{k+1}(k+1)!(2x+1)^{-k-2}$$ ***(A1)***

$$\Rightarrow\frac{d^{k+1}y}{dx^{k+1}}=(-1)^{k+1}\frac{2^{k+1}(k+1)!}{(2x+1)^{k+2}}$$ ***A1***

hence if the result is true for $n=k$, it is true for $n=k+1$
since the result is true for $n=1$, the result is proved by mathematical induction ***R1***

Note: Only award final ***R1*** if all the ***M*** marks have been gained.

[9 marks]

Total [13 marks]

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MARKSCHEME

November 2011

MATHEMATICS

Higher Level

Paper 1

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc.*
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty ***(AP)*** *no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award A0. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. area of triangle $=\frac{1}{2}(2x)^2\sin\frac{\pi}{3}$ ***(M1)***

$=x^2\sqrt{3}$ ***A1***

Note: A $0.5\times$ base $\times$ height calculation is acceptable.

area of sector $=\frac{\theta}{2}r^2=\frac{\pi}{6}r^2$ ***(M1)A1***

area of triangle is twice the area of the sector

$\Rightarrow 2\left(\frac{\pi}{6}r^2\right)=x^2\sqrt{3}$ ***M1***

$\Rightarrow r=x\sqrt{\frac{3\sqrt{3}}{\pi}}$ or equivalent ***A1***

[6 marks]

2. $\mathrm{i}=\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}$ ***(A1)***

$z_1=\mathrm{i}^{\frac{1}{3}}=\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)^{\frac{1}{3}}=\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\quad\left(=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$ ***M1A1***

$z_2=\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\quad\left(=-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$ ***(M1)A1***

$z_3=\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)=-\mathrm{i}$ ***A1***

Note: Accept exponential and cis forms for intermediate results, but not the final roots.

Note: Accept the method based on expanding $(\mathrm{a}+\mathrm{b})^3$. ***M1*** for attempt, ***M1*** for equating real and imaginary parts, ***A1*** for finding $\mathrm{a}=0$ and $\mathrm{b}=\frac{1}{2}$, then ***A1A1A1*** for the roots.

[6 marks]

3. tree diagram *(M1)*

$$P(I|D) = \frac{P(D|I) \times P(I)}{P(D)}$$ *(M1)*

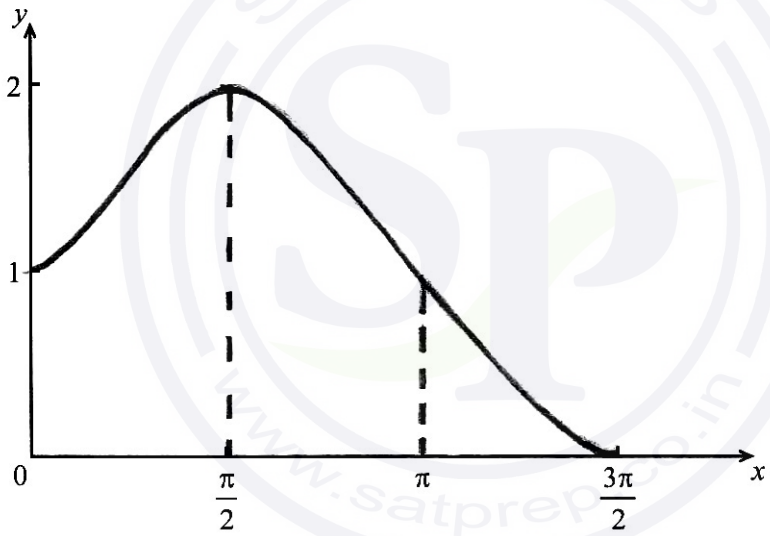
$$= \frac{0.1 \times 0.2}{0.1 \times 0.2 + 0.8 \times 0.75}$$ *A1A1A1*

$$\left(= \frac{0.02}{0.62}\right) = \frac{1}{31}$$ *A1*

Note: Alternative presentation of results: ***M1*** for labelled tree; ***A1*** for initial branching probabilities, 0.2 and 0.8; ***A1*** for at least the relevant second branching probabilities, 0.1 and 0.75; ***A1*** for the 'infected' end-point probabilities, 0.02 and 0.6; ***M1A1*** for the final conditional probability calculation.

[6 marks]

4. (a)



A1

(b) $(1+\sin x)^2 = 1 + 2\sin x + \sin^2 x$

$= 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)$ *A1*

$= \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$ *AG*

continued …

Question 4 continued

(c) $V = \pi\int_0^{\frac{3\pi}{2}} (1+\sin x)^2 \mathrm{d}x$ ***(M1)***

$= \pi\int_0^{\frac{3\pi}{2}} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right)\mathrm{d}x$

$= \pi\left[\frac{3}{2}x - 2\cos x - \frac{\sin 2x}{4}\right]_0^{\frac{3\pi}{2}}$ ***A1***

$= \frac{9\pi^2}{4} + 2\pi$ ***A1A1***

[6 marks]

5. $\mathrm{P}(A) = \frac{\pi}{25\pi}\times\frac{1}{2} = \frac{1}{50}$ ***(M1)A1***

$\mathrm{P}(B) = \frac{8\pi}{25\pi}\times\frac{1}{2} = \frac{4}{25}$ ***A1***

$\mathrm{P}(C) = \frac{16\pi}{25\pi}\times\frac{1}{2} = \frac{8}{25}$ ***A1***

Note:	The ***M1*** is for the use of 3 areas

$\mathrm{E}(X) = (0.5\times 0) + \frac{1}{50}\times 10 + \frac{4}{25}\times 6 + \frac{8}{25}\times 3 = \frac{106}{50}$ (= 2.12) ***M1A1***

Note:	The final ***M1*** is available if the probabilities are incorrect but sum to 1 or

[6 marks]

6. proposition is true for $n=1$ since $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(1-x)^2}$ ***M1***

$=\frac{1!}{(1-x)^2}$ ***A1***

Note: Must see the 1! for the ***A1***.

assume true for $n=k$, $k\in\mathbb{Z}^+$, *i.e.* $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}=\frac{k!}{(1-x)^{k+1}}$ ***M1***

consider $\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=\frac{\mathrm{d}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)}{\mathrm{d}x}$ ***(M1)***

$=(k+1)k!(1-x)^{-(k+1)-1}$ ***A1***

$=\frac{(k+1)!}{(1-x)^{k+2}}$ ***A1***

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers ***R1***

Note: The final ***R1*** is only available if at least 4 of the previous marks have been awarded.

[7 marks]

7. to find the points of intersection of the two curves

$-x^2+2=x^3-x^2-bx+2$ ***M1***

$x^3-bx=x(x^2-b)=0$

$\Rightarrow x=0;\ x=\pm\sqrt{b}$ ***A1A1***

$A_1=\int_{-\sqrt{b}}^{0}\left[(x^3-x^2-bx+2)-(-x^2+2)\right]\mathrm{d}x\left(=\int_{-\sqrt{b}}^{0}(x^3-bx)\,\mathrm{d}x\right)$ ***M1***

$=\left[\frac{x^4}{4}-\frac{bx^2}{2}\right]_{-\sqrt{b}}^{0}$

$=-\left(\frac{(-\sqrt{b})^4}{4}-\frac{b(-\sqrt{b})^2}{2}\right)=-\frac{b^2}{4}+\frac{b^2}{2}=\frac{b^2}{4}$ ***A1***

$A_2=\int_{0}^{\sqrt{b}}\left[(-x^2+2)-(x^3-x^2-bx+2)\right]\mathrm{d}x$ ***M1***

continued …

Question 7 continued

$= \int_0^{\sqrt{b}} (-x^3 + bx)\,\mathrm{d}x$

$= \left[-\frac{x^4}{4} + \frac{bx^2}{2}\right]_0^{\sqrt{b}} = \frac{b^2}{4}$ ***A1***

therefore $A_1 = A_2 = \frac{b^2}{4}$ ***AG***

[7 marks]

8. (a) angle APB is a right angle

$\Rightarrow \cos\theta = \frac{\mathrm{AP}}{4} \Rightarrow \mathrm{AP} = 4\cos\theta$ ***A1***

Note: Allow correct use of cosine rule.

arc $\mathrm{PB} = 2\times 2\theta = 4\theta$ ***A1***

$t = \frac{\mathrm{AP}}{3} + \frac{\mathrm{PB}}{6}$ ***M1***

Note: Allow use of their AP and their PB for the ***M1***.

$\Rightarrow t = \frac{4\cos\theta}{3} + \frac{4\theta}{6} = \frac{4\cos\theta}{3} + \frac{2\theta}{3} = \frac{2}{3}(2\cos\theta + \theta)$ ***AG***

(b) $\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{2}{3}(-2\sin\theta + 1)$ ***A1***

$\frac{2}{3}(-2\sin\theta + 1) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ (or 30 degrees) ***A1***

(c) $\frac{\mathrm{d}^2 t}{\mathrm{d}\theta^2} = -\frac{4}{3}\cos\theta < 0 \quad \left(\text{at } \theta = \frac{\pi}{6}\right)$ ***M1***

$\Rightarrow t$ is maximized at $\theta = \frac{\pi}{6}$ ***R1***

time needed to walk along arc AB is $\frac{2\pi}{6}$ (≈ 1 hour)

time needed to row from A to B is $\frac{4}{3}$ (≈ 1.33 hour)

hence, time is minimized in walking from A to B ***R1***

[8 marks]

9. (a) for the equation to have real roots

$(y-1)^2 - 4y(y-1) \geq 0$ ***M1***

$\Rightarrow 3y^2 - 2y - 1 \leq 0$

(by sign diagram, or algebraic method) ***M1***

$-\frac{1}{3} \leq y \leq 1$ ***A1A1***

Note: Award first ***A1*** for $-\frac{1}{3}$ and 1, and second ***A1*** for inequalities. These are independent marks.

(b) $f : x \to \frac{x+1}{x^2+x+1} \Rightarrow x+1 = yx^2 + yx + y$ ***(M1)***

$\Rightarrow 0 = yx^2 + (y-1)x + (y-1)$ ***A1***

hence, from (a) range is $-\frac{1}{3} \leq y \leq 1$ ***A1***

(c) a value for y would lead to 2 values for x from (a) ***R1***

Note: Do not award ***R1*** if (b) has not been tackled.

[8 marks]

SECTION B

10. (a) $k\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$ ***M1***

$k[-\cos x]_0^{\frac{\pi}{2}} = 1$

$k = 1$ ***A1***

[2 marks]

(b) $E(X) = \int_0^{\frac{\pi}{2}} x \sin x \, dx$ ***M1***

integration by parts ***M1***

$[-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$ ***A1A1***

$= 1$ ***A1***

[5 marks]

(c) $\int_0^M \sin x \, dx = \frac{1}{2}$ ***M1***

$[-\cos x]_0^M = \frac{1}{2}$ ***A1***

$\cos M = \frac{1}{2}$

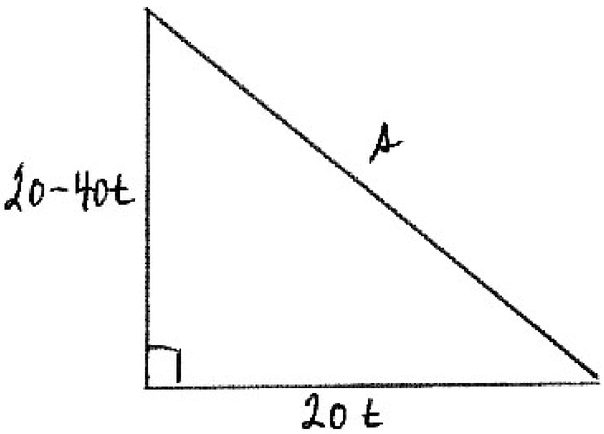
$M = \frac{\pi}{3}$ ***A1***

Note: accept $\arccos \frac{1}{2}$

[3 marks]

Total [10 marks]

11. (a)



(M1)

$s^2 = (20t)^2 + (20 - 40t)^2$ ***M1***

$s^2 = 2000t^2 - 1600t + 400$ ***A1***

to minimize s it is enough to minimize s^2

$f'(t) = 4000t - 1600$ ***A1***

setting $f'(t)$ equal to 0 ***M1***

$4000t - 1600 = 0 \Rightarrow t = \frac{2}{5}$ or 24 minutes ***A1***

$f''(t) = 4000 > 0$ ***M1***

$\Rightarrow$ at $t = \frac{2}{5}$, $f(t)$ is minimized

hence, the ships are closest at 12:24 ***A1***

Note: accept solution based on s.

[8 marks]

(b) $f\left(\frac{2}{5}\right) = \sqrt{80}$ ***M1A1***

since $\sqrt{80} < 9$, the captains can see one another ***R1***

[3 marks]

Total [11 marks]

12. (a) (i) $|\boldsymbol{a}-\boldsymbol{b}|=|\boldsymbol{a}+\boldsymbol{b}|$

$\Rightarrow (\boldsymbol{a}-\boldsymbol{b})\bullet(\boldsymbol{a}-\boldsymbol{b})=(\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{a}+\boldsymbol{b})$ ***(M1)***

$\Rightarrow |\boldsymbol{a}|^2-2\boldsymbol{a}\bullet\boldsymbol{b}+|\boldsymbol{b}|^2=|\boldsymbol{a}|^2+2\boldsymbol{a}\bullet\boldsymbol{b}+|\boldsymbol{b}|^2$ ***A1***

$\Rightarrow 4\boldsymbol{a}\bullet\boldsymbol{b}=0\Rightarrow \boldsymbol{a}\bullet\boldsymbol{b}=0$ ***A1***

therefore ***a*** and ***b*** are perpendicular ***R1***

Note: Allow use of 2-d components.

Note: Do not condone sloppy vector notation, so we must see something to the effect that $|\boldsymbol{c}|^2=\boldsymbol{c.c}$ is clearly being used for the ***M1***.

Note: Allow a correct geometric argument, for example that the diagonals of a parallelogram have the same length only if it is a rectangle.

(ii) $|\boldsymbol{a}\times\boldsymbol{b}|^2=(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***M1A1***

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\bullet\boldsymbol{b})^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***M1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***A1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}|^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\bullet\boldsymbol{b})^2$ ***AG***

[8 marks]

(b) (i) area of triangle $=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ ***(M1)***

$=\frac{1}{2}|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})|$ ***A1***

$=\frac{1}{2}|\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{b}\times-\boldsymbol{a}+-\boldsymbol{a}\times\boldsymbol{c}+-\boldsymbol{a}\times-\boldsymbol{a}|$ ***A1***

$\boldsymbol{b}\times-\boldsymbol{a}=\boldsymbol{a}\times\boldsymbol{b}$; $\boldsymbol{c}\times\boldsymbol{a}=-\boldsymbol{a}\times\boldsymbol{c}$; $-\boldsymbol{a}\times-\boldsymbol{a}=0$ ***M1***

hence, area of triangle is $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}|$ ***AG***

(ii) D is the foot of the perpendicular from B to AC

area of triangle $\text{ABC}=\frac{1}{2}|\overrightarrow{AC}||\overrightarrow{BD}|$ ***A1***

therefore

$\frac{1}{2}|\overrightarrow{AC}||\overrightarrow{BD}|=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ ***M1***

hence, $|\overrightarrow{BD}|=\frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{|\overrightarrow{AC}|}$ ***A1***

$=\frac{|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}|}{|\boldsymbol{c}-\boldsymbol{a}|}$ ***AG***

[7 marks]

Total [15 marks]

13. (a) $\frac{dy}{dx} = \frac{e}{\ln e}(2+2) = 4e$ *A1*

at $(2, e)$ the tangent line is $y - e = 4e(x-2)$ *M1*

hence $y = 4ex - 7e$ *A1*

[3 marks]

(b) $\frac{dy}{dx} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}dy = (x+2)dx$ *M1*

$\int \frac{\ln y}{y}dy = \int (x+2)dx$

using substitution $u = \ln y$; $du = \frac{1}{y}dy$ *(M1)(A1)*

$\Rightarrow \int \frac{\ln y}{y}dy = \int u\,du = \frac{1}{2}u^2$ *(A1)*

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$ *A1A1*

at $(2, e)$, $\frac{(\ln e)^2}{2} = 6 + c$ *M1*

$\Rightarrow c = -\frac{11}{2}$ *A1*

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

$\ln y = \pm\sqrt{x^2 + 4x - 11} \Rightarrow y = e^{\pm\sqrt{x^2+4x-11}}$ *M1A1*

since $y > 1$, $f(x) = e^{\sqrt{x^2+4x-11}}$ *R1*

[11 marks]

Note: ***M1*** for attempt to make y the subject.

(c) **EITHER**

$x^2 + 4x - 11 > 0$ *A1*

using the quadratic formula *M1*

critical values are $\frac{-4 \pm \sqrt{60}}{2}$ $\left(= -2 \pm \sqrt{15}\right)$ *A1*

using a sign diagram or algebraic solution *M1*

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ *A1A1*

OR

$x^2 + 4x - 11 > 0$ *A1*

by methods of completing the square *M1*

$(x+2)^2 > 15$ *A1*

$\Rightarrow x + 2 < -\sqrt{15}$ or $x + 2 > \sqrt{15}$ *(M1)*

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ *A1A1*

[6 marks]

continued …

Question 13 continued

(d) $f(x) = f'(x) \Rightarrow f(x) = \dfrac{f(x)}{\ln f(x)}(x+2)$ ***M1***

$\Rightarrow \ln\left(f(x)\right) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2+4x-11}\right)$ ***A1***

$\Rightarrow (x+2)^2 = x^2+4x-11 \Rightarrow x^2+4x+4 = x^2+4x-11$ ***A1***

$\Rightarrow 4 = -11$, hence $f(x) \neq f'(x)$ ***R1AG***

[4 marks]

Total [24 marks]