N20/5/MATHL/HP2/ENG/TZ0/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2020

Mathematics

Higher level

Paper 2

22 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log (a-b)$	Do not award the final ***A1***

3 *N* marks

Award ***N*** *marks for* ***correct*** *answers where there is* ***no*** *working*.

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5\ \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. attempt to find $\hat{AOB}$ by right-angled trigonometry or the cosine rule ***(M1)***

EITHER

$\hat{AOB} = 2\arcsin\left(\frac{5.5}{15}\right)$ ***A1***

OR

$\hat{AOB} = \arccos\left(\frac{15^2 + 15^2 - 11^2}{2 \times 15 \times 15}\right)$ ***A1***

THEN

$= 0.750847...\ (= 43.0204...^{\circ})$

Note: Award ***(M1)A1*** for correct calculation of $\hat{AOB}$ or $\frac{1}{2}\hat{AOB}$

shaded area = area of sector- area of triangle $\left(= \frac{1}{2}r^2(\theta - \sin\theta)\right)$ ***(M1)***

$= \frac{1}{2} \times 15^2 \times (0.750847... - \sin 0.750847...)$ ***(A1)***

$= 7.72$ (cm²) ***A1***

[5 marks]

2. let X be the random variable "number of books Jenna reads per week."

then $X \sim \text{Po}(2.6)$

$\text{P}(X \geq 4) = 0.264\ (0.263998\ldots)$ ***(M1)(A1)***

$0.263998... \times 52$ ***(M1)***

$= 13.7$ ***A1***

Note: Accept 14 weeks.

[4 marks]

3. (a) the principal axis is $\frac{5+(-1)}{2}(=2)$

so $p = 2$ ***A1***

the amplitude is $\frac{5-(-1)}{2}(=3)$

so $q = 3$ ***A1***

EITHER

one period is $2\left(-\frac{3\pi}{4}-\left(-\frac{9\pi}{4}\right)\right)$ ***(M1)***

$=3\pi$

$\Rightarrow \frac{2\pi}{r}=3\pi$

OR

Substituting a point eg $-1=2+\sin\left(-\frac{3\pi}{4}r\right)$

$\sin\left(-\frac{3\pi}{4}r\right)=-1\Rightarrow -\frac{3\pi}{4}r=\ldots-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\ldots$

Choice of correct solution $-\frac{3\pi}{4}r=-\frac{\pi}{2}$ ***(M1)***

THEN

$\Rightarrow r=\frac{2}{3}$ ***A1***

$\left(\Rightarrow y=2+3\sin\left(\frac{2x}{3}\right)\right)$

Note: *q* and *r* can be both given as negatives for full marks

[4 marks]

(b) roots are $x=-1.09459\ldots, x=-3.617797\ldots$ ***(A1)***

$$\int_{-3.617797...}^{-1.09459...}\left(2+3\sin\left(\frac{2x}{3}\right)\right)\mathrm{d}x$$ ***(M1)***

$=-1.66\left(=-1.66179...\right)$ ***(A1)***

so area $=1.66$ (units^2) ***A1***

[4 marks]

Total [8 marks]

4. use of Binomial expansion to find a term in either $\left(\frac{1}{3x^2}-\frac{x}{2}\right)^9$, $\left(\frac{1}{3x^{7/3}}-\frac{x^{2/3}}{2}\right)^9$,

$\left(\frac{1}{3}-\frac{x^3}{2}\right)^9$, $\left(\frac{1}{3x^3}-\frac{1}{2}\right)^9$ or $\left(2-3x^3\right)^9$ ***(M1)(A1)***

> **Note:** Award ***M1*** for a product of three terms including a binomial coefficient and powers of the two terms, and ***A1*** for a correct expression of a term in the expansion.

finding the powers required to be 2 and 7 ***(M1)(A1)***

constant term is ${}^9C_2\times\left(\frac{1}{3}\right)^2\times\left(-\frac{1}{2}\right)^7$ ***(M1)***

> **Note:** Ignore all x's in student's expression.

therefore term independent of x is $-\frac{1}{32}$ $(= -0.03125)$ ***A1***

[6 marks]

5. (a) (i) people's holidays are independent of each other ***R1***

the proportion is constant (at 0.15) ***R1***

(ii) $X \sim \mathrm{B}(16, 0.15)$

$\mathrm{P}(X \geq 3) = 0.439$ ***(M1)A1***

[4 marks]

(b) probability of at least one $= 1 -$ probability of none

$\Rightarrow 1 - 0.85^n > 0.999$ **OR** $0.85^n < 0.001$ ***(A1)***

attempt to solve inequality ***(M1)***
$n \geq 42.503...$

so least possible $n = 43$ ***A1***

[3 marks]
Total [7 marks]

6. $n=1$: $\text{LHS}=\frac{\mathrm{d}\left(x\mathrm{e}^{px}\right)}{\mathrm{d}x}=xp\mathrm{e}^{px}+\mathrm{e}^{px}=\left(px+1\right)\mathrm{e}^{px}$, $\text{RHS}=p^{0}\left(px+1\right)\mathrm{e}^{px}$

$\text{LHS}=\text{RHS}$ so true for $n=1$: ***A1***

> **Note:** Award ***A1*** if $n=0$ is proved.

assume proposition true for $n=k$, i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x\mathrm{e}^{px}\right)=p^{k-1}\left(px+k\right)\mathrm{e}^{px}$ ***M1***

> **Notes:** Do not award ***M1*** if using *n* instead of *k*.
> Assumption of truth must be present.
> Subsequent marks are not dependent on this ***M1*** mark.

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x\mathrm{e}^{px}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x\mathrm{e}^{px}\right)\right)$ **(*M1*)**

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(p^{k-1}\left(px+k\right)\mathrm{e}^{px}\right)$ ***M1***

$=p^{k-1}\left(px+k\right)p\mathrm{e}^{px}+\mathrm{e}^{px}\left(p^{k}\right)$

$=p^{k}\left(px+k\right)\mathrm{e}^{px}+\mathrm{e}^{px}\left(p^{k}\right)$ ***A1***

> **Note:** Award ***A1*** for correct derivative.

$=p^{k}\left(px+k+1\right)\mathrm{e}^{px}$ ***A1***

$=p^{((k+1)-1)}\left(px+\left(k+1\right)\right)\mathrm{e}^{px}$

> **Note:** The final ***A1*** can be awarded for either of the two lines above.

hence true for $n=1$ and $n=k$ true $\Rightarrow n=k+1$ true ***R1***

therefore true for all $n\in\mathbb{Z}^{+}$

> **Note:** Only award the final ***R1*** if the three method marks have been awarded.

[7 marks]

7. (a) identifying two or three possible cases ***(M1)***

total number of possible groups is $\binom{7}{5}+\binom{7}{4}\binom{5}{1}+\binom{7}{3}\binom{5}{2}$ ***(A1)(A1)***

Note: Award ***A1*** for any two correct cases, ***A1*** for the other one.

$=21+(35\times5)+(35\times10)$

$=546$ ***A1***

[4 marks]

(b) **METHOD 1**

identifying at least two of the three possible cases- Gary goes, Gerwyn goes or neither goes ***(M1)***

total number of possible groups is $\binom{10}{5}+\binom{10}{4}+\binom{10}{4}$ ***(A1)***

$=252+210+210$

$=672$ ***A1***

[3 marks]

METHOD 2

identifying the overall number of groups and no. of cases where both Gary and Gerwyn go. ***(M1)***

total number of possible groups is $\binom{12}{5}-\binom{10}{3}$ ***(A1)***

$=792-120$

$=672$ ***A1***

[3 marks]
Total [7 marks]

8. (a) valid attempt to use chain rule or quotient rule ***(M1)***

$$\frac{dy}{dx} = \frac{-10e^{-0.5x}}{\left(3-2e^{-0.5x}\right)^2} \text{ OR } \frac{dy}{dx} = -10e^{-0.5x}\left(3-2e^{-0.5x}\right)^{-2}$$ ***A1A1***

[3 marks]

Note: Award ***A1*** for numerator and ***A1*** for denominator, or ***A1*** for each part if the second alternative given.

(b) valid attempt to use chain rule $\left(\text{eg } \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}\right)$ (***M1)***

$$\frac{dx}{dt} = -0.1 \div \frac{-10e^{-2}}{\left(3-2e^{-2}\right)^2} \quad (= -0.1 \div -0.181676\ldots)$$ or equivalent (***A1)***

$$= 0.550428\ldots$$

$$\frac{dx}{dt} = 0.550 \text{ (ms}^{-1}\text{)}$$ ***A1***

[3 marks]

Total [6 marks]

Section B

9. (a) $X \sim \mathrm{N}(102, 8^2)$

$\mathrm{P}(X < 100) = 0.401$ ***(M1)A1***

[2 marks]

(b) $\mathrm{P}(X > w) = 0.444$ ***(M1)***

$\Rightarrow w = 103\,(\text{g})$ ***A1***

[2 marks]

(c) $\mathrm{P}(X > 110 \mid X > 105) = \dfrac{\mathrm{P}(X > 110 \cap X > 105)}{\mathrm{P}(X > 105)}$ ***(M1)***

$= \dfrac{\mathrm{P}(X > 110)}{\mathrm{P}(X > 105)}$ ***(A1)***

$= \dfrac{0.15865...}{0.35383...}$

$= 0.448$ ***A1***

[3 marks]

(d) **EITHER**

$\mathrm{P}(90 < X < 114) = 0.866...$ ***(A1)***

OR

$\mathrm{P}(-1.5 < Z < 1.5) = 0.866...$ ***(A1)***

THEN

$0.866... \times 500$ ***(M1)***

$= 433$ ***A1***

[3 marks]

(e) $p = \mathrm{P}(X < 95) = 0.19078...$ ***(A1)***

recognising $Y \sim \mathrm{B}(80, p)$ ***(M1)*** now using $Y \sim \mathrm{B}(80, 0.19078...)$ ***(M1)***

$\mathrm{P}(Y \geq 20) = 0.116$ ***A1***

[4 marks]
Total [14 marks]

10. (a) $3(1-3\lambda)-(2-\lambda)+(-2+4\lambda)=-13$ (***M1***)

$\lambda = 3$ (***A1***)

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + 3\begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \\ 10 \end{pmatrix}$$ (***M1***)

so $\mathrm{P}(-8, -1, 10)$ ***A1***

Note: Do not award the final ***A1*** if a vector given instead of coordinates

[4 marks]

(b) **METHOD 1**

$$\boldsymbol{r} = \mu\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

substituting into equation of the plane ***M1***

$9\mu + \mu + \mu = -13$

$\mu = -\frac{13}{11} (= -1.18...)$ ***A1***

$$\text{distance} = \frac{13\sqrt{3^2 + (-1)^2 + 1^2}}{11}$$ ***(M1)***

$$= \frac{13}{\sqrt{11}} \left(= \frac{13\sqrt{11}}{11} = 3.92 \right)$$ ***A1***

[4 marks]

METHOD 2

choice of any point on the plane, eg $(-8,\ -1,\ 10)$ to use in distance formula (***M1***)

so distance $= \dfrac{\begin{pmatrix}-8\\-1\\10\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\-1\end{pmatrix}}{\sqrt{(-3)^2+1^2+(-1)^2}}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$= \dfrac{24-1-10}{\sqrt{11}}$

$= \dfrac{13}{\sqrt{11}}\left(= \dfrac{13\sqrt{11}}{11} = 3.92\right)$ ***A1***

[4 marks]

(c) **EITHER**

identify two vectors (***A1***)

eg, $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-3\\-1\\4\end{pmatrix}$

$\boldsymbol{n} = \begin{pmatrix}1\\2\\-2\end{pmatrix}\times\begin{pmatrix}-3\\-1\\4\end{pmatrix} = \begin{pmatrix}6\\2\\5\end{pmatrix}$ (***M1***)

OR

identify three points in the plane (***A1***)

eg $\lambda = 0,1$ gives $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\2\end{pmatrix}$

solving system of equations (***M1***)

THEN

$\Pi_2 : \boldsymbol{r}.\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} = 0$ ***A1***

Note*:* Accept $6x + 2y + 5z = 0$.

[3 marks]

(d) vector normal to Π_1 is eg $\boldsymbol{n_1} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

vector normal to Π_2 is eg $\boldsymbol{n_2} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$ ***(A1)***

required angle is θ, where $\cos\theta = \dfrac{\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}}{\sqrt{11}\sqrt{65}}$ ***M1A1***

$\cos\theta = \dfrac{21}{\sqrt{11}\sqrt{65}} = 0.785...$ ***(A1)***

$\theta = 0.667526...$

$\theta = 0.668 \left(= 38.2^\circ\right)$ ***A1***

Note: Award the penultimate ***(A1)*** but not the final ***A1*** for the obtuse angle $2.47406\ldots$ or 142° .

[5 marks]
Total [16 marks]

11. (a) $\frac{\pi}{6}(=0.524)$ ***A1***

$\frac{\pi}{3}(=1.05)$ ***A1***

[2 marks]

(b) attempt to use integration by parts ***M1***

$s=\int \mathrm{e}^{-3t}\sin 6t \ \mathrm{d}t$

EITHER

$=-\frac{\mathrm{e}^{-3t}\sin 6t}{3}-\int -2\mathrm{e}^{-3t}\cos 6t \ \mathrm{d}t$ ***A1***

$=-\frac{\mathrm{e}^{-3t}\sin 6t}{3}-\left(\frac{2\mathrm{e}^{-3t}\cos 6t}{3}-\int -4\mathrm{e}^{-3t}\sin 6t \ \mathrm{d}t\right)$ ***A1***

$=-\frac{\mathrm{e}^{-3t}\sin 6t}{3}-\left(\frac{2\mathrm{e}^{-3t}\cos 6t}{3}+4s\right)$

$5s=\frac{-3\mathrm{e}^{-3t}\sin 6t-6\mathrm{e}^{-3t}\cos 6t}{9}$ ***M1***

OR

$=-\frac{\mathrm{e}^{-3t}\cos 6t}{6}-\int \frac{1}{2}\mathrm{e}^{-3t}\cos 6t \ \mathrm{d}t$ ***A1***

$=-\frac{\mathrm{e}^{-3t}\cos 6t}{6}-\left(\frac{\mathrm{e}^{-3t}\sin 6t}{12}+\int \frac{1}{4}\mathrm{e}^{-3t}\sin 6t \ \mathrm{d}t\right)$ ***A1***

$=-\frac{\mathrm{e}^{-3t}\cos 6t}{6}-\left(\frac{\mathrm{e}^{-3t}\sin 6t}{12}+\frac{1}{4}s\right)$

$\frac{5}{4}s=\frac{-2\mathrm{e}^{-3t}\cos 6t-\mathrm{e}^{-3t}\sin 6t}{12}$ ***M1***

THEN

$s=-\frac{\mathrm{e}^{-3t}\left(\sin 6t+2\cos 6t\right)}{15}(+c)$ ***A1***

at $t=0, s=0 \Rightarrow 0=-\frac{2}{15}+c$ ***M1***

$c=\frac{2}{15}$ ***A1***

$$s=\frac{2}{15}-\frac{e^{-3t}\left(\sin 6t+2\cos 6t\right)}{15}$$

[7 marks]

(c) **EITHER**

substituting $t=\frac{\pi}{6}$ into their equation for s ***(M1)***

$$\left(s=\frac{2}{15}-\frac{e^{-\frac{\pi}{2}}\left(\sin\pi+2\cos\pi\right)}{15}\right)$$

OR

using GDC to find maximum value ***(M1)***

OR

evaluating $\int_0^{\frac{\pi}{6}} v\,dt$ ***(M1)***

THEN

$$=0.161\left(=\frac{2}{15}\left(1+e^{-\frac{\pi}{2}}\right)\right)$$ ***A1***

[2 marks]

(d) **METHOD 1**

EITHER

distance required $= \int_0^{1.5} \left| e^{-3t} \sin 6t \right| \, dt$ ***(M1)***

OR

distance required $= \int_0^{\frac{\pi}{6}} e^{-3t} \sin 6t \, dt + \left| \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^{-3t} \sin 6t \, dt \right| + \int_{\frac{\pi}{3}}^{1.5} e^{-3t} \sin 6t \, dt$ ***(M1)***

$(= 0.16105... + 0.033479... + 0.006806...)$

THEN

$= 0.201$ (m) ***A1***

METHOD 2

using successive minimum and maximum values on the displacement graph ***(M1)***

$0.16105... + (0.16105... - 0.12757...) + (0.13453... - 0.12757...)$

$= 0.201$ (m) ***A1***

[2 marks]

(e) (i) valid attempt to find $\frac{dv}{dt}$ using product rule and set $\frac{dv}{dt} = 0$ ***M1***

$\frac{dv}{dt} = e^{-3t} 6\cos 6t - 3e^{-3t} \sin 6t$ ***A1***

$\frac{dv}{dt} = 0 \Rightarrow \tan 6t = 2$ ***AG***

(ii) attempt to evaluate t_1, t_2, t_3 in exact form ***M1***

$6t_1 = \arctan 2 \left(\Rightarrow t_1 = \frac{1}{6} \arctan 2 \right)$

$6t_2 = \pi + \arctan 2 \left(\Rightarrow t_2 = \frac{\pi}{6} + \frac{1}{6} \arctan 2 \right)$

$6t_3 = 2\pi + \arctan 2 \left(\Rightarrow t_3 = \frac{\pi}{3} + \frac{1}{6} \arctan 2 \right)$ ***A1***

Note: The ***A1*** is for any two consecutive correct, or showing that $6t_2 = \pi + 6t_1$ or $6t_3 = \pi + 6t_2$.

showing that $\sin 6t_{n+1} = -\sin 6t_n$

eg $\tan 6t = 2 \Rightarrow \sin 6t = \pm \frac{2}{\sqrt{5}}$ ***M1A1***

showing that $\frac{e^{-3t_{n+1}}}{e^{-3t_n}} = e^{-\frac{\pi}{2}}$ ***M1***

eg $e^{-3\left(\frac{\pi}{6}+k\right)} \div e^{-3k} = e^{-\frac{\pi}{2}}$

Note: Award the ***A1*** for any two consecutive terms.

$\frac{v_3}{v_2} = \frac{v_2}{v_1} = -e^{-\frac{\pi}{2}}$ ***AG***

[7 marks]
Total [20 marks]

Markscheme

November 2019

Mathematics

Higher level

Paper 2

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Examples

	Correct answer seen	**Further working seen**	**Action**
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Section A

1. $u_1r^3 = -70$, $u_1r^6 = 8.75$ ***(M1)***

$r^3 = \frac{8.75}{-70} = -0.125$ ***(A1)***

$\Rightarrow r = -0.5$ ***(A1)***

valid attempt to find u_2 ***(M1)***

for example: $u_1 = \frac{-70}{-0.125} = 560$

$u_2 = 560 \times -0.5$

$= -280$ ***A1***

[5 marks]

2. (a) $X \sim \text{Po}(1.3)$

$\text{P}(X \geq 2) = 0.373$ ***(M1)A1***

[2 marks]

(b) $V \sim \text{B}(5, 0.373)$ ***(M1)A1***

Note: Award ***(M1)*** for recognition of binomial or equivalent, ***A1*** for correct parameters.

$\text{P}(V = 4) = 0.0608$ ***(M1)A1***

[4 marks]

Total [6 marks]

3. (a) $f(1) = 0$ ***(A1)***

$f(0) = -1$ ***A1***

[2 marks]

(b) $a = f(3)$ ***(M1)***

$\Rightarrow a = 4$ ***A1***

[2 marks]

(c) domain is $-2 \leq x \leq 6$ ***A1***

range is $-6 \leq y \leq 10$ ***A1***

[2 marks]

Total [6 marks]

4. (a) each arc has length $r\theta = 6 \times \frac{\pi}{3} = 2\pi \left(= 6.283...\right)$ ***(M1)***

perimeter is therefore $6\pi \left(= 18.8\right) (\text{cm})$ ***A1***

[2 marks]

(b) area of sector, s, is $\frac{1}{2} r^2 \theta = 18 \times \frac{\pi}{3} = 6\pi \left(= 18.84...\right)$ ***(A1)***

area of triangle, t, is $\frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3} \left(= 15.58...\right)$ ***(M1)(A1)***

> **Note:** area of segment, k, is 3.261… implies area of triangle

finding $3s - 2t$ or $3k + t$ or similar

area $= 3s - 2t = 18\pi - 18\sqrt{3} \left(= 25.4\right) (\text{cm}^2)$ ***(M1)A1***

[5 marks]

Total [7 marks]

5. attempt to find coefficients in binomial expansion ***(M1)***

coefficient of x^2 : $\binom{n}{2} \times 2^{n-2}$; coefficient of x^3 : $\binom{n}{3} \times 2^{n-3}$ ***A1A1***

> **Note:** Condone terms given rather than coefficients.
> Terms may be seen in an equation such as that below.

$\binom{n}{3} \times 2^{n-3} = 4\binom{n}{2} \times 2^{n-2}$ ***(A1)***

attempt to solve equation using GDC or algebraically ***(M1)***

$\binom{n}{3} = 8\binom{n}{2}$

$\frac{n!}{3!(n-3)!} = \frac{8n!}{2!(n-2)!}$

$\frac{1}{3} = \frac{8}{n-2}$

$n = 26$ ***A1***

[6 marks]

6. **METHOD 1**

one other root is $3-\mathrm{i}$ ***A1***
let third root be α ***(M1)***
considering sum or product of roots ***(M1)***
sum of roots $=6+\alpha=\frac{37}{a}$ ***A1***
product of roots $=10\alpha=\frac{10}{a}$ ***A1***
hence $a=6$ ***A1***

[6 marks]

METHOD 2

one other root is $3-\mathrm{i}$ ***A1***
quadratic factor will be $z^2-6z+10$ ***(M1)A1***
$P(z)=az^3-37z^2+66z-10=(z^2-6z+10)(az-1)$ ***M1***
comparing coefficients ***(M1)***
hence $a=6$ ***A1***

[6 marks]

METHOD 3

substitute $3+\mathrm{i}$ into $P(z)$ ***(M1)***
$a(18+26\mathrm{i})-37(8+6\mathrm{i})+66(3+\mathrm{i})-10=0$ ***(M1)A1***
equating real or imaginary parts or dividing ***M1***
$18a-296+198-10=0$ or $26a-222+66=0$ or $\frac{10-66(3+\mathrm{i})+37(8+6\mathrm{i})}{18+26\mathrm{i}}$ ***A1***
hence $a=6$ ***A1***

[6 marks]

7. $T\sim \mathrm{N}(11.6, 0.8^2)$

$\mathrm{P}(T<10.7\,|\,T<11)$ ***(M1)***

$=\frac{\mathrm{P}(T<10.7\cap T<11)}{\mathrm{P}(T<11)}$ ***(M1)***

$=\frac{\mathrm{P}(T<10.7)}{\mathrm{P}(T<11)}$ ***(M1)***

$\mathrm{P}\,(T<10.7)=0.1302\ldots$ ***(A1)***

$\mathrm{P}\,(T<11)=0.2266\ldots$ ***(A1)***

$\mathrm{P}\,(T<10.7\,|\,T<11)=0.575$ ***A1***

Note: Accept only 0.575.

[6 marks]

8. (a) **METHOD 1**

$10!-2\times 9!\,(=2903040)$ ***(A1)(A1)A1***

> **Note:** Award ***A1*** for $10!$, ***A1*** for $2\times 9!$, ***A1*** for final answer.

METHOD 2

$9\times 8\times 8!$ ***(A1)(A1)A1***

> **Note:** Award ***A1*** for 9×8 or equivalent, ***A1*** for $8!$ and ***A1*** for answer.

[3 marks]

(b) **METHOD 1**

$8\times 7\times 8!\,(=2257920)$ ***(A1)A1***

> **Note:** Award ***(A1)*** for 8×7, ***A1*** for final answer.

METHOD 2

$10!-2\times 8!-2\times 2\times 7\times 8!$

> **Note:** Award ***A1*** for $10!$ minus EITHER subtracted terms and ***A1*** for final correct answer.

[2 marks]

(c) **METHOD 1**

$8\times 7\times (8!-2\times 7!)\,(=1693440)$ ***(A1)(A1)A1***

> **Note:** Award ***(A1)*** for 8×7, ***(A1)*** for $2\times 7!$, ***A1*** for final answer.
> $(8!-2\times 7!)$ can be replaced by $6\times 7!$ *or* ${}^7P_2\times 6!$ which may be awarded the second ***A1***.

METHOD 2

their answer to (a) $-2\times 8!-2\times 2\times 7\times 8!$ ***(A1)(A1)A1***

> **Note:** Award ***A1*** for subtracting each of the terms and ***A1*** for final answer.

METHOD 3

their answer to (b) $-2\times 7\times 8!$ or equivalent ***(A1)A2***

> **Note:** Award ***A1*** for the subtraction and ***A2*** for final answer.

[3 marks]

Total [8 marks]

Section B

9. (a) (i) $A(7.47, 2.28)$ and $B(43.4, -2.45)$ ***A1A1A1A1***

(ii) maximum speed is 2.45 (m s^{-1}) ***A1***

[5 marks]

(b) (i) $v = 0 \Rightarrow t_1 = 25.1\,(\text{s})$ ***(M1)A1***

(ii) $\int_0^{t_1} v\,\text{d}t$ ***(M1)***

$= 41.0\,(\text{m})$ ***A1***

(iii) $a = \frac{\text{d}v}{\text{d}t}$ at $t = t_1 = 25.1$ ***(M1)***

$a = -0.200$ (m s^{-2}) ***A1***

Note: Accept $a = -0.2$.

[6 marks]

(c) attempt to integrate between 0 and 30 ***(M1)***

Note: An unsupported answer of 38.6 can imply integrating from 0 to 30.

EITHER

$\int_0^{30} |v|\,\text{d}t$ ***(A1)***

OR

$41.0 - \int_{t_1}^{30} v\,\text{d}t$ ***(A1)***

THEN

$= 43.3\,(\text{m})$ ***A1***

[3 marks]

Total [14 marks]

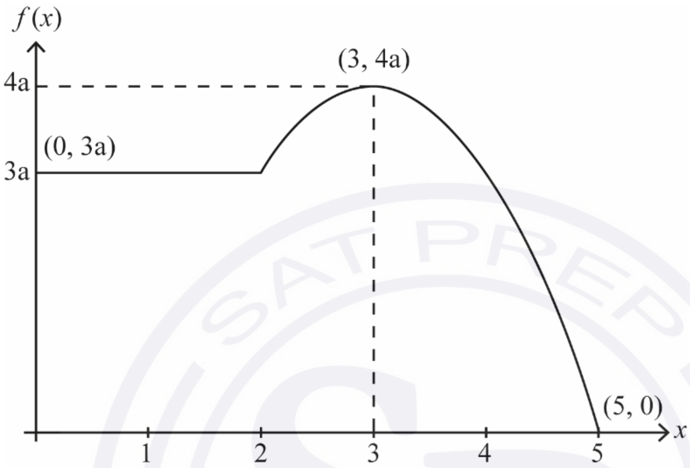
10. (a) $\left(\mathrm{P}(1 < X < 3) =\right) \int_1^2 3a \, \mathrm{d}x + a\int_2^3 -x^2 + 6x - 5 \, \mathrm{d}x$ ***(M1)(A1)(A1)***

$= 3a + \frac{11}{3}a$

$= \frac{20}{3}a \, (= 6.67a)$ ***A1***

[4 marks]

(b)



A4

award ***A1*** for $(0, 3a)$, ***A1*** for continuity at $(2, 3a)$, ***A1*** for maximum at $(3, 4a)$, ***A1*** for $(5, 0)$

Note: Award ***A3*** if correct four points are not joined by a straight line and a quadratic curve.

[4 marks]

(c) (i) $\mathrm{P}(0 \le X \le 5) = 6a + a\int_2^5 -x^2 + 6x - 5 \, \mathrm{d}x$ ***(M1)***

$= 15a$ ***(A1)***

$15a = 1$ ***(M1)***

$\Rightarrow a = \frac{1}{15} \, (= 0.0667)$ ***A1***

(ii) $\mathrm{E}(X) = \frac{1}{5}\int_0^2 x \, \mathrm{d}x + \frac{1}{15}\int_2^5 -x^3 + 6x^2 - 5x \, \mathrm{d}x$ ***(M1)(A1)***

$= 2.35$ ***A1***

continued…

Question 10 continued

(iii) attempt to use $\int_0^m f(x)\mathrm{d}x = 0.5$ ***(M1)***

$0.4 + a\int_2^m -x^2 + 6x - 5\,\mathrm{d}x = 0.5$ ***(A1)***

$a\int_2^m -x^2 + 6x - 5\,\mathrm{d}x = 0.1$

attempt to solve integral using GDC and/or analytically ***(M1)***

$\frac{1}{15}\left[-\frac{1}{3}x^3 + 3x^2 - 5x\right]_2^m = 0.1$

$m = 2.44$ ***A1***

[11 marks]

Total [19 marks]

11. (a) (i) valid attempt to differentiate implicitly ***(M1)***

$4x = 3\sin^2 y \cos y \frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{3\sin^2 y \cos y}$ ***A1***

(ii) at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{3\sin^2 y \cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ ***(M1)***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8}{3\sqrt{3}}\,(= -1.54)$ ***A1***

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ ***(M1)A1***

Note: Accept $y = -1.54x + 3$.

[8 marks]

(b) $x = \sqrt{\frac{1}{2}\sin^3 y}$ ***(M1)***

$\int_0^{\pi} \sqrt{\frac{1}{2}\sin^3 y}\,\mathrm{d}y$ ***(A1)***

$= 1.24$ ***A1***

[3 marks]

continued…

Question 11 continued

(c) use of volume $= \int \pi x^2 \mathrm{d}y$ ***(M1)***

$= \int_0^{\pi} \frac{1}{2}\pi \sin^3 y \, \mathrm{d}y$ ***A1***

$= \frac{1}{2}\pi \int_0^{\pi} \left(\sin y - \sin y \cos^2 y\right) \mathrm{d}y$

Note: Condone absence of limits up to this point.

reasonable attempt to integrate ***(M1)***

$= \frac{1}{2}\pi \left[-\cos y + \frac{1}{3}\cos^3 y \right]_0^{\pi}$ ***A1A1***

Note: Award ***A1*** for correct limits (not to be awarded if previous ***M1*** has not been awarded) and ***A1*** for correct integrand.

$= \frac{1}{2}\pi\left(1 - \frac{1}{3}\right) - \frac{1}{2}\pi\left(-1 + \frac{1}{3}\right)$ ***A1***

$= \frac{2\pi}{3}$ ***AG***

Note: Do not accept decimal answer equivalent to $\frac{2\pi}{3}$.

[6 marks]

Total [17 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2019

Mathematics

Higher level

Paper 2

Instructions to Examiners

Abbreviations

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

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Section A

1. **METHOD 1**

equation of tangent is $y = 22.167...\ x - 14.778...$ **OR** $y - 7.389... = 22.167...\ (x-1)$

(M1)(A1)

meets the x-axis when $y = 0$

$x = 0.667$

meets x -axis at $(0.667, 0)\left(=\left(\frac{2}{3}, 0\right)\right)$ ***A1A1***

> **Note:** Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates $(x, 0)$ given.

METHOD 2

Attempt to differentiate ***(M1)***

$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$

when $x = 1,\ \frac{dy}{dx} = 3e^2$ ***(M1)***

equation of the tangent is $y - e^2 = 3e^2(x-1)$

$y = 3e^2x - 2e^2$

meets x-axis at $x = \frac{2}{3}$

$\left(\frac{2}{3}, 0\right)$ ***A1A1***

> **Note:** Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates $(x, 0)$ given.

Total [4 marks]

2. (a) $z = 2e^{\frac{\pi}{4}i}\left(= 2e^{0.785i}\right)$ ***A1***

> **Note:** Accept all answers in the form $2e^{\left(\frac{\pi}{4}+2\pi n\right)i}$.

$z = 2e^{\frac{5\pi}{4}i}\left(= 2e^{3.93i}\right)$ **OR** $z = 2e^{-\frac{3\pi}{4}i}\left(= 2e^{-2.36i}\right)$ ***(M1)A1***

> **Note:** Accept all answers in the form $2e^{\left(-\frac{3\pi}{4}+2\pi n\right)i}$.

> **Note:** Award ***M1A0*** for correct answers in the incorrect form, *eg* $-2e^{\frac{\pi}{4}i}$.

[3 marks]

continued…

Question 2 continued

(b) $z = 1.41 + 1.41\text{i}$, $z = -1.41 - 1.41\text{i}$ ***A1A1***
$\left(z = \sqrt{2} + \sqrt{2}\text{i}, z = -\sqrt{2} - \sqrt{2}\text{i}\right)$

[2 marks]

Total [5 marks]

3. (a) (i) 6.75 ***A1***

(ii) 2.22 ***A1***

[2 marks]

(b) (i) 8.75 ***A1***

(ii) 2.22 ***A1***

[2 marks]

(c) the order is $3, 4, 6, 7, 7, 8, 9, 10$
median is currently 7 ***A1***

Note: This can be indicated by a diagram/list, rather than actually stated.

with 9 numbers the middle value (median) will be the 5^{th} value ***R1***
which will correspond to 7 regardless of whether the position of the median moves up or down ***R1***

Note: Accept answers using data $5, 6, 8, 9, 9, 10, 11, 12$ (ie from part (b)).

[3 marks]

Total [7 marks]

4. (a) $f(x) \geq 3$ ***A1***

[1 mark]

(b) $x = \sec y + 2$ ***(M1)***

Note: Exchange of variables can take place at any point.

$\cos y = \dfrac{1}{x-2}$ ***(A1)***

$f^{-1}(x) = \arccos\left(\dfrac{1}{x-2}\right)$, $x \geq 3$ ***A1A1***

Note: Allow follow through from (a) for last ***A1*** mark which is independent of earlier marks in (b).

[4 marks]

Total [5 marks]

5. **METHOD 1**

write as $\int 1 \times (\ln x)^2 \mathrm{d}x$ ***(M1)***

$= x(\ln x)^2 - \int x \times \frac{2(\ln x)}{x}\mathrm{d}x \left(= x(\ln x)^2 - \int 2\ln x\right)$ ***M1A1***

$= x(\ln x)^2 - 2x\ln x + \int 2\mathrm{d}x$ ***(M1)(A1)***

$= x(\ln x)^2 - 2x\ln x + 2x + c$ ***A1***

METHOD 2

let $u = \ln x$ ***M1***

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int u^2 \mathrm{e}^u \mathrm{d}u$ ***A1***

$= u^2\mathrm{e}^u - \int 2u\mathrm{e}^u \mathrm{d}u$ ***M1***

$= u^2\mathrm{e}^u - 2u\mathrm{e}^u + \int 2\mathrm{e}^u \mathrm{d}u$ ***A1***

$= u^2\mathrm{e}^u - 2u\mathrm{e}^u + 2\mathrm{e}^u + c$

$= x(\ln x)^2 - 2x\ln x + 2x + c$ ***M1A1***

METHOD 3

Setting up $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \ln x$ ***M1***

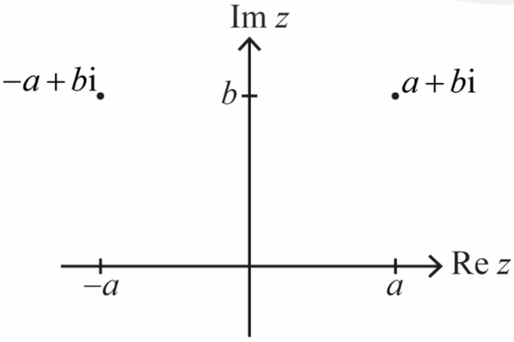
$\ln x(x\ln x - x) - \int (\ln x - 1)\mathrm{d}x$ ***M1A1***

$= x(\ln x)^2 - x\ln x - (x\ln x - x) + x + c$ ***M1A1***

$= x(\ln x)^2 - 2x\ln x + 2x + c$ ***A1***

Total [6 marks]

6. (a)



A1

Note: Award ***A1*** for z in first quadrant and z-2a its reflection in the y-axis.

[1 mark]

continued…

Question 6 continued

(b) (i) $\pi - \theta$ (or any equivalent) ***A1***

(ii) $\arg\left(\frac{z}{z-2a}\right) = \arg(z) - \arg(z-2a)$ ***(M1)***

$= 2\theta - \pi$ (or any equivalent) ***A1***

[3 marks]

(c) **METHOD 1**

if $\text{Re}\left(\frac{z}{z-2a}\right) = 0$ then $2\theta - \pi = \frac{n\pi}{2}$, ($n$ odd) ***(M1)***

$-\pi < 2\theta - \pi < 0 \Rightarrow n = -1$

$2\theta - \pi = -\frac{\pi}{2}$ ***(A1)***

$\theta = \frac{\pi}{4}$ ***A1***

METHOD 2

$\frac{a+b\text{i}}{-a+b\text{i}} = \frac{b^2 - a^2 - 2ab\text{i}}{a^2 + b^2}$ ***M1***

$\text{Re}\left(\frac{z}{z-2a}\right) = 0 \Rightarrow b^2 - a^2 = 0$

$b = a$ ***A1***

$\theta = \frac{\pi}{4}$ ***A1***

Note: Accept any equivalent, *eg* $\theta = -\frac{7\pi}{4}$.

[3 marks]

Total [7 marks]

7. volume $= \pi\int_0^9 \left(y^{\frac{1}{2}} + 1\right)^2 \text{d}y - \pi\int_1^9 (y-1)\,\text{d}y$ ***(M1)(M1)(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of formula for rotating about y-axis, ***(M1)*** for finding at least one inverse, ***(M1)*** for subtracting volumes, ***(A1)(A1)*** for each correct expression, including limits.

$= 268.6\ldots - 100.5\ldots (85.5\pi - 32\pi)$

$= 168 (= 53.5\pi)$ ***A2***

Total [7 marks]

8. (a) $x < -0.414,\ x > 2.41$ ***A1A1***

$\left(x < 1-\sqrt{2},\ x > 1+\sqrt{2}\right)$

> **Note:** Award ***A1*** for -0.414, 2.41 and ***A1*** for correct inequalities.

[2 marks]

(b) check for $n = 3$,

$16 > 9$ so true when $n = 3$ ***A1***

assume true for $n = k$

$2^{k+1} > k^2$ ***M1***

> **Note:** Award ***M0*** for statements such as "let $n = k$".

> **Note:** Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

prove true for $n = k+1$

$2^{k+2} = 2 \times 2^{k+1}$

$> 2k^2$ ***M1***

$= k^2 + k^2$ ***(M1)***

$> k^2 + 2k + 1$ (from part (a)) ***A1***

which is true for $k \geq 3$ ***R1***

> **Note:** Only award the **A1** or the **R1** if it is clear why. Alternate methods are possible.

$= (k+1)^2$

hence if true for $n = k$ true for $n = k+1$, true for $n = 3$ so true for all $n \geq 3$ ***R1***

> **Note:** Only award the final ***R1*** provided at least three of the previous marks are awarded.

[7 marks]

Total [9 marks]

Section B

9. (a) (i) use of formula or Venn diagram ***(M1)***

$0.72 + 0.45 - 1$ ***(A1)***

$= 0.17$ ***A1***

(ii) $0.72 - 0.17 = 0.55$ ***A1***

[4 marks]

(b) (i) $200 \times 0.45 = 90$ ***A1***

(ii) let X be the number of customers who order cake

$X \sim \mathrm{B}(200, 0.45)$ ***(M1)***

$\mathrm{P}(X > 100) = \mathrm{P}(X \geq 101)\left(= 1 - \mathrm{P}(X \leq 100)\right)$ ***(M1)***

$= 0.0681$ ***A1***

[4 marks]

(c) (i) $0.46 \times 0.8 = 0.368$ ***A1***

(ii) **METHOD 1**

$0.368 + 0.54 \times \mathrm{P}(S \mid F) = 0.72$ ***M1A1A1***

Note: Award ***M1*** for an appropriate tree diagram. Award ***A1*** for LHS, ***A1*** for RHS.

$\mathrm{P}(S \mid F) = 0.652$ ***A1***

METHOD 2

$\mathrm{P}(S \mid F) = \dfrac{\mathrm{P}(S \cap F)}{\mathrm{P}(F)}$ ***(M1)***

$= \dfrac{0.72 - 0.368}{0.54}$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$\mathrm{P}(S \mid F) = 0.652$ ***A1***

[5 marks]

Total [13 marks]

10. (a) $3, -3$ ***A1A1***

[2 marks]

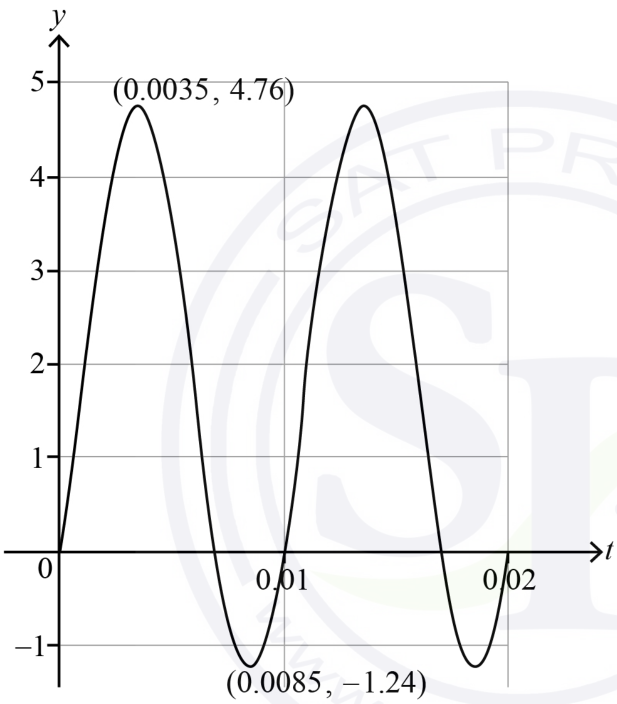
(b) stretch parallel to the y-axis (with x-axis invariant), scale factor $\frac{2}{3}$ ***A1***

translation of $\begin{pmatrix} -0.003 \\ 0 \end{pmatrix}$ (shift to the left by 0.003) ***A1***

Note: Can be done in either order.

[2 marks]

(c)



correct shape over correct domain with correct endpoints ***A1***
first maximum at $(0.0035, 4.76)$ ***A1***
first minimum at $(0.0085, -1.24)$ ***A1***

[3 marks]

(d) $p \geq 3$ between $t = 0.0016762$ and 0.0053238 and $t = 0.011676$ and 0.015324 ***(M1)(A1)***

Note: Award ***M1A1*** for either interval.

$= 0.00730$ ***A1***

[3 marks]

(e) $p_{av} = \frac{1}{0.007}\int_0^{0.007} 6\sin(100\pi t)\sin\left(100\pi(t + 0.003)\right)\mathrm{d}t$ ***(M1)***

$= 2.87$ ***A1***

[2 marks]

continued…

Question 10 continued

(f) in each cycle the area under the t axis is smaller than area above the t axis ***R1***
the curve begins with the positive part of the cycle ***R1***

[2 marks]

(g) $a = \frac{4.76-(-1.24)}{2}$ ***(M1)***

$a = 3.00$ ***A1***

$d = \frac{4.76+(-1.24)}{2}$

$d = 1.76$ ***A1***

$b = \frac{2\pi}{0.01}$

$b = 628 (= 200\pi)$ ***A1***

$c = 0.0035 - \frac{0.01}{4}$ ***(M1)***

$c = 0.00100$ ***A1***

[6 marks]

Total [20 marks]

11. (a) recognition of the other root $= -d\text{i}$ ***(A1)***

$\log_2 a + \log_2 b + \log_2 c + d\text{i} - d\text{i} = 3$ ***M1A1***

Note: Award ***M1*** for sum of the roots, ***A1*** for 3. Award ***A0M1A0*** for just $\log_2 a + \log_2 b + \log_2 c = 3$.

$\log_2 abc = 3$ ***(M1)***

$\Rightarrow abc = 2^3$ ***A1***

$abc = 8$ ***AG***

[5 marks]

continued…

Question 11 continued

(b) **METHOD 1**

let the geometric series be u_1, u_1r, u_1r^2

$(u_1r)^3 = 8$ ***M1***

$u_1r = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$

$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***

$b = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

[3 marks]

(c) **METHOD 1**

product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***

$r_1 \times r_2 = -8$ ***A1***

sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***

$r_1 + r_2 = 2$ ***A1***

solving simultaneously ***(M1)***

$r_1 = -2$, $r_2 = 4$ ***A1A1***

METHOD 2

product of the roots $\log_2 a \times \log_2 b \times \log_2 c \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***M1A1***

$\log_2 a \times \log_2 b \times \log_2 c = -8$ ***A1***

EITHER

a, b, c can be written as $\frac{2}{r}$, 2, $2r$ ***M1***

$\left(\log_2 \frac{2}{r}\right)(\log_2 2)(\log_2 2r) = -8$

attempt to solve ***M1***

$(1 - \log_2 r)(1 + \log_2 r) = -8$

$\log_2 r = \pm 3$

$r = \frac{1}{8}, 8$ ***A1A1***

continued…

Question 11 continued

OR

a, b, c can be written as $a, 2, \frac{4}{a}$ ***M1***

$(\log_2 a)(\log_2 2)\left(\log_2 \frac{4}{a}\right) = -8$

attempt to solve ***M1***

$a = \frac{1}{4}, 16$ ***A1A1***

THEN

a, and c are $\frac{1}{4}, 16$ ***(A1)***

roots are $-2,\ 4$ ***A1***

[9 marks]

Total [17 marks]

M19/5/MATHL/HP2/ENG/TZ2/XX/M

Markscheme

May 2019

Mathematics

Higher level

Paper 2

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the “must be seen” marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).
A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. attempt to apply cosine rule ***M1***

$\cos A = \dfrac{5^2 + 11^2 - 14^2}{2 \times 5 \times 11} = -0.4545...$

$\Rightarrow A = 117.03569...^\circ$

$\Rightarrow A = 117.0^\circ$ ***A1***

attempt to apply sine rule or cosine rule: ***M1***

$\dfrac{\sin 117.03569...^\circ}{14} = \dfrac{\sin B}{11}$

$\Rightarrow B = 44.4153...^\circ$

$\Rightarrow B = 44.4^\circ$ ***A1***

$C = 180^\circ - A - B$

$C = 18.5^\circ$ ***A1***

Note: Candidates may attempt to find angles in any order of their choosing.

[5 marks]

2. (a) $X \sim N(820, 230^2)$ ***(M1)***

Note: Award ***M1*** for an attempt to use normal distribution. Accept labelled normal graph.

$\Rightarrow P(X > 1000) = 0.217$ ***A1***

[2 marks]

(b) $Y \sim B(24, 0.217...)$ ***(M1)***

Note: Award ***M1*** for recognition of binomial distribution with parameters.

$P(Y \le 10) - P(Y \le 4)$ ***(M1)***

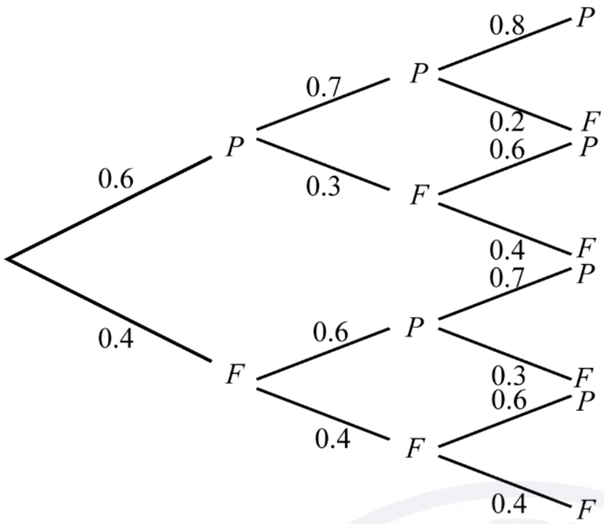
Note: Award ***M1*** for an attempt to find $P(5 \le Y \le 10)$ or $P(Y \le 10) - P(Y \le 4)$.

$= 0.613$ ***A1***

[3 marks]

Total [5 marks]

3. (a)



A1A1A1

Note: Award ***A1*** for each correct column of probabilities.

[3 marks]

(b) probability (at least twice) $=$

EITHER

$(0.6\times0.7\times0.8)+(0.6\times0.7\times0.2)+(0.6\times0.3\times0.6)+(0.4\times0.6\times0.7)$ ***(M1)***

OR

$(0.6\times0.7) + (0.6\times0.3\times0.6) + (0.4\times0.6\times0.7)$ ***(M1)***

Note: Award ***M1*** for summing all required probabilities.

THEN

$=0.696$ ***A1***

[2 marks]

(c) P(passes third paper given only one paper passed before)

$$=\frac{\text{P (passes third AND only one paper passed before)}}{\text{P (passes once in first two papers)}}$$ ***(M1)***

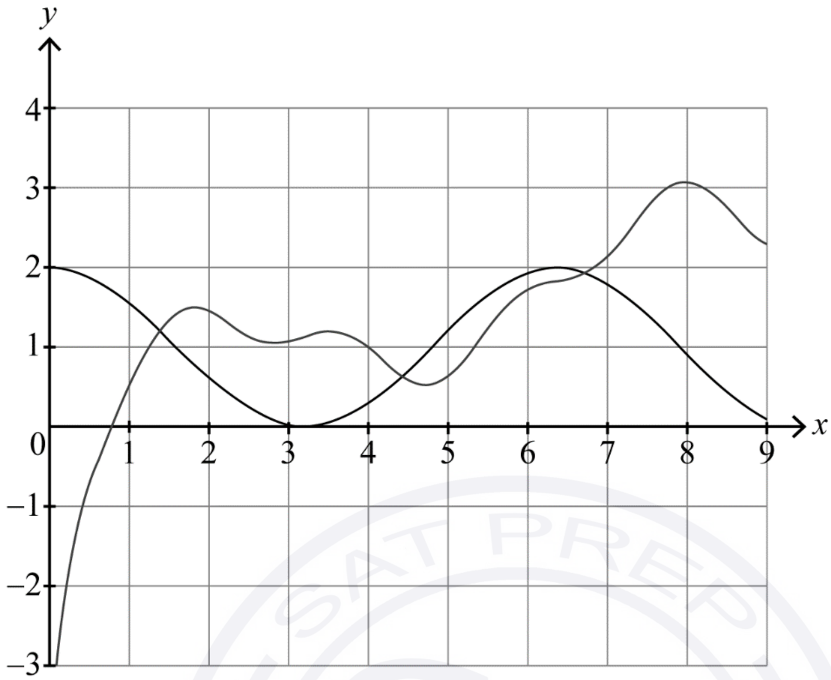
$$=\frac{(0.6\times0.3\times0.6)+(0.4\times0.6\times0.7)}{(0.6\times0.3)+(0.4\times0.6)}$$ ***A1***

$=0.657$ ***A1***

[3 marks]

Total [8 marks]

4. (a)



A1A1

Note: Award ***A1*** for each correct curve, showing all local max & mins.

Note: Award ***A0A0*** for the curves drawn in degrees.

[2 marks]

(b) $x = 1.35, 4.35, 6.64$ ***(M1)***

Note: Award ***M1*** for attempt to find points of intersections between two curves.

$0 < x < 1.35$ ***A1***

Note: Accept $x < 1.35$.

$4.35 < x < 6.64$ ***A1A1***

Note: Award ***A1*** for correct endpoints, ***A1*** for correct inequalities.

Note: Award ***M1FTA1FTA0FTA0FT*** for $0 < x < 7.31$.

Note: Accept $x < 7.31$.

[4 marks]

Total [6 marks]

5. (a) **METHOD 1**

$\text{LHS} = \frac{1+\sin 2x}{\cos 2x} = \frac{1+2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ***M1***

$= \frac{\left(\cos^2 x + \sin^2 x\right) + 2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ***M1***

$= \frac{\left(\cos x + \sin x\right)^2}{\left(\cos x + \sin x\right)\left(\cos x - \sin x\right)}$ ***A1***

$= \frac{\cos x + \sin x}{\cos x - \sin x}$

$= \frac{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}$ ***A1***

$= \frac{1+\tan x}{1-\tan x}$ ***AG***

Note: Candidates may start with RHS, apply MS in reverse.

[4 marks]

METHOD 2

$\text{LHS} = \frac{1+\sin 2x}{\cos 2x} = \frac{1+2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ***M1***

dividing numerator and denominator by $\cos^2 x$ ***M1***

$= \frac{\sec^2 x + 2\tan x}{1-\tan^2 x}$

$= \frac{1+\tan^2 x + 2\tan x}{1-\tan^2 x}$ ***A1***

$= \frac{(\tan x + 1)^2}{(1-\tan x)(1+\tan x)}$ ***A1***

$= \frac{1+\tan x}{1-\tan x}$ ***AG***

Note: Candidates may start with RHS; apply MS in reverse.

[4 marks]

(b) valid attempt to solve $\frac{1+\tan x}{1-\tan x} = \sqrt{3}$ ***(M1)***

$\tan x = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$x = 0.262\left(=\frac{\pi}{12}\right),\ x = 3.40\left(=\frac{13\pi}{12}\right)$ ***A1***

Note: Award ***M1A0*** if only one correct solution is given.

[2 marks]

Total [6 marks]

6. attempt to integrate a to find v ***M1***

$v = \int a \ \mathrm{d}t = \int (2t-1) \ \mathrm{d}t$

$= t^2 - t + c$ ***A1***

$s = \int v \ \mathrm{d}t = \int (t^2 - t + c) \ \mathrm{d}t$

$= \frac{t^3}{3} - \frac{t^2}{2} + ct + d$ ***A1***

attempt at substitution of given values ***(M1)***

at $t=6$, $18.25 = 72 - 18 + 6c + d$

at $t=15$, $922.75 = 1125 - 112.5 + 15c + d$

solve simultaneously: ***(M1)***

$c = -6$; $d = 0.25$ ***A1***

$\Rightarrow s = \frac{t^3}{3} - \frac{t^2}{2} - 6t + \frac{1}{4}$

[6 marks]

7. $n=1 \Rightarrow S_1 = u_1$, so true for $n=1$ ***R1***

assume true for $n=k$, ie. $S_k = \frac{u_1(1-r^k)}{1-r}$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".

Note: Subsequent marks after the first ***M1*** are independent of this mark and can be awarded.

$S_{k+1} = S_k + u_1 r^k$ ***M1***

$S_{k+1} = \frac{u_1(1-r^k)}{1-r} + u_1 r^k$ ***A1***

$S_{k+1} = \frac{u_1(1-r^k)}{1-r} + \frac{u_1 r^k (1-r)}{1-r}$

$S_{k+1} = \frac{u_1 - u_1 r^k + u_1 r^k - r u_1 r^k}{1-r}$ ***A1***

$S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$ ***A1***

true for $n=1$ and if true for $n=k$ then true for $n=k+1$,
the statement is true for any positive integer (or equivalent). ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[7 marks]

8. (a) **METHOD 1**

$w^3 = 8\mathrm{i}$

writing $8\mathrm{i} = 8\left(\cos\left(\frac{\pi}{2}+2\pi k\right)+\mathrm{i}\sin\left(\frac{\pi}{2}+2\pi k\right)\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to find cube roots of w using modulus-argument form.

cube roots $w = 2\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)+\mathrm{i}\sin\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)\right)$ ***(M1)***

ie. $w=\sqrt{3}+\mathrm{i},-\sqrt{3}+\mathrm{i},-2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w=1.73+\mathrm{i}$ and $w=-1.73+\mathrm{i}$.

[4 marks]

METHOD 2

$w^3+(2\mathrm{i})^3=0$

$(w+2\mathrm{i})\left(w^2-2w\mathrm{i}-4\right)=0$ ***M1***

$w=\frac{2\mathrm{i}\pm\sqrt{12}}{2}$ ***M1***

$w=\sqrt{3}+\mathrm{i},-\sqrt{3}+\mathrm{i},-2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w=1.73+\mathrm{i}$ and $w=-1.73+\mathrm{i}$.

[4 marks]

(b) $w_1=-2\mathrm{i}$

$\frac{z}{z-\mathrm{i}}=-2\mathrm{i}$ ***M1***

$z=-2\mathrm{i}(z-\mathrm{i})$

$z(1+2\mathrm{i})=-2$

$z=\frac{-2}{1+2\mathrm{i}}$ ***A1***

$z=-\frac{2}{5}+\frac{4}{5}\mathrm{i}$ ***A1***

Note: Accept $a=-\frac{2}{5}, b=\frac{4}{5}$.

[3 marks]

Total [7 marks]

Section B

9. (a) **METHOD 1**

attempt to find roots or factors *(M1)*

roots are $-3,\ 1,\ (4+\mathrm{i}),(4-\mathrm{i})$ *A1A1*

Note: Award ***A1*** for each pair of roots or factors, real and complex.

attempt to form quadratic *M1*

$(z-1)(z+3)=z^2+2z-3$ *A1*

$(z-(4+\mathrm{i}))(z-(4-\mathrm{i}))$

$=z^2-(4-\mathrm{i})z-(4+\mathrm{i})z+17$ *(A1)*

$=z^2-8z+17$ *A1*

$z^4-6z^3-2z^2+58z-51=(z^2-8z+17)(z^2+2z-3)$

[7 marks]

METHOD 2

attempt to find roots or factors *(M1)*

real roots are $-3,\ 1$ (or real factors $(z+3),(z-1)$) *A1*

attempt to form quadratic *M1*

$(z-1)(z+3)=z^2+2z-3$ *A1*

$z^4-6z^3-2z^2+58z-51=\left[z^2+2z-3\right]\left[z^2+kz+17\right]$

equate coefficients of z^2 *M1*

$-2=2k-3+17$ *A1*

solve to give $k=-8$ *A1*

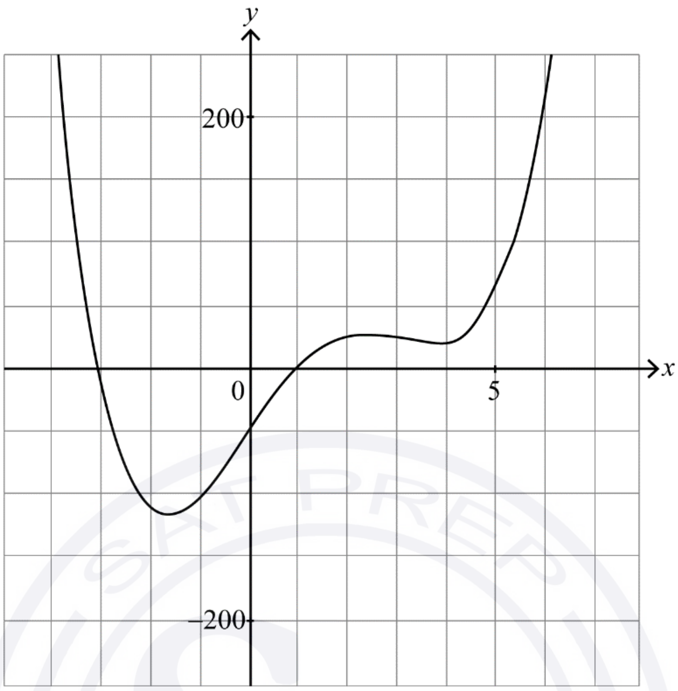
$z^4-6z^3-2z^2+58z-51=(z^2-8z+17)(z^2+2z-3)$

[7 marks]

continued…

Question 9 continued

(b)



shape **A1**

x-axis intercepts at $(-3, 0)$, $(1, 0)$ and y-axis intercept at $(0, -51)$ **A1A1**

minimum points at $(-1.62, -118)$ and $(3.72, 19.7)$ **A1A1**

maximum point at $(2.40, 26.9)$ **A1**

Note: Coordinates may be seen on the graph or elsewhere.

Note: Accept -3, 1 and -51 marked on the axes.

[6 marks]

(c) from graph, $19.7 \le k \le 26.9$ **A1A1**

Note: Award ***A1*** for correct endpoints and ***A1*** for correct inequalities.

[2 marks]

Total [15 marks]

10. (a) $X \sim \text{Po}(2.1)$

$\text{P}(X = 0) = 0.122 \left(= \text{e}^{-2.1}\right)$ ***(M1)A1***

[2 marks]

continued…

Question 10 continued

(b)

y	0	1	2	3	4
$\mathrm{P}(Y=y)$	0.122... $\left(=\mathrm{e}^{-2.1}\right)$	0.257... $\left(=\mathrm{e}^{-2.1}2.1\right)$	0.270... $\left(=\frac{\mathrm{e}^{-2.1}2.1^2}{2!}\right)$	0.189... $\left(=\frac{\mathrm{e}^{-2.1}2.1^3}{3!}\right)$	0.161...

A1A1A1A1

Note: Award ***A1*** for each correct probability for $Y=1, 2, 3, 4$. Accept 0.162 for $\mathrm{P}(Y=4)$.

[4 marks]

(c) $\mathrm{E}(Y)=\sum y\mathrm{P}(Y=y)$ ***(M1)***

$=1\times0.257...+2\times0.270...+3\times0.189...+4\times0.161...$ ***(A1)***

$=2.01$ ***A1***

[3 marks]

(d) let T be the no of days per year that Steffi does not visit

$T\sim\mathrm{B}(365, 0.122...)$ ***(M1)***

require $0.45\le\mathrm{P}(T\le n)<0.55$ ***(M1)***

$\mathrm{P}(T\le44)=0.51$

$n=44$ ***A1***

[3 marks]

(e) **METHOD 1**

let V be the discrete random variable "number of times Steffi is not fed per day"

$\mathrm{E}(V)=1\times\mathrm{P}(X=5)+2\times\mathrm{P}(X=6)+3\times\mathrm{P}(X=7)+\cdots$ ***M1***

$=1\times0.0416...+2\times0.0145...+3\times0.00437...+\cdots$ ***A1***

$=0.083979...$ ***A1***

expected no of occasions per year $>0.083979...\times365=30.7$ ***A1***

hence Steffi can expect not to be fed on at least 30 occasions ***AG***

Note: Candidates may consider summing more than three terms in their calculation for $\mathrm{E}(V)$.

[4 marks]

METHOD 2

$\mathrm{E}(X)-\mathrm{E}(Y)=0.0903\ldots$ ***M1A1***

$0.0903\ldots\times365$ ***M1***

$=33.0>30$ ***A1AG***

[4 marks]

Total [16 marks]

11. (a) **METHOD 1**

for example

$\vec{PQ} = \begin{pmatrix} -1 \\ -5 \\ 8 \end{pmatrix}, \ \vec{PR} = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$ ***A1A1***

$\vec{PQ} \times \vec{PR} = 33\boldsymbol{i} + 11\boldsymbol{j} + 11\boldsymbol{k}$ ***(M1)A1***

$\boldsymbol{r.n = a.n}$

$33x + 11y + 11z = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 33 \\ 11 \\ 11 \end{pmatrix} = 22$ ***(M1)***

$\Rightarrow 3x + y + z = 2$ or equivalent ***A1***

[6 marks]

METHOD 2

assume plane can be written as $ax + by + cz = 1$ ***M1***

substituting each set of coordinates gives the system of equations:

$a + 6b - 7c = 1$

$0a + b + c = 1$

$2a + 0b - 4c = 1$ ***A1***

solving by GDC ***(M1)***

$a = \frac{3}{2}, \ b = \frac{1}{2}, \ c = \frac{1}{2}$ ***A1A1A1***

$\Rightarrow \frac{3}{2}x + \frac{1}{2}y + \frac{1}{2}z = 1$ or equivalent

[6 marks]

(b) **METHOD 1**

substitution of equation of line into both equations of planes ***M1***

$3\left(\frac{5}{4} + \frac{\lambda}{2}\right) + \lambda + \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 2$ ***A1***

$\left(\frac{5}{4} + \frac{\lambda}{2}\right) - 3\lambda - \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 3$ ***A1***

[3 marks]

continued…

Question 11 continued

METHOD 2

adding Π_1 and Π_2 gives $4x - 2y = 5$ ***M1***

given $y = \lambda \Rightarrow x = \frac{5}{4} + \frac{\lambda}{2}$ ***A1***

$z = 2 - y - 3x = -\frac{7}{4} - \frac{5\lambda}{2}$ ***A1***

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$$ ***AG***

[3 marks]

METHOD 3

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$$ ***A1***

$$= 4\begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$$ ***R1***

common point $\frac{5}{4} - 3(0) - \left(-\frac{7}{4}\right) = 3$ and $-3\left(\frac{5}{4}\right) - 0 - \left(-\frac{7}{4}\right) = -2$ ***A1***

[3 marks]

(c) normal to Π_3 is perpendicular to direction of L

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 0$$ ***A1***

$\Rightarrow a + 2b - 5c = 0$ ***AG***

[1 mark]

continued…

Question 11 continued

(d) (i) substituting $\begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix}$ into Π_3: ***M1***

$\frac{5a}{4} - \frac{7c}{4} = 1$ ***A1***

$5a - 7c = 4$ ***AG***

(ii) attempt to find scalar products for Π_1 and Π_3, Π_2 and Π_3 and equating ***M1***

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = \frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***M1***

Note: Accept $3a+b+c = a-3b-c$.

$\Rightarrow a + 2b + c = 0$ ***A1***

attempt to solve $a+2b+c=0$, $a+2b-5c=0$, $5a-7c=4$ ***M1***

$\Rightarrow a = \frac{4}{5},\ b = -\frac{2}{5},\ c = 0$ ***A1***

hence equation is $\frac{4x}{5} - \frac{2y}{5} = 1$

for second equation:

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = -\frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***(M1)***

$\Rightarrow 2a - b = 0$

attempt to solve $2a-b=0$, $a+2b-5c=0$, $5a-7c=4$

$\Rightarrow a = -2,\ b = -4,\ c = -2$ ***A1***

hence equation is $-2x - 4y - 2z = 1$

[9 marks]

Total [19 marks]

Markscheme

November 2018

Mathematics

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2018"**. It is essential that you read this document before you start marking. In particular, please note the following.

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $u_4 = u_1 r^3 \Rightarrow -2.916 = 4r^3$ ***(A1)***

solving, $r = -0.9$ ***(M1)A1***

[3 marks]

(b) $S_\infty = \dfrac{4}{1-(-0.9)}$ ***(M1)***

$= \dfrac{40}{19} (= 2.11)$ ***A1***

[2 marks]

Total [5 marks]

2. $f'(x) = \int \left(15\sqrt{x} + \dfrac{1}{(x+1)^2}\right) \mathrm{d}x = 10x^{\frac{3}{2}} - \dfrac{1}{x+1} \ (+c)$ ***(M1)A1A1***

Note: ***A1*** for first term, ***A1*** for second term. Withhold one ***A1*** if extra terms are seen.

$f(x) = \int \left(10x^{\frac{3}{2}} - \dfrac{1}{x+1} + c\right) \mathrm{d}x = 4x^{\frac{5}{2}} - \ln(x+1) + cx + d$ ***A1***

Note: Allow FT from incorrect $f'(x)$ if it is of the form $f'(x) = Ax^{\frac{3}{2}} + \dfrac{B}{x+1} + c$.

Accept $\ln|x+1|$.

attempt to use at least one boundary condition in their $f(x)$ ***(M1)***

$x = 0, y = -4$

$\Rightarrow d = -4$ ***A1***

$x = 1, y = 0$

$\Rightarrow 0 = 4 - \ln 2 + c - 4$

$\Rightarrow c = \ln 2 (= 0.693)$ ***A1***

$f(x) = 4x^{\frac{5}{2}} - \ln(x+1) + x\ln 2 - 4$

[7 marks]

3. (a) use of inverse normal (implied by $\pm 0.1509\ldots$ or $\pm 1.554\ldots$) ***(M1)***

$\mathrm{P}(X < 16) = 0.56$

$\Rightarrow \dfrac{16 - \mu}{\sigma} = 0.1509\ldots$ ***(A1)***

$\mathrm{P}(X < 17) = 0.94$

$\Rightarrow \dfrac{17 - \mu}{\sigma} = 1.554\ldots$ ***(A1)***

attempt to solve a pair of simultaneous equations ***(M1)***

$\mu = 15.9$, $\sigma = 0.712$ ***A1A1***

[6 marks]

(b) correctly shaded diagram or intent to find $\mathrm{P}(X \geq 15)$ ***(M1)***

$= 0.895$ ***A1***

Note: Accept answers rounding to 0.89 or 0.90. Award ***M1A0*** for the answer 0.9.

[2 marks]

Total [8 marks]

4. METHOD 1

$\left(x + \dfrac{3}{x^2}\right)^5 = \ldots + \begin{pmatrix}5\\2\end{pmatrix} x^2 \left(\dfrac{3}{x^2}\right)^3 + \ldots$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, a power of x, and a power of $\dfrac{3}{x^2}$, ***A1*** for correct binomial coefficient, ***A1*** for correct powers.

$= \ldots + 10 \times \dfrac{27}{x^4} + \ldots \left(= \ldots + \dfrac{270}{x^4} + \ldots\right)$ ***(A1)***

constant term is $x^4\left(\dfrac{270}{x^4}\right)$

$= 270$ ***A1***

continued…

Question 4 continued

METHOD 2

EITHER

the general term is $x^4\binom{5}{r}x^r\left(\frac{3}{x^2}\right)^{5-r}$ ***(M1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, power(s) of x, and a power of $\frac{3}{x^2}$.

$=\binom{5}{r}\times 3^{5-r}\times\frac{x^{r+4}}{x^{10-2r}}\left(=\binom{5}{r}\times 3^{5-r}x^{3r-6}\right)$

constant term occurs when $r=2$ ***(A1)***

OR

the general term is $x^4\binom{5}{5-r}x^{5-r}\left(\frac{3}{x^2}\right)^{r}$ ***(M1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, power(s) of x, and a power of $\frac{3}{x^2}$.

$=\binom{5}{5-r}\times 3^{r}\times\frac{x^{9-r}}{x^{2r}}\left(=\binom{5}{5-r}\times 3^{r}x^{9-3r}\right)$

constant term occurs when $r=3$ ***(A1)***

continued…

Question 4 continued

THEN

$\binom{5}{2}(3)^3$ *(A1)*

$= 270$ ***A1***

[5 marks]

5. METHOD 1

$\dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \dfrac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ *(A1)*

$= \dfrac{9x^2h+9xh^2+3h^3-h}{h}$ ***A1***

cancelling h ***M1***

$= 9x^2+9xh+3h^2-1$

then $\lim_{h\to 0}\left(9x^2+9xh+3h^2-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

METHOD 2

$\dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \dfrac{3\left((x+h)^3-x^3\right)+\left(x-(x+h)\right)}{h}$ *(A1)*

$= \dfrac{3h\left(\left(x+h\right)^2+x\left(x+h\right)+x^2\right)-h}{h}$ ***A1***

cancelling h ***M1***

$= 3\left(\left(x+h\right)^2+x\left(x+h\right)+x^2\right)-1$

then $\lim_{h\to 0}\left(3\left(\left(x+h\right)^2+x\left(x+h\right)+x^2\right)-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

[5 marks]

6. (a) attempt to substitute $x = 5$ and set equal to zero, or use of long / synthetic division ***(M1)***

$2 \times 5^4 - 15 \times 5^3 + a \times 5^2 + 5b + c = 0$ ***A1***

$(\Rightarrow 25a + 5b + c = 625)$

[2 marks]

(b) 0 ***A1***

[1 mark]

(c) **EITHER**

attempt to solve $P'(5) = 0$ ***(M1)***

$\Rightarrow 8 \times 5^3 - 45 \times 5^2 + 4 \times 5 + b = 0$

OR

$\left(x^2 - 10x + 25\right)\left(2x^2 + \alpha x + \beta\right) = 2x^4 - 15x^3 + 2x^2 + bx + c$ ***(M1)***

comparing coefficients gives $\alpha = 5, \beta = 2$

THEN

$b = 105$ ***A1***

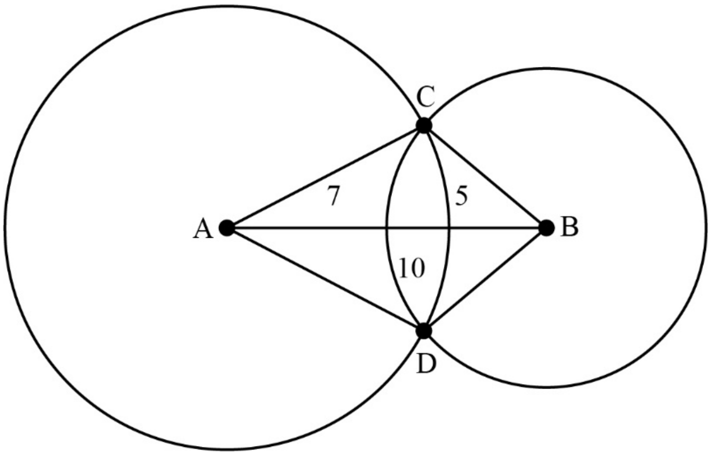
$\therefore c = 625 - 25 \times 2 - 525$

$c = 50$ ***A1***

[3 marks]

Total [6 marks]

7.



use of cosine rule ***(M1)***

$\text{C}\hat{\text{A}}\text{B} = \arccos\left(\dfrac{49 + 100 - 25}{2 \times 7 \times 10}\right) = 0.48276\ldots (= 27.660\ldots^\circ)$ ***(A1)***

$\text{C}\hat{\text{B}}\text{A} = \arccos\left(\dfrac{25 + 100 - 49}{2 \times 5 \times 10}\right) = 0.70748\ldots (= 40.535\ldots^\circ)$ ***(A1)***

attempt to subtract triangle area from sector area ***(M1)***

area $= \dfrac{1}{2} \times 49\left(2\text{C}\hat{\text{A}}\text{B} - \sin 2\text{C}\hat{\text{A}}\text{B}\right) + \dfrac{1}{2} \times 25\left(2\text{C}\hat{\text{B}}\text{A} - \sin 2\text{C}\hat{\text{B}}\text{A}\right)$

$= 3.5079\ldots + 5.3385\ldots$ ***(A1)***

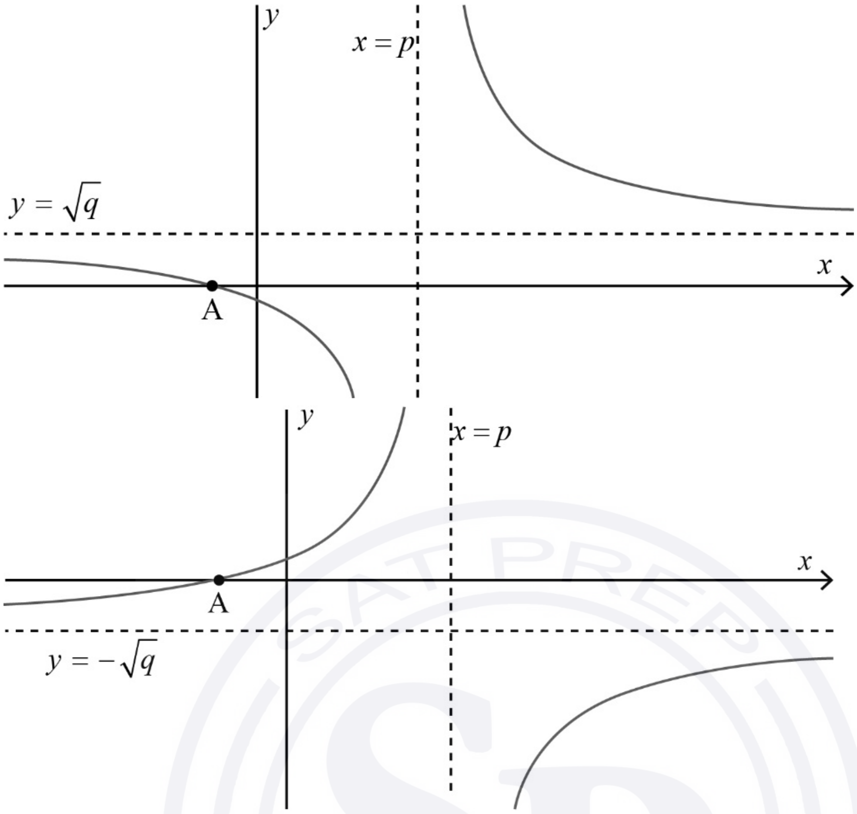
Note: Award this ***A1*** for either of these two values.

$= 8.85\left(\text{km}^2\right)$ ***A1***

Note: Accept all answers that round to 8.8 or 8.9.

[6 marks]

8. (a)



either graph passing through (or touching) A **A1**
correct shape and vertical asymptote with correct equation for either graph **A1**
correct horizontal asymptote with correct equation for either graph **A1**
two completely correct sketches **A1**

[4 marks]

(b) $a\left(-\frac{1}{2}\right)+1=0 \Rightarrow a=2$ **A1**

from horizontal asymptote, $\left(\frac{a}{b}\right)^2=\frac{4}{9}$ ***(M1)***

$\frac{a}{b}=\pm\frac{2}{3} \Rightarrow b=\pm 3$ **A1**

from vertical asymptote, $b\left(\frac{4}{3}\right)+c=0$

$b=3,\ c=-4$ or $b=-3,\ c=4$ **A1**

[4 marks]

Total [8 marks]

Section B

9. (a) **METHOD 1**

$$f'(x) = \frac{\frac{2(x-3)}{x} - (2\ln x + 1)}{(x-3)^2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right)$$ ***(M1)A1A1A1***

Note: Award ***M1*** for attempt at quotient rule, ***A1A1*** for numerator and ***A1*** for denominator.

METHOD 2

$$f(x) = (2\ln x + 1)(x-3)^{-1}$$ ***(A1)***

$$f'(x) = \left(\frac{2}{x}\right)(x-3)^{-1} - (2\ln x + 1)(x-3)^{-2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right)$$ ***(M1)A1A1***

Note: Award ***M1*** for attempt at product rule, ***A1*** for first term, ***A1*** for second term.

[4 marks]

(b) finding turning point of $y = f'(x)$ or finding root of $y = f''(x)$ ***(M1)***

$x = 0.899$ ***A1***

$y = f(0.899048\ldots) = -0.375$ ***(M1)A1***

$(0.899, -0.375)$

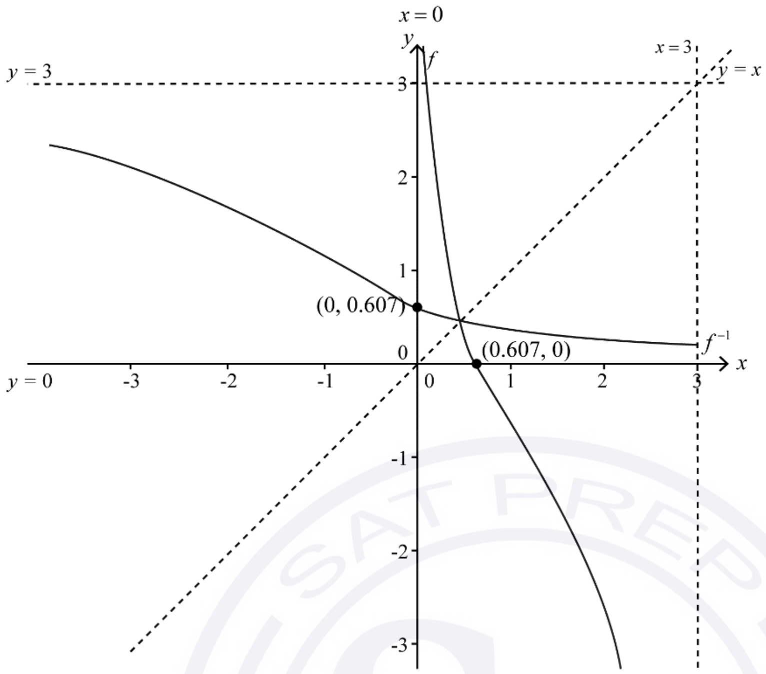
Note: Do not accept $x = 0.9$. Accept y-coordinates rounding to -0.37 or -0.375 but not -0.38.

[4 marks]

continued…

Question 9 continued

(c)



(i) smooth curve over the correct domain which does not cross the y-axis and is concave down for $x > 1$ ***A1***

x-intercept at 0.607 ***A1***

equations of asymptotes given as $x = 0$ and $x = 3$ (the latter must be drawn) ***A1A1***

[4 marks]

(ii) attempt to reflect graph of f in $y = x$ ***(M1)***

smooth curve over the correct domain which does not cross the x-axis and is concave down for $y > 1$ ***A1***

y-intercept at 0.607 ***A1***

equations of asymptotes given as $y = 0$ and $y = 3$ (the latter must be drawn) ***A1***

Note: For ***FT*** from (i) to (ii) award max ***M1A0A1A0***.

[4 marks]

(d) solve $f(x) = f^{-1}(x)$ or $f(x) = x$ to get $x = 0.372$ ***(M1)A1***

$0 < x < 0.372$ ***A1***

Note: Do not award ***FT*** marks.

[3 marks]

Total [19 marks]

10. (a) (i) $\mathrm{P}(X < 60)$
$= \mathrm{P}(X \le 59)$ ***(M1)***
$= 0.102$ ***A1***

(ii) standard deviation $= \sqrt{70}\ (= 8.37)$ ***(M1)A1***

[4 marks]

(b) (i) use of midpoints (accept consistent use of 45, 55 etc.) ***(M1)***

$$\frac{44.5 \times 2 + 54.5 \times 15 + 64.5 \times 40 + 74.5 \times 53 + 94.5 + 104.5 \times 3 + 114.5 \times 6}{2 + 15 + 40 + 53 + 0 + 1 + 3 + 6}$$

(M1)

$= \dfrac{8530}{120} (= 71.1)$ ***A1***

Note: If 45, 55, etc. are used consistently instead of midpoints (implied by the answer 71.58…) award ***M1M1A0.***

(ii) 13.9 ***(M1)A1***

[5 marks]

(c) valid reason given to include the examples below ***R1***
variance is 192 which is not close to the mean (accept not equal to)
standard deviation too high (using parts (a)(ii) and (b)(ii))
relative frequency of $X \le 59$ is 0.142 which is too high (using part (a)(i))
Poisson would give a frequency of roughly 14 for $80 \le X \le 89$

Note: Reasons which do not use values found in previous parts must be backed up with numerical evidence.

[1 mark]

(d) $\mathrm{P}(Y > 10) = 0.99$
$1 - \mathrm{P}(Y \le 10) = 0.99 \Rightarrow \mathrm{P}(Y \le 10) = 0.01$ ***(M1)***
attempt to solve a correct equation ***(M1)***
$\lambda = 20.1$ ***A1***

[3 marks]

continued…

Question 10 continued

(e) in 1 day, no of emails is $X \sim \text{Po}(\lambda)$
in 2 days, no of emails is $Y \sim \text{Po}(2\lambda)$ ***(A1)***
$\text{P}(10 \text{ on first day} \mid 20 \text{ in 2 days})$ ***(M1)***

$$= \frac{\text{P}(X=10) \times \text{P}(X=10)}{\text{P}(Y=20)}$$ ***(M1)***

$$= \frac{\left(\frac{\lambda^{10} e^{-\lambda}}{10!}\right)^2}{\frac{(2\lambda)^{20} e^{-2\lambda}}{20!}}$$ ***A1***

$$= \frac{\lambda^{20} e^{-2\lambda}}{2^{20} \lambda^{20} e^{-2\lambda}} \times \frac{20!}{(10!)^2}$$ ***A1***

$$= \frac{20!}{2^{20}(10!)^2}$$

which is independent of λ ***AG***

[5 marks]

Total [18 marks]

11. (a) **METHOD 1**

use of tan ***(M1)***

$$\tan\theta_p = \frac{1}{p}$$ ***(A1)***

$$\theta_p = \arctan\left(\frac{1}{p}\right)$$ ***A1***

METHOD 2

$$\text{AP} = \sqrt{p^2+1}$$ ***(A1)***

use of sin, cos, sine rule or cosine rule using the correct length of AP ***(M1)***

$\theta_p = \arcsin\left(\frac{1}{\sqrt{p^2+1}}\right)$ or $\theta_p = \arccos\left(\frac{p}{\sqrt{p^2+1}}\right)$ ***A1***

[3 marks]

continued…

Question 11 continued

(b) $\text{QR} = 1 \Rightarrow r = q + 1$ ***(A1)***

Note: This may be seen anywhere.

$\tan\theta_p = \tan\left(\theta_q + \theta_r\right)$

attempt to use compound angle formula for tan ***M1***

$\tan\theta_p = \dfrac{\tan\theta_q + \tan\theta_r}{1 - \tan\theta_q \tan\theta_r}$ ***(A1)***

$\dfrac{1}{p} = \dfrac{\frac{1}{q} + \frac{1}{r}}{1 - \left(\frac{1}{q}\right)\left(\frac{1}{r}\right)}$ ***(M1)***

$\dfrac{1}{p} = \dfrac{\frac{1}{q} + \frac{1}{q+1}}{1 - \left(\frac{1}{q}\right)\left(\frac{1}{q+1}\right)}$ or $p = \dfrac{1 - \left(\frac{1}{q}\right)\left(\frac{1}{q+1}\right)}{\frac{1}{q} + \frac{1}{q+1}}$ ***A1***

$\dfrac{1}{p} = \dfrac{q + q + 1}{q(q+1) - 1}$ ***M1***

Note: Award ***M1*** for multiplying top and bottom by $q(q+1)$.

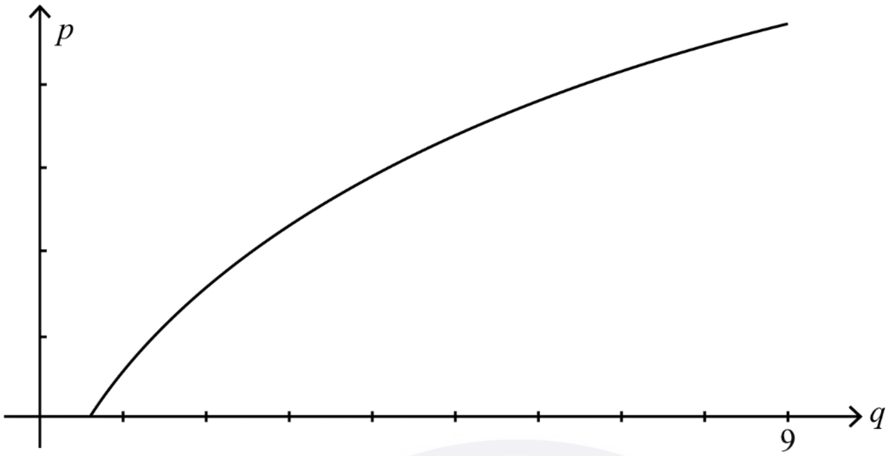
$p = \dfrac{q^2 + q - 1}{2q + 1}$ ***AG***

[6 marks]

continued…

Question 11 continued

(c)



increasing function with positive q-intercept **A1**

Note: Accept curves which extend beyond the domain shown above.

$(0.618 <)\ q < 9$ ***(A1)***

$\Rightarrow$ range is $(0 <)\ p < 4.68$ ***(A1)***

$0 < p < 4.68$ ***A1***

[4 marks]

Total [13 marks]

Markscheme

May 2018

Mathematics

Higher level

Paper 2

14 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2018"**. It is essential that you read this document before you start marking. In particular, please note the following.

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $u_1 + 2d = 1407$, $u_1 + 9d = 1183$ ***(M1)(A1)***

$u_1 = 1471$, $d = -32$ ***A1A1***

[4 marks]

(b) $1471 + (n-1)(-32) > 0$ ***(M1)***

$\Rightarrow n < \frac{1471}{32} + 1$

$n < 46.96\ldots$ ***(A1)***

so 46 positive terms ***A1***

[3 marks]

Total [7 marks]

2. **METHOD 1**

$\alpha + \beta = 5$, $\alpha\beta = -7$ ***(M1)(A1)***

Note: Award ***M1A0*** if only one equation obtained.

$(\alpha + 1) + (\beta + 1) = 5 + 2 = 7$ ***A1***

$(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1$ ***(M1)***

$= -7 + 5 + 1 = -1$

$p = -7$, $q = -1$ ***A1A1***

METHOD 2

$\alpha = \frac{5 + \sqrt{53}}{2} = 6.1\ldots$; $\beta = \frac{5 - \sqrt{53}}{2} = -1.1\ldots$ ***(M1)(A1)***

$\alpha + 1 = \frac{7 + \sqrt{53}}{2} = 7.1\ldots$; $\beta + 1 = \frac{7 - \sqrt{53}}{2} = -0.1\ldots$ ***A1***

$(x - 7.14\ldots)(x + 0.14\ldots) = x^2 - 7x - 1$ ***(M1)***

$p = -7$, $q = -1$ ***A1A1***

Note: Exact answers only.

[6 marks]

3. $\tan(x + \pi) = \tan x \left(= \frac{\sin x}{\cos x} \right)$ ***(M1)A1***

$\cos\left(x - \frac{\pi}{2} \right) = \sin x$ ***(M1)A1***

Note: The two ***M1***'s can be awarded for observation or for expanding.

$\tan(x + \pi)\cos\left(x - \frac{\pi}{2} \right) = \frac{\sin^2 x}{\cos x}$ ***A1***

[5 marks]

4. (a) $\text{P}(L \geq 5) = 0.910$ ***(M1)A1***

[2 marks]

(b) X is the number of wolves found to be at least 5 years old
recognising binomial distribution ***M1***
$X \sim \text{B}(8, 0.910\ldots)$
$\text{P}(X > 6) = 1 - \text{P}(X \leq 6)$ ***(M1)***
$= 0.843$ ***A1***

Note: Award ***M1A0*** for finding $\text{P}(X \geq 6)$.

[3 marks]

Total [5 marks]

5. (a) $2x^3 - 3x + 1 = Ax\left(x^2 + 1\right) + Bx + C$
$A = 2$, $C = 1$, ***A1***
$A + B = -3 \Rightarrow B = -5$ ***A1***

[2 marks]

(b) $\int \frac{2x^3 - 3x + 1}{x^2 + 1}\text{d}x = \int\left(2x - \frac{5x}{x^2 + 1} + \frac{1}{x^2 + 1}\right)\text{d}x$ ***M1M1***

Note: Award ***M1*** for dividing by $(x^2 + 1)$ to get $2x$, ***M1*** for separating the $5x$ and 1.

$= x^2 - \frac{5}{2}\ln\left(x^2 + 1\right) + \arctan x (+c)$ ***(M1)A1A1***

Note: Award ***(M1)A1*** for integrating $\frac{5x}{x^2 + 1}$, ***A1*** for the other two terms.

[5 marks]

Total [7 marks]

6. X is number of squirrels in reserve
$X \sim \text{Po}(179.2)$ ***A1***

Note: Award ***A1*** if 179.2 or 56×3.2 seen or implicit in future calculations.

recognising conditional probability ***M1***
$\text{P}\left(X > 190 \mid X \geq 168\right)$
$= \frac{\text{P}(X > 190)}{\text{P}(X \geq 168)}\left(= \frac{0.19827\ldots}{0.80817\cdots}\right)$ ***(A1)(A1)***
$= 0.245$ ***A1***

[5 marks]

7. (a) **EITHER**

2019: $2500 \times 0.93 + 250 = 2575$ ***(M1)A1***

2020: $2575 \times 0.93 + 250$ ***M1***

OR

2020: $2500 \times 0.93^2 + 250(0.93 + 1)$ ***M1M1A1***

Note: Award ***M1*** for starting with 2500, ***M1*** for multiplying by 0.93 and adding 250 twice. ***A1*** for correct expression. Can be shown in recursive form.

THEN

$(= 2644.75) = 2645$ ***AG***

[3 marks]

(b) 2020: $2500 \times 0.93^2 + 250(0.93 + 1)$

2042: $2500 \times 0.93^{24} + 250\left(0.93^{23} + 0.93^{22} + \ldots + 1\right)$ ***(M1)(A1)***

$= 2500 \times 0.93^{24} + 250\dfrac{\left(0.93^{24} - 1\right)}{(0.93 - 1)}$ ***(M1)(A1)***

$= 3384$ ***A1***

Note: If recursive formula used, award ***M1*** for $u_n = 0.93\, u_{n-1} + 250$ **and** u_0 or u_1 seen (can be awarded if seen in part (a)). Then award ***M1A1*** for attempt to find u_{24} or u_{25} respectively (different term if other than 2500 used) (***M1A0*** if incorrect term is being found) and ***A2*** for correct answer.

Note: Accept all answers that round to 3380.

[5 marks]

Total [8 marks]

8. **METHOD 1**

let p have no pets, q have one pet and r have two pets ***(M1)***

$p + q + r + 2 = 25$ ***(A1)***

$0p + 1q + 2r + 6 = 18$ ***A1***

Note: Accept a statement that there are a total of 12 pets.

attempt to use variance equation, or evidence of trial and error ***(M1)***

$\dfrac{0p + 1q + 4r + 18}{25} - \left(\dfrac{18}{25}\right)^2 = \left(\dfrac{24}{25}\right)^2$ ***(A1)***

attempt to solve a system of linear equations ***(M1)***

$p = 14$ ***A1***

continued…

Question 8 continued

METHOD 2

x	0	1	2	3
$\mathrm{P}(X = x)$	p	q	r	$\frac{2}{25}$

(M1)

$p + q + r + \frac{2}{25} = 1$ ***(A1)***

$q + 2r + \frac{6}{25} = \frac{18}{25} \left(\Rightarrow q + 2r = \frac{12}{25} \right)$ ***A1***

$q + 4r + \frac{18}{25} - \left(\frac{18}{25} \right)^2 = \frac{576}{625} \left(\Rightarrow q + 4r = \frac{18}{25} \right)$ ***(M1)(A1)***

$q = \frac{6}{25}$, $r = \frac{3}{25}$ ***(M1)***

$p = \frac{14}{25}$ ***A1***

so 14 have no pets

[7 marks]

Section B

9. (a) differentiating implicitly: ***M1***

$2xy + x^2 \dfrac{dy}{dx} = -4y^3 \dfrac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for each side.

if $\dfrac{dy}{dx} = 0$ then either $x = 0$ or $y = 0$ ***M1A1***

$x = 0 \Rightarrow$ two solutions for $y\left(y = \pm\sqrt[4]{5}\right)$ ***R1***

$y = 0$ not possible (as $0 \neq 5$) ***R1***

hence exactly two points ***AG***

Note: For a solution that only refers to the graph giving two solutions at $x = 0$ and no solutions for $y = 0$ award ***R1*** only.

[7 marks]

(b) at $(2, 1)$ $4 + 4\dfrac{dy}{dx} = -4\dfrac{dy}{dx}$ ***M1***

$\dfrac{dy}{dx} = -\dfrac{1}{2}$ ***(A1)***

gradient of normal is 2 ***M1***

$1 = 4 + c$ ***(M1)***

equation of normal is $y = 2x - 3$ ***A1***

[5 marks]

(c) substituting ***(M1)***

$x^2(2x-3) = 5 - (2x-3)^4$ or $\left(\dfrac{y+3}{2}\right)^2 y = 5 - y^4$ ***(A1)***

$x = 0.724$ ***A1***

[3 marks]

continued…

Question 9 continued

(d) recognition of two volumes ***(M1)***

volume 1 $= \pi\int_1^{\sqrt[4]{5}} \frac{5-y^4}{y}\,\mathrm{d}y (= 1.01\pi = 3.178\ldots)$ ***M1A1A1***

Note: Award ***M1*** for attempt to use $\pi\int x^2\mathrm{d}y$, ***A1*** for limits, ***A1*** for $\frac{5-y^4}{y}$. Condone omission of π at this stage.

volume 2

EITHER

$= \frac{1}{3}\pi \times 2^2 \times 4 (= 16.75\ldots)$ ***(M1)(A1)***

OR

$= \pi\int_{-3}^{1}\left(\frac{y+3}{2}\right)^2 \mathrm{d}y \left(= \frac{16\pi}{3} = 16.75...\right)$ ***(M1)(A1)***

THEN

total volume $= 19.9$ ***A1***

[7 marks]

Total [22 marks]

10. (a) $a\left[\int_0^{0.5} 3x\,\mathrm{d}x + \int_{0.5}^{2}(2-x)\mathrm{d}x\right] = 1$ ***M1***

Note: Award the ***M1*** for the total integral equalling 1, or equivalent.

$a\left(\frac{3}{2}\right) = 1$ ***(M1)A1***

$a = \frac{2}{3}$ ***AG***

[3 marks]

continued…

Question 10 continued

(b) **EITHER**

$\int_0^{0.5} 2x\,dx + \frac{2}{3}\int_{0.5}^{1} (2 - x)\,dx$ ***(M1)(A1)***

$= \frac{2}{3}$ ***A1***

OR

$\frac{2}{3}\int_1^2 (2 - x)\,dx = \frac{1}{3}$ ***(M1)***

so $P(X < 1) = \frac{2}{3}$ ***(M1)A1***

[3 marks]

(c) $P(s < X < 0.8) = \int_s^{0.5} 2x\,dx + \frac{2}{3}\int_{0.5}^{0.8} (2 - x)\,dx$ ***M1A1***

$= \left[x^2\right]_s^{0.5} + 0.27$

$0.25 - s^2 + 0.27$ ***(A1)***

$P(2s < X < 0.8) = \frac{2}{3}\int_{2s}^{0.8} (2 - x)\,dx$ ***A1***

$= \frac{2}{3}\left[2x - \frac{x^2}{2}\right]_{2s}^{0.8}$

$\frac{2}{3}\left(1.28 - (4s - 2s^2)\right)$

equating

$0.25 - s^2 + 0.27 = \frac{4}{3}\left(1.28 - (4s - 2s^2)\right)$ ***(A1)***

attempt to solve for s ***(M1)***

$s = 0.274$ ***A1***

[7 marks]

Total [13 marks]

11. (a) $\boldsymbol{r}_A = \boldsymbol{r}_B$ ***(M1)***

$2 - t = -0.5t \Rightarrow t = 4$ ***A1***

checking $t = 4$ satisfies $4 + t = 3.2 + 1.2t$ and $-1 - 0.15t = -2 + 0.1t$ ***R1***

$\mathrm{P}(-2, 8, -1.6)$ ***A1***

Note: Do not award final ***A1*** if answer given as column vector.

[4 marks]

(b) (i) $0.9 \times \begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}$ ***A1***

Note: Accept use of cross product equalling zero.

hence in the same direction ***AG***

(ii) $\begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ -1.6 \end{pmatrix}$ ***M1***

Note: The ***M1*** can be awarded for any one of the resultant equations.

$\Rightarrow t = \dfrac{40}{9} = 4.44\ldots$ ***A1***

[3 marks]

(c) (i) $\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 2 - t \\ 4 + t \\ -1 - 0.15t \end{pmatrix} - \begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix}$ ***(M1)(A1)***

$= \begin{pmatrix} 2 - 0.55t \\ 0.8 - 0.08t \\ 1 - 0.24t \end{pmatrix}$ ***(A1)***

Note: Accept $\boldsymbol{r}_B - \boldsymbol{r}_A$.

distance $D = \sqrt{(2 - 0.55t)^2 + (0.8 - 0.08t)^2 + (1 - 0.24t)^2}$ ***M1A1***

$(= \sqrt{8.64 - 2.688t + 0.317t^2})$

(ii) minimum when $\dfrac{\mathrm{d}D}{\mathrm{d}t} = 0$ ***(M1)***

$t = 3.83$ ***A1***

(iii) 0.511 (km) ***A1***

[8 marks]

Total [15 marks]

Markscheme

May 2018

Mathematics

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2018"**. It is essential that you read this document before you start marking. In particular, please note the following.

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log (a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Note: Accept answers that round to the correct 2sf unless otherwise stated in the markscheme.

Section A

1. (a) $z = \frac{(2+7\text{i})}{(6+2\text{i})} \times \frac{(6-2\text{i})}{(6-2\text{i})}$ ***(M1)***

$= \frac{26+38\text{i}}{40} \left(= \frac{13+19\text{i}}{20} = 0.65 + 0.95\text{i} \right)$ ***A1***

[2 marks]

(b) attempt to use $|z| = \sqrt{a^2+b^2}$ ***(M1)***

$|z| = \sqrt{\frac{53}{40}} \left(= \frac{\sqrt{530}}{20} \right)$ or equivalent ***A1***

Note: ***A1*** is only awarded for the correct exact value.

[2 marks]

(c) **EITHER**

$\arg z = \arg(2+7\text{i}) - \arg(6+2\text{i})$ ***(M1)***

OR

$\arg z = \arctan\left(\frac{19}{13}\right)$ ***(M1)***

THEN

$\arg z = 0.9707$ (radians) ($= 55.6197$ degrees) ***A1***

Note: Only award the last ***A1*** if 4 decimal places are given.

[2 marks]

Total [6 marks]

2. **METHOD 1**

substitute each of $x = 1, 2$ and 3 into the quartic and equate to zero ***(M1)***

$p + q + r = -7$

$4p + 2q + r = -11$ or equivalent ***(A2)***

$9p + 3q + r = -29$

Note: Award ***A2*** for all three equations correct, ***A1*** for two correct.

attempting to solve the system of equations ***(M1)***

$p = -7, q = 17, r = -17$ ***A1***

Note: Only award ***M1*** when some numerical values are found when solving algebraically or using GDC.

continued…

Question 2 continued

METHOD 2

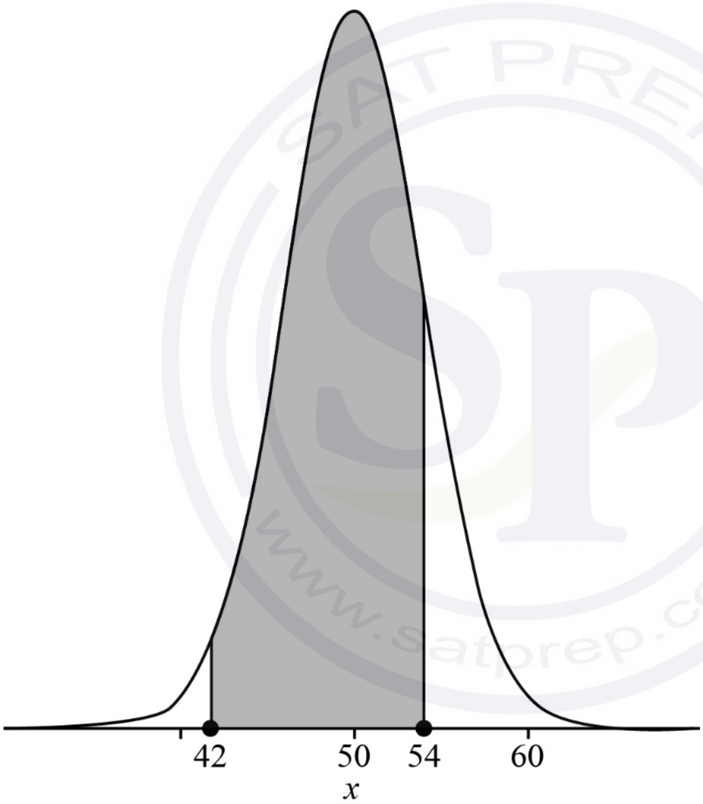
attempt to find fourth factor *(M1)*
$(x-1)$ ***A1***
attempt to expand $(x-1)^2(x-2)(x-3)$ ***M1***
$= x^4 - 7x^3 + 17x^2 - 17x + 6 \; (p = -7,\ q = 17,\ r = -17)$ ***A2***

Note: Award ***A2*** for all three values correct, ***A1*** for two correct.

Note: Accept long / synthetic division.

[5 marks]

3. (a)



normal curve centred on 50 ***A1***
vertical lines at $x = 42$ and $x = 54$, with shading in between ***A1***

[2 marks]

(b) $\text{P}(42 < X < 54) \; (= \text{P}(-2 < Z < 1))$ *(M1)*
$= 0.819$ ***A1***

[2 marks]

continued…

Question 3 continued

(c) $P(\mu - k\sigma < X < \mu + k\sigma) = 0.5 \Rightarrow P(X < \mu + k\sigma) = 0.75$ ***(M1)***

$k = 0.674$ ***A1***

Note: Award ***M1A0*** for $k = -0.674$.

[2 marks]

Total [6 marks]

4. (a) (i) **METHOD 1**

$\text{PC} = \frac{\sqrt{3}}{2}$ or 0.8660 ***(M1)***

$\text{PM} = \frac{1}{2}\text{PC} = \frac{\sqrt{3}}{4}$ or 0.4330 ***(A1)***

$\text{AM} = \sqrt{\frac{1}{4} + \frac{3}{16}}$

$= \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

Note: Award ***M1*** for attempting to solve triangle AMP.

METHOD 2

using the cosine rule

$\text{AM}^2 = 1^2 + \left(\frac{\sqrt{3}}{4}\right)^2 - 2 \times \frac{\sqrt{3}}{4} \times \cos(30°)$ ***M1A1***

$\text{AM} = \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

(ii) $\tan(\text{A}\hat{\text{M}}\text{P}) = \frac{2}{\sqrt{3}}$ or equivalent ***(M1)***

$= 0.857$ ***A1***

[5 marks]

continued…

Question 4 continued

(b) **EITHER**

$\frac{1}{2}\text{AM}^2\left(2\text{A}\hat{\text{M}}\text{P} - \sin(2\text{A}\hat{\text{M}}\text{P})\right)$ ***(M1)A1***

OR

$\frac{1}{2}\text{AM}^2 \times 2\text{A}\hat{\text{M}}\text{P} - \frac{\sqrt{3}}{8}$ ***(M1)A1***

$= 0.158(\text{m}^2)$ ***A1***

Note: Award ***M1*** for attempting to calculate area of a sector minus area of a triangle.

[3 marks]

Total [8 marks]

5. (a) $\begin{pmatrix} 3n+1 \\ 3n-2 \end{pmatrix} = \frac{(3n+1)!}{(3n-2)!3!}$ ***(M1)***

$= \frac{(3n+1)3n(3n-1)}{3!}$ ***A1***

$= \frac{9}{2}n^3 - \frac{1}{2}n$ or equivalent ***A1***

[3 marks]

(b) attempt to solve $\frac{9}{2}n^3 - \frac{1}{2}n > 10^6$ ***(M1)***

$n > 60.57\ldots$ ***(A1)***

Note: Allow equality.

$\Rightarrow n = 61$ ***A1***

[3 marks]

Total [6 marks]

6. let P_n be the statement: $(1-a)^n > 1-na$ for some $n \in \mathbb{Z}^+, n \ge 2,$ where $0 < a < 1$

consider the case $n=2$: $(1-a)^2 = 1-2a+a^2$ ***M1***

$>1-2a$ because $a^2 > 0$. Therefore P_2 is true ***R1***

assume P_n is true for some $n = k$

$(1-a)^k > 1-ka$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on this ***M1***.

EITHER

consider $(1-a)^{k+1} = (1-a)(1-a)^k$ ***M1***

$>1-(k+1)a+ka^2$ ***A1***

$>1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

OR

multiply both sides by $(1-a)$ (which is positive) ***M1***

$(1-a)^{k+1} > (1-ka)(1-a)$

$(1-a)^{k+1} > 1-(k+1)a+ka^2$ ***A1***

$(1-a)^{k+1} > 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

THEN

P_2 is true and P_k is true $\Rightarrow P_{k+1}$ is true so P_n true for all $n \ge 2$ (or equivalent) ***R1***

Note: Only award the last ***R1*** if at least four of the previous marks are gained including the ***A1***.

[7 marks]

7. (a) attempt to solve $v(t) = 0$ for t or equivalent ***(M1)***

$t_1 = 0.441(\text{s})$ ***A1***

[2 marks]

(b) (i) $a(t) = \frac{dv}{dt} = -e^{-t} - 16te^{-2t} + 16t^2e^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

(ii) $a(t_1) = -2.28\left(\text{ms}^{-2}\right)$ ***A1***

[3 marks]

Total [5 marks]

8. (a) $np = 3.5$ ***(A1)***

$p \le 1 \Rightarrow$ least $n = 4$ ***A1***

[2 marks]

continued…

Question 8 continued

(b) $(1-p)^n + np(1-p)^{n-1} = 0.09478$ ***M1A1***

attempt to solve above equation with $np = 3.5$ ***(M1)***

$n = 12,\ p = \frac{7}{24}\ (= 0.292)$ ***A1A1***

Note: Do not accept n as a decimal.

[5 marks]

Total [7 marks]

Section B

9. (a) (i) $X \sim \text{Po}(5.3)$

$\text{P}(X = 4) = \text{e}^{-5.3}\dfrac{5.3^4}{4!}$ ***(M1)***

$= 0.164$ ***A1***

(ii) **METHOD 1**

listing probabilities (table or graph) ***M1***

mode $X = 5$ (with probability 0.174) ***A1***

Note: Award ***M0A0*** for 5 (taxis) or mode $= 5$ with no justification.

METHOD 2

mode is the integer part of mean ***R1***

$E(X) = 5.3 \Rightarrow \text{mode} = 5$ ***A1***

Note: Do not allow ***R0A1***.

(iii) attempt at conditional probability ***(M1)***

$\dfrac{\text{P}(X = 7)}{\text{P}(X \geq 6)}$ or equivalent $\left(= \dfrac{0.1163...}{0.4365...}\right)$ ***A1***

$= 0.267$ ***A1***

[7 marks]

(b) **METHOD 1**

the possible arrivals are $(2,0)$, $(1,1)$, $(0,2)$ ***(A1)***

$Y \sim \text{Po}(0.65)$ ***A1***

attempt to compute, using sum and product rule, ***(M1)***

$0.070106\ldots \times 0.52204\ldots + 0.026455\ldots \times 0.33932\ldots + 0.0049916\ldots \times 0.11028\ldots$ ***(A1)(A1)***

Note: Award ***A1*** for one correct product and ***A1*** for two other correct products.

$= 0.0461$ ***A1***

[6 marks]

continued…

Question 9 continued

METHOD 2

recognising a sum of 2 independent Poisson variables *eg* $Z = X + Y$ ***R1***

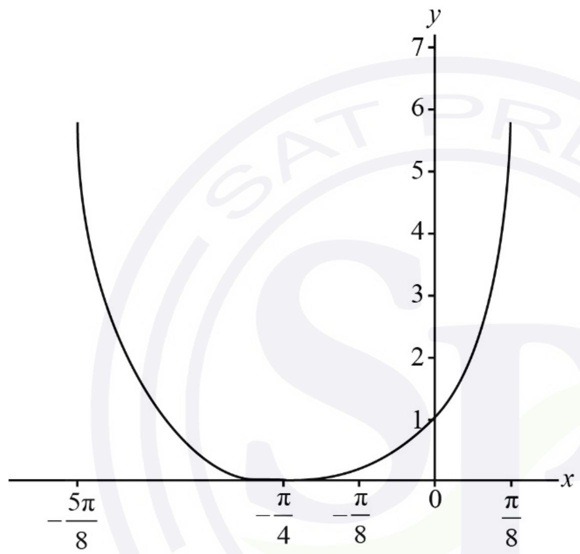
$\lambda = 5.3 + \frac{1.3}{2}$ ***A1***

$\mathrm{P}(Z = 2) = 0.0461$ ***(M1)A3***

[6 marks]

Total [13 marks]

10. (a) (i)



A1A1

A1 for correct concavity, many to one graph, symmetrical about the midpoint of the domain and with two axes intercepts.

> **Note:** Axes intercepts and scales not required.

A1 for correct domain

(ii) for each value of x there is a unique value of $f(x)$ ***A1***

> **Note:** Accept "passes the vertical line test" or equivalent.

(iii) no inverse because the function fails the horizontal line test or equivalent ***R1***

> **Note:** No **FT** if the graph is in degrees (one-to-one).

(iv) the expression is not valid at either of $x = \frac{\pi}{4}$ $\left(\text{or } -\frac{3\pi}{4}\right)$ ***R1***

[5 marks]

continued…

Question 10 continued

(b) **METHOD 1**

$$f(x) = \frac{\tan\left(x + \frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4} - x\right)}$$ ***M1***

$$= \frac{\dfrac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}}{\dfrac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}}$$ ***M1A1***

$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

METHOD 2

$$f(x) = \tan\left(x + \frac{\pi}{4}\right)\tan\left(\frac{\pi}{2} - \frac{\pi}{4} + x\right)$$ ***(M1)***

$$= \tan^2\left(x + \frac{\pi}{4}\right)$$ ***A1***

$$g(t) = \left(\frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}\right)^2$$ ***A1***

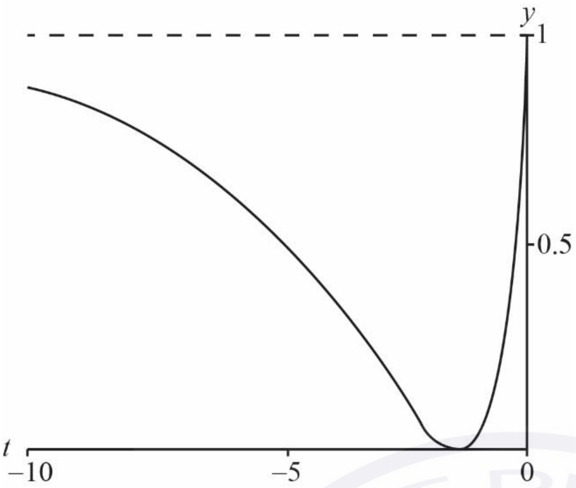
$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

[3 marks]

continued…

Question 10 continued

(c)



for $t \leq 0$, correct concavity with two axes intercepts and with asymptote $y = 1$ ***A1***

t intercept at $(-1, 0)$ ***A1***

y intercept at $(0, 1)$ ***A1***

[3 marks]

(d) (i) **METHOD 1**

α, β satisfy $\frac{(1+t)^2}{(1-t)^2} = k$ ***M1***

$1 + t^2 + 2t = k(1 + t^2 - 2t)$ ***A1***

$(k-1)t^2 - 2(k+1)t + (k-1) = 0$ ***A1***

attempt at using quadratic formula ***M1***

$\alpha, \beta = \frac{k + 1 \pm 2\sqrt{k}}{k-1}$ or equivalent ***A1***

METHOD 2

α, β satisfy $\frac{1+t}{1-t} = (\pm)\sqrt{k}$ ***M1***

$t + \sqrt{k}t = \sqrt{k} - 1$ ***M1***

$t = \frac{\sqrt{k}-1}{\sqrt{k}+1}$ (or equivalent) ***A1***

$t - \sqrt{k}t = -\left(\sqrt{k}+1\right)$ ***M1***

$t = \frac{\sqrt{k}+1}{\sqrt{k}-1}$ (or equivalent) ***A1***

so for *eg*, $\alpha = \frac{\sqrt{k}-1}{\sqrt{k}+1}$, $\beta = \frac{\sqrt{k}+1}{\sqrt{k}-1}$

continued…

Question 10 continued

(ii) $\alpha + \beta = 2\frac{(k+1)}{(k-1)}\left(= -2\frac{(1+k)}{(1-k)}\right)$ ***A1***

since $1 + k > 1 - k$ ***R1***

$\alpha + \beta < -2$ ***AG***

Note: Accept a valid graphical reasoning.

[7 marks]

Total [18 marks]

11. (a) attempt at implicit differentiation ***M1***

$1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})\sin(xy) = 0$ ***A1M1A1***

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{dy}{dx} = -1 - y\sin(xy)$ or equivalent ***A1***

$\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$ ***AG***

[5 marks]

(b) (i) **EITHER**

when $xy = -\frac{\pi}{2}$, $\cos xy = 0$ ***M1***

$\Rightarrow x + y = 0$ ***(A1)***

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent ***M1***

$x - \frac{\pi}{2x} = 0$ ***(A1)***

THEN

therefore $x^2 = \frac{\pi}{2}\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ ***A1***

$P\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right), Q\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ or $P(1.25, -1.25)$, $Q(-1.25, 1.25)$ ***A1***

continued…

Question 11 continued

(ii) $m_1 = -\left(\dfrac{1 - \sqrt{\dfrac{\pi}{2}} \times -1}{1 + \sqrt{\dfrac{\pi}{2}} \times -1}\right)$ ***M1A1***

$m_2 = -\left(\dfrac{1 + \sqrt{\dfrac{\pi}{2}} \times -1}{1 - \sqrt{\dfrac{\pi}{2}} \times -1}\right)$ ***A1***

$m_1 m_2 = 1$ ***AG***

Note: Award ***M1A0A0*** if decimal approximations are used.

Note: No ***FT*** applies.

[7 marks]

(c) equate derivative to -1 ***M1***

$(y - x)\sin(xy) = 0$ ***(A1)***

$y = x$, $\sin(xy) = 0$ ***R1***

in the first case, attempt to solve $2x = \cos\left(x^2\right)$ ***M1***

$(0.486, 0.486)$ ***A1***

in the second case, $\sin(xy) = 0 \Rightarrow xy = 0$ and $x + y = 1$ ***(M1)***

$(0,1)$, $(1,0)$ ***A1***

[7 marks]

Total [19 marks]

Markscheme

November 2017

Mathematics

Higher level

Paper 2

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2017"**. It is essential that you read this document before you start marking.
In particular, please note the following.

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. let b be the cost of one banana, k the cost of one kiwifruit, and m the cost of one melon
attempt to set up three linear equations ***(M1)***
$2b + 3k + 4m = 658$
$5b + 2k + 8m = 1232$
$5b + 4k = 300$ ***(A1)***
attempt to solve three simultaneous equations ***(M1)***
$b = 36,\ k = 30,\ m = 124$
banana costs (\$)0.36, kiwifruit costs (\$)0.30, melon costs (\$)1.24 ***A1***

[4 marks]

2. (a) $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

$\Rightarrow 0.75 = \dfrac{0.6}{\mathrm{P}(B)}$ ***(M1)***

$\Rightarrow \mathrm{P}(B) \left(= \dfrac{0.6}{0.75} \right) = 0.8$ ***A1***

[2 marks]

(b) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
$\Rightarrow 0.95 = \mathrm{P}(A) + 0.8 - 0.6$ ***(M1)***
$\Rightarrow \mathrm{P}(A) = 0.75$ ***A1***

[2 marks]

(c) **METHOD 1**

$\mathrm{P}(A' \mid B) = \dfrac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} = \dfrac{0.2}{0.8} = 0.25$ ***A1***

$\mathrm{P}(A' \mid B) = \mathrm{P}(A')$ ***R1***
hence A' and B are independent ***AG***

Note:	If there is evidence that the student has calculated $\mathrm{P}(A' \cap B) = 0.2$ by assuming independence in the first place, award ***A0R0***.

continued…

Question 2 continued

METHOD 2

EITHER

$P(A) = P(A|B)$ ***A1***

OR

$P(A) \times P(B) = 0.75 \times 0.80 = 0.6 = P(A \cap B)$ ***A1***

THEN

A and B are independent ***R1***

hence A' and B are independent ***AG***

METHOD 3

$P(A') \times P(B) = 0.25 \times 0.80 = 0.2$ ***A1***

$P(A') \times P(B) = P(A' \cap B)$ ***R1***

hence A' and B are independent ***AG***

[2 marks]

Total [6 marks]

3. METHOD 1

area = (four sector areas radius 9) + (four sector areas radius 3) ***(M1)***

$= 4\left(\frac{1}{2}9^2\frac{\pi}{9}\right) + 4\left(\frac{1}{2}3^2\frac{7\pi}{18}\right)$ ***(A1)(A1)***

$= 18\pi + 7\pi$

$= 25\pi\left(= 78.5\text{cm}^2\right)$ ***A1***

METHOD 2

area =

(area of circle radius 3) + (four sector areas radius 9) – (four sector areas radius 3) ***(M1)***

$\pi 3^2 + 4\left(\frac{1}{2}9^2\frac{\pi}{9}\right) - 4\left(\frac{1}{2}3^2\frac{\pi}{9}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for the second term and ***A1*** for the third term.

$= 9\pi + 18\pi - 2\pi$

$= 25\pi\left(= 78.5\text{cm}^2\right)$ ***A1***

Note: Accept working in degrees.

[4 marks]

4. let X be the random variable “amount of caffeine content in coffee”

$P(X > 120) = 0.2$, $P(X > 110) = 0.6$ ***(M1)***

$(\Rightarrow P(X < 120) = 0.8$, $P(X < 110) = 0.4)$

Note: Award ***M1*** for at least one correct probability statement.

$$\frac{120-\mu}{\sigma} = 0.84162\ldots, \quad \frac{110-\mu}{\sigma} = -0.253347\ldots$$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for attempt to find at least one appropriate z-value.

$120 - \mu = 0.84162\sigma$, $110 - \mu = -0.253347\sigma$

attempt to solve simultaneous equations ***(M1)***

$\mu = 112$, $\sigma = 9.13$ ***A1***

[6 marks]

5. attempt to use tan, or sine rule, in triangle BXN or BXS ***(M1)***

$$\text{NX} = 80\tan 55^\circ \left(= \frac{80}{\tan 35^\circ} = 114.25\right)$$ ***(A1)***

$$\text{SX} = 80\tan 65^\circ \left(= \frac{80}{\tan 25^\circ} = 171.56\right)$$ ***(A1)***

Attempt to use cosine rule ***M1***

$\text{SN}^2 = 171.56^2 + 114.25^2 - 2 \times 171.56 \times 114.25\cos 70^\circ$ ***(A1)***

$\text{SN} = 171(\text{m})$ ***A1***

Note: Award final ***A1*** only if the correct answer has been given to 3 significant figures.

[6 marks]

6. (a) let X be the number of bananas eaten in one day

$X \sim \text{Po}(0.2)$

$P(X \geq 1) = 1 - P(X = 0)$ ***(M1)***

$= 0.181\left(= 1 - e^{-0.2}\right)$ ***A1***

[2 marks]

(b) **EITHER**

let Y be the number of bananas eaten in one week

$Y \sim \text{Po}(1.4)$ ***(A1)***

$P(Y = 0) = 0.246596\ldots\left(= e^{-1.4}\right)$ ***(A1)***

OR

let Z be the number of days in one week at least one banana is eaten

$Z \sim B(7, 0.181\ldots)$ ***(A1)***

$P(Z = 0) = 0.246596\ldots$ ***(A1)***

continued…

Question 6 continued

THEN

$52 \times 0.246596\ldots$ ***(M1)***

$= 12.8\left(= 52\mathrm{e}^{-1.4}\right)$ ***A1***

[4 marks]

Total [6 marks]

7. **METHOD 1**

let roots be α and 3α ***(M1)***

sum of roots $(4\alpha) = \frac{8}{7}$ ***M1***

$\Rightarrow \alpha = \frac{2}{7}$ ***A1***

EITHER

product of roots $\left(3\alpha^2\right) = \frac{p}{7}$ ***M1***

$p = 21\alpha^2 = 21 \times \frac{4}{49}$

OR

$7\left(\frac{2}{7}\right)^2 - 8\left(\frac{2}{7}\right) + p = 0$ ***M1***

$\frac{4}{7} - \frac{16}{7} + p = 0$

THEN

$\Rightarrow p = \frac{12}{7}(=1.71)$ ***A1***

METHOD 2

$x = \frac{8 \pm \sqrt{64 - 28p}}{14}$ ***(M1)***

$\frac{8 + \sqrt{64 - 28p}}{14} = 3\left(\frac{8 - \sqrt{64 - 28p}}{14}\right)$ ***M1A1***

$8 + \sqrt{64 - 28p} = 24 - 3\sqrt{64 - 28p} \Rightarrow \sqrt{64 - 28p} = 4$ ***(M1)***

$p = \frac{12}{7}(=1.71)$ ***A1***

[5 marks]

8. **EITHER**

$x^2 = 2\sec\theta$

$2x\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$ ***M1A1***

$\displaystyle\int \frac{dx}{x\sqrt{x^4-4}}$

$\displaystyle = \int \frac{\sec\theta\tan\theta d\theta}{2\sec\theta\sqrt{4\sec^2\theta - 4}}$ ***M1A1***

OR

$x = \sqrt{2}(\sec\theta)^{\frac{1}{2}}\left(=\sqrt{2}(\cos\theta)^{-\frac{1}{2}}\right)$

$\dfrac{dx}{d\theta} = \dfrac{\sqrt{2}}{2}(\sec\theta)^{\frac{1}{2}}\tan\theta\left(=\dfrac{\sqrt{2}}{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta\right)$ ***M1A1***

$\displaystyle\int \frac{dx}{x\sqrt{x^4-4}}$

$\displaystyle = \int \frac{\sqrt{2}(\sec\theta)^{\frac{1}{2}}\tan\theta d\theta}{2\sqrt{2}(\sec\theta)^{\frac{1}{2}}\sqrt{4\sec^2\theta-4}}\left(=\int \frac{\sqrt{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta d\theta}{2\sqrt{2}(\cos\theta)^{-\frac{1}{2}}\sqrt{4\sec^2\theta-4}}\right)$ ***M1A1***

THEN

$\displaystyle = \frac{1}{2}\int \frac{\tan\theta d\theta}{2\tan\theta}$ ***(M1)***

$\displaystyle = \frac{1}{4}\int d\theta$

$= \dfrac{\theta}{4} + c$ ***A1***

$x^2 = 2\sec\theta \Rightarrow \cos\theta = \dfrac{2}{x^2}$ ***M1***

Note: This ***M1*** may be seen anywhere, including a sketch of an appropriate triangle.

so $\dfrac{\theta}{4} + c = \dfrac{1}{4}\arccos\left(\dfrac{2}{x^2}\right) + c$ ***AG***

[7 marks]

9. (a) $12!(=479001600)$ ***A1***

[1 mark]

(b) **METHOD 1**

$8 \times 2 = 16$ ways of sitting Helen and Nicky, $10!$ ways of sitting everyone else ***(A1)***

$16 \times 10!$

$= 58060800$ ***A1***

METHOD 2

$8 \times 1 \times 10!(=29030400)$ ways if Helen sits in the front or back row

$4 \times 2 \times 10!(=29030400)$ ways if Helen sits in the middle row ***(A1)***

Note: Award ***A1*** for one correct value.

2×29030400

$= 58060800$ ***A1***

[2 marks]

(c) **METHOD 1**

$9 \times 2 \times 10!(=65318400)$ ways if Helen and Nicky sit next to each other ***(A1)***

attempt to subtract from total number of ways ***(M1)***

$12! - 9 \times 2 \times 10!$

$= 413683200$ ***A1***

METHOD 2

$6 \times 10 \times 10!(=217728000)$ ways if Helen sits in column 1 or 4 ***(A1)***

$6 \times 9 \times 10!(=195955200)$ ways if Helen sits in column 2 or 3 ***(A1)***

$217728000 + 195955200$

$= 413683200$ ***A1***

[3 marks]

Total [6 marks]

Section B

10. (a) (i) attempt to use quotient rule or product rule ***M1***

$$f'(x) = \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x}\left(= \frac{1}{2\sqrt{x}\sin x} - \frac{\sqrt{x}\cos x}{\sin^2 x}\right)$$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2\sqrt{x}\sin x}$ or equivalent and ***A1*** for $-\frac{\sqrt{x}\cos x}{\sin^2 x}$ or equivalent.

setting $f'(x) = 0$ ***M1***

$\frac{\sin x}{2\sqrt{x}} - \sqrt{x}\cos x = 0$

$\frac{\sin x}{2\sqrt{x}} = \sqrt{x}\cos x$ or equivalent ***A1***

$\tan x = 2x$ ***AG***

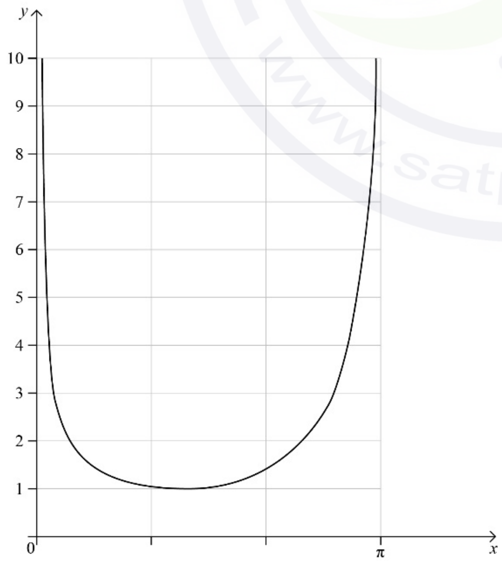
(ii) $x = 1.17$

$0 < x \le 1.17$ ***A1A1***

Note: Award ***A1*** for $0 < x$ and ***A1*** for $x \le 1.17$. Accept $x < 1.17$.

[7 marks]

(b)



concave up curve over correct domain with one minimum point above the *x*-axis. ***A1***

approaches $x = 0$ asymptotically ***A1***

approaches $x = \pi$ asymptotically ***A1***

Note: For the final ***A1*** an asymptote must be seen, and π must be seen on the x-axis or in an equation.

[3 marks]

continued…

Question 10 continued

(c) $f'(x)\left(=\dfrac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\sqrt{x}\cos x}{\sin^2 x}\right)=1$ ***(A1)***

attempt to solve for x ***(M1)***

$x = 1.96$ ***A1***

$y = f(1.96...)$

$= 1.51$ ***A1***

[4 marks]

(d) $V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{x\,\mathrm{d}x}{\sin^2 x}$ ***(M1)(A1)***

Note: ***M1*** is for an integral of the correct squared function (with or without limits and/or π).

$= 2.68 (= 0.852\pi)$ ***A1***

[3 marks]

Total [17 marks]

11. (a) (i) $f'(x) = 4\sin x\cos x + 14\cos 2x + \sec^2 x$ (or equivalent) ***(M1)A1***

(ii)

A1A1A1A1

Note: Award ***A1*** for correct behaviour at $x = 0$, ***A1*** for correct domain and correct behaviour for $x \to \dfrac{\pi}{2}$, ***A1*** for two clear intersections with x-axis and minimum point, ***A1*** for clear maximum point.

continued…

Question 11 continued

(iii) $x = 0.0736$ ***A1***
$x = 1.13$ ***A1***

[8 marks]

(b) (i) attempt to write $\sin x$ in terms of u only ***(M1)***

$\sin x = \dfrac{u}{\sqrt{1+u^2}}$ ***A1***

(ii) $\cos x = \dfrac{1}{\sqrt{1+u^2}}$ ***(A1)***

attempt to use $\sin 2x = 2\sin x\cos x \left(= 2\dfrac{u}{\sqrt{1+u^2}}\dfrac{1}{\sqrt{1+u^2}}\right)$ ***(M1)***

$\sin 2x = \dfrac{2u}{1+u^2}$ ***A1***

(iii) $2\sin^2 x + 7\sin 2x + \tan x - 9 = 0$

$\dfrac{2u^2}{1+u^2} + \dfrac{14u}{1+u^2} + u - 9\ (= 0)$ ***M1***

$\dfrac{2u^2 + 14u + u\left(1+u^2\right) - 9\left(1+u^2\right)}{1+u^2} = 0$ (or equivalent) ***A1***

$u^3 - 7u^2 + 15u - 9 = 0$ ***AG***

[7 marks]

(c) $u = 1$ or $u = 3$ ***(M1)***
$x = \arctan(1)$ ***A1***
$x = \arctan(3)$ ***A1***

Note: Only accept answers given the required form.

[3 marks]

Total [18 marks]

12. (a) 150000×1.035^{20} ***(M1)(A1)***

$= \$298468$ ***A1***

Note: Only accept answers to the nearest dollar. Accept \$298469.

[3 marks]

(b) attempt to look for a pattern by considering 1 year, 2 years *etc* ***(M1)***
recognising a geometric series with first term P and common ratio 1.02 ***(M1)***

EITHER

$P + 1.02P + \ldots + 1.02^{19}P \left(= P\left(1 + 1.02 + \ldots + 1.02^{19}\right)\right)$ ***A1***

OR

explicitly identify $u_1 = P$, $r = 1.02$ and $n = 20$ (may be seen as S_{20}). ***A1***

THEN

$$S_{20} = \frac{\left(1.02^{20} - 1\right)P}{\left(1.02 - 1\right)}$$ ***AG***

[3 marks]

(c) $24.297\ldots P = 298468$ ***(M1)(A1)***
$P = 12284$ ***A1***

Note: Accept answers which round to 12284.

[3 marks]

(d) (i) **METHOD 1**

$Q\left(1.028^n\right) = 5000\left(1 + 1.028 + 1.028^2 + 1.028^3 + \ldots + 1.028^{n-1}\right)$ ***M1A1***

$$Q = \frac{5000\left(1 + 1.028 + 1.028^2 + 1.028^3 + \ldots + 1.028^{n-1}\right)}{1.028^n}$$ ***A1***

$$= \frac{5000}{1.028} + \frac{5000}{1.028^2} + \ldots + \frac{5000}{1.028^n}$$ ***AG***

continued…

Question 12 continued

METHOD 2

the initial value of the first withdrawal is $\frac{5000}{1.028}$ ***A1***

the initial value of the second withdrawal is $\frac{5000}{1.028^2}$ ***R1***

the investment required for these two withdrawals is $\frac{5000}{1.028}+\frac{5000}{1.028^2}$ ***R1***

$Q=\frac{5000}{1.028}+\frac{5000}{1.028^2}+\ldots+\frac{5000}{1.028^n}$ ***AG***

(ii) sum to infinity is $\frac{\frac{5000}{1.028}}{1-\frac{1}{1.028}}$ ***(M1)(A1)***

$=178571.428\ldots$

so minimum amount is $\$178572$ ***A1***

Note: Accept answers which round to $\$178571$ or $\$178572$.

[6 marks]

Total [15 marks]

Markscheme

May 2017

Mathematics

Higher level

Paper 2

19 pages

Instructions to Examiners

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Section A

1. (a) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***M1***

$0.5 = k + 3k - k^2$ ***A1***

$k^2 - 4k + 0.5 = 0$

$k = 0.129$ ***A1***

Note: Do not award the final ***A1*** if two solutions are given.

[3 marks]

(b) use of $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$ or alternative ***(M1)***

$\mathrm{P}(A' \cap B) = 3k - k^2$ ***(A1)***

$= 0.371$ ***A1***

[3 marks]

Total [6 marks]

2. (a) $y + x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1A1A1***

Note: Award ***A1*** for the first two terms, ***A1*** for the third term and the 0.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{1 - xy}$ ***A1***

Note: Accept $\frac{-y^2}{\ln y}$.

Note: Accept $\frac{-y}{x - \frac{1}{y}}$.

[4 marks]

(b) $m_T = \frac{\mathrm{e}^2}{1 - \mathrm{e} \times \frac{2}{\mathrm{e}}}$ ***(M1)***

$m_T = -\mathrm{e}^2$ ***(A1)***

$y - \mathrm{e} = -\mathrm{e}^2 x + 2\mathrm{e}$

$-\mathrm{e}^2 x - y + 3\mathrm{e} = 0$ or equivalent ***A1***

Note: Accept $y = -7.39x + 8.15$.

[3 marks]

Total [7 marks]

3. METHOD 1

$^{8}C_{r}\left(\frac{1}{x}\right)^{8-r}(5x)^{r} = {}^{8}C_{r}(5)^{r}x^{2r-8}$ ***(M1)***

$r = 5$ ***(A1)***

$^{8}C_{5} \times 5^{5} = {}^{7}C_{4}a^{3} \times 5^{4}$ ***M1A1***

$56 \times 5 = 35a^{3}$

$a^{3} = 8$ ***(A1)***

$a = 2$ ***A1***

METHOD 2

attempt to expand both binomials ***M1***

$175000x^{2}$ ***A1***

$21875a^{3}x^{4}$ ***A1***

$175000 = 21875a^{3}$ ***M1***

$a^{3} = 8$ ***(A1)***

$a = 2$ ***A1***

[6 marks]

4. (a) **METHOD 1**

$2\arcsin(x-1)-\frac{\pi}{4}=\frac{\pi}{4}$ ***(M1)***

$x=1+\frac{1}{\sqrt{2}}(=1.707\ldots)$ ***(A1)***

$\int_0^{1+\frac{1}{\sqrt{2}}}\frac{\pi}{4}-\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1A1***

Note: Award ***M1*** for an attempt to find the difference between two functions, ***A1*** for all correct.

METHOD 2

when $x=0,\ y=\frac{-5\pi}{4}\ (=-3.93)$ ***A1***

$x=1+\sin\left(\frac{4y+\pi}{8}\right)$ ***M1A1***

Note: Award ***M1*** for an attempt to find the inverse function.

$\int_{\frac{-5\pi}{4}}^{\frac{\pi}{4}}\left(1+\sin\left(\frac{4y+\pi}{8}\right)\right)\mathrm{d}y$ ***A1***

METHOD 3

$\left|\int_0^{1.38\ldots}\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x\right|+\int_0^{1.71\ldots}\frac{\pi}{4}\mathrm{d}x-\int_{1.38\ldots}^{1.71\ldots}\left(2\arcsin(x-1)-\frac{\pi}{4}\right)\mathrm{d}x$ ***M1A1A1A1***

Note: Award ***M1*** for considering the area below the x-axis and above the x-axis and ***A1*** for each correct integral.

[4 marks]

(b) area = 3.30 (square units) ***A2***

[2 marks]

Total [6 marks]

5. (a) $\lambda = 4 \times 0.5$ ***(M1)***

$\lambda = 2$ ***(A1)***

$\mathrm{P}(X \le 2) = 0.677$ ***A1***

[3 marks]

(b) $Y \sim B(10,\, 0.677)$ ***(M1)(A1)***

$\mathrm{P}(Y = 7) = 0.263$ ***A1***

Note: Award ***M1*** for clear recognition of binomial distribution.

[3 marks]

Total [6 marks]

6. (a) $x = \frac{\pi}{4}$ ***A1***

$x = \frac{5\pi}{4},\ x = -\frac{3\pi}{4}$ ***A1***

[2 marks]

(b) reflection in the y-axis results in $y = \tan\left(-x + \frac{\pi}{4}\right)\left(= \cot\left(x + \frac{\pi}{4}\right)\right)$ ***(A1)***

vertical stretch gives $y = \frac{1}{2}\tan\left(-x + \frac{\pi}{4}\right)\left(= \frac{1}{2}\cot\left(x + \frac{\pi}{4}\right)\right)$ ***(A1)***

translation

$y = \frac{1}{2}\tan\left[-\left(x - \frac{\pi}{4} - \frac{\pi}{4}\right)\right] - 3$

$= \frac{1}{2}\tan\left(-x + \frac{\pi}{2}\right) - 3\left(= \frac{1}{2}\cot(x) - 3\right)$ ***A1A1***

Notes: Award the ***A1***s independently of each other.
Do not penalize the absence of $y =$.

[4 marks]

Total [6 marks]

7. METHOD 1

$\vec{AB} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$ ***(A1)***

$\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***M1A1***

$= \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ ***A1***

$x - y - z = k$ ***M1***

$k = 3$

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

METHOD 2

let plane Π be $ax + by + cz = d$

attempt to form one or more simultaneous equations: ***M1***

$a + 2b - c = 0$ (1) ***A1***

$6a + 2b + c = d$ (2)

$3a - b + c = d$ (3) ***A1***

Note: Award second ***A1*** for equations (2) and (3).

attempt to solve ***M1***

EITHER

using GDC gives $a = \frac{d}{3},\ b = -\frac{d}{3},\ c = -\frac{d}{3}$ ***(A1)***

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

OR

row reduction ***M1***

equation of plane Π is $x - y - z = 3$ or equivalent ***A1***

[6 marks]

8. (a) area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ ***M1A1***

$V =$ area of segment $\times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ ***A1***

[3 marks]

(b) **METHOD 1**

$\frac{dV}{dt} = \frac{5}{4}(1 - \cos\theta)\frac{d\theta}{dt}$ ***M1A1***

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{d\theta}{dt}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128 \text{ (rad } s^{-1})$ ***A1***

METHOD 2

$\frac{d\theta}{dt} = \frac{d\theta}{dV} \times \frac{dV}{dt}$ ***(M1)***

$\frac{dV}{d\theta} = \frac{5}{4}(1 - \cos\theta)$ ***A1***

$\frac{d\theta}{dt} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128 \left(\frac{4}{3125}\right) \text{(rad } s^{-1})$ ***A1***

[4 marks]

Total [7 marks]

Section B

9. (a) $T \sim N(196, 24^2)$

$P(T < 180) = 0.252$ ***(M1)A1***

[2 marks]

(b) $P(T < T_1) = 0.05$ ***(M1)***

$T_1 = 157$ ***A1***

[2 marks]

(c) $F \sim N(210, \sigma^2)$

$P(F < 235) = 0.79$ ***(M1)***

$\frac{235 - 210}{\sigma} = 0.806421$ or equivalent ***(M1)(A1)***

$\sigma = 31.0$ ***A1***

[4 marks]

Total [8 marks]

10. (a) $p^2 = 12^2 + r^2 - 2 \times 12 \times r \times \cos(30^\circ)$ ***M1A1***

$r^2 - 12\sqrt{3}\,r + 144 - p^2 = 0$ ***AG***

[2 marks]

(b) **EITHER**

$r^2 - 12\sqrt{3}r + 80 = 0$ ***(M1)***

OR

using the sine rule ***(M1)***

THEN

$PQ = 5.10\text{ (cm)}$ or ***A1***

$PQ = 15.7\text{ (cm)}$ ***A1***

[3 marks]

(c) area $= \frac{1}{2} \times 12 \times 5.1008\ldots \times \sin(30^\circ)$ ***M1A1***

$= 15.3\,(\text{cm}^2)$ ***A1***

[3 marks]

continued…

Question 10 continued

(d) **METHOD 1**

EITHER

$r^2 - 12\sqrt{3}r + 144 - p^2 = 0$

discriminant $= \left(12\sqrt{3}\right)^2 - 4 \times \left(144 - p^2\right)$ ***M1***

$= 4\left(p^2 - 36\right)$ ***A1***

$\left(p^2 - 36\right) > 0$ ***M1***

$p > 6$ ***A1***

OR

construction of a right angle triangle ***(M1)***

$12 \sin 30^\circ = 6$ ***M1(A1)***

hence for two triangles $p > 6$ ***R1***

THEN

$p < 12$ ***A1***

$144 - p^2 > 0$ to ensure two positive solutions or valid geometric argument ***R1***

$\therefore\ 6 < \mathrm{p} < 12$ ***A1***

METHOD 2

diagram showing two triangles ***(M1)***

$12 \sin 30^o = 6$ ***M1A1***

one right angled triangle when $p = 6$ ***(A1)***

$\therefore p > 6$ for two triangles ***R1***

$p < 12$ for two triangles ***A1***

$6 < p < 12$ ***A1***

[7 marks]

Total [15 marks]

11. (a) $v(15) = \dfrac{98}{\sqrt{1+(15-10)^2}}$ ***(M1)***

$v(15) = 19.2\ (\text{ms}^{-1})$ ***A1***

[2 marks]

(b) $\displaystyle\int_0^{10} 9.8t\,\text{d}t$ ***(M1)***

$= 490\ (\text{m})$ ***A1***

[2 marks]

(c) $\dfrac{98}{\sqrt{1+(t-10)^2}} = 2.8$ ***(M1)***

$t = 44.985\ldots(\text{s})$ ***A1***

$h = 490 + \displaystyle\int_{10}^{44.9\ldots} \frac{98}{\sqrt{1+(t-10)^2}}\,\text{d}t$ ***(M1)(A1)***

$h = 906(\text{m})$ ***A1***

[5 marks]

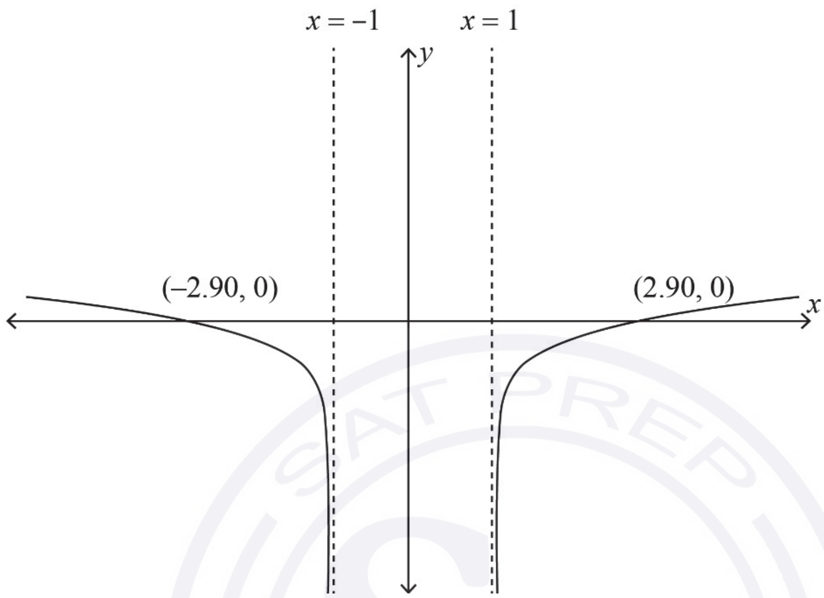
Total [9 marks]

12. (a) $x^2 - 1 > 0$ ***(M1)***

$x < -1$ or $x > 1$ ***A1***

[2 marks]

(b)



shape ***A1***

$x = 1$ and $x = -1$ ***A1***

x-intercepts ***A1***

[3 marks]

(c) **EITHER**

f is symmetrical about the y-axis ***R1***

OR

$f(-x) = f(x)$ ***R1***

[1 mark]

(d) **EITHER**

f is not one-to-one function ***R1***

OR

horizontal line cuts twice ***R1***

Note: Accept any equivalent correct statement.

[1 mark]

continued…

Question 12 continued

(e) $x = -1 + \ln\left(\sqrt{y^2 - 1}\right)$ ***M1***

$e^{2x+2} = y^2 - 1$ ***M1***

$g^{-1}(x) = \sqrt{e^{2x+2} + 1},\ x \in \mathbb{R}$ ***A1A1***

[4 marks]

(f) $g'(x) = \dfrac{1}{\sqrt{x^2 - 1}} \times \dfrac{2x}{2\sqrt{x^2 - 1}}$ ***M1A1***

$g'(x) = \dfrac{x}{x^2 - 1}$ ***A1***

[3 marks]

(g) (i) $g'(x) = \dfrac{x}{x^2 - 1} = 0 \Rightarrow x = 0$ ***M1***

which is not in the domain of g (hence no solutions to $g'(x) = 0$) ***R1***

(ii) $\left(g^{-1}\right)'(x) = \dfrac{e^{2x+2}}{\sqrt{e^{2x+2} + 1}}$ ***M1***

as $e^{2x+2} > 0 \Rightarrow \left(g^{-1}\right)'(x) > 0$ so no solutions to $\left(g^{-1}\right)'(x) = 0$ ***R1***

Note: Accept: equation $e^{2x+2} = 0$ has no solutions.

[4 marks]

Total [18 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2017

Mathematics

Higher level

Paper 2

Instructions to Examiners

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AG Answer given in the question and so no marks are awarded.

Using the markscheme

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Note: Accept all numerical answers which round correctly to the correct 2 sf answer unless stated otherwise. Do not accept any answer given to 2 sf unless stated otherwise.

Section A

1. (a) $\text{P}(5 \text{ or more}) = \frac{29}{75} (= 0.387)$ ***(M1)A1***

[2 marks]

(b) mean score $= \frac{2 \times 3 + 3 \times 15 + 4 \times 28 + 5 \times 17 + 6 \times 9 + 7 \times 3}{75}$ ***(M1)***

$= \frac{323}{75} (= 4.31)$ ***A1***

[2 marks]

Total [4 marks]

2. (a) **METHOD 1**

$4x^2 + y^2 = 7$

$8x + 2y\frac{\text{d}y}{\text{d}x} = 0$ ***(M1)(A1)***

$\frac{\text{d}y}{\text{d}x} = -\frac{4x}{y}$

Note: Award ***M1A1*** for finding $\frac{\text{d}y}{\text{d}x} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{y}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x - 1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)$ $\left(y = 0.433x + 1.30\right)$

continued…

Question 2 continued

METHOD 2

$4x^2 + y^2 = 7$

$y = \sqrt{7-4x^2}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{\sqrt{7-4x^2}}$ ***(A1)***

Note: Award ***M1A1*** for finding $\frac{\mathrm{d}y}{\mathrm{d}x} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{\sqrt{7-4x^2}}{4x}$ ***(M1)***

hence gradient of normal at $\left(1, \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4} (= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x-1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right) \left(y = 0.433x + 1.30\right)$

[6 marks]

(b) Use of $V = \pi \int_0^{\frac{\sqrt{7}}{2}} y^2 \mathrm{d}x$

$V = \pi \int_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) \mathrm{d}x$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark. Do not condone absence of or multiples of π.

$= 19.4 \left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

Total [9 marks]

3. (a) $P(X < 250) = 0.0228$ ***(M1)A1***

[2 marks]

(b) $\frac{250 - \mu}{1.5} = -2.878\ldots$ ***(M1)(A1)***

$\Rightarrow \mu = 254.32$ ***A1***

Notes: Only award ***A1*** here if the correct 2dp answer is seen. Award ***M0*** for use of 1.5^2.

[3 marks]

(c) $\frac{250 - 253}{\sigma} = -2.878\ldots$ ***(A1)***

$\Rightarrow \sigma = 1.04$ ***A1***

[2 marks]

Total [7 marks]

4. (a) $k^2 - k - 12 < 0$

$(k-4)(k+3) < 0$ ***(M1)***

$-3 < k < 4$ ***A1***

[2 marks]

(b) $\cos B = \frac{2^2 + c^2 - 4^2}{4c}$ $\left(\text{or } 16 = 2^2 + c^2 - 4c\cos B\right)$ ***M1***

$\Rightarrow \frac{c^2 - 12}{4c} < \frac{1}{4}$ ***A1***

$\Rightarrow c^2 - c - 12 < 0$

from result in (a)

$0 < \text{AB} < 4$ or $-3 < \text{AB} < 4$ ***(A1)***

but AB must be at least 2

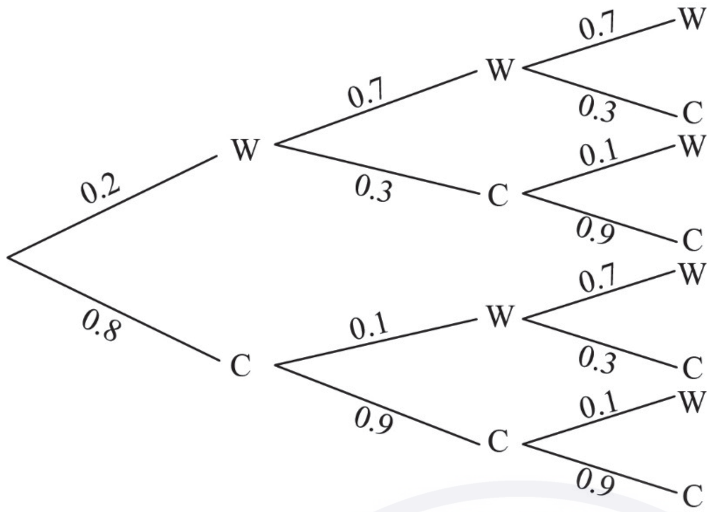
$\Rightarrow 2 < AB < 4$ ***A1***

Note: Allow $\leq AB$ for either of the final two ***A*** marks.

[4 marks]

Total [6 marks]

5. (a)



M1A2

Note: Award ***M1*** for 3 stage tree-diagram, ***A2*** for $0.8, 0.9, 0.3$ probabilities correctly placed.

[3 marks]

(b) $0.2 \times 0.7 \times 0.3 + 0.2 \times 0.3 \times 0.9 + 0.8 \times 0.1 \times 0.3 + 0.8 \times 0.9 \times 0.9 = 0.768$

(M1)A1

[2 marks]

(c) $\text{P(1st July is calm | 3rd July is windy)} = \dfrac{\text{P(1st July is calm and 3rd July is windy)}}{\text{P(3rd July is windy)}}$

(M1)

$= \dfrac{0.8 \times 0.1 \times 0.7 + 0.8 \times 0.9 \times 0.1}{1 - 0.768}$

OR $\dfrac{0.8 \times 0.1 \times 0.7 + 0.8 \times 0.9 \times 0.1}{0.2 \times 0.7 \times 0.7 + 0.2 \times 0.3 \times 0.1 + 0.8 \times 0.1 \times 0.7 + 0.8 \times 0.9 \times 0.1}$

OR $\dfrac{0.128}{0.232}$ ***(A1)(A1)***

Note: Award ***A1*** for correct numerator, ***A1*** for correct denominator.

$= 0.552$ ***A1***

[4 marks]

Total [9 marks]

6. $\log_{10}\frac{1}{2\sqrt{2}}(p+2q)=\frac{1}{2}\left(\log_{10}p+\log_{10}q\right)$

$\log_{10}\frac{1}{2\sqrt{2}}(p+2q)=\frac{1}{2}\log_{10}pq$ ***(M1)***

$\log_{10}\frac{1}{2\sqrt{2}}(p+2q)=\log_{10}(pq)^{\frac{1}{2}}$ ***(M1)***

$\frac{1}{2\sqrt{2}}(p+2q)=(pq)^{\frac{1}{2}}$ ***(A1)***

$(p+2q)^2=8pq$

$p^2+4pq+4q^2=8pq$

$p^2-4pq+4q^2=0$

$(p-2q)^2=0$ ***M1***

hence $p=2q$ ***A1***

[5 marks]

7. **METHOD 1**

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$

$(\boldsymbol{a} \times \boldsymbol{b}) - (\boldsymbol{b} \times \boldsymbol{c}) = 0$

$(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \boldsymbol{b}) = 0$ ***M1A1***

$(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b} = 0$ ***A1***

$(\boldsymbol{a} + \boldsymbol{c})$ is parallel to $\boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***R1AG***

Note:	Condone absence of arrows, underlining, or other otherwise "correct" vector notation throughout this question.

Note:	Allow "is in the same direction to", for the final ***R*** mark.

METHOD 2

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$$ ***M1A1***

$$a_2b_3 - a_3b_2 = b_2c_3 - b_3c_2 \Rightarrow b_3\left(a_2 + c_2\right) = b_2\left(a_3 + c_3\right)$$
$$a_3b_1 - a_1b_3 = b_3c_1 - b_1c_3 \Rightarrow b_1\left(a_3 + c_3\right) = b_3\left(a_1 + c_1\right)$$
$$a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 \Rightarrow b_2\left(a_1 + c_1\right) = b_1\left(a_2 + c_2\right)$$

$$\frac{\left(a_1 + c_1\right)}{b_1} = \frac{\left(a_2 + c_2\right)}{b_2} = \frac{\left(a_3 + c_3\right)}{b_3} = s$$ ***A1***

$$\Rightarrow a_1 + c_1 = sb_1$$
$$\Rightarrow a_2 + c_2 = sb_2$$
$$\Rightarrow a_3 + c_3 = sb_3$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = s\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$ ***A1***

$\Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***AG***

[4 marks]

8. **METHOD 1**

consideration of all papers

all papers may be sat in $18!$ ways ***A1***

number of ways of positioning "pairs" of science subjects $= 3! \times 17!$ ***A1***

but this includes two copies of each "triple" ***(R1)***

number of ways of positioning "triplets" of science subjects $= 3! \times 16!$ ***A1***

hence number of arrangements is $18! - 3! \times 17! + 3! \times 16!$ ***M1A1***

$\left(= 4.39 \times 10^{15}\right)$

METHOD 2

consideration of all the non-science papers ***(M1)***

hence all non-science papers can be sat in $15!$ ways ***A1***

there are $16 \times 15 \times 14 \ (= 3360)$ ways of positioning the three science papers ***(M1)A1***

hence the number of arrangements is $16 \times 15 \times 14 \times 15! \left(= 4.39 \times 10^{15}\right)$ ***(M1)A1***

METHOD 3

consideration of all papers

all papers may be sat in $18!$ ways ***A1***

number of ways of positioning exactly two science subjects $= 3! \times 15! \times 16 \times 15$ ***M1A1***

number of ways of positioning "triplets" of science subjects $= 3! \times 16!$ ***A1***

hence number of arrangements is $18! - 3! \times 16! - 3! \times 15! \times 16 \times 15$ ***M1A1***

$\left(= 4.39 \times 10^{15}\right)$

[6 marks]

Section B

9. (a) $\vec{BC} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ *(A1)*

$\boldsymbol{r} = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

(or $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$) *(M1)A1*

Note: Do not award ***A1*** unless $\boldsymbol{r} =$ or equivalent correct notation seen.

[3 marks]

(b) attempt to write in parametric form using two different parameters **AND** equate *M1*

$2\mu = 2 - \lambda$

$\mu = -1 + 4\lambda$

$-2\mu = 2 + \lambda$ *A1*

attempt to solve first pair of simultaneous equations for two parameters *M1*

solving first two equations gives $\lambda = \frac{4}{9}, \mu = \frac{7}{9}$ *(A1)*

substitution of these two values in third equation *(M1)*

since the values do not fit, the lines do not intersect *R1*

Note: Candidates may note that adding the first and third equations immediately leads to a contradiction and hence they can immediately deduce that the lines do not intersect.

[6 marks]

(c) **METHOD 1**

plane is of the form $\boldsymbol{r}\bullet(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = d$ *(A1)*

$d = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})\bullet(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -1$ *(M1)*

hence Cartesian form of plane is $2x + y - 2z = -1$ *A1*

METHOD 2

plane is of the form $2x + y - 2z = d$ *(A1)*

substituting $(1, 3, 3)$ (to find gives $2 + 3 - 6 = -1$) *(M1)*

hence Cartesian form of plane is $2x + y - 2z = -1$ *A1*

[3 marks]

continued…

Question 9 continued

(d) **METHOD 1**

attempt scalar product of direction vector BC with normal to plane ***M1***

$(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -2 + 4 - 2$

$= 0$ ***A1***

hence BC lies in Π_1 ***AG***

METHOD 2

substitute eqn of line into plane ***M1***

line $r = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$. Plane $\pi_1 : 2x + y - 2z = -1$

$2(2-\lambda) + (-1+4\lambda) - 2(2+\lambda)$

$= -1$ ***A1***

hence BC lies in Π_1 ***AG***

Note: Candidates may also just substitute $2i - j + 2k$ into the plane since they are told C lies on π_1.

Note: Do not award ***A1FT***.

[2 marks]

(e) **METHOD 1**

applying scalar product to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ ***M1***

$(2\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 0$ ***A1***

$(2\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$ ***A1***

METHOD 2

attempt to find cross product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ ***M1***

plane Π_2 has normal $\overrightarrow{OA} \times \overrightarrow{OB} = -8\boldsymbol{j} - 4\boldsymbol{k}$ ***A1***

since $-8\boldsymbol{j} - 4\boldsymbol{k} = -4(2\boldsymbol{j} + \boldsymbol{k})$, $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2 ***R1***

[3 marks]

(f) plane Π_3 has normal $\overrightarrow{OA} \times \overrightarrow{OC} = 9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$ ***A1***

[1 mark]

continued…

Question 9 continued

(g) attempt to use dot product of normal vectors *(M1)*

$\cos\theta = \frac{(2\boldsymbol{j} + \boldsymbol{k})\bullet(9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k})}{|2\boldsymbol{j} + \boldsymbol{k}||9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}|}$ *(M1)*

$= \frac{-11}{\sqrt{5}\sqrt{170}}(= -0.377...)$ *(A1)*

Note: Accept $\frac{11}{\sqrt{5}\sqrt{170}}$.

acute angle between planes $= 67.8^\circ \left(= 1.18^c\right)$ ***A1***

[4 marks]

Total [22 marks]

10. (a) $\int_0^4 \left(\frac{x^2}{a} + b\right)\mathrm{d}x = 1 \Rightarrow \left[\frac{x^3}{3a} + bx\right]_0^4 = 1 \Rightarrow \frac{64}{3a} + 4b = 1$ ***M1A1***

$\int_2^4 \left(\frac{x^2}{a} + b\right)\mathrm{d}x = 0.75 \Rightarrow \frac{56}{3a} + 2b = 0.75$ ***M1A1***

Note: $\int_0^2 \left(\frac{x^2}{a} + b\right)\mathrm{d}x = 0.25 \Rightarrow \frac{8}{3a} + 2b = 0.25$ could be seen/used in place of either of the above equations.

evidence of an attempt to solve simultaneously
(or check given a, b values are consistent) ***M1***

$a = 32,\ b = \frac{1}{12}$ ***AG***

[5 marks]

(b) $\mathrm{E}(X) = \int_0^4 x\left(\frac{x^2}{32} + \frac{1}{12}\right)\mathrm{d}x$ *(M1)*

$\mathrm{E}(X) = \frac{8}{3}(= 2.67)$ ***A1***

[2 marks]

(c) $\mathrm{E}\left(X^2\right) = \int_0^4 x^2\left(\frac{x^2}{32} + \frac{1}{12}\right)\mathrm{d}x$ *(M1)*

$\mathrm{Var(X)} = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2 = \frac{16}{15}(= 1.07)$ ***A1***

[2 marks]

continued…

Question 10 continued

(d) $\int_0^m \left(\frac{x^2}{32}+\frac{1}{12}\right)\mathrm{d}x = 0.5$ ***(M1)***

$\frac{m^3}{96}+\frac{m}{12}=0.5\left(\Rightarrow m^3+8m-48=0\right)$ ***(A1)***

$m = 2.91$ ***A1***

[3 marks]

(e) $Y \sim B(8,\ 0.75)$ ***(M1)***

$E(Y) = 8 \times 0.75 = 6$ ***A1***

[2 marks]

(f) $\mathrm{P}(Y \geq 3) = 0.996$ ***A1***

[1 mark]

Total [15 marks]

11. (a) $g(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$

$g(1) = 0 \Rightarrow a + b = 8$ ***M1A1***

$g(-1) = 0 \Rightarrow -a + b = -6$ ***A1***

$\Rightarrow a = 7,\ \mathrm{b} = 1$ ***A1***

[4 marks]

(b) $3x^4 + 7x^3 + x^2 - 7x - 4 = \left(x^2-1\right)\left(px^2 + qx + r\right)$

attempt to equate coefficients ***(M1)***

$p = 3,\ q = 7,\ r = 4$ ***(A1)***

$3x^4 + 7x^3 + x^2 - 7x - 4 = \left(x^2-1\right)\left(3x^2 + 7x + 4\right)$

$= (x-1)(x+1)^2(3x+4)$ ***A1***

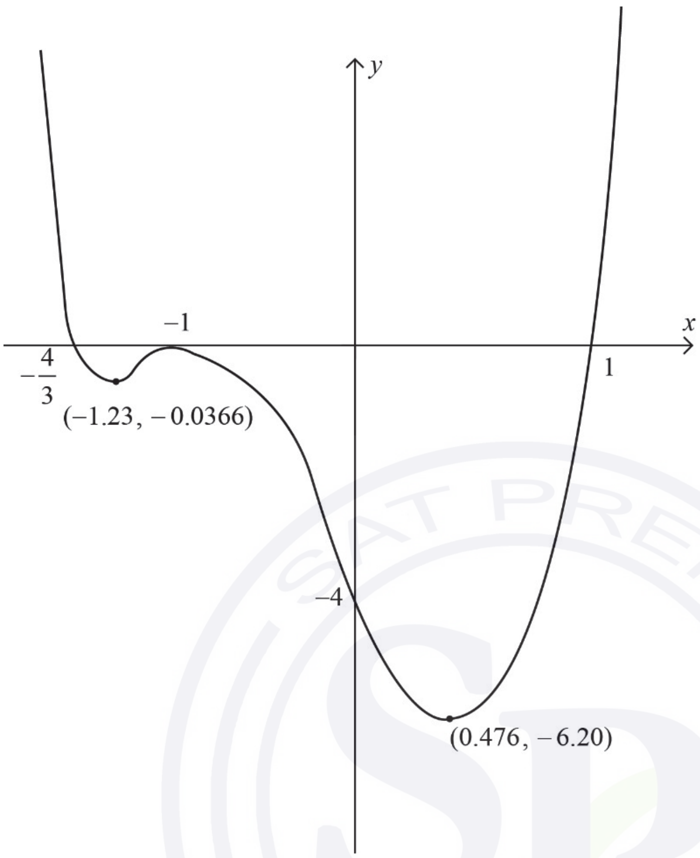
Note: Accept any equivalent valid method.

[3 marks]

continued…

Question 11 continued

(c)



A1 for correct shape (*ie* with correct number of max/min points)
A1 for correct x and y intercepts
A1 for correct maximum and minimum points

[3 marks]

(d) $c > 0$ ***A1***
$-6.20 < c < -0.0366$ ***A1A1***

Note:	Award ***A1*** for correct end points and ***A1*** for correct inequalities.

Note:	If the candidate has misdrawn the graph and omitted the first minimum point, the maximum mark that may be awarded is ***A1FTA0A0*** for $c > -6.20$ seen.

[3 marks]

Total [13 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2016

Mathematics

Higher level

Paper 2

23 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2016"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log (a-b)$	Do not award the final ***A1***

3 ***N*** marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $\mathrm{E}\left(X^2\right) = \sum x^2 \cdot \mathrm{P}(X = x) = 10.37$ $(= 10.4$ 3 sf) ***(M1)A1***

[2 marks]

(b) **METHOD 1**

$\mathrm{sd}(X) = 1.44069\ldots$ ***(M1)(A1)***

$\mathrm{Var}(X) = 2.08$ $(= 2.0756)$ ***A1***

METHOD 2

$\mathrm{E}(X) = 2.88$ $(= 0.06 + 0.27 + 0.5 + 0.98 + 0.63 + 0.44)$ ***(A1)***

use of $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2$ ***(M1)***

Note: Award ***(M1)*** only if $\left(\mathrm{E}(X)\right)^2$ is used correctly.

$\left(\mathrm{Var}(X) = 10.37 - 8.29\right)$

$\mathrm{Var}(X) = 2.08$ $(= 2.0756)$ ***A1***

Note: Accept 2.11.

METHOD 3

$\mathrm{E}(X) = 2.88$ $(= 0.06 + 0.27 + 0.5 + 0.98 + 0.63 + 0.44)$ ***(A1)***

use of $\mathrm{Var}(X) = \mathrm{E}\left(\left(X - \mathrm{E}(X)\right)^2\right)$ ***(M1)***

$\left(0.679728 + \ldots + 0.549152\right)$

$\mathrm{Var}(X) = 2.08$ $(= 2.0756)$ ***A1***

[3 marks]

Total [5 marks]

2. $\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

EITHER

$\theta = \arccos\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \quad \left(\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$ ***(M1)***

$= \arccos\left(\frac{2 + 0 - 1}{\sqrt{3}\sqrt{5}}\right) \quad \left(\cos\theta = \frac{2 + 0 - 1}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arccos\left(\frac{1}{\sqrt{15}}\right) \quad \left(\cos\theta = \frac{1}{\sqrt{15}}\right)$

OR

$\theta = \arcsin\left(\frac{|\boldsymbol{n}_1 \times \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \quad \left(\sin\theta = \frac{|\boldsymbol{n}_1 \times \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$ ***(M1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right) \quad \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{15}}\right) \quad \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{15}}\right)$

THEN

$= 75.0^\circ$ (or 1.31) ***A1***

[5 marks]

3. (a) **METHOD 1**

$\mathrm{P}(X = x+1) = \dfrac{\mu^{x+1}}{(x+1)!}\mathrm{e}^{-\mu}$ ***A1***

$= \dfrac{\mu}{x+1} \times \dfrac{\mu^{x}}{x!}\mathrm{e}^{-\mu}$ ***M1A1***

$= \dfrac{\mu}{x+1} \times \mathrm{P}(X = x)$ ***AG***

METHOD 2

$\dfrac{\mu}{x+1} \times \mathrm{P}(X = x) = \dfrac{\mu}{x+1} \times \dfrac{\mu^{x}}{x!}\mathrm{e}^{-\mu}$ ***A1***

$= \dfrac{\mu^{x+1}}{(x+1)!}\mathrm{e}^{-\mu}$ ***M1A1***

$= \mathrm{P}(X = x+1)$ ***AG***

METHOD 3

$\dfrac{\mathrm{P}(X = x+1)}{\mathrm{P}(X = x)} = \dfrac{\dfrac{\mu^{x+1}}{(x+1)!}\mathrm{e}^{-\mu}}{\dfrac{\mu^{x}}{x!}\mathrm{e}^{-\mu}}$ ***(M1)***

$= \dfrac{\mu^{x+1}}{\mu^{x}} \times \dfrac{x!}{(x+1)!}$ ***A1***

$= \dfrac{\mu}{x+1}$ ***A1***

and so $\mathrm{P}(X = x+1) = \dfrac{\mu}{x+1} \times \mathrm{P}(X = x)$ ***AG***

[3 marks]

(b) $\mathrm{P}(X = 3) = \dfrac{\mu}{3} \cdot \mathrm{P}(X = 2)$ $\left(0.112777 = \dfrac{\mu}{3} \cdot 0.241667\right)$ ***A1***

attempting to solve for μ ***(M1)***

$\mu = 1.40$ ***A1***

[3 marks]

Total [6 marks]

4. attempting a valid method to obtain the required term in the expansion ***(M1)***

Note: Valid methods include an attempt to expand, noting the behaviour of the powers of x, use of the general binomial expansion term, use of a ratio etc.

identifying the correct term ***(A1)***

$\binom{12}{8} \times 4^4 \times \left(-\frac{3}{2}\right)^8 \quad \left(= 495 \times 4^4 \times \left(-\frac{3}{2}\right)^8\right)$ ***M1A1***

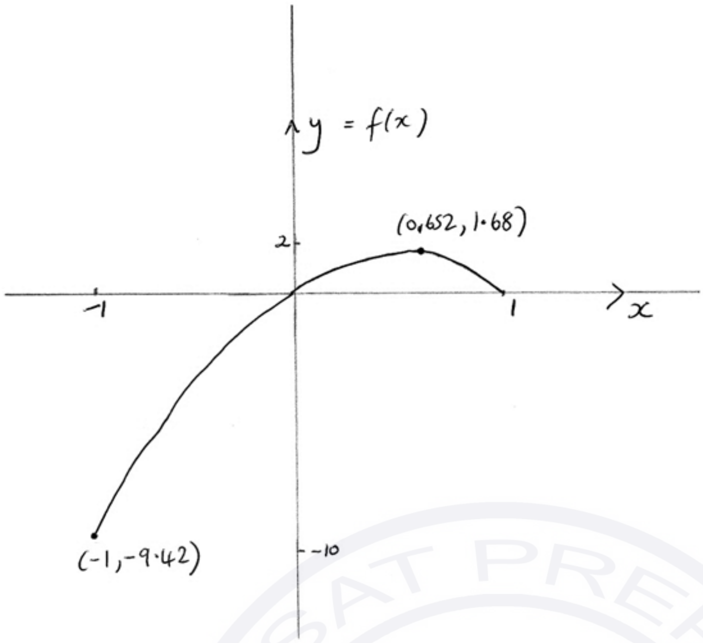
Note: Accept $\binom{12}{4}$.

Note: Award ***M1*** for the product of a binomial coefficient, a power of 4 and either a power of $-\frac{3}{2}$ or $\frac{3}{2}$.

$= 3247695$ ***A1***

[5 marks]

5. (a)



correct shape passing through the origin and correct domain ***A1***

Note: Endpoint coordinates are not required. The domain can be indicated by -1 and 1 marked on the x-axis.

$(0.652, 1.68)$ ***A1***

two correct intercepts (coordinates not required) ***A1***

Note: A graph passing through the origin is sufficient for $(0, 0)$.

[3 marks]

(b) $[-9.42, 1.68]$ (or $[-3\pi, 1.68]$) ***A1A1***

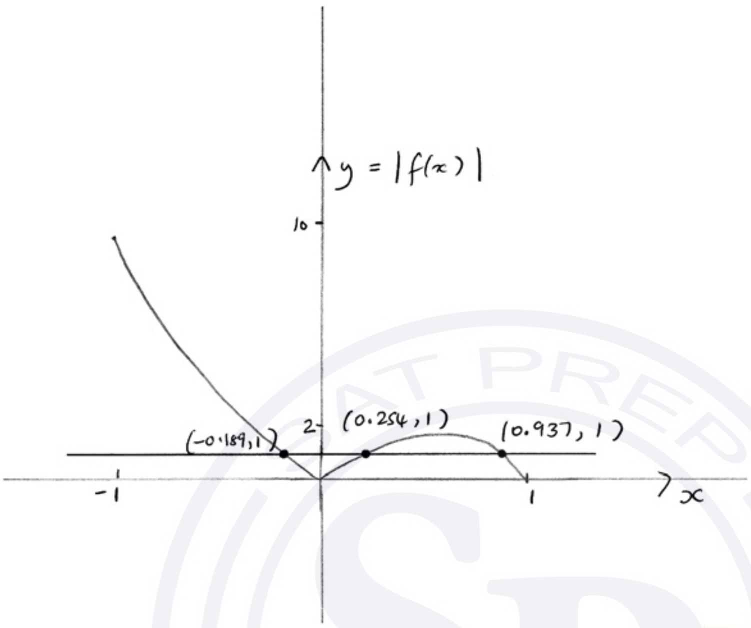
Note: Award ***A1A0*** for open or semi-open intervals with correct endpoints. Award ***A1A0*** for closed intervals with one correct endpoint.

[2 marks]

continued…

Question 5 continued

(c) attempting to solve either $|3x \arccos(x)| > 1$ (or equivalent) or $|3x \arccos(x)| = 1$ (or equivalent) (*eg*. graphically) ***(M1)***



$x = -0.189,\ 0.254,\ 0.937$ ***(A1)***

$-1 \le x < -0.189$ or $0.254 < x < 0.937$ ***A1A1***

Note: Award ***A0*** for $x < -0.189$.

[4 marks]

Total [9 marks]

6. **METHOD 1**

substituting for x and attempting to solve for y (or vice versa) ***(M1)***

$y = (\pm)\ 0.11821\ldots$ ***(A1)***

EITHER

$145x + 143y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0\ \left(\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{145x}{143y}\right)$ ***M1A1***

OR

$145x\dfrac{\mathrm{d}x}{\mathrm{d}t} + 143y\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***M1A1***

THEN

attempting to find $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{145(3.2\times10^{-3})}{143((\pm)\ 0.11821\ldots)}\times(7.75\times10^{-5})\right)$ ***(M1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

METHOD 2

$y = (\pm)\sqrt{\dfrac{1-72.5x^2}{71.5}}$ ***M1A1***

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = (\pm)0.0274\ldots$ ***(M1)(A1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = (\pm)0.0274\ldots\times7.75\times10^{-5}$ ***(M1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13\times10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider $\pm$.

[6 marks]

7. (a) **METHOD 1**

let $\text{AC} = x$

$3^2 = x^2 + 4^2 - 8x\cos\frac{\pi}{9}$ ***M1A1***

attempting to solve for x ***(M1)***

$x = 1.09,\ 6.43$ ***A1A1***

METHOD 2

let $\text{AC} = x$

using the sine rule to find a value of C ***M1***

$4^2 = x^2 + 3^2 - 6x\cos(152.869\ldots°) \Rightarrow x = 1.09$ ***(M1)A1***

$4^2 = x^2 + 3^2 - 6x\cos(27.131\ldots°) \Rightarrow x = 6.43$ ***(M1)A1***

METHOD 3

let $\text{AC} = x$

using the sine rule to find a value of B and a value of C ***M1***

obtaining $B = 132.869\ldots°,\ 7.131\ldots°$ and $C = 27.131\ldots°,\ 152.869\ldots°$ ***A1***

($B = 2.319\ldots,\ 0.124\ldots$ and $C = 0.473\ldots,\ 2.668\ldots$)

attempting to find a value of x using the cosine rule ***(M1)***

$x = 1.09,\ 6.43$ ***A1A1***

Note: Award ***M1A0(M1)A1A0*** for one correct value of x

[5 marks]

(b) $\frac{1}{2}\times 4\times 6.428\ldots\times\sin\frac{\pi}{9}$ and $\frac{1}{2}\times 4\times 1.088\ldots\times\sin\frac{\pi}{9}$ ***(A1)***

($4.39747\ldots$ and $0.744833\ldots$)

let D be the difference between the two areas

$D = \frac{1}{2}\times 4\times 6.428\ldots\times\sin\frac{\pi}{9} - \frac{1}{2}\times 4\times 1.088\ldots\times\sin\frac{\pi}{9}$ ***(M1)***

$(D = 4.39747\ldots - 0.744833\ldots)$

$= 3.65\left(\text{cm}^2\right)$ ***A1***

[3 marks]

Total [8 marks]

8. (a) $\mathrm{P}(X < 42.52) = 0.6940$ ***(M1)***

either $\mathrm{P}\left(Z < \frac{30.31-\mu}{\sigma}\right) = 0.1180$ or $\mathrm{P}\left(Z < \frac{42.52-\mu}{\sigma}\right) = 0.6940$ ***(M1)***

$\frac{30.31-\mu}{\sigma} = \underbrace{\Phi^{-1}(0.1180)}_{-1.1850\ldots}$ ***(A1)***

$\frac{42.52-\mu}{\sigma} = \underbrace{\Phi^{-1}(0.6940)}_{0.5072\ldots}$ ***(A1)***

attempting to solve simultaneously ***(M1)***

$\mu = 38.9$ and $\sigma = 7.22$ ***A1***

[6 marks]

(b) $\mathrm{P}(\mu - 1.2\sigma < X < \mu + 1.2\sigma)$ (or equivalent *eg*. $2\mathrm{P}(\mu < X < \mu + 1.2\sigma)$) ***(M1)***

$= 0.770$ ***A1***

Note: Award ***(M1)A1*** for $\mathrm{P}(-1.2 < Z < 1.2) = 0.770$.

[2 marks]

Total [8 marks]

9. (a) $A = 2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2$ ***M1A1A1***

Note: Award ***M1A1A1*** for alternative correct expressions *eg*. $A = 4\left(\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)r^2 + \frac{1}{2}\theta r^2$.

[3 marks]

(b) **METHOD 1**

consider for example triangle ADM where M is the midpoint of BD ***M1***

$\sin\frac{\alpha}{4} = \frac{1}{4}$ ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

METHOD 2

attempting to use the cosine rule (to obtain $1 - \cos\frac{\alpha}{2} = \frac{1}{8}$) ***M1***

$\sin\frac{\alpha}{4} = \frac{1}{4}$ (obtained from $\sin\frac{\alpha}{4} = \sqrt{\frac{1-\cos\frac{\alpha}{2}}{2}}$) ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

METHOD 3

$\sin\left(\frac{\pi}{2} - \frac{\alpha}{4}\right) = 2\sin\frac{\alpha}{2}$ where $\frac{\theta}{2} = \frac{\pi}{2} - \frac{\alpha}{4}$

$\cos\frac{\alpha}{4} = 4\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}$ ***M1***

Note: Award ***M1*** either for use of the double angle formula or the conversion from sine to cosine.

$\frac{1}{4} = \sin\frac{\alpha}{4}$ ***A1***

$\frac{\alpha}{4} = \arcsin\frac{1}{4}$

$\alpha = 4\arcsin\frac{1}{4}$ ***AG***

[2 marks]

continued…

Question 9 continued

(c) (from triangle ADM), $\theta = \pi - \frac{\alpha}{2}$ $\left(= \pi - 2\arcsin\frac{1}{4} = 2\arccos\frac{1}{4} = 2.6362\ldots\right)$ ***A1***

attempting to solve $2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2 = 4$

with $\alpha = 4\arcsin\frac{1}{4}$ and $\theta = \pi - \frac{\alpha}{2}$ $\left(= 2\arccos\frac{1}{4}\right)$ for r ***(M1)***

$r = 1.69$ ***A1***

[3 marks]

Total [8 marks]

Section B

10. (a) attempting to solve either $2e^x - 1 = 0$ or $2e^x - 1 \neq 0$ for x ***(M1)***

$D = \mathbb{R} \setminus \{-\ln 2\}$ (or equivalent *eg* $x \neq -\ln 2$) ***A1***

> **Note:** Accept $D = \mathbb{R} \setminus \{-0.693\}$ or equivalent *eg* $x \neq -0.693$.

[2 marks]

(b) considering $\lim\limits_{x \to -\ln 2} f(x)$ ***(M1)***

$x = -\ln 2$ $(x = -0.693)$ ***A1***

considering one of $\lim\limits_{x \to -\infty} f(x)$ or $\lim\limits_{x \to +\infty} f(x)$ ***M1***

$\lim\limits_{x \to -\infty} f(x) = -2 \Rightarrow y = -2$ ***A1***

$\lim\limits_{x \to +\infty} f(x) = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$ ***A1***

> **Note:** Award ***A0A0*** for $y = -2$ and $y = -\frac{1}{2}$ stated without any justification.

[5 marks]

(c) $f'(x) = \dfrac{-e^x(2e^x - 1) - 2e^x(2 - e^x)}{(2e^x - 1)^2}$ ***M1A1A1***

$= -\dfrac{3e^x}{(2e^x - 1)^2}$ ***AG***

[3 marks]

(d) $f'(x) < 0$ (for all $x \in D$) $\Rightarrow f$ is (strictly) decreasing ***R1***

> **Note:** Award ***R1*** for a statement such as $f'(x) \neq 0$ and so the graph of f has no turning points.

one branch is above the upper horizontal asymptote and the other branch is below the lower horizontal asymptote ***R1***

f has an inverse ***AG***

$-\infty < x < -2 \cup -\frac{1}{2} < x < \infty$ ***A2***

> **Note:** Award ***A2*** if the domain of the inverse is seen in either part (d) or in part (e).

[4 marks]

continued…

Question 10 continued

(e) $x = \dfrac{2 - e^y}{2e^y - 1}$ ***M1***

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$2xe^y - x = 2 - e^y$ ***M1***

$e^y(2x + 1) = x + 2$ ***A1***

$f^{-1}(x) = \ln\left(\dfrac{x + 2}{2x + 1}\right)$ $\left(f^{-1}(x) = \ln(x + 2) - \ln(2x + 1)\right)$ ***A1***

[4 marks]

(f) use of $V = \pi\int_a^b x^2\,dy$ ***(M1)***

$= \pi\int_0^1 \left(\ln\left(\dfrac{y + 2}{2y + 1}\right)\right)^2 dy$ ***(A1)(A1)***

Note: Award ***(A1)*** for the correct integrand and ***(A1)*** for the limits.

$= 0.331$ ***A1***

[4 marks]

Total [22 marks]

11. (a) $\mathrm{P}(X = 3) = (0.1)^3$ ***A1***

$= 0.001$ ***AG***

$\mathrm{P}(X = 4) = \mathrm{P}\left(VV\overline{V}V\right) + \mathrm{P}\left(V\overline{V}VV\right) + \mathrm{P}\left(\overline{V}VVV\right)$ ***(M1)***

$= 3 \times (0.1)^3 \times 0.9$ (or equivalent) ***A1***

$= 0.0027$ ***AG***

[3 marks]

(b) **METHOD 1**

attempting to form equations in a and b ***M1***

$\frac{9 + 3a + b}{2000} = \frac{1}{1000}$ $(3a + b = -7)$ ***A1***

$\frac{16 + 4a + b}{2000} \times \frac{9}{10} = \frac{27}{10000}$ $(4a + b = -10)$ ***A1***

attempting to solve simultaneously ***(M1)***

$a = -3,\ b = 2$ ***A1***

METHOD 2

$\mathrm{P}(X = n) = \binom{n-1}{2} \times 0.1^3 \times 0.9^{n-3}$ ***M1***

$= \frac{(n-1)(n-2)}{2000} \times 0.9^{n-3}$ ***(M1)A1***

$= \frac{n^2 - 3n + 2}{2000} \times 0.9^{n-3}$ ***A1***

$a = -3,\ b = 2$ ***A1***

Note: Condone the absence of 0.9^{n-3} in the determination of the values of a and b.

[5 marks]

continued…

Question 11 continued

(c) **METHOD 1**

EITHER

$\mathrm{P}(X = n) = \dfrac{n^2 - 3n + 2}{2000} \times 0.9^{n-3}$ ***(M1)***

OR

$\mathrm{P}(X = n) = \dbinom{n-1}{2} \times 0.1^3 \times 0.9^{n-3}$ ***(M1)***

THEN

$= \dfrac{(n-1)(n-2)}{2000} \times 0.9^{n-3}$ ***A1***

$\mathrm{P}(X = n-1) = \dfrac{(n-2)(n-3)}{2000} \times 0.9^{n-4}$ ***A1***

$\dfrac{\mathrm{P}(X = n)}{\mathrm{P}(X = n-1)} = \dfrac{(n-1)(n-2)}{(n-2)(n-3)} \times 0.9$ ***A1***

$= \dfrac{0.9(n-1)}{n-3}$ ***AG***

METHOD 2

$\dfrac{\mathrm{P}(X = n)}{\mathrm{P}(X = n-1)} = \dfrac{\dfrac{n^2 - 3n + 2}{2000} \times 0.9^{n-3}}{\dfrac{(n-1)^2 - 3(n-1) + 2}{2000} \times 0.9^{n-4}}$ ***(M1)***

$= \dfrac{0.9\left(n^2 - 3n + 2\right)}{\left(n^2 - 5n + 6\right)}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$= \dfrac{0.9(n-1)(n-2)}{(n-2)(n-3)}$ ***A1***

$= \dfrac{0.9(n-1)}{n-3}$ ***AG***

[4 marks]

continued…

Question 11 continued

(d) (i) attempting to solve $\frac{0.9(n-1)}{n-3}=1$ for n ***M1***

$n=21$ ***A1***

$\frac{0.9(n-1)}{n-3}<1 \Rightarrow n>21$ ***R1***

$\frac{0.9(n-1)}{n-3}>1 \Rightarrow n<21$ ***R1***

X has two modes ***AG***

Note: Award ***R1R1*** for a clearly labelled graphical representation of the two inequalities (using $\frac{\mathrm{P}(X=n)}{\mathrm{P}(X=n-1)}$).

(ii) the modes are 20 and 21 ***A1***

[5 marks]

(e) **METHOD 1**

$Y \sim \mathrm{B}(x, 0.1)$ ***(A1)***

attempting to solve $\mathrm{P}(Y \geq 3) > 0.5$ (or equivalent *eg* $1-\mathrm{P}(Y \leq 2) > 0.5$) for x ***(M1)***

Note: Award ***(M1)*** for attempting to solve an equality (obtaining $x=26.4$).

$x=27$ ***A1***

METHOD 2

$\sum_{n=0}^{x} \mathrm{P}(X=n) > 0.5$ ***(A1)***

attempting to solve for x ***(M1)***

$x=27$ ***A1***

[3 marks]

Total [20 marks]

12. (a) $A_1 = 1.004x$ ***A1***

$A_2 = 1.004(1.004x + x)$ ***A1***

$= 1.004^2 x + 1.004x$ ***AG***

Note: Accept an argument in words for example, first deposit has been in for two months and second deposit has been in for one month.

[2 marks]

(b) (i) $A_3 = 1.004\left(1.004^2 x + 1.004x + x\right) = 1.004^3 x + 1.004^2 x + 1.004x$ ***(M1)A1***

$A_4 = 1.004^4 x + 1.004^3 x + 1.004^2 x + 1.004x$ ***A1***

(ii) $A_{120} = \left(1.004^{120} + 1.004^{119} + \ldots + 1.004\right)x$ ***(A1)***

$= \dfrac{1.004^{120} - 1}{1.004 - 1} \times 1.004x$ ***M1A1***

$= 251\left(1.004^{120} - 1\right)x$ ***AG***

[6 marks]

(c) $A_{216} = 251\left(1.004^{216} - 1\right)x \quad \left(= x\sum_{t=1}^{216} 1.004^t\right)$ ***A1***

[1 mark]

(d) $251\left(1.004^{216} - 1\right)x = 20000 \Rightarrow x = 58.22\ldots$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for $251\left(1.004^{216} - 1\right)x > 20000$, ***(M1)*** for attempting to solve and ***(A1)*** for $x > 58.22\ldots$.

$x = 59$ ***A1***

Note: Accept $x = 58$. Accept $x \geq 59$.

[4 marks]

(e) $r = 1.004^{12} \ (= 1.049\ldots)$ ***(M1)***

$15000\, r^n - 1000\, \dfrac{r^n - 1}{r - 1} = 0 \Rightarrow n = 27.8\ldots$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for the equation (with their value of r), ***(M1)*** for attempting to solve for n and ***(A1)*** for $n = 27.8\ldots$.

$n = 28$ ***A1***

Note: Accept $n = 27$.

[5 marks]

Total [18 marks]

Markscheme

May 2016

Mathematics

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2016"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
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Section A

1. (a) $\vec{OA} \times \vec{OB} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$ ***(M1)A1***

> **Note:** ***M1A0*** can be awarded for attempt at a correct method **shown**, or correct method implied by the digits 4, 4, 2 found in the correct order.

[2 marks]

(b) area $= \frac{1}{2}\sqrt{4^2 + 4^2 + 2^2} = 3$ ***M1A1***

[2 marks]

Total [4 marks]

2. (a) $(x+2)^2 - 6$ ***A1A1***

[2 marks]

(b) $(g \circ f)(x) = (x+2)^2 - 6$ ***(M1)***

$\Rightarrow g(x) = x^2 - 6$ ***A1***

[2 marks]

Total [4 marks]

3. (a) $v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{e}^{-t}}{2 - \mathrm{e}^{-t}} \left(= \frac{1}{2\mathrm{e}^{t} - 1} \text{ or } -1 + \frac{2}{2 - \mathrm{e}^{-t}} \right)$ ***M1A1***

[2 marks]

(b) $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{-\mathrm{e}^{-t}\left(2 - \mathrm{e}^{-t}\right) - \mathrm{e}^{-t} \times \mathrm{e}^{-t}}{\left(2 - \mathrm{e}^{-t}\right)^2} \left(= \frac{-2\mathrm{e}^{-t}}{\left(2 - \mathrm{e}^{-t}\right)^2} \right)$ ***M1A1***

> **Note:** If simplified in part (a) award ***(M1)A1*** for $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{-2\mathrm{e}^{t}}{\left(2\mathrm{e}^{t} - 1\right)^2}$.

> **Note:** Award ***M1A1*** for $a = -\mathrm{e}^{-t}\left(2 - \mathrm{e}^{-t}\right)^{-2}\left(\mathrm{e}^{-t}\right) - \mathrm{e}^{-t}\left(2 - \mathrm{e}^{-t}\right)^{-1}$.

[2 marks]

(c) $a = -2\ (\mathrm{ms}^{-2})$ ***A1***

[1 mark]

Total [5 marks]

4. attempting to use the area of sector formula (including for a semicircle) ***M1***

semi-circle $\frac{1}{2}\pi \times 5^2 = \frac{25\pi}{2} = 39.26990817\ldots$ ***(A1)***

angle in smaller sector is $\pi - \theta$ ***(A1)***

area of sector $= \frac{1}{2} \times 2^2 \times (\pi - \theta)$ ***(A1)***

attempt to total a sum of areas of regions to 44 ***(M1)***

$2(\pi - \theta) = 44 - 39.26990817\ldots$

$\theta = 0.777 \left(= \frac{29\pi}{4} - 22 \right)$ ***A1***

Note: Award all marks except the final ***A1*** for correct working in degrees.

Note: Attempt to solve with goat inside triangle should lead to nonsense answer and so should only receive a maximum of the two ***M*** marks.

[6 marks]

5. (a) $f(-x) = \frac{3(-x)^2 + 10}{(-x)^2 - 4}$ ***A1***

$= \frac{3x^2 + 10}{x^2 - 4} = f(x)$

$f(x) = f(-x)$ ***R1***

hence this is an even function ***AG***

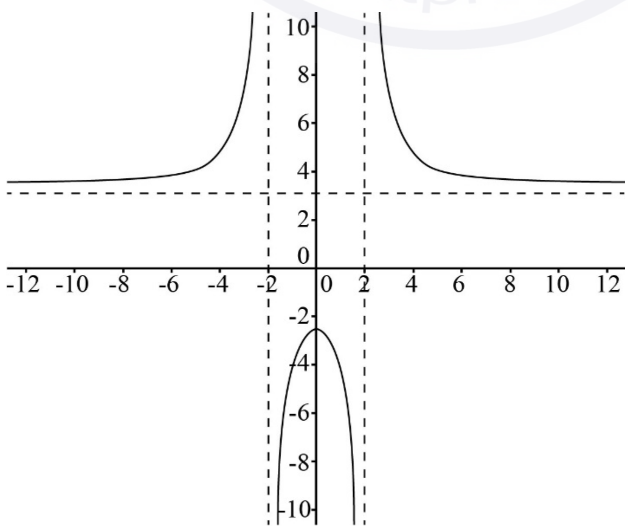
Note: Award ***A1R1*** for the statement, all the powers are even hence $f(x) = f(-x)$.

Note: Just stating all the powers are even is ***A0R0***.

Note: Do not accept arguments based on the symmetry of the graph.

[2 marks]

(b) (i)



correct shape in 3 parts which are asymptotic and symmetrical ***A1***

correct vertical asymptotes clear at 2 and -2 ***A1***

correct horizontal asymptote clear at 3 ***A1***

continued…

Question 5 continued

(ii) $f(x) > 3$ ***A1***
$f(x) \le -2.5$ ***A1***

[5 marks]

Total [7 marks]

6. let the heights of the students be X

$\mathrm{P}(X < 1.62) = 0.4$, $\mathrm{P}(X > 1.79) = 0.25$ ***M1***

Note: Award ***M1*** for either of the probabilities above.

$\mathrm{P}\left(Z < \frac{1.62 - \mu}{\sigma}\right) = 0.4$, $\mathrm{P}\left(Z < \frac{1.79 - \mu}{\sigma}\right) = 0.75$ ***M1***

Note: Award ***M1*** for either of the expressions above.

$\frac{1.62 - \mu}{\sigma} = -0.2533\ldots$, $\frac{1.79 - \mu}{\sigma} = 0.6744\ldots$ ***M1A1***

Note: ***A1*** for both values correct.

$\mu = 1.67(\mathrm{m})$, $\sigma = 0.183(\mathrm{m})$ ***A1A1***

Note: Accept answers that round to $1.7(\mathrm{m})$ and $0.18(\mathrm{m})$.

Note: Accept answers in centimetres.

[6 marks]

7. (a) $a = 420.65$ ***A1***

$390.94 = a \times 2^b$ ***M1***

$2^b = \frac{390.94}{420.65} = 0.929...$ ***A1***

$b = -0.10567$ ***A1***

[4 marks]

(b) $N = 8$ $T = 337.67$ ***A1***

Note: Accept 5sf answers between 337.44 and 337.67.

[1 mark]

(c) $N = 8$ Percentage error 1.29% ***A1***

Note: Accept negative values of the above.

[1 mark]

continued…

Question 7 continued

(d) likely not to be a good fit for larger values of N **R1**
likely to be quite a good fit for values close to 8 **R1**

[2 marks]

Total [8 marks]

8. $a^2 + 4a - b = 2$ **M1A1**

EITHER

$a^2 + 4a - (b + 2) = 0$

as a is real $\Rightarrow 16 + 4(b + 2) \geq 0$ **M1A1**

OR

$b = a^2 + 4a - 2$ **M1**

$= (a + 2)^2 - 6$ ***(A1)***

THEN

$b \geq -6$

hence smallest possible value for b is -6 **A1**

[5 marks]

9. (a) other two roots are $c - \mathrm{i}$ and $2 - \mathrm{i}d$ **A1**

[1 mark]

(b) **METHOD 1**

use of sum of roots ***(M1)***

$2c + 4 = 10$

$c = 3$ **A1**

use of product of roots **M1**

product is $(c + \mathrm{i})(c - \mathrm{i})(2 + \mathrm{i}d)(2 - \mathrm{i}d)$ **A1**

$\left(c^2 + 1\right)\left(4 + d^2\right)\left[= 10\left(4 + d^2\right)\right] = 50$ **A1**

Note: The line above can be awarded if they have used their value of c.

$d = 1$ **A1**

continued…

Question 9 continued

METHOD 2

$z^4 - 10z^3 + az^2 + bz + 50 = \left(z^2 - 2cz + c^2 + 1\right)\left(z^2 - 4z + 4 + d^2\right)$ ***M1A1***

compare constant terms or coefficients of z^3 ***(M1)***

$4 + 2c = 10$

$\left(c^2 + 1\right)\left(4 + d^2\right) = 50$ ***A1***

$c = 3,\ d = 1$ ***A1A1***

[6 marks]

Total [7 marks]

10. $\mathrm{P}(3 \text{ in the first hour}) = \dfrac{\lambda^3 e^{-\lambda}}{3!}$ ***A1***

number to arrive in the four hours follows $Po(4\lambda)$ ***M1***

$\mathrm{P}(5 \text{ arrive in total}) = \dfrac{(4\lambda)^5 e^{-4\lambda}}{5!}$ ***A1***

attempt to find $\mathrm{P}(2 \text{ arrive in the next three hours})$ ***M1***

$= \dfrac{(3\lambda)^2 e^{-3\lambda}}{2!}$ ***A1***

use of conditional probability formula ***M1***

$\mathrm{P}(3 \text{ in the first hour given 5 in total}) = \dfrac{\dfrac{\lambda^3 e^{-\lambda}}{3!} \times \dfrac{(3\lambda)^2 e^{-3\lambda}}{2!}}{\dfrac{(4\lambda)^5 e^{-4\lambda}}{5!}}$ ***A1***

$\dfrac{\left(\dfrac{9}{2!3!}\right)}{\left(\dfrac{4^5}{5!}\right)} = \dfrac{45}{512} = 0.0879$ ***A1***

[8 marks]

Section B

11. (a) valid method *eg*, sketch of curve or critical values found ***(M1)***

$x < -2.24$, $x > 2.24$, ***A1***

$-1 < x < 0.8$ ***A1***

Note: Award ***M1A1A0*** for correct intervals but with inclusive inequalities.

[3 marks]

(b) (i) $(1.67, -5.14)$, $(-1.74, -3.71)$ ***A1A1***

Note: Award ***A1A0*** for any two correct terms.

(ii) $f'(x) = 4x^3 + 0.6x^2 - 11.6x - 1$

$f''(x) = 12x^2 + 1.2x - 11.6 = 0$ ***(M1)***

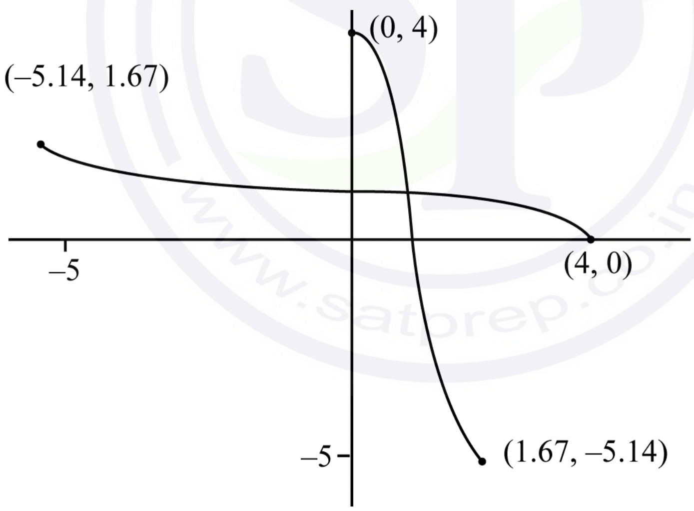
-1.03, 0.934 ***A1A1***

Note: ***M1*** should be awarded if graphical method to find zeros of $f''(x)$ or turning points of $f'(x)$ is shown.

[5 marks]

(c) (i) 1.67 ***A1***

(ii)



M1A1A1

Note: Award ***M1*** for reflection of their $y = f(x)$ in the line $y = x$ provided their f is one-one.

A1 for (0, 4), (4,0) (Accept axis intercept values) ***A1*** for the other two sets of coordinates of other end points

(iii) $x = f(1)$ ***M1***

$= -1.6$ ***A1***

[6 marks]

continued…

Question 11 continued

(d) (i) $y = 2\sin(x-1) - 3$

$x = 2\sin(y-1) - 3$ ***(M1)***

$\left(g^{-1}(x) =\right) \arcsin\left(\dfrac{x+3}{2}\right) + 1$ ***A1***

$-5 \le x \le -1$ ***A1A1***

> **Note:** Award ***A1*** for -5 and -1, and ***A1*** for correct inequalities if numbers are reasonable.

(ii) $f^{-1}\left(g(x)\right) < 1$

$g(x) > -1.6$ ***(M1)***

$x > g^{-1}(-1.6) = 1.78$ ***(A1)***

> **Note:** Accept = in the above.

$1.78 < x \le \dfrac{\pi}{2} + 1$ ***A1A1***

> **Note:** ***A1*** for $x > 1.78$ (allow $\ge$) and ***A1*** for $x \le \dfrac{\pi}{2} + 1$.

[8 marks]

Total [22 marks]

12. (a) $a^2 = 5 - 1$ ***(M1)***

$a = 2$ ***A1***

[2 marks]

(b) $2y\dfrac{dy}{dx} - \left(2x\dfrac{dy}{dx} + 2y\right) = -e^x$ ***M1A1A1A1***

> **Note:** Award ***M1*** for an attempt at implicit differentiation, ***A1*** for each part.

$\dfrac{dy}{dx} = \dfrac{2y - e^x}{2(y-x)}$ ***AG***

[4 marks]

(c) at $x = 0$, $\dfrac{dy}{dx} = \dfrac{3}{4}$ ***(A1)***

finding the negative reciprocal of a number ***(M1)***

gradient of normal is $-\dfrac{4}{3}$

$y = -\dfrac{4}{3}x + 2$ ***A1***

[3 marks]

continued…

Question 12 continued

(d) substituting linear expression (*M1*)

$\left(-\frac{4}{3}x+2\right)^2-2x\left(-\frac{4}{3}x+2\right)+e^x-5=0$ or equivalent

$x=1.56$ ***(M1)A1***

$y=-0.0779$ ***A1***

$(1.56\ ,-0.0779)$

[4 marks]

(e) $\frac{dv}{dx}=3y^2\frac{dy}{dx}$ ***M1A1***

$\frac{dv}{dx}=3\times4\times\frac{3}{4}=9$ ***A1***

[3 marks]

Total [16 marks]

13. (a) $E(X)=1\times\frac{1}{6}+2\times\frac{2}{6}+3\times\frac{3}{6}=\frac{14}{6}\left(=\frac{7}{3}=2.33\right)$ ***(M1)A1***

[2 marks]

(b) (i) $3\times P(113)+3\times P(122)$ ***(M1)***

$3\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{2}+3\times\frac{1}{6}\times\frac{1}{3}\times\frac{1}{3}=\frac{7}{72}(=0.0972)$ ***A1***

Note: Award ***M1*** for attempt to find at least four of the cases.

(ii) recognising 111 as a possibility (implied by $\frac{1}{216}$) ***(M1)***

recognising 112 and 113 as possibilities (implied by $\frac{2}{216}$ and $\frac{3}{216}$) ***(M1)***

seeing the three arrangements of 112 and 113 ***(M1)***

$P(111)+3\times P(112)+3\times P(113)$

$=\frac{1}{216}+\frac{6}{216}+\frac{9}{216}=\frac{16}{216}\left(=\frac{2}{27}=0.0741\right)$ ***A1***

[6 marks]

continued…

Question 13 continued

(c) let the number of twos be X, $X \sim B\left(10, \frac{1}{3}\right)$ ***(M1)***

$\mathrm{P}(X < 4) = \mathrm{P}(X \le 3) = 0.559$ ***(M1)A1***

[3 marks]

(d) let n be the number of balls drawn

$\mathrm{P}(X \ge 1) = 1 - \mathrm{P}(X = 0)$ ***M1***

$= 1 - \left(\frac{2}{3}\right)^n > 0.95$ ***M1***

$\left(\frac{2}{3}\right)^n < 0.05$

$n = 8$ ***A1***

[3 marks]

(e) $8p_1 = 4.8 \Rightarrow p_1 = \frac{3}{5}$ ***(M1)A1***

$8p_2(1 - p_2) = 1.5$ ***(M1)***

$p_2^2 - p_2 - 0.1875 = 0$ ***(M1)***

$p_2 = \frac{1}{4}$ $\left(\text{or } \frac{3}{4}\right)$ ***A1***

reject $\frac{3}{4}$ as it gives a total greater than one

$\mathrm{P}(1 \text{ or } 2) = \frac{17}{20}$ or $\mathrm{P}(3) = \frac{3}{20}$ ***(A1)***

recognising LCM as 20 so min total number is 20 ***(M1)***

the least possible number of 3's is 3 ***A1***

[8 marks]

Total [22 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2016

Mathematics

Higher level

Paper 2

23 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

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A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2016"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.

- If a part is completely wrong, stamp ***A0*** by the final answer.

- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

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Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

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Section A

1. $AC^2 = 7.8^2 + 10.4^2$ ***(M1)***

$AC = 13$ ***(A1)***

use of cosine rule *eg*, $\cos\left(A\hat{B}C\right) = \dfrac{6.5^2 + 9.1^2 - 13^2}{2(6.5)(9.1)}$ ***M1***

$A\hat{B}C = 111.804\ldots^\circ \ (= 1.95134\ldots)$ ***(A1)***

$= 112^\circ$ ***A1***

[5 marks]

2. (a) $P(0 \le X \le 2) = 0.242$ ***(M1)A1***

[2 marks]

(b) **METHOD 1**

$P(|X| > 1) = P(X < -1) + P(X > 1)$ ***(M1)***

$= 0.02275\ldots + 0.84134\ldots$ ***(A1)***

$= 0.864$ ***A1***

METHOD 2

$P(|X| > 1) = 1 - P(-1 < X < 1)$ ***(M1)***

$= 1 - 0.13590\ldots$ ***(A1)***

$= 0.864$ ***A1***

[3 marks]

(c) $c = 3.30$ ***(M1)A1***

[2 marks]

Total [7 marks]

3. **METHOD 1**

$\ln \frac{y}{x} = 2 \Rightarrow -\ln x + \ln y = 2$ ***A1***

$\ln x^2 + \ln y^3 = 7 \Rightarrow 2\ln x + 3\ln y = 7$ ***(M1)A1***

attempting to solve for x and y (to obtain $\ln x = \frac{1}{5}$ and $\ln y = \frac{11}{5}$) ***(M1)***

$x = e^{\frac{1}{5}}\ (= 1.22)$ ***A1***

$y = e^{\frac{11}{5}}\ (= 9.03)$ ***A1***

METHOD 2

$\ln \frac{y}{x} = 2 \Rightarrow y = e^2 x$ ***A1***

$\ln x^2 + \ln e^6 x^3 = 7$ ***(M1)A1***

attempting to solve for x ***(M1)***

$x = e^{\frac{1}{5}}\ (= 1.22)$ ***A1***

$y = e^{\frac{11}{5}}\ (= 9.03)$ ***A1***

METHOD 3

$\ln \frac{y}{x} = 2 \Rightarrow y = e^2 x$ ***A1***

$\ln x^2 + \ln y^3 = 7 \Rightarrow \ln\left(x^2 y^3\right) = 7$ ***A1***

$x^2 y^3 = e^7$ ***(M1)***

substituting $y = e^2 x$ into $x^2 y^3 = e^7$ (to obtain $e^6 x^5 = e^7$) ***M1***

$x = e^{\frac{1}{5}}\ (= 1.22)$ ***A1***

$y = e^{\frac{11}{5}}\ (= 9.03)$ ***A1***

[6 marks]

4. $ar + ar^2 = 96$ ***A1***

Note: Award ***A1*** for any valid equation involving a and r, *eg*, $\frac{a(1-r^3)}{1-r} - a = 96$.

$\frac{a}{1-r} = 500$ ***A1***

EITHER

attempting to eliminate a to obtain $500r(1 - r^2) = 96$ (or equivalent in unsimplified form) ***(M1)***

OR

attempting to obtain $a = \frac{96}{r + r^2}$ and $a = 500(1 - r)$ ***(M1)***

THEN

attempting to solve for r ***(M1)***

$r = 0.2\left(= \frac{1}{5}\right)$ or $r = 0.885\left(= \frac{\sqrt{97} - 1}{10}\right)$ ***A1A1***

[6 marks]

5. $x = \sqrt{\frac{1-y}{1+y}}$ ***M1***

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$x^2 = \frac{1-y}{1+y}$

$x^2 + x^2 y = 1 - y$ ***M1***

Note: Award ***M1*** for attempting to make y the subject.

$y\left(1 + x^2\right) = 1 - x^2$ ***(A1)***

$f^{-1}(x) = \frac{1 - x^2}{1 + x^2},\ x \ge 0$ ***A1A1***

Note: Award ***A1*** only if $f^{-1}(x)$ is seen. Award ***A1*** for the domain.

the range of f^{-1} is $-1 < f^{-1}(x) \le 1$ ***A1***

Note: Accept correct alternative notation *eg*. $-1 < y \le 1$.

[6 marks]

6. (a) $X \sim \mathrm{Po}(0.5)$ ***(A1)***

$\mathrm{P}(X \ge 1) = 0.393\ \left(= 1 - \mathrm{e}^{-0.5}\right)$ ***(M1)A1***

[3 marks]

(b) $\mathrm{P}(X = 0) = 0.607\ldots$ ***(A1)***

$\mathrm{E}(P) = (0.607\ldots \times 5) - (0.393\ldots \times 3)$ ***(M1)***

the expected profit is $\$1.85$ per glass sheet ***A1***

[3 marks]

(c) $Y \sim \mathrm{Po}(2)$ ***(M1)***

$\mathrm{P}(Y = 0) = 0.135\ \left(= \mathrm{e}^{-2}\right)$ ***A1***

[2 marks]

Total [8 marks]

7. (a) $3x^2 + 3y^2 \frac{dy}{dx} = 4\left(y + x\frac{dy}{dx}\right)$ ***M1A1***

$\left(3y^2 - 4x\right)\frac{dy}{dx} = 4y - 3x^2$ ***A1***

$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$ ***AG***

[3 marks]

(b) $\frac{dy}{dx} = 0 \Rightarrow 4y - 3x^2 = 0$ ***(M1)***

substituting $x = k$ and $y = \frac{3}{4}k^2$ into $x^3 + y^3 = 4xy$ ***M1***

$k^3 + \frac{27}{64}k^6 = 3k^3$ ***A1***

attempting to solve $k^3 + \frac{27}{64}k^6 = 3k^3$ for k ***(M1)***

$k = 1.68\left(= \frac{4}{3}\sqrt[3]{2}\right)$ ***A1***

Note: Condone substituting $y = \frac{3}{4}x^2$ into $x^3 + y^3 = 4xy$ and solving for x.

[5 marks]

Total [8 marks]

8. (a) $\frac{dv}{ds} = \frac{\cos s}{\sin^2 s + 1}$ ***M1A1***

$a = v\frac{dv}{ds}$ ***(M1)***

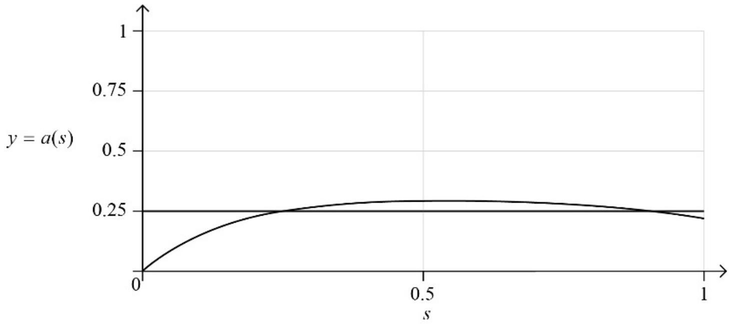
$a = \frac{\arctan(\sin s)\cos s}{\sin^2 s + 1}$ ***A1***

[4 marks]

continued…

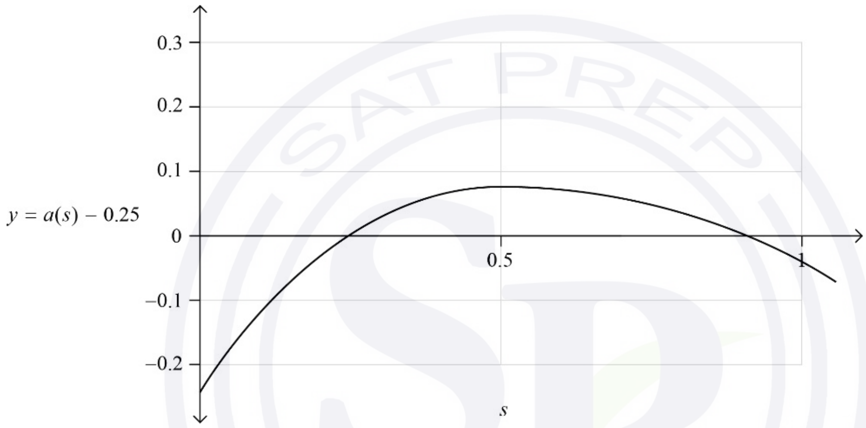
Question 8 continued

(b) **EITHER**



(M1)

OR



(M1)

THEN

$s = 0.296,\ 0.918$ (m) ***A1***

[2 marks]

Total [6 marks]

9. (a) (i) **METHOD 1**

$$\left|\vec{OC}\right|^2 = \vec{OC}\cdot\vec{OC}$$

$= (\boldsymbol{a} + \boldsymbol{b})\cdot(\boldsymbol{a} + \boldsymbol{b})$ ***A1***

$= \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{a} + \boldsymbol{b}\cdot\boldsymbol{b}$ ***A1***

$= |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

continued…

Question 9 continued

METHOD 2

$|\vec{OC}|^2 = |\vec{OA}|^2 + |\vec{OB}|^2 - 2|\vec{OA}||\vec{OB}|\cos(O\hat{A}C)$ ***A1***

$|\vec{OA}||\vec{OB}|\cos(O\hat{A}C) = -(\boldsymbol{a}\cdot\boldsymbol{b})$ ***A1***

$|\vec{OC}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

(ii) **METHOD 1**

$|\vec{AB}|^2 = \vec{AB}\cdot\vec{AB}$

$= (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a})$ ***A1***

$= \boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a}$ ***A1***

$= |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

METHOD 2

$|\vec{AB}|^2 = |\vec{AC}|^2 + |\vec{BC}|^2 - 2|\vec{AC}||\vec{BC}|\cos(A\hat{C}B)$ ***A1***

$|\vec{AC}||\vec{BC}|\cos(A\hat{C}B) = \boldsymbol{a}\cdot\boldsymbol{b}$ ***A1***

$|\vec{AB}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

[4 marks]

(b) $|\vec{OC}| = |\vec{AB}| \Rightarrow |\vec{OC}|^2 = |\vec{AB}|^2 \Rightarrow |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$ ***R1(M1)***

Note: Award ***R1*** for $|\vec{OC}| = |\vec{AB}| \Rightarrow |\vec{OC}|^2 = |\vec{AB}|^2$ and ***(M1)*** for $|\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$.

$\boldsymbol{a}\cdot\boldsymbol{b} = 0$ ***A1***

hence OACB is a rectangle (***a*** and ***b*** both non-zero)
with adjacent sides at right angles ***R1***

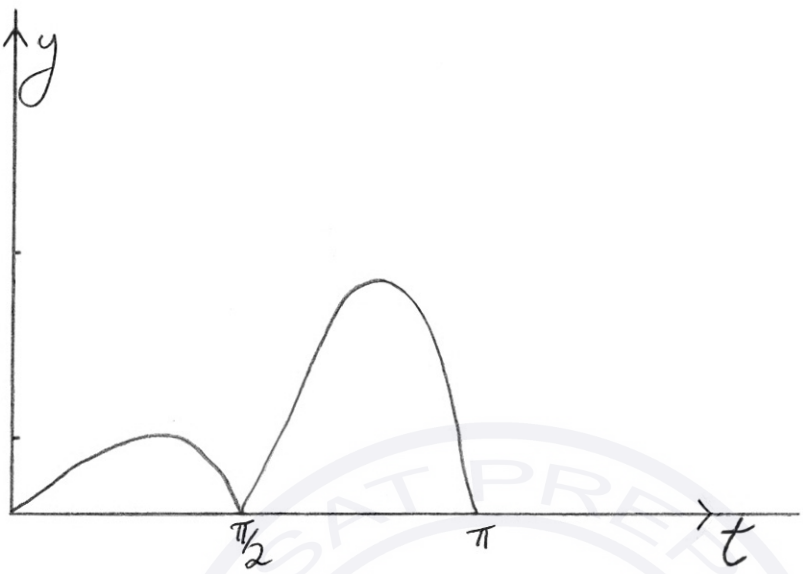
Note: Award ***R1(M1)A0R1*** if the dot product has not been used.

[4 marks]

Total [8 marks]

Section B

10.



(a) two enclosed regions ($0 \le t \le \frac{\pi}{2}$ and $\frac{\pi}{2} \le t \le \pi$) bounded by the curve and the t-axis ***A1***

correct non-symmetrical shape for $0 \le t \le \frac{\pi}{2}$ and

$\frac{\pi}{2} <$ mode of $T < \pi$ clearly apparent ***A1***

[2 marks]

(b) mode $= 2.46$ ***A1***

[1 mark]

(c) $\mathrm{E}(T) = \frac{1}{\pi}\int_0^{\pi} t^2 \left|\sin 2t\right| \mathrm{d}t$ ***(M1)***

$= 2.04$ ***A1***

[2 marks]

continued…

Question 10 continued

(d) **EITHER**

$$\text{Var}(T) = \int_0^{\pi} (t - 2.03788\ldots)^2 \left(\frac{t|\sin 2t|}{\pi}\right) \text{d}t$$ ***(M1)(A1)***

OR

$$\text{Var}(T) = \int_0^{\pi} t^2 \left(\frac{t|\sin 2t|}{\pi}\right) \text{d}t - (2.03788\ldots)^2$$ ***(M1)(A1)***

THEN

$\text{Var}(T) = 0.516$ ***A1***

[3 marks]

(e) $\frac{1}{\pi}\int_{2.03788\ldots}^{2.456590\ldots} t|\sin 2t|\,\text{d}t = 0.285$ ***(M1)A1***

[2 marks]

(f) (i) attempting integration by parts ***(M1)***

($u = t$, $\text{d}u = \text{d}t$, $\text{d}v = \sin 2t\ \text{d}t$ and $v = -\frac{1}{2}\cos 2t$)

$$\frac{1}{\pi}\left[t\left(-\frac{1}{2}\cos 2t\right)\right]_0^T - \frac{1}{\pi}\int_0^T \left(-\frac{1}{2}\cos 2t\right) \text{d}t$$ ***A1***

Note: Award ***A1*** if the limits are not included.

$$= \frac{\sin 2T}{4\pi} - \frac{T\cos 2T}{2\pi}$$ ***A1***

(ii) $\frac{\sin \pi}{4\pi} - \frac{\frac{\pi}{2}\cos \pi}{2\pi} = \frac{1}{4}$ ***A1***

as $\text{P}\left(0 \le T \le \frac{\pi}{2}\right) = \frac{1}{4}$ (or equivalent), then the lower quartile of T is $\frac{\pi}{2}$ ***R1AG***

[5 marks]

Total [15 marks]

11. (a) **EITHER**

$\alpha = \arctan\frac{7}{10} - \arctan\frac{5}{10}$ $(= 34.992\ldots^\circ - 26.5651\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \hat{\text{APT}} - \hat{\text{BPT}}$, ***(A1)*** for a correct $\hat{\text{APT}}$ and ***(A1)*** for a correct $\hat{\text{BPT}}$.

OR

$\alpha = \arctan 2 - \arctan\frac{10}{7}$ $(= 63.434\ldots^\circ - 55.008\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \hat{\text{PBT}} - \hat{\text{PAT}}$, ***(A1)*** for a correct $\hat{\text{PBT}}$ and ***(A1)*** for a correct $\hat{\text{PAT}}$.

OR

$\alpha = \arccos\left(\frac{125 + 149 - 4}{2 \times \sqrt{125} \times \sqrt{149}}\right)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of cosine rule, ***(A1)*** for a correct numerator and ***(A1)*** for a correct denominator.

THEN

$= 8.43^\circ$ ***A1***

[4 marks]

continued…

Question 11 continued

(b) **EITHER**

$$\tan\alpha = \frac{\frac{7}{x} - \frac{5}{x}}{1 + \left(\frac{7}{x}\right)\left(\frac{5}{x}\right)}$$ ***M1A1A1***

> **Note:** Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2}{x}}{1 + \frac{35}{x^2}}$$ ***M1***

OR

$$\tan\alpha = \frac{\frac{x}{5} - \frac{x}{7}}{1 + \left(\frac{x}{5}\right)\left(\frac{x}{7}\right)}$$ ***M1A1A1***

> **Note:** Award ***M1*** for use of $\tan(A - B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$$= \frac{\frac{2x}{35}}{1 + \frac{x^2}{35}}$$ ***M1***

OR

$$\cos\alpha = \frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***M1A1***

> **Note:** Award ***M1*** for either use of the cosine rule or use of $\cos(A - B)$.

$$\sin\alpha = \frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}$$ ***A1***

$$\tan\alpha = \frac{\frac{2x}{\sqrt{(x^2 + 25)(x^2 + 49)}}}{\frac{x^2 + 35}{\sqrt{(x^2 + 25)(x^2 + 49)}}}$$ ***M1***

THEN

$$\tan\alpha = \frac{2x}{x^2 + 35}$$ ***AG***

[4 marks]

continued…

Question 11 continued

(c) (i) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = \frac{2(x^2+35)-(2x)(2x)}{(x^2+35)^2} \left(= \frac{70-2x^2}{(x^2+35)^2}\right)$ ***M1A1A1***

Note: Award ***M1*** for attempting product or quotient rule differentiation, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0 \Rightarrow 70 - 2x^2 = 0$ ***(M1)***

$x = \sqrt{35}(\mathrm{m})(= 5.9161\ldots(\mathrm{m}))$ ***A1***

$\tan\alpha = \frac{1}{\sqrt{35}}(= 0.16903\ldots)$ ***(A1)***

OR

attempting to locate the stationary point on the graph of

$\tan\alpha = \frac{2x}{x^2+35}$ ***(M1)***

$x = 5.9161\ldots\ (\mathrm{m})\left(= \sqrt{35}\,(\mathrm{m})\right)$ ***A1***

$\tan\alpha = 0.16903\ldots\left(= \frac{1}{\sqrt{35}}\right)$ ***(A1)***

THEN

$\alpha = 9.59^\circ$ ***A1***

continued…

Question 11 continued

METHOD 2

EITHER

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right) \Rightarrow \frac{d\alpha}{dx} = \frac{70-2x^2}{\left(x^2+35\right)^2+4x^2}$ ***M1***

$\frac{d\alpha}{dx} = 0 \Rightarrow x = \sqrt{35}(\text{m})\left(= 5.9161\ldots(\text{m})\right)$ ***A1***

OR

attempting to locate the stationary point on the graph of

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right)$ ***(M1)***

$x = 5.9161\ldots(\text{m}) \left(= \sqrt{35}\,(\text{m})\right)$ ***A1***

THEN

$\alpha = 0.1674\ldots\left(= \arctan\frac{1}{\sqrt{35}}\right)$ ***(A1)***

$= 9.59^{\circ}$ ***A1***

(iii) $\frac{d^2}{dx^2}(\tan\alpha) = \frac{\left(x^2+35\right)^2(-4x)-(2)(2x)\left(x^2+35\right)\left(70-2x^2\right)}{\left(x^2+35\right)^4} \left(=\frac{4x\left(x^2-105\right)}{\left(x^2+35\right)^3}\right)$

M1A1

substituting $x = \sqrt{35}$ $(= 5.9161\ldots)$ into $\frac{d^2}{dx^2}(\tan\alpha)$ ***M1***

$\frac{d^2}{dx^2}(\tan\alpha) < 0$ $(= -0.004829\ldots)$ and so $\alpha = 9.59^{\circ}$ is the maximum value of α ***R1***

α never exceeds 10° ***AG***

[11 marks]

continued…

Question 11 continued

(d) attempting to solve $\frac{2x}{x^2+35} \geq \tan 7^\circ$ ***(M1)***

> **Note:** Award ***(M1)*** for attempting to solve $\frac{2x}{x^2+35} = \tan 7^\circ$.

$x = 2.55$ and $x = 13.7$ ***(A1)***

$2.55 \leq x \leq 13.7$ (m) ***A1***

[3 marks]

Total [22 marks]

12. (a) (i) $\frac{1}{4\left(\frac{e^x+e^{-x}}{2}\right) - 2\left(\frac{e^x-e^{-x}}{2}\right)}$ ***(M1)***

$= \frac{1}{2\left(e^x+e^{-x}\right) - \left(e^x-e^{-x}\right)}$ ***(A1)***

$= \frac{1}{e^x+3e^{-x}}$ ***A1***

$= \frac{e^x}{e^{2x}+3}$ ***AG***

(ii) $u = e^x \Rightarrow du = e^x dx$ ***A1***

$\int \frac{e^x}{e^{2x}+3} dx = \int \frac{1}{u^2+3} du$ ***M1***

(when $x = 0$, $u = 1$ and when $x = \ln 3$, $u = 3$)

$\int_1^3 \frac{1}{u^2+3} du = \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3$ ***M1A1***

$\left(= \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{e^x}{\sqrt{3}}\right)\right]_0^{\ln 3}\right)$

$= \frac{\pi\sqrt{3}}{9} - \frac{\pi\sqrt{3}}{18}$ ***(M1)***

$= \frac{\pi\sqrt{3}}{18}$ ***A1***

[9 marks]

continued…

Question 12 continued

(b) (i) $(n+1)e^{2x} - 2ke^{x} + (n-1) = 0$ ***M1A1***

$e^{x} = \dfrac{2k \pm \sqrt{4k^2 - 4\left(n^2 - 1\right)}}{2(n+1)}$ ***M1***

$x = \ln\left(\dfrac{k \pm \sqrt{k^2 - n^2 + 1}}{n+1}\right)$ ***M1A1***

(ii) for two real solutions, we require $k > \sqrt{k^2 - n^2 + 1}$ ***R1***

and we also require $k^2 - n^2 + 1 > 0$ ***R1***

$k^2 > n^2 - 1$ ***A1***

$\Rightarrow k > \sqrt{n^2 - 1}\ \ (k \in \mathbb{R}^+)$ ***AG***

[8 marks]

(c) (i) **METHOD 1**

$t(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

$t'(x) = \dfrac{\left(e^{x} + e^{-x}\right)^2 - \left(e^{x} - e^{-x}\right)^2}{\left(e^{x} + e^{-x}\right)^2}$ ***M1A1***

$t'(x) = \dfrac{\left(\dfrac{e^{x} + e^{-x}}{2}\right)^2 - \left(\dfrac{e^{x} - e^{-x}}{2}\right)^2}{\left(\dfrac{e^{x} + e^{-x}}{2}\right)^2}$ ***A1***

$= \dfrac{\left[f(x)\right]^2 - \left[g(x)\right]^2}{\left[f(x)\right]^2}$ ***AG***

METHOD 2

$t'(x) = \dfrac{f(x)g'(x) - g(x)f'(x)}{\left[f(x)\right]^2}$ ***M1A1***

$g'(x) = f(x)$ and $f'(x) = g(x)$ ***A1***

$= \dfrac{\left[f(x)\right]^2 - \left[g(x)\right]^2}{\left[f(x)\right]^2}$ ***AG***

continued…

Question 12 continued

METHOD 3

$t(x) = \left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)^{-1}$

$t'(x) = 1 - \dfrac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$ ***M1A1***

$= 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$ ***A1***

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

METHOD 4

$t'(x) = \dfrac{g'(x)}{f(x)} - \dfrac{g(x)f'(x)}{[f(x)]^2}$ ***M1A1***

$g'(x) = f(x)$ and $f'(x) = g(x)$ gives $t'(x) = 1 - \dfrac{[g(x)]^2}{[f(x)]^2}$ ***A1***

$= \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

(ii) **METHOD 1**

$[f(x)]^2 > [g(x)]^2$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \dfrac{e^x + e^{-x}}{2}$ and $g(x) = \dfrac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

continued…

Question 12 continued

METHOD 2

$[f(x)]^2 - [g(x)]^2 = 1$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

> **Note:** Award as above for use of either $f(x) = \frac{e^x + e^{-x}}{2}$ and $g(x) = \frac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 3

$t'(x) = \frac{4}{(e^x + e^{-x})^2}$

$(e^x + e^{-x})^2 > 0$ ***M1A1***

$\frac{4}{(e^x + e^{-x})^2} > 0$ ***R1***

hence $t'(x) > 0$, $x \in \mathbb{R}$ ***AG***

[6 marks]

Total [23 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2015

Mathematics

Higher level

Paper 2

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2015"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N*** marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark,*
but award all others so that the candidate only loses one mark.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $0.818 = 0.65 + 0.48 - \mathrm{P}(A \cap B)$ ***(M1)***

$\mathrm{P}(A \cap B) = 0.312$ ***A1***

[2 marks]

(b) $\mathrm{P}(A)\,\mathrm{P}(B) = 0.312\left(= 0.48 \times 0.65\right)$ ***A1***

since $\mathrm{P}(A)\,\mathrm{P}(B) = \mathrm{P}(A \cap B)$ then A and B are independent ***R1***

Note: Only award the ***R1*** if numerical values are seen. Award ***A1R1*** for a correct conditional probability approach.

[2 marks]

Total [4 marks]

2. using technology and/or by elimination (*eg* ref on GDC) ***(M1)***

$x = 1.89\left(= \frac{17}{9}\right),\ y = 1.67\left(= \frac{5}{3}\right),\ z = -2.22\left(= \frac{-20}{9}\right)$ ***A1A1A1***

[4 marks]

3. (a) $\frac{0 \cdot 4 + 1 \cdot k + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 3 + 8 \cdot 1}{13 + k} = 1.95 \quad \left(\frac{k + 32}{k + 13} = 1.95\right)$ ***(M1)***

attempting to solve for k ***(M1)***

$k = 7$ ***A1***

[3 marks]

(b) (i) $\frac{7 + 32 + 22}{7 + 13 + 1} = 2.90\left(= \frac{61}{21}\right)$ ***(M1)A1***

(ii) standard deviation $= 4.66$ ***A1***

Note: Award ***A0*** for 4.77.

[3 marks]

Total [6 marks]

4. (a) (i) $A = -3$ ***A1***

(ii) period $= \frac{2\pi}{B}$ ***(M1)***

$B = 2$ ***A1***

Note: Award as above for $A = 3$ and $B = -2$.

(iii) $C = 2$ ***A1***

[4 marks]

(b) $x = 1.74,\ 2.97$ $\left(x = \frac{1}{2}\left(\pi + \arcsin\frac{1}{3}\right), \frac{1}{2}\left(2\pi - \arcsin\frac{1}{3}\right)\right)$ ***(M1)A1***

[2 marks]

Note: Award ***(M1)A0*** if extra correct solutions *eg* $(-1.40,\ -0.170)$ are given outside the domain $0 \le x \le \pi$.

Total [6 marks]

5. (a) (i) area $= \int_2^4 \sqrt{y-2}\,\mathrm{d}y$ ***M1A1***

(ii) $= 1.886$ (4 sf only) ***A1***

[3 marks]

(b) volume $= \pi\int_2^4 (y-2)\,\mathrm{d}y$ ***(M1)***

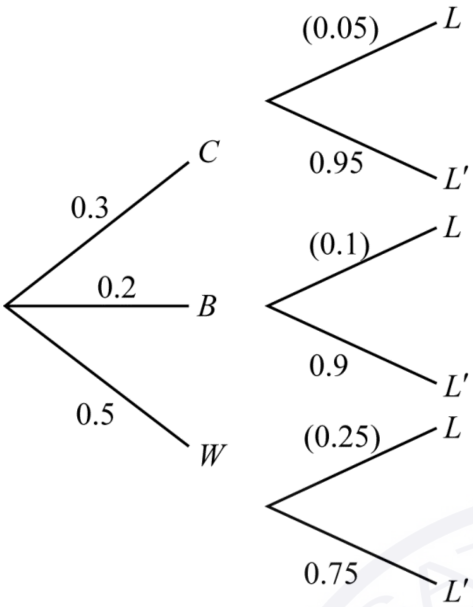
$= \pi\left[\frac{y^2}{2} - 2y\right]_2^4$ ***(A1)***

$= 2\pi$ (exact only) ***A1***

[3 marks]

Total [6 marks]

6. **EITHER**



M1A1A1

> **Note:** Award ***M1*** for a two-level tree diagram, ***A1*** for correct first level probabilities, and ***A1*** for correct second level probabilities.

OR

$$\mathrm{P}(B \mid L') = \frac{\mathrm{P}(L' \mid B)\,\mathrm{P}(B)}{\mathrm{P}(L' \mid B)\,\mathrm{P}(B) + \mathrm{P}(L' \mid C)\,\mathrm{P}(C) + \mathrm{P}(L' \mid W)\,\mathrm{P}(W)}\left(= \frac{\mathrm{P}(B \cap L')}{\mathrm{P}(L')}\right)$$ ***(M1)(A1)(A1)***

THEN

$$\mathrm{P}(B \mid L') = \frac{0.9 \times 0.2}{0.9 \times 0.2 + 0.95 \times 0.3 + 0.75 \times 0.5}\left(= \frac{0.18}{0.84}\right)$$ ***M1A1***

$$= 0.214\left(= \frac{3}{14}\right)$$ ***A1***

[6 marks]

7. $21 = \frac{1}{2} \cdot 6 \cdot 11 \cdot \sin A$ ***(M1)***

$\sin A = \frac{7}{11}$ ***(A1)***

EITHER

$\hat{A} = 0.6897..., 2.452... \left(\hat{A} = \arcsin\frac{7}{11}, \pi - \arcsin\frac{7}{11} = 39.521...^\circ, 140.478...^\circ \right)$ ***(A1)***

OR

$\cos A = \pm\frac{6\sqrt{2}}{11} (= \pm 0.771...)$ ***(A1)***

THEN

$\text{BC}^2 = 6^2 + 11^2 - 2 \cdot 6 \cdot 11 \cos A$ ***(M1)***

$\text{BC} = 16.1 \text{ or } 7.43$ ***A1A1***

Note: Award ***M1A1A0M1A1A0*** if only one correct solution is given.

[6 marks]

8. (a) $A\int_1^5 \sin(\ln x)\,\text{d}x = 1$ ***(M1)***

$A = 0.323$ (3 dp only) ***A1***

[2 marks]

(b) either a graphical approach or $f'(x) = \frac{\cos(\ln x)}{x} = 0$ ***(M1)***

$x = 4.81 \left(= \text{e}^{\frac{\pi}{2}} \right)$ ***A1***

Note: Do not award ***A1FT*** for a candidate working in degrees.

[2 marks]

(c) $\text{P}(X \le 3 | X \ge 2) = \frac{\text{P}(2 \le X \le 3)}{\text{P}(X \ge 2)} \left(= \frac{\int_2^3 \sin(\ln(x))\text{d}x}{\int_2^5 \sin(\ln(x))\text{d}x} \right)$ ***(M1)***

$= 0.288$ ***A1***

Note: Do not award ***A1FT*** for a candidate working in degrees.

[2 marks]

Total [6 marks]

9. (a) $t_1 = 1.77(\text{s}) \left(= \sqrt{\pi}(\text{s})\right)$ and $t_2 = 2.51(\text{s}) \left(= \sqrt{2\pi}(\text{s})\right)$ ***A1A1***

[2 marks]

(b) (i) attempting to find (graphically or analytically) the first t_{max} ***(M1)***

$t = 1.25(\text{s}) \left(= \sqrt{\frac{\pi}{2}}(\text{s})\right)$ ***A1***

(ii) attempting to find (graphically or analytically) the first t_{min} ***(M1)***

$t = 2.17(\text{s}) \left(= \sqrt{\frac{3\pi}{2}}(\text{s})\right)$ ***A1***

[4 marks]

(c) distance travelled $= \left|\int_{1.772\ldots}^{2.506\ldots} 1 - \text{e}^{-\sin t^2}\,\text{d}t\right|$ (or equivalent) ***(M1)***

$= 0.711(\text{m})$ ***A1***

[2 marks]

Total [8 marks]

10. (a) $\boldsymbol{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ ***A1***

$\boldsymbol{b} = \frac{1}{3}\left(\begin{pmatrix} 4 \\ 16 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} \frac{5}{3} \\ 4 \end{pmatrix}$ ***(M1)A1***

[3 marks]

(b) **METHOD 1**

Roderick must signal in a direction vector perpendicular to Ed's path. ***(M1)***

the equation of the signal is $\boldsymbol{s} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ (or equivalent) ***A1***

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \frac{t}{3}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ ***M1***

$\frac{5}{3}t + 12\lambda = 12$ and $4t - 5\lambda = 5$ ***M1***

$t = 2.13 \left(= \frac{360}{169}\right)$ ***A1***

[5 marks]

METHOD 2

$\begin{pmatrix} 5 \\ 12 \end{pmatrix} \cdot \left(\begin{pmatrix} 11 \\ 9 \end{pmatrix} - \begin{pmatrix} -1 + \frac{5}{3}t \\ 4 + 4t \end{pmatrix}\right) = 0$ (or equivalent) ***M1A1A1***

Note: Award the ***M1*** for an attempt at a scalar product equated to zero, ***A1*** for the first factor and ***A1*** for the complete second factor.

attempting to solve for *t* ***(M1)***

$t = 2.13 \left(= \frac{360}{169}\right)$ ***A1***

[5 marks]

continued…

Question 10 continued

METHOD 3

$x=\sqrt{\left(12-\frac{5t}{3}\right)^2+(5-4t)^2}$ (or equivalent) $\left(x^2=\left(12-\frac{5t}{3}\right)^2+(5-4t)^2\right)$ ***M1A1A1***

Note: Award ***M1*** for use of Pythagoras' theorem, ***A1*** for $\left(12-\frac{5t}{3}\right)^2$ and ***A1*** for $(5-4t)^2$.

attempting (graphically or analytically) to find *t* such that $\frac{\mathrm{d}x}{\mathrm{d}t}=0\left(\frac{\mathrm{d}\left(x^2\right)}{\mathrm{d}t}=0\right)$ ***(M1)***

$t=2.13\left(=\frac{360}{169}\right)$ ***A1***

[5 marks]

METHOD 4

$\cos\theta=\frac{\begin{pmatrix}12\\5\end{pmatrix}\cdot\begin{pmatrix}5\\12\end{pmatrix}}{\left|\begin{pmatrix}12\\5\end{pmatrix}\right|\left|\begin{pmatrix}5\\12\end{pmatrix}\right|}=\frac{120}{169}$ ***M1A1***

Note: Award ***M1*** for attempting to calculate the scalar product.

$\frac{120}{13}=\frac{t}{3}\left|\begin{pmatrix}5\\12\end{pmatrix}\right|$ (or equivalent) ***(A1)***

attempting to solve for *t* ***(M1)***

$t=2.13\left(=\frac{360}{169}\right)$ ***A1***

[5 marks]

Total [8 marks]

Section B

11. (a) (i) let W be the weight of a worker and $W \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$\mathrm{P}\left(Z < \frac{62-\mu}{\sigma}\right) = 0.3$ and $\mathrm{P}\left(Z < \frac{98-\mu}{\sigma}\right) = 0.75$ ***(M1)***

$\frac{62-\mu}{\sigma} = \Phi^{-1}(0.3)\ (= -0.524\ldots)$ and

$\frac{98-\mu}{\sigma} = \Phi^{-1}(0.75)\ (= 0.674\ldots)$

or linear equivalents ***A1A1***

(ii) attempting to solve simultaneously ***(M1)***

$\mu = 77.7,\ \sigma = 30.0$ ***A1A1***

[6 marks]

(b) $\mathrm{P}(W > 100) = 0.229$ ***A1***

[1 mark]

(c) let X represent the number of workers over $100\,\mathrm{kg}$ in a lift of ten passengers

$X \sim \mathrm{B}(10,\ 0.229\ldots)$ ***(M1)***

$\mathrm{P}(X \geq 4) = 0.178$ ***A1***

[2 marks]

continued…

Question 11 continued

(d) $\text{P}(X < 4 | X \geq 1) = \dfrac{\text{P}(1 \leq X \leq 3)}{\text{P}(X \geq 1)}$ ***M1(A1)***

Note: Award the ***M1*** for a clear indication of conditional probability.

$= 0.808$ ***A1***

[3 marks]

(e) $L \sim \text{Po}(50)$ ***(M1)***

$\text{P}(L > 60) = 1 - \text{P}(L \leq 60)$ ***(M1)***

$= 0.0722$ ***A1***

[3 marks]

(f) 400 workers require at least 40 elevators ***(A1)***

$\text{P}(L \geq 40) = 1 - \text{P}(L \leq 39)$ ***(M1)***

$= 0.935$ ***A1***

[3 marks]

Total [18 marks]

Note: For Q12(a) (i) – (iii) and (b) (ii), award ***A1*** for correct endpoints and , if correct, award ***A1*** for a closed interval.
Further, award ***A1A0*** for one correct endpoint and a closed interval.

12. (a) (i) $-4 \le y \le -2$ ***A1A1***

(ii) $-5 \le y \le -1$ ***A1A1***

(iii) $-3 \le 2x - 6 \le 5$ ***(M1)***

Note: Award ***M1*** for $f\left(2x-6\right)$.

$3 \le 2x \le 11$

$\frac{3}{2} \le x \le \frac{11}{2}$ ***A1A1***

[7 marks]

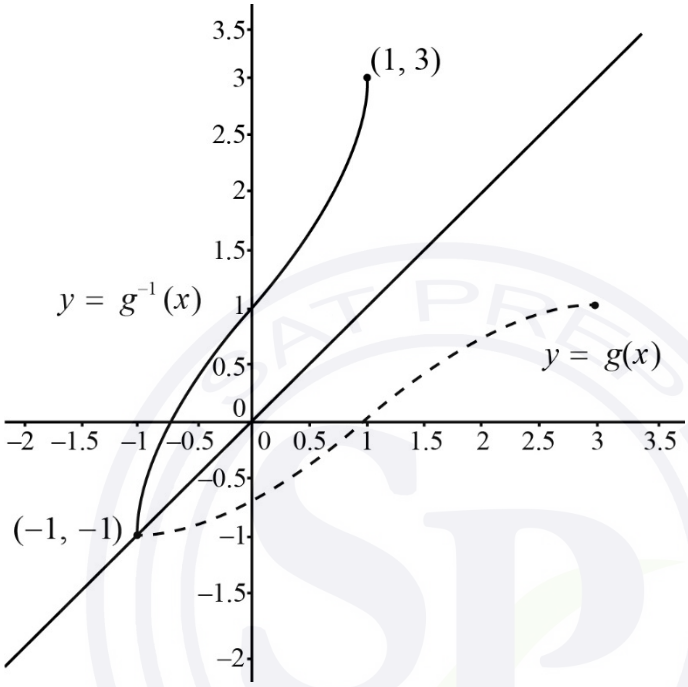
continued…

Question 12 continued

(b) (i) any valid argument *eg* f is not one to one, f is many to one, fails horizontal line test, not injective **R1**

(ii) largest domain for the function $g(x)$ to have an inverse is $[-1, 3]$ **A1A1**

(iii)



y-intercept indicated (coordinates not required) **A1**
correct shape **A1**
coordinates of end points $(1, 3)$ and $(-1, -1)$ **A1**

Note: Do not award any of the above marks for a graph that is not one to one.

[6 marks]

continued…

Question 12 continued

(c) (i) $y = \dfrac{2x - 5}{x + d}$

$(x + d)\,y = 2x - 5$ ***M1***

Note: Award ***M1*** for attempting to rearrange x and y in a linear expression.

$x(y - 2) = -dy - 5$ ***(A1)***

$x = \dfrac{-dy - 5}{y - 2}$ ***(A1)***

Note: x and y can be interchanged at any stage

$h^{-1}(x) = \dfrac{-dx - 5}{x - 2}$ ***A1***

Note: Award ***A1*** only if $h^{-1}(x)$ is seen.

(ii) self Inverse $\Rightarrow h(x) = h^{-1}(x)$

$\dfrac{2x - 5}{x + d} \equiv \dfrac{-dx - 5}{x - 2}$ ***(M1)***

$d = -2$ ***A1***

(iii) **METHOD 1**

$\dfrac{2k(x) - 5}{k(x) - 2} = \dfrac{2x}{x + 1}$ ***(M1)***

$k(x) = \dfrac{x + 5}{2}$ ***A1***

METHOD 2

$h^{-1}\left(\dfrac{2x}{x + 1}\right) = \dfrac{2\left(\dfrac{2x}{x+1}\right) - 5}{\dfrac{2x}{x+1} - 2}$ ***(M1)***

$k(x) = \dfrac{x + 5}{2}$ ***A1***

[8 marks]

Total [21 marks]

13. (a) $f'(x) = 30\mathrm{e}^{-\frac{x^2}{400}} \cdot -\frac{2x}{400}\left(= -\frac{3x}{20}\mathrm{e}^{-\frac{x^2}{400}}\right)$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f''(x) = -\frac{3}{20}\mathrm{e}^{-\frac{x^2}{400}} + \frac{3x^2}{4000}\mathrm{e}^{-\frac{x^2}{400}}\left(= \frac{3}{20}\mathrm{e}^{-\frac{x^2}{400}}\left(\frac{x^2}{200} - 1\right)\right)$ ***M1A1***

Note: Award ***M1*** for attempting to use the product rule.

[4 marks]

(b) the roof function has maximum gradient when $f''(x) = 0$ ***(M1)***

Note: Award ***(M1)*** for attempting to find $f''\left(-\sqrt{200}\right)$.

EITHER
$= 0$ ***A1***

OR
$f''(x) = 0 \Rightarrow x = \pm\sqrt{200}$ ***A1***

THEN
valid argument for maximum such as reference to an appropriate graph or change in the sign of $f''(x)$ *eg* $f''(-15) = 0.010...(> 0)$ and $f''(-14) = -0.001...(< 0)$ ***R1***

$\Rightarrow x = -\sqrt{200}$ ***AG***

[3 marks]

continued…

Question 13 continued

(c) $A = 2a \cdot 30\mathrm{e}^{-\frac{a^2}{400}} \left(= 60a\mathrm{e}^{-\frac{a^2}{400}} = -400f'(a) \right)$ ***(M1)(A1)***

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}a} = 60a\mathrm{e}^{-\frac{a^2}{400}} \cdot -\frac{a}{200} + 60\mathrm{e}^{-\frac{a^2}{400}} = 0 \Rightarrow a = \sqrt{200} \quad \left(-400f''(a) = 0 \Rightarrow a = \sqrt{200}\right)$

M1A1

OR

by symmetry *eg* $a = -\sqrt{200}$ found in (b) or $A_{\max}$ coincides with $f''(a) = 0$ ***R1***

$\Rightarrow a = \sqrt{200}$ ***A1***

THEN

$A_{\max} = 60 \cdot \sqrt{200}\mathrm{e}^{-\frac{200}{400}}$ ***M1***

$= 600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$ ***AG***

[5 marks]

(d) (i) perimeter $= 4a + 60\mathrm{e}^{-\frac{a^2}{400}}$ ***A1A1***

Note: Condone use of x.

(ii) $I(a) = \frac{4a + 60\mathrm{e}^{-\frac{a^2}{400}}}{60a\mathrm{e}^{-\frac{a^2}{400}}}$ ***(A1)***

graphing $I(a)$ or other valid method to find the minimum ***(M1)***

$a = 12.6$ ***A1***

(iii) area under roof $= \int_{-20}^{20} 30\mathrm{e}^{-\frac{x^2}{400}}\,\mathrm{d}x$ ***M1***

$= 896.18\ldots$ ***(A1)***

area of living space $= 60 \cdot (12.6\ldots) \cdot \mathrm{e}^{-\frac{(12.6\ldots)^2}{400}} = 508.56\ldots$ ***(A1)***

percentage of empty space $= 43.3\%$ ***A1***

[9 marks]

Total [21 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2015

Mathematics

Higher level

Paper 2

21 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2015"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
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- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

> *Students must always use correct mathematical notation, not calculator notation.*

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. $\int_{-1}^{1} \pi\left(e^{-x^2}\right)^2 dx \quad \left(\int_{-1}^{1} \pi e^{-2x^2} dx \text{ or } \int_{0}^{1} 2\pi e^{-2x^2} dx\right)$ ***(M1)(A1)(A1)***

> **Note:** Award ***M1*** for integral involving the function given; ***A1*** for correct limits; ***A1*** for π and $\left(e^{-x^2}\right)^2$

$= 3.758249... = 3.76$ ***A1***

[4 marks]

2. (a) $X \sim N\left(210, 22^2\right)$

$P\left(X < 180\right) = 0.0863$ ***(M1)A1***

[2 marks]

(b) $P\left(X < T\right) = 0.9 \Rightarrow T = 238$ (mins) ***(M1)A1***

[2 marks]

Total [4 marks]

3. (a) $W \sim B(1000, 0.1)$ (accept $C_k^{1000}\left(0.1\right)^k\left(0.9\right)^{1000-k}$) ***A1A1***

> **Note:** First ***A1*** is for recognizing the binomial, second ***A1*** for both parameters if stated explicitly in this part of the question.

[2 marks]

(b) $\mu (= 1000 \times 0.1) = 100$ ***A1***

[1 mark]

(c) $P(W > 89) = P(W \geq 90) \quad \left(= 1 - P(W \leq 89)\right)$ ***(M1)***

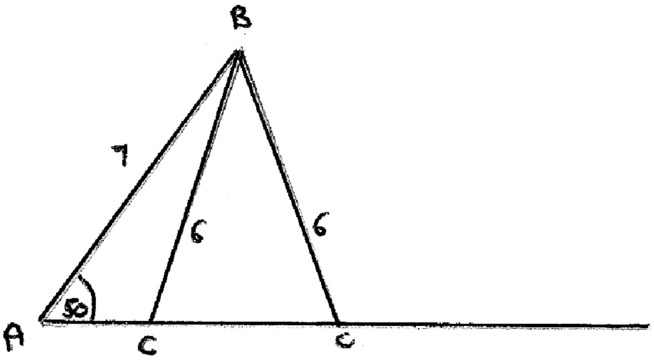
$= 0.867$ ***A1***

> **Notes:** Award ***M0A0*** for 0.889

[2 marks]

Total [5 marks]

4.



METHOD 1

$\frac{6}{\sin 50} = \frac{7}{\sin C} \Rightarrow \sin C = \frac{7\sin 50}{6}$ ***(M1)***

$C = 63.344...$ ***(A1)***

or $C = 116.655...$ ***(A1)***

$B = 13.344...$ (or $B = 66.656...$) ***(A1)***

area $= \frac{1}{2} \times 6 \times 7 \times \sin 13.344...$ (or $\frac{1}{2} \times 6 \times 7 \times \sin 66.656...$) ***(M1)***

$4.846...$ (or $= 19.281...$)

so answer is 4.85 (cm^2) ***A1***

METHOD 2

$6^2 = 7^2 + b^2 - 2 \times 7b\cos 50$ ***(M1)(A1)***

$b^2 - 14b\cos 50 + 13 = 0$ or equivalent method to solve the above equation ***(M1)***

$b = 7.1912821...$ or $b = 1.807744...$ ***(A1)***

area $= \frac{1}{2} \times 7 \times 1.8077...\sin 50 = 4.846...$ ***(M1)***

(or $\frac{1}{2} \times 7 \times 7.1912821...\sin 50 = 19.281...$)

so answer is 4.85 (cm^2) ***A1***

METHOD 3

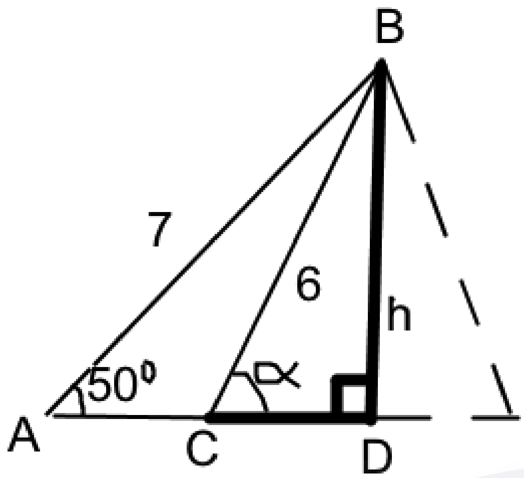


Diagram showing triangles ACB and ADB *(M1)*

$h = 7\sin(50) = 5.3623\ldots$ (cm) *(M1)*

$\alpha = \arcsin\dfrac{h}{6} = 63.3442\ldots$ *(M1)*

$\text{AC} = \text{AD} - \text{CD} = 7\cos 50 - 6\cos\alpha = 1.8077\ldots$ (cm) *(M1)*

area $= \dfrac{1}{2} \times 1.8077\ldots \times 5.3623\ldots$ *(M1)*

$= 4.85$ (cm^2) *A1*

Total [6 marks]

5. $V = 200\pi r^2$ ***(A1)***

> **Note:** Allow $V = \pi h r^2$ if value of h is substituted later in the question.

EITHER

$\frac{\mathrm{d}V}{\mathrm{d}t} = 200\pi 2r \frac{\mathrm{d}r}{\mathrm{d}t}$ ***M1A1***

> **Note:** Award ***M1*** for an attempt at implicit differentiation.

at $r = 2$ we have $30 = 200\pi 4 \frac{\mathrm{d}r}{\mathrm{d}t}$ ***M1***

OR

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}r}}$ ***M1***

$\frac{\mathrm{d}V}{\mathrm{d}r} = 400\pi r$ ***M1***

$r = 2$ we have $\frac{\mathrm{d}V}{\mathrm{d}r} = 800\pi$ ***A1***

THEN

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{30}{800\pi}\left(= \frac{3}{80\pi} = 0.0119\right)$ $(\mathrm{cm\,s^{-1}})$ ***A1***

Total [5 marks]

6. $f'(x) = 3x^2 + \mathrm{e}^x$ ***A1***

> **Note:** Accept labelled diagram showing the graph $y = f'(x)$ above the x-axis; do not accept unlabelled graphs nor graph of $y = f(x)$.

EITHER

this is always > 0 ***R1***
so the function is (strictly) increasing ***R1***
and thus 1–1 ***A1***

OR

this is always > 0 (accept $\neq 0$) ***R1***
so there are no turning points ***R1***
and thus 1–1 ***A1***

> **Note:** ***A1*** is dependent on the first ***R1***.

Total [4 marks]

7. (a) $2\frac{e^{-m}m^4}{4!} = \frac{e^{-m}m^5}{5!}$ ***M1A1***

$\frac{2}{4!} = \frac{m}{5!}$ or other simplification ***M1***

> **Note:** accept a labelled graph showing clearly the solution to the equation. Do not accept simple verification that $m = 10$ is a solution.

$\Rightarrow m = 10$ ***AG***

[3 marks]

(b) $P(X = 6 \mid X \le 11) = \frac{P(X = 6)}{P(X \le 11)}$ ***(M1) (A1)***

$= \frac{0.063055...}{0.696776...}$ ***(A1)***

$= 0.0905$ ***A1***

[4 marks]

Total [7 marks]

8. (a) require $\begin{pmatrix} 4 \\ \lambda \\ 10 \end{pmatrix} = s\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ ***(M1)***

$\Rightarrow 4 = 2s \Rightarrow s = 2 \Rightarrow \lambda = 6$ ***A1***

> **Note:** Accept cross product solution.

[2 marks]

(b) require $\boldsymbol{v} \bullet \boldsymbol{w} = 2 \times 4 + 3 \times \lambda + 5 \times 10 = 0 \Rightarrow 3\lambda = -58 \Rightarrow \lambda = \frac{-58}{3} (-19.3)$ ***M1A1***

[2 marks]

(c) $\boldsymbol{v} \bullet \boldsymbol{w} = 2 \times 4 + 3 \times \lambda + 5 \times 10 = \sqrt{2^2 + 3^2 + 5^2} \times \sqrt{4^2 + \lambda^2 + 10^2} \times \cos 10^\circ$ ***(M1)(A1)***

$58 + 3\lambda = \sqrt{38} \times \sqrt{116 + \lambda^2} \times \cos 10^\circ$

$\lambda = 3.73$ or 8.76 ***A1A1***

[4 marks]

Total [8 marks]

9. $x = 0 \Rightarrow y = 1$ ***(A1)***

$y'(0) = 1.367879\ldots$ ***(M1)(A1)***

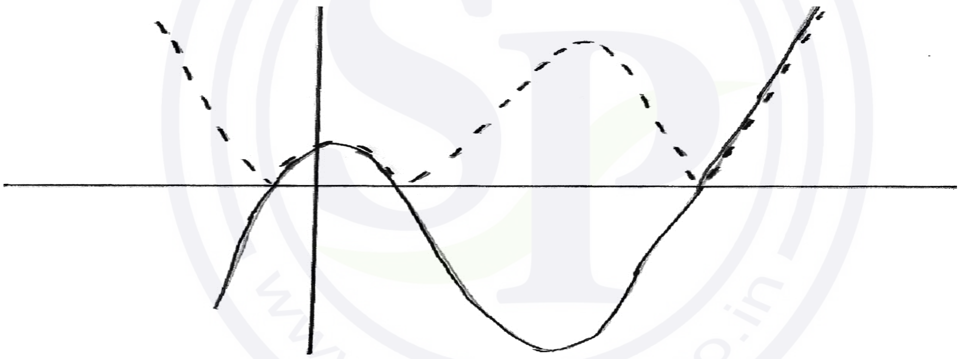
Note: The exact answer is $y'(0) = \frac{e+1}{e} = 1 + \frac{1}{e}$.

so gradient of normal is $\frac{-1}{1.367879\ldots} (= -0.731058\ldots)$ ***(M1)(A1)***
equation of normal is $y = -0.731058\ldots x + c$ ***(M1)***
gives $y = -0.731x + 1$ ***A1***

Note: The exact answer is $y = -\frac{e}{e+1}x + 1$.
Accept $y - 1 = -0.731058\ldots(x - 0)$

Total [7 marks]

10. (a)



as roots of $f(x) = 0$ are $-1, 1, 5$ ***(M1)***

solution is $]-\infty, -1[\cup]1, 5[$ ($x < -1$ or $1 < x < 5$) ***A1A1***

Note: Award ***A1A0*** for closed intervals.

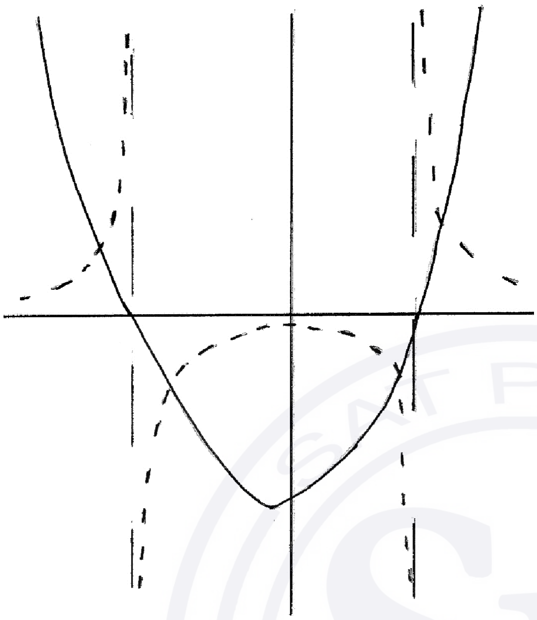
[3 marks]

continued…

Question 10 continued

(b) METHOD 1

(graphs of $g(x)$ and $\frac{1}{g(x)}$)



roots of $g(x)=0$ are -3 and 2 ***(M1)(A1)***

Notes: Award ***M1*** if quadratic graph is drawn or two roots obtained. Roots may be indicated anywhere *eg* asymptotes on graph or in inequalities below.

the intersections of the graphs $g(x)$ and of $1/g(x)$
are $-3.19, -2.79, 1.79, 2.19$ ***(M1)(A1)***

Note: Award ***A1*** for at least one of the values above seen anywhere.

solution is $]-3.19,-3[\cup]-2.79,1.79[\cup]2,2.19[$
($-3.19 < x < -3$ or $-2.79 < x < 1.79$ or $2 < x < 2.19$) ***A1A1A1***

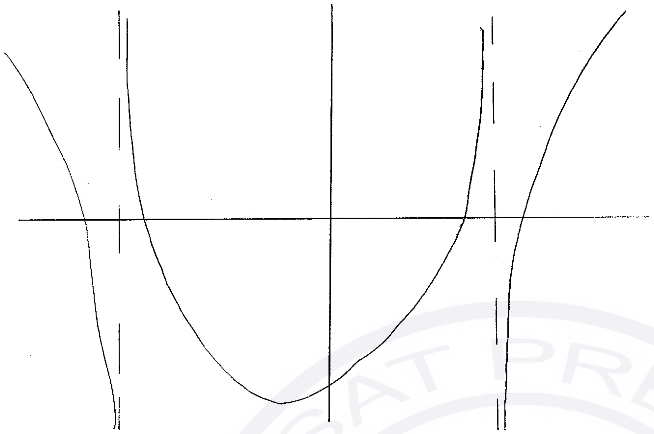
Note: Award ***A1A1A0*** for closed intervals.

continued…

Question 10 continued

METHOD 2

(graph of $g(x) - \frac{1}{g(x)}$)



asymptotes at $x = -3$ and $x = 2$ ***(M1)(A1)***

Note: May be indicated on the graph.

roots of graph are $-3.19, -2.79, 1.79, 2.19$ ***(M1)(A1)***

Note: Award ***A1*** for at least one of the values above seen anywhere.

solution is (when graph is negative)
$]-3.19,-3[\cup]-2.79,1.79[\cup]2,2.19[$
($-3.19 < x < -3$ or $-2.79 < x < 1.79$ or $2 < x < 2.19$) ***A1A1A1***

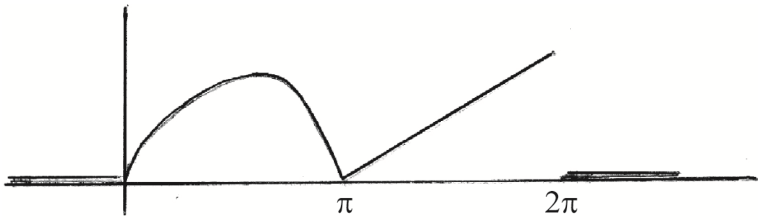
Note: Award ***A1A1A0*** for closed intervals.

[7 marks]

Total [10 marks]

Section B

11. (a)



Award ***A1*** for sine curve from 0 to π, award ***A1*** for straight line from π to 2π ***A1A1***

[2 marks]

(b) $\int_0^{\pi} \frac{\sin x}{4} \mathrm{d}x = \frac{1}{2}$ ***(M1)A1***

[2 marks]

(c) **METHOD 1**

require $\frac{1}{2} + \int_{\pi}^{2\pi} a(x-\pi)\,\mathrm{d}x = 1$ ***(M1)***

$\Rightarrow \frac{1}{2} + a\left[\frac{(x-\pi)^2}{2}\right]_{\pi}^{2\pi} = 1$ (or $\frac{1}{2} + a\left[\frac{x^2}{2} - \pi x\right]_{\pi}^{2\pi} = 1$) ***A1***

$\Rightarrow a\frac{\pi^2}{2} = \frac{1}{2}$ ***A1***

$\Rightarrow a = \frac{1}{\pi^2}$ ***AG***

Note:	Must obtain the exact value. Do not accept answers obtained with calculator.

METHOD 2

$0.5 +$ area of triangle $= 1$ ***R1***

area of triangle $= \frac{1}{2}\pi \times a\pi = 0.5$ ***M1A1***

Note:	Award ***M1*** for correct use of area formula $= 0.5$, ***A1*** for $a\pi$.

$a = \frac{1}{\pi^2}$ ***AG***

[3 marks]

(d) median is π ***A1***

[1 mark]

continued…

Question 11 continued

(e) $\mu = \int_0^{\pi} x \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} x \cdot \frac{x-\pi}{\pi^2} \mathrm{d}x$ ***(M1)(A1)***

$= 3.40339... = 3.40$ (or $\frac{\pi}{4} + \frac{5\pi}{6} = \frac{13}{12}\pi$) ***A1***

[3 marks]

(f) For $\mu = 3.40339...$

EITHER

$\sigma^2 = \int_0^{\pi} x^2 \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} x^2 \cdot \frac{x-\pi}{\pi^2} \mathrm{d}x - \mu^2$ ***(M1)(A1)***

OR

$\sigma^2 = \int_0^{\pi} (x-\mu)^2 \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} (x-\mu)^2 \cdot \frac{x-\pi}{\pi^2} \mathrm{d}x$ ***(M1)(A1)***

THEN

$= 3.866277... = 3.87$ ***A1***

[3 marks]

(g) $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{\frac{3\pi}{2}} \frac{x-\pi}{\pi^2} \mathrm{d}x = 0.375$ (or $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$) ***(M1)A1***

[2 marks]

(h) $\mathrm{P}\left(\pi \le X \le 2\pi \middle| \frac{\pi}{2} \le X \le \frac{3\pi}{2} \right) = \frac{\mathrm{P}\left(\pi \le X \le \frac{3\pi}{2} \right)}{\mathrm{P}\left(\frac{\pi}{2} \le X \le \frac{3\pi}{2} \right)}$ ***(M1)(A1)***

$= \frac{\int_{\pi}^{\frac{3\pi}{2}} \frac{(x-\pi)}{\pi^2} \mathrm{d}x}{0.375} = \frac{0.125}{0.375}$ (or $= \frac{\frac{1}{8}}{\frac{3}{8}}$ from diagram areas) ***(M1)***

$= \frac{1}{3}$ (0.333) ***A1***

[4 marks]

Total [20 marks]

12. (a) (i) $(\cos\theta + \mathrm{i}\sin\theta)^5$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta +$
$10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$ ***A1A1***

$(= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta -$
$10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta)$

Note: Award first ***A1*** for correct binomial coefficients.

(ii) $(\mathrm{cis}\theta)^5 = \mathrm{cis}5\theta = \cos 5\theta + \mathrm{i}\sin 5\theta$ ***M1***

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta +$
$5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$ ***A1***

Note: Previous line may be seen in (i)

equating imaginary terms ***M1***

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ ***AG***

(iii) equating real terms

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ***A1***

[6 marks]

(b) $(r\,\mathrm{cis}\alpha)^5 = 1 \Rightarrow r^5\,\mathrm{cis}5\alpha = 1\,\mathrm{cis}0$ ***M1***

$r^5 = 1 \Rightarrow r = 1$ ***A1***

$5\alpha = 0 \pm 360k,\ k \in \mathbb{Z} \Rightarrow \alpha = 72k$ ***(M1)***

$\alpha = 72^\circ$ ***A1***

Note: Award ***M1A0*** if final answer is given in radians.

[4 marks]

(c) use of $\sin(5\times 72) = 0$ **OR** the imaginary part of 1 is 0 ***(M1)***

$0 = 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha$ ***A1***

$\sin\alpha \neq 0 \Rightarrow 0 = 5\left(1-\sin^2\alpha\right)^2 - 10\left(1-\sin^2\alpha\right)\sin^2\alpha + \sin^4\alpha$ ***M1***

Note: Award ***M1*** for replacing $\cos^2\alpha$.

$0 = 5\left(1 - 2\sin^2\alpha + \sin^4\alpha\right) - 10\sin^2\alpha + 10\sin^4\alpha + \sin^4\alpha$ ***A1***

Note: Award ***A1*** for any correct simplification.

so $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$ ***AG***

[4 marks]

continued…

Question 12 continued

(d) $\sin^2\alpha = \frac{20 \pm \sqrt{400-320}}{32}$ ***M1A1***

$\sin\alpha = \pm\sqrt{\frac{20 \pm \sqrt{80}}{32}}$

$\sin\alpha = \frac{\pm\sqrt{10 \pm 2\sqrt{5}}}{4}$ ***A1***

Note: Award ***A1*** regardless of signs. Accept equivalent forms with integral denominator, simplification may be seen later.

as $72 > 60$, $\sin 72 > \frac{\sqrt{3}}{2} = 0.866\ldots$ so we have to take both positive signs (or equivalent argument) ***R1***

Note: Allow verification of correct signs with calculator if clearly stated

$\sin 72 = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$ ***A1***

[5 marks]

Total [19 marks]

13. (a) (i) $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -10 \ (\mathrm{m\,s^{-2}})$ ***A1***

(ii) $t = 10 \Rightarrow v = -100 \ (\mathrm{m\,s^{-1}})$ ***A1***

(iii) $s = \int -10t\,\mathrm{d}t = -5t^2 \ (+c)$ ***M1A1***

$s = 1000$ for $t = 0 \Rightarrow c = 1000$ ***(M1)***

$s = -5t^2 + 1000$ ***A1***

at $t = 10,\ s = 500$ (m) ***AG***

Note: Accept use of definite integrals.

[6 marks]

(b) $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{(-10-5v)}$ ***A1***

[1 mark]

(c) **METHOD 1**

$t = \int \frac{1}{-10-5v}\,\mathrm{d}v = -\frac{1}{5}\ln(-10-5v)(+c)$ ***M1A1***

Note: Accept equivalent forms using modulus signs.

$t = 10,\ v = -100$

$10 = -\frac{1}{5}\ln(490) + c$ ***M1***

$c = 10 + \frac{1}{5}\ln(490)$ ***A1***

$t = 10 + \frac{1}{5}\ln 490 - \frac{1}{5}\ln(-10-5v)$ ***A1***

Note: Accept equivalent forms using modulus signs.

$t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

continued…

Question 13 continued

METHOD 2

$t=\int \frac{1}{-10-5v}\mathrm{d}v=-\frac{1}{5}\int \frac{1}{2+v}\mathrm{d}v=-\frac{1}{5}\ln\left|2+v\right|(+c)$ ***M1A1***

Note: Accept equivalent forms.

$t=10,\ v=-100$

$10=-\frac{1}{5}\ln\left|-98\right|+c$ ***M1***

Note: If $\ln(-98)$ is seen do not award further A marks.

$c=10+\frac{1}{5}\ln 98$ ***A1***

$t=10+\frac{1}{5}\ln 98-\frac{1}{5}\ln\left|2+v\right|$ ***A1***

Note: Accept equivalent forms.

$t=10+\frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

[5 marks]

(d) $5(t-10)=\ln\frac{98}{(-2-v)}$

$\frac{2+v}{98}=-\mathrm{e}^{-5(t-10)}$ ***(M1)***

$v=-2-98\mathrm{e}^{-5(t-10)}$ ***A1***

[2 marks]

(e) $\frac{\mathrm{d}s}{\mathrm{d}t}=-2-98\mathrm{e}^{-5(t-10)}$

$s=-2t+\frac{98}{5}\mathrm{e}^{-5(t-10)}(+k)$ ***M1A1***

at $t=10,\ s=500\Rightarrow 500=-20+\frac{98}{5}+k\Rightarrow k=500.4$ ***M1A1***

$s=-2t+\frac{98}{5}\mathrm{e}^{-5(t-10)}+500.4$ ***A1***

Note: Accept use of definite integrals.

[5 marks]

continued…

Question 13 continued

(f) $t = 250$ for $s = 0$ ***(M1)A1***

[2 marks]

Total [21 marks]

Markscheme

May 2015

Mathematics

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2015"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.

- If a part is completely wrong, stamp ***A0*** by the final answer.

- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark,*
but award all others so that the candidate only loses one mark.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $A = \frac{1}{2} \times 5 \times 12 \times \sin 100^{\circ}$ *(M1)*

$= 29.5 \left(\text{cm}^2\right)$ *A1*

[2 marks]

(b) $\text{AC}^2 = 5^2 + 12^2 - 2 \times 5 \times 12 \times \cos 100^{\circ}$ *(M1)*

therefore $\text{AC} = 13.8 \text{ (cm)}$ *A1*

[2 marks]

Total [4 marks]

2. (a) $\begin{pmatrix} 11 \\ 4 \end{pmatrix} = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330$ *(M1)A1*

[2 marks]

(b) $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \frac{5 \times 4}{2 \times 1} \times \frac{6 \times 5}{2 \times 1}$ *M1*

$= 150$ *A1*

[2 marks]

(c) **METHOD1**

number of ways all men $= \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5$

$330 - 5 = 325$ *M1A1*

Note: Allow ***FT*** from answer obtained in part (a).

[2 marks]

continued…

Question 2 continued.

METHOD 2

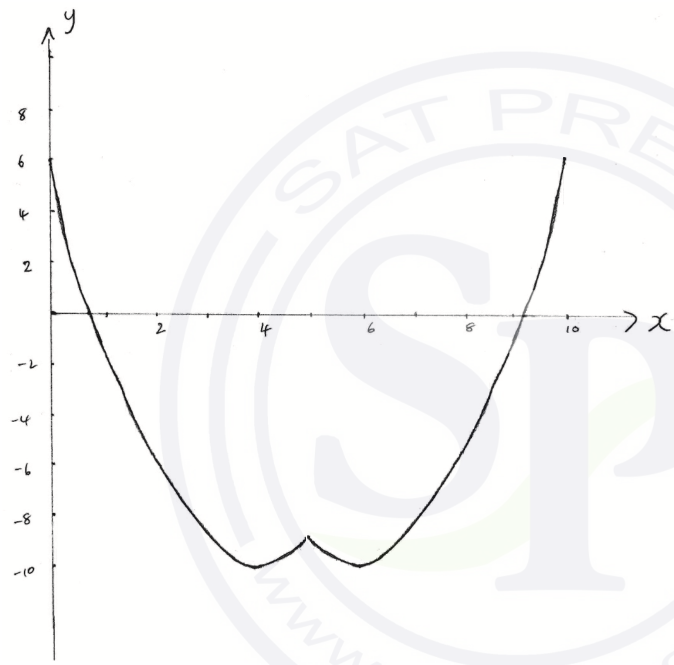
$\binom{6}{1}\binom{5}{3}+\binom{6}{2}\binom{5}{2}+\binom{6}{3}\binom{5}{1}+\binom{6}{4}$ ***M1***

=325 ***A1***

[2 marks]

Total [6 marks]

3. (a)



general shape including 2 minimums, cusp ***A1A1***
correct domain and symmetrical about the middle $(x = 5)$ ***A1***

[3 marks]

(b) $x = 9.16$ or $x = 0.838$ ***A1A1***

[2 marks]

Total [5 marks]

4. (a) (i) $X \sim Po(5)$

$P(X \geq 8) = 0.133$ ***(M1)A1***

(ii) $7 \times 0.133\ldots$ ***M1***

≈ 0.934 days ***A1***

Note: Accept "1 day".

[4 marks]

(b) $7 \times 5 = 35\ (Y \sim Po(35))$ ***(A1)***

$P(Y \leq 29) = 0.177$ ***(M1)A1***

[3 marks]

Total [7 marks]

5. (a) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} 2(0)+2b \\ -2a-1(0) \\ b-2a \end{pmatrix} = \begin{pmatrix} 2b \\ -2a \\ b-2a \end{pmatrix}$ ***(M1)(A1)***

$\begin{pmatrix} 2b \\ -2a \\ b-2a \end{pmatrix} = \begin{pmatrix} 4 \\ b \\ c \end{pmatrix}$ ***(M1)***

$\Rightarrow a = -1,\ b = 2,\ c = 4$ ***A2***

Note: Award ***A1*** for two correct.

[5 marks]

(b) $\boldsymbol{n} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$ ***(A1)***

$\Rightarrow 4x + 2y + 4z = 0\ (2x + y + 2z = 0)$ ***A1***

[2 marks]

Total [7 marks]

6. (a) **EITHER**

$y = \ln(x-a) + b = \ln(5x+10)$ ***(M1)***

$y = \ln(x-a) + \ln c = \ln(5x+10)$

$y = \ln\left(c(x-a)\right) = \ln(5x+10)$ ***(M1)***

OR

$y = \ln(5x+10) = \ln\left(5(x+2)\right)$ ***(M1)***

$y = \ln(5) + \ln(x+2)$ ***(M1)***

THEN

$a = -2,\ b = \ln 5$ ***A1A1***

Note: Accept graphical approaches.

Note: Accept $a = 2,\ b = 1.61$

[4marks]

(b) $V = \pi\int_{e}^{2e}\left[\ln(5x+10)\right]^2 \mathrm{d}x$ ***(M1)***

$= 99.2$ ***A1***

[2marks]

Total [6 marks]

7. (a)

$$2x + y + 6z = 0$$
$$4x + 3y + 14z = 4$$
$$2x - 2y + (\alpha - 2)z = \beta - 12$$

attempt at row reduction ***M1***

eg $R_2 - 2R_1$ and $R_3 - R_1$

$$2x + y + 6z = 0$$
$$y + 2z = 4$$
$$-3y + (\alpha - 8)z = \beta - 12$$ ***A1***

eg $R_3 + 3R_2$

$$2x + y + 6z = 0$$
$$y + 2z = 4$$ ***A1***
$$(\alpha - 2)z = \beta$$

(i) no solutions if $\alpha = 2$, $\beta \neq 0$ ***A1***

(ii) one solution if $\alpha \neq 2$ ***A1***

(iii) infinite solutions if $\alpha = 2$, $\beta = 0$ ***A1***

Note: Accept alternative methods e.g. determinant of a matrix

Note: Award ***A1A1A0*** if all three consistent with their reduced form, ***A1A0A0*** if two or one answer consistent with their reduced form.

[6 marks]

(b) $y + 2z = 4 \Rightarrow y = 4 - 2z$ ***A1***

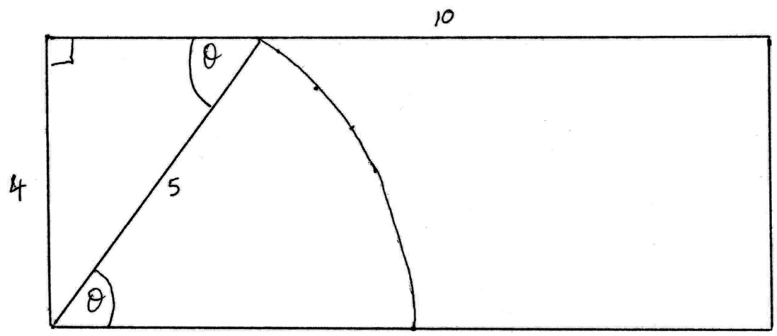
$2x = -y - 6z = 2z - 4 - 6z = -4z - 4 \Rightarrow x = -2z - 2$ ***A1***

therefore Cartesian equation is $\frac{x+2}{-2} = \frac{y-4}{-2} = \frac{z}{1}$ or equivalent ***A1***

[3 marks]

Total [9 marks]

8. (a)



EITHER

area of triangle $=\frac{1}{2}\times 3\times 4\ (=6)$ ***A1***

area of sector $=\frac{1}{2}\arcsin\left(\frac{4}{5}\right)\times 5^2\ (=11.5911\ldots)$ ***A1***

OR

$\int_0^4 \sqrt{25-x^2}\,dx$ ***M1A1***

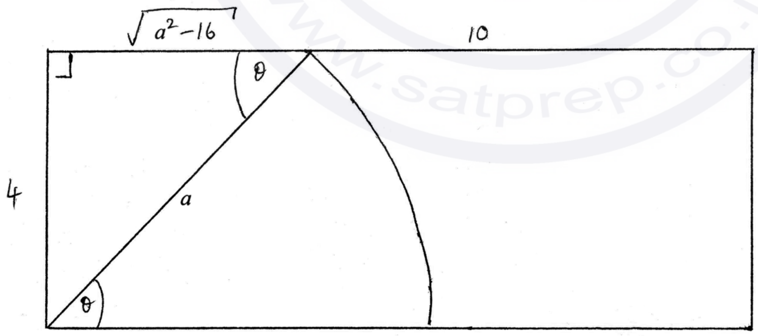
THEN

total area $=17.5911\ldots\,m^2$ ***(A1)***

percentage $=\frac{17.5911\ldots}{40}\times 100=44\%$ ***A1***

[4 marks]

(b) **METHOD 1**



area of triangle $=\frac{1}{2}\times 4\times\sqrt{a^2-16}$ ***A1***

$\theta=\arcsin\left(\frac{4}{a}\right)$ ***(A1)***

area of sector $=\frac{1}{2}r^2\theta=\frac{1}{2}a^2\arcsin\left(\frac{4}{a}\right)$ ***A1***

therefore total area $=2\sqrt{a^2-16}+\frac{1}{2}a^2\arcsin\left(\frac{4}{a}\right)=20$ ***A1***

rearrange to give: $a^2\arcsin\left(\frac{4}{a}\right)+4\sqrt{a^2-16}=40$ ***AG***

continued…

Question 8 continued

METHOD 2

$$\int_0^4 \sqrt{a^2 - x^2}\, dx = 20$$ ***M1***

use substitution $x = a\sin\theta,\ \frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta$

$$\int_0^{\arcsin\left(\frac{4}{a}\right)} a^2 \cos^2\theta \mathrm{d}\theta = 20$$

$$\frac{a^2}{2} \int_0^{\arcsin\left(\frac{4}{a}\right)} \left(\cos 2\theta + 1\right) \mathrm{d}\theta = 20$$ ***M1***

$$a^2 \left[\left(\frac{\sin 2\theta}{2} + \theta\right)\right]_0^{\arcsin\left(\frac{4}{a}\right)} = 40$$ ***A1***

$$a^2 \left[\left(\sin\theta\cos\theta + \theta\right)\right]_0^{\arcsin\left(\frac{4}{a}\right)} = 40$$

$$a^2 \arcsin\left(\frac{4}{a}\right) + a^2\left(\frac{4}{a}\right)\sqrt{\left(1 - \left(\frac{4}{a}\right)^2\right)} = 40$$ ***A1***

$$a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40$$ ***AG***

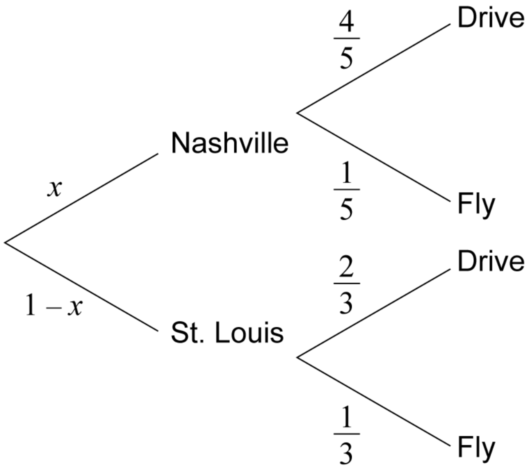
[4 marks]

(c) solving using $\text{GDC} \Rightarrow a = 5.53\,\text{cm}$ ***A2***

[2 marks]

Total [10 marks]

9.



(a) attempt to set up the problem using a tree diagram and/or an equation, with the unknown x ***M1***

$$\frac{4}{5}x+\frac{2}{3}(1-x)=\frac{13}{18}$$ ***A1***

$$\frac{4x}{5}-\frac{2x}{3}=\frac{13}{18}-\frac{2}{3}$$

$$\frac{2x}{15}=\frac{1}{18}$$

$$x=\frac{5}{12}$$ ***A1***

[3 marks]

(b) attempt to set up the problem using conditional probability ***M1***

EITHER

$$\frac{\frac{5}{12}\times\frac{1}{5}}{1-\frac{13}{18}}$$ ***A1***

OR

$$\frac{\frac{5}{12}\times\frac{1}{5}}{\frac{1}{12}+\frac{7}{36}}$$ ***A1***

THEN

$$=\frac{3}{10}$$ ***A1***

[3 marks]

Total [6 marks]

Section B

10. (a) (i) $P(110 < X < 130) = = 0.49969... = 0.500 = 50.0\%$ ***(M1)A1***

Note: Accept 50

Note: Award ***M1A0*** for 0.50 (0.500)

(ii) $P(X > 130) = (1 - 0.707\ldots) = 0.293\ldots$ ***M1***

expected number of turnips $= 29.3$ ***A1***

Note: Accept 29.

(iii) no of turnips weighing more than 130 is $Y \sim B(100,\, 0.293)$ ***M1***

$P(Y \geq 30) = 0.478$ ***A1***

[6 marks]

(b) (i) $X \sim N\left(144,\, \sigma^2\right)$

$P(X \leq 130) = \dfrac{1}{15} = 0.0667$ ***(M1)***

$P\left(Z \leq \dfrac{130 - 144}{\sigma}\right) = 0.0667$

$\dfrac{14}{\sigma} = 1.501$ ***(A1)***

$\sigma = 9.33\,g$ ***A1***

(ii) $P(X > 150 \mid X > 130) = \dfrac{P(X > 150)}{P(X > 130)}$ ***M1***

$= \dfrac{0.26008\ldots}{1 - 0.06667} = 0.279$ ***A1***

expected number of turnips $= 55.7$ ***A1***

[6 marks]

Total [12 marks]

11. (a) attempt at implicit differentiation ***M1***

$$2x - 5x\frac{dy}{dx} - 5y + 2y\frac{dy}{dx} = 0$$ ***A1A1***

> **Note:** ***A1*** for differentiation of $x^2 - 5xy$, ***A1*** for differentiation of y^2 and 7.

$$2x - 5y + \frac{dy}{dx}(2y - 5x) = 0$$

$$\frac{dy}{dx} = \frac{5y - 2x}{2y - 5x}$$ ***AG***

[3 marks]

(b) $\frac{dy}{dx} = \frac{5\times1 - 2\times6}{2\times1 - 5\times6} = \frac{1}{4}$ ***A1***

gradient of normal $= -4$ ***A1***

equation of normal $y = -4x + c$ ***M1***

substitution of $(6, 1)$

$y = -4x + 25$ ***A1***

> **Note:** Accept $y - 1 = -4(x - 6)$

[4 marks]

(c) setting $\frac{5y - 2x}{2y - 5x} = 1$ ***M1***

$y = -x$ ***A1***

substituting into original equation ***M1***

$x^2 + 5x^2 + x^2 = 7$ ***(A1)***

$7x^2 = 7$

$x = \pm 1$ ***A1***

points $(1, -1)$ and $(-1, 1)$ ***(A1)***

distance $= \sqrt{8}\left(= 2\sqrt{2}\right)$ ***(M1)A1***

[8 marks]

Total [15 marks]

12. (a) **METHOD 1**

$s = \int \left(9t - 3t^2\right) \mathrm{d}t = \frac{9}{2}t^2 - t^3 (+c)$ ***(M1)***

$t = 0,\ s = 3 \Rightarrow c = 3$ ***(A1)***

$t = 4 \Rightarrow s = 11$ ***A1***

[3 marks]

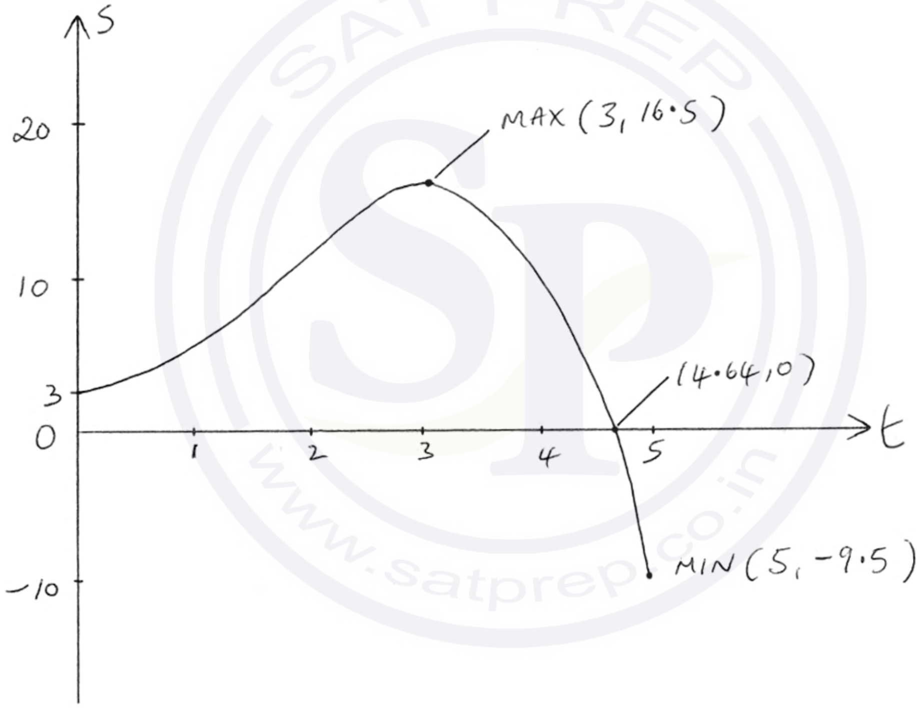
METHOD 2

$s = 3 + \int_0^4 \left(9t - 3t^2\right) dt$ ***(M1)(A1)***

$s = 11$ ***A1***

[3 marks]

(b) $s = 3 + \frac{9}{2}t^2 - t^3$ ***(A1)***



correct shape over correct domain ***A1***

maximum at $(3, 16.5)$ ***A1***

t intercept at 4.64, *s* intercept at 3 ***A1***

minimum at $(5, -9.5)$ ***A1***

[5 marks]

continued…

Question 12 continued

(c) $-9.5 = a + b\cos 2\pi$

$16.5 = a + b\cos 3\pi$ ***(M1)***

Note: Only award ***M1*** if two simultaneous equations are formed over the correct domain.

$a = \frac{7}{2}$ ***A1***

$b = -13$ ***A1***

[3 marks]

(d) at t_1:

$3 + \frac{9}{2}t^2 - t^3 = 3$ ***(M1)***

$t^2\left(\frac{9}{2} - t\right) = 0$

$t_1 = \frac{9}{2}$ ***A1***

solving $\frac{7}{2} - 13\cos\frac{2\pi t}{5} = 3$ ***(M1)***

$\text{GDC} \Rightarrow t_2 = 6.22$ ***A1***

Note: Accept graphical approaches.

[4 marks]

Total [15 marks]

13. (a) L_1 and L_2 are not parallel, since $\begin{pmatrix}-1\\1\\2\end{pmatrix} \neq k\begin{pmatrix}2\\1\\6\end{pmatrix}$ ***R1***

if they meet, then $1-\lambda = 1+2\mu$ and $2+\lambda = 2+\mu$ ***M1***

solving simultaneously $\Rightarrow \lambda = \mu = 0$ ***A1***

$2+2\lambda = 4+6\mu \Rightarrow 2 \neq 4$ contradiction, ***R1***

so lines are skew ***AG***

Note: Do not award the second ***R1*** if their values of parameters are incorrect.

[4 marks]

(b) $\begin{pmatrix}-1\\1\\2\end{pmatrix} \bullet \begin{pmatrix}2\\1\\6\end{pmatrix} (=11) = \sqrt{6}\sqrt{41}\cos\theta$ ***M1A1***

$\cos\theta = \dfrac{11}{\sqrt{246}}$ ***(A1)***

$\theta = 45.5^\circ$ (0.794 radians) ***A1***

[4 marks]

(c) (i) $\begin{pmatrix}-1\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\6\end{pmatrix} = \begin{pmatrix}6-2\\4+6\\-1-2\end{pmatrix}$ ***(M1)***

$= \begin{pmatrix}4\\10\\-3\end{pmatrix} = 4\boldsymbol{i} + 10\boldsymbol{j} - 3\boldsymbol{k}$ ***A1***

continued…

Question 13 continued

(ii) **METHOD 1**

let P be the intersection of L_1 and L_3
let Q be the intersection of L_2 and L_3

$$\vec{OP}=\begin{pmatrix}1-\lambda\\2+\lambda\\2+2\lambda\end{pmatrix}\quad \vec{OQ}=\begin{pmatrix}1+2\mu\\2+\mu\\4+6\mu\end{pmatrix}$$ ***M1***

therefore $\vec{PQ}=\vec{OQ}-\vec{OP}=\begin{pmatrix}2\mu+\lambda\\\mu-\lambda\\2+6\mu-2\lambda\end{pmatrix}$ ***M1A1***

$$\begin{pmatrix}2\mu+\lambda\\\mu-\lambda\\2+6\mu-2\lambda\end{pmatrix}=t\begin{pmatrix}4\\10\\-3\end{pmatrix}$$ ***M1***

$$2\mu+\lambda-4t=0$$
$$\mu-\lambda-10t=0$$
$$6\mu-2\lambda+3t=-2$$

solving simultaneously ***(M1)***

$\lambda=\frac{32}{125}(0.256),\ \mu=-\frac{28}{125}(-0.224)$ ***A1***

Note: Award ***A1*** for either correct λ or μ.

EITHER

therefore $\vec{OP}=\begin{pmatrix}1-\lambda\\2+\lambda\\2+2\lambda\end{pmatrix}=\begin{pmatrix}\frac{93}{125}\\\frac{282}{125}\\\frac{314}{125}\end{pmatrix}=\begin{pmatrix}0.744\\2.256\\2.512\end{pmatrix}$ ***A1***

therefore $L_3:\boldsymbol{r}_3=\begin{pmatrix}0.744\\2.256\\2.512\end{pmatrix}+\alpha\begin{pmatrix}4\\10\\-3\end{pmatrix}$ ***A1***

continued…

Question 13 continued

OR

therefore $\vec{OQ} = \begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix} = \begin{pmatrix} \frac{69}{125} \\ \frac{222}{125} \\ \frac{332}{125} \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

[10 marks]

METHOD 2

$L_3 : r_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$

forming two equations as intersections with L_1 and L_2

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t_2 \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$ ***M1A1A1***

Note: Only award ***M1A1A1*** if two different parameters t_1, t_2 used.

attempting to solve simultaneously ***M1***

$\lambda = \frac{32}{125}(0.256),\ \mu = -\frac{28}{125}(-0.224)$ ***A1***

Note: Award ***A1*** for either correct λ or μ.

continued…

Question 13 continued

EITHER

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1A1***

OR

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$$ ***A1***

therefore $L_3 : \boldsymbol{r}_3 = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

Total [18 marks]

N14/5/MATHL/HP2/ENG/TZ0/XX/M

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MARKSCHEME

November 2014

MATHEMATICS

Higher Level

Paper 2

22 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2014"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.

- If a part is completely wrong, stamp ***A0*** by the final answer.

- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. $\boldsymbol{n}_1 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ ***(A1)(A1)***

use of $\cos\theta = \frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ ***(M1)***

$\cos\theta = \frac{7}{\sqrt{21}\sqrt{19}} \left(= \frac{7}{\sqrt{399}}\right)$ ***(A1)(A1)***

> **Note:** Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$\theta = 69^{\circ}$ ***A1***

> **Note:** Award ***A1*** for 111°.

Total [6 marks]

2. (a) $\mathrm{P}(X > x) = 0.99 \ (= \mathrm{P}(X < x) = 0.01)$ ***(M1)***

$\Rightarrow x = 54.6(\text{cm})$ ***A1***

[2 marks]

(b) $\mathrm{P}(60.15 \le X \le 60.25)$ ***(M1)(A1)***

$= 0.0166$ ***A1***

[3 marks]

Total [5 marks]

3. use of $\mu = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ to obtain $\frac{2+x+y+10+17}{5} = 8$ ***(M1)***

$x + y = 11$ ***A1***

EITHER

use of $\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}$ to obtain $\frac{(-6)^2 + (x-8)^2 + (y-8)^2 + 2^2 + 9^2}{5} = 27.6$ ***(M1)***

$(x-8)^2 + (y-8)^2 = 17$ ***A1***

OR

use of $\sigma^2 = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$ to obtain $\frac{2^2 + x^2 + y^2 + 10^2 + 17^2}{5} - 8^2 = 27.6$ ***(M1)***

$x^2 + y^2 = 65$ ***A1***

THEN

attempting to solve the two equations ***(M1)***

$x = 4$ and $y = 7$ (only as $x < y$) ***A1*** ***N4***

Note: Award ***A0*** for $x = 7$ and $y = 4$.

Note: Award ***(M1)A1(M0)A0(M1)A1*** for $x + y = 11 \Rightarrow x = 4$ and $y = 7$.

Total [6 marks]

4. **METHOD 1**

attempt to set up (diagram, vectors) ***(M1)***

correct distances $x = 15t,\ y = 20t$ ***(A1)(A1)***

the distance between the two cyclists at time t is $s = \sqrt{(15t)^2 + (20t)^2} = 25t$ (km) ***A1***

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25$ ($\mathrm{km\,h^{-1}}$) ***A1***

hence the rate is independent of time ***AG***

METHOD 2

attempting to differentiate $x^2 + y^2 = s^2$ implicitly ***(M1)***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2s\frac{\mathrm{d}s}{\mathrm{d}t}$ ***(A1)***

the distance between the two cyclists at time t is $\sqrt{(15t)^2 + (20t)^2} = 25t$ (km) ***(A1)***

$2(15t)(15) + 2(20t)(20) = 2(25t)\frac{\mathrm{d}s}{\mathrm{d}t}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their equation involving $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25$ ($\mathrm{km\,h^{-1}}$) ***A1***

hence the rate is independent of time ***AG***

METHOD 3

$s = \sqrt{x^2 + y^2}$ ***(A1)***

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}}{\sqrt{x^2 + y^2}}$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to differentiate the expression for s.

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{(15t)(15) + (20t)(20)}{\sqrt{(15t)^2 + (20t)^2}}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25$ ($\mathrm{km\ h^{-1}}$) ***A1***

hence the rate is independent of time ***AG***

Total [5 marks]

5. (a) attempting to find a normal to π *eg* $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}$ ***(M1)***

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = 17\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$ ***M1***

$2x - 2y + z = 4$ (or equivalent) ***A1***

[4 marks]

(b) $l_3 : \boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, t \in \mathbb{R}$ ***(A1)***

attempting to solve $\begin{pmatrix} 4+2t \\ -2t \\ 8+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$ for t *ie* $9t + 16 = 4$ for t ***M1***

$t = -\dfrac{4}{3}$ ***A1***

$\left(\dfrac{4}{3}, \dfrac{8}{3}, \dfrac{20}{3}\right)$ ***A1***

[4 marks]

Total [8 marks]

6. using $p(a) = -7$ to obtain $3a^3 + a^2 + 5a + 7 = 0$ ***M1A1***

$(a+1)(3a^2 - 2a + 7) = 0$ ***(M1)(A1)***

> **Note:** Award ***M1*** for a cubic graph with correct shape and ***A1*** for clearly showing that the above cubic crosses the horizontal axis at $(-1, 0)$ only.

$a = -1$ ***A1***

EITHER

showing that $3a^2 - 2a + 7 = 0$ has no real (two complex) solutions for *a* ***R1***

OR

showing that $3a^3 + a^2 + 5a + 7 = 0$ has one real (and two complex) solutions for *a* ***R1***

> **Note:** Award ***R1*** for solutions that make specific reference to an appropriate graph.

Total [6 marks]

7. (a) using $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$ to form $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$ ***(M1)***

$a(a+6d) = (a+2d)^2$ ***A1***

$2d(2d - a) = 0$ (or equivalent) ***A1***

since $d \neq 0 \Rightarrow d = \frac{a}{2}$ ***AG***

[3 marks]

(b) substituting $d = \frac{a}{2}$ into $a + 6d = 3$ and solving for *a* and *d* ***(M1)***

$a = \frac{3}{4}$ and $d = \frac{3}{8}$ ***(A1)***

$r = \frac{1}{2}$ ***A1***

$$\frac{n}{2}\left(2 \times \frac{3}{4} + (n-1)\frac{3}{8}\right) - \frac{3\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \geq 200$$ ***(A1)***

attempting to solve for *n* ***(M1)***

$n \geq 31.68\ldots$

so the least value of *n* is 32 ***A1***

[6 marks]

Total [9 marks]

8. (a) $3-\frac{t}{2}=0 \Rightarrow t=6\,(\text{s})$ ***(M1)A1***

[2 marks]

> **Note:** Award ***A0*** if either $t=-0.236$ or $t=4.24$ or both are stated with $t=6$.

(b) let d be the distance travelled before coming to rest

$d=\int_0^4 5-(t-2)^2\,\mathrm{d}t+\int_4^6 3-\frac{t}{2}\mathrm{d}t$ ***(M1)(A1)***

> **Note:** Award ***M1*** for two correct integrals even if the integration limits are incorrect. The second integral can be specified as the area of a triangle.

$d=\frac{47}{3}\,(=15.7)\,(\text{m})$ ***(A1)***

attempting to solve $\int_6^T\left(\frac{t}{2}-3\right)\mathrm{d}t=\frac{47}{3}$ (or equivalent) for T ***M1***

$T=13.9\,(\text{s})$ ***A1***

[5 marks]

Total [7 marks]

9. (a) each triangle has area $\frac{1}{8}x^2 \sin\frac{2\pi}{n}$ (use of $\frac{1}{2}ab\sin C$) ***(M1)***

there are n triangles so $A = \frac{1}{8}nx^2 \sin\frac{2\pi}{n}$ ***A1***

$C = \frac{4\left(\frac{1}{8}nx^2 \sin\frac{2\pi}{n}\right)}{\pi x^2}$ ***A1***

so $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$ ***AG***

[3 marks]

(b) attempting to find the least value of n such that $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.99$ ***(M1)***

$n = 26$ ***A1***

attempting to find the least value of n such that $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$ ***(M1)***

$n = 21$ (and so a regular polygon with 21 sides) ***A1***

> **Note:** Award ***(M0)A0(M1)A1*** if $\frac{n}{2\pi}\sin\frac{2\pi}{n} > 0.99$ is not considered and $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)} > 0.99$ is correctly considered.
>
> Award ***(M1)A1(M0)A0*** for $n = 26$.

[4 marks]

(c) **EITHER**

for even and odd values of n, the value of C seems to increase towards the limiting value of the circle $(C = 1)$ *ie* as n increases, the polygonal regions get closer and closer to the enclosing circular region ***R1***

OR

the differences between the odd and even values of n illustrate that this measure of compactness is not a good one. ***R1***

[1 mark]

Total [8 marks]

SECTION B

10. (a) use of $A=\frac{1}{2}qr\sin\theta$ to obtain $A=\frac{1}{2}(x+2)(5-x)^2\sin 30^\circ$ ***M1***

$=\frac{1}{4}(x+2)(25-10x+x^2)$ ***A1***

$A=\frac{1}{4}(x^3-8x^2+5x+50)$ ***AG***

[2 marks]

(b) (i) $\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{4}\left(3x^2-16x+5\right)=\frac{1}{4}(3x-1)(x-5)$ ***A1***

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{4}\left(3\left(\frac{1}{3}\right)^2-16\left(\frac{1}{3}\right)+5\right)=0$ ***M1A1***

OR

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{4}\left(3\left(\frac{1}{3}\right)-1\right)\left(\left(\frac{1}{3}\right)-5\right)=0$ ***M1A1***

THEN

so $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***AG***

METHOD 2

solving $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ for x ***M1***

$-2<x<5\Rightarrow x=\frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***AG***

METHOD 3

a correct graph of $\frac{\mathrm{d}A}{\mathrm{d}x}$ versus x ***M1***

the graph clearly showing that $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***AG***

[3 marks]

continued…

Question 10 continued

(c) (i) $\frac{d^2A}{dx^2} = \frac{1}{2}(3x-8)$ ***A1***

for $x = \frac{1}{3}$, $\frac{d^2A}{dx^2} = -3.5\,(<0)$ ***R1***

so $x = \frac{1}{3}$ gives the maximum area of triangle PQR ***AG***

(ii) $A_{\max} = \frac{343}{27}\,(=12.7)\,(\text{cm}^2)$ ***A1***

(iii) $\text{PQ} = \frac{7}{3}\,(\text{cm})$ and $\text{PR} = \left(\frac{14}{3}\right)^2\,(\text{cm})$ ***(A1)***

$\text{QR}^2 = \left(\frac{7}{3}\right)^2 + \left(\frac{14}{3}\right)^4 - 2\left(\frac{7}{3}\right)\left(\frac{14}{3}\right)^2 \cos 30^\circ$ ***(M1)(A1)***

$= 391.702\ldots$

$\text{QR} = 19.8\,(\text{cm})$ ***A1***

[7 marks]

Total [12 marks]

11. (a) (i) $\mathrm{P}(X=0)=0.549\left(=\mathrm{e}^{-0.6}\right)$ ***A1***

(ii) $\mathrm{P}(X\geq 3)=1-\mathrm{P}(X\leq 2)$ ***(M1)***
$\mathrm{P}(X\geq 3)=0.0231$ ***A1***

[3 marks]

(b) **EITHER**

using $Y\sim \mathrm{Po}(3)$ ***(M1)***

OR

using $(0.549)^5$ ***(M1)***

THEN

$\mathrm{P}(Y=0)=0.0498\left(=\mathrm{e}^{-3}\right)$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) $\mathrm{P}(X=0)$ (most likely number of complaints received is zero) ***A1***

EITHER

calculating $\mathrm{P}(X=0)=0.549$ and $\mathrm{P}(X=1)=0.329$ ***M1A1***

OR

sketching an appropriate (discrete) graph of $\mathrm{P}(X=x)$ against x ***M1A1***

OR

finding $\mathrm{P}(X=0)=e^{-0.6}$ and stating that $\mathrm{P}(X=0)>0.5$ ***M1A1***

OR

using $\mathrm{P}(X=x)=\mathrm{P}(X=x-1)\times\dfrac{\mu}{x}$ where $\mu<1$ ***M1A1***

[3 marks]

(d) $\mathrm{P}(X=0)=0.8(\Rightarrow e^{-\lambda}=0.8)$ ***(A1)***

$\lambda=0.223\left(=\ln\dfrac{5}{4},=-\ln\dfrac{4}{5}\right)$ ***A1***

[2 marks]

Total [10 marks]

12. (a) $\text{P}(\text{Ava wins on her first turn}) = \frac{1}{3}$ ***A1***

[1 mark]

(b) $\text{P}(\text{Barry wins on his first turn}) = \left(\frac{2}{3}\right)^2$ ***(M1)***

$= \frac{4}{9} (= 0.444)$ ***A1***

[2 marks]

(c) $\text{P}(\text{Ava wins in one of her first three turns})$

$= \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3}$ ***M1A1A1***

> **Note:** Award ***M1*** for adding probabilities, award ***A1*** for a correct second term and award ***A1*** for a correct third term.
> Accept a correctly labelled tree diagram, awarding marks as above.

$= \frac{103}{243} (= 0.424)$ ***A1***

[4 marks]

(d) $\text{P}(\text{Ava eventually wins}) = \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \ldots$ ***(A1)***

using $S_\infty = \frac{a}{1-r}$ with $a = \frac{1}{3}$ and $r = \frac{2}{9}$ ***(M1)(A1)***

> **Note:** Award ***(M1)*** for using $S_\infty = \frac{a}{1-r}$ and award ***(A1)*** for $a = \frac{1}{3}$ and $r = \frac{2}{9}$.

$= \frac{3}{7} (= 0.429)$ ***A1***

[4 marks]

Total [11 marks]

13. (a) attempting to use $V=\pi\int_a^b x^2\mathrm{d}y$ ***(M1)***

attempting to express x^2 in terms of *y ie* $x^2=4(y+16)$ ***(M1)***

for $y=h$, $V=4\pi\int_0^h y+16\,\mathrm{d}y$ ***A1***

$V=4\pi\left(\frac{h^2}{2}+16h\right)$ ***AG***

[3 marks]

(b) (i) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}h}=4\pi(h+16)$ ***(A1)***

$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{4\pi(h+16)}\times\frac{-250\sqrt{h}}{\pi(h+16)}$ ***M1A1***

Note: Award ***M1*** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$ ***AG***

METHOD 2

$\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (implicit differentiation) ***(M1)***

$\frac{-250\sqrt{h}}{\pi(h+16)}=4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (or equivalent) ***A1***

$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{4\pi(h+16)}\times\frac{-250\sqrt{h}}{\pi(h+16)}$ ***M1A1***

$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$ ***AG***

(ii) $\frac{\mathrm{d}t}{\mathrm{d}h}=-\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$ ***A1***

$t=\int-\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\,\mathrm{d}h$ ***(M1)***

$t=\int-\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\,\mathrm{d}h$ ***A1***

$t=\frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}}+32h^{\frac{1}{2}}+256h^{-\frac{1}{2}}\right)\mathrm{d}h$ ***AG***

continued…

Question 13 continued

(iii) **METHOD 1**

$t = \frac{-4\pi^2}{250}\int_{48}^{0}\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$ ***(M1)***

$t = 2688.756\ldots\text{(s)}$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$t = \frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}} + \frac{64}{3}h^{\frac{3}{2}} + 512h^{\frac{1}{2}}\right) + c$

when $t = 0, h = 48 \Rightarrow c = 2688.756\ldots\left(c = \frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}} + \frac{64}{3}\times 48^{\frac{3}{2}} + 512\times 48^{\frac{1}{2}}\right)\right)$ ***(M1)***

when $h = 0$, $t = 2688.756\ldots\left(t = \frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}} + \frac{64}{3}\times 48^{\frac{3}{2}} + 512\times 48^{\frac{1}{2}}\right)\right)\text{(s)}$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

[10 marks]

(c) **EITHER**

the depth stabilises when $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ *ie* $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ ***R1***

attempting to solve $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ for h ***(M1)***

OR

the depth stabilises when $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$ *ie* $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ for h ***(M1)***

THEN

$h = 5.06\text{ (cm)}$ ***A1***

[3 marks]

Total [16 marks]

14. (a) **METHOD 1**

squaring both equations **_M1_**

$9\sin^2 B + 24\sin B\cos C + 16\cos^2 C = 36$ **_(A1)_**

$9\cos^2 B + 24\cos B\sin C + 16\sin^2 C = 1$ **_(A1)_**

adding the equations and using $\cos^2\theta + \sin^2\theta = 1$ to obtain

$9 + 24\sin(B+C) + 16 = 37$ **_M1_**

$24(\sin B\cos C + \cos B\sin C) = 12$ **_A1_**

$24\sin(B+C) = 12$ **_(A1)_**

$\sin(B+C) = \frac{1}{2}$ **_AG_**

METHOD 2

substituting for $\sin B$ and $\cos B$ to obtain

$\sin(B+C) = \left(\frac{6-4\cos C}{3}\right)\cos C + \left(\frac{1-4\sin C}{3}\right)\sin C$ **_M1_**

$= \frac{6\cos C + \sin C - 4}{3}$ (or equivalent) **_A1_**

substituting for $\sin C$ and $\cos C$ to obtain

$\sin(B+C) = \sin B\left(\frac{6-3\sin B}{4}\right) + \cos B\left(\frac{1-3\cos B}{4}\right)$ **_M1_**

$= \frac{\cos B + 6\sin B - 3}{4}$ (or equivalent) **_A1_**

Adding the two equations for $\sin(B+C)$:

$2\sin(B+C) = \frac{(18\sin B + 24\cos C) + (4\sin C + 3\cos B) - 25}{12}$ **_A1_**

$\sin(B+C) = \frac{36+1-25}{24}$ **_(A1)_**

$\sin(B+C) = \frac{1}{2}$ **_AG_**

METHOD 3

substituting for $\sin B$ and $\sin C$ to obtain

$\sin(B+C) = \left(\frac{6-4\cos C}{3}\right)\cos C + \cos B\left(\frac{1-3\cos B}{4}\right)$ **_M1_**

substituting for $\cos B$ and $\cos C$ to obtain

$\sin(B+C) = \sin B\left(\frac{6-3\sin B}{4}\right) + \left(\frac{1-4\sin C}{3}\right)\sin C$ **_M1_**

Adding the two equations for $\sin(B+C)$:

$2\sin(B+C) = \frac{6\cos C + \sin C - 4}{3} + \frac{6\sin B + \cos B - 3}{4}$ (or equivalent) **_A1A1_**

$2\sin\left(B+C\right)=\frac{(18\sin B+24\cos C)+(4\sin C+3\cos B)-25}{12}$ ***A1***

$\sin\left(B+C\right)=\frac{36+1-25}{24}$ ***(A1)***

$\sin\left(B+C\right)=\frac{1}{2}$ ***AG***

[6 marks]

(b) $\sin A=\sin\left(180^{\circ}-(B+C)\right)$ so $\sin A=\sin(B+C)$ ***R1***

$\sin(B+C)=\frac{1}{2}\Rightarrow\sin A=\frac{1}{2}$ ***A1***

$\Rightarrow A=30^{\circ}$ or $A=150^{\circ}$ ***A1***

if $A=150^{\circ}$, then $B<30^{\circ}$ ***R1***

for example, $3\sin B+4\cos C<\frac{3}{2}+4<6$, *ie* a contradiction ***R1***

only one possible value ($A=30^{\circ}$) ***AG***

[5 marks]

Total [11 marks]

M14/5/MATHL/HP2/ENG/TZ1/XX/M

MARKSCHEME

May 2014

MATHEMATICS

Higher Level

Paper 2

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2014**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. **METHOD 1**

substituting
$-5+12\mathrm{i}+a(2+3\mathrm{i})+b=0$ ***(A1)***
equating real or imaginary parts ***(M1)***
$12+3a=0 \Rightarrow a=-4$ ***A1***
$-5+2a+b=0 \Rightarrow b=13$ ***A1***

METHOD 2

other root is $2-3\mathrm{i}$ ***(A1)***
considering either the sum or product of roots or multiplying factors ***(M1)***
$4=-a$ (sum of roots) so $a=-4$ ***A1***
$13=b$ (product of roots) ***A1***

[4 marks]

2. $X : \mathrm{N}\left(100, \sigma^2\right)$
$\mathrm{P}(X<124)=0.68$ ***(M1)(A1)***
$\frac{24}{\sigma}=0.4676....$ ***(M1)***
$\sigma=51.315\ldots$ ***(A1)***
variance $=2630$ ***A1***

[5 marks]

Notes: Accept use of $\mathrm{P}(X<124.5)=0.68$ leading to variance $=2744$.

3. the number of ways of allocating presents to the first child is $\begin{pmatrix}7\\3\end{pmatrix}\left(\text{or } \begin{pmatrix}7\\2\end{pmatrix}\right)$ ***(A1)***

multiplying by $\begin{pmatrix}4\\2\end{pmatrix}\left(\text{or } \begin{pmatrix}5\\3\end{pmatrix} \text{ or } \begin{pmatrix}5\\2\end{pmatrix}\right)$ ***(M1)(A1)***

Note: Award ***M1*** for multiplication of combinations.

$\begin{pmatrix}7\\3\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}=210$ ***A1***

[4 marks]

4. (a) $$\begin{cases} x+2y-z=2 \\ 2x+y+z=1 \\ -x+4y+az=4 \end{cases}$$

$$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ 6y+(a-1)z=6 \end{cases}$$ ***M1A1***

$$\rightarrow \begin{cases} x+2y-z=2 \\ -3y+3z=-3 \\ (a+5)z=0 \end{cases}$$ ***A1***

(or equivalent)

if not a unique solution then $a=-5$ ***A1***

Note: The first ***M1*** is for attempting to eliminate a variable, the first ***A1*** for obtaining two expression in just two variables (plus a), and the second ***A1*** for obtaining an expression in just a and one other variable

[4 marks]

(b) if $a=-5$ there are an infinite number of solutions as last equation always true ***R1***

and if $a \neq -5$ there is a unique solution ***R1***

hence always a solution ***AG***

[2 marks]

Total [6 marks]

5. (a) $\frac{\pi}{2}(1.57)$, $\frac{3\pi}{2}(4.71)$ ***A1A1***

hence the coordinates are $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ ***A1***

[3 marks]

(b) (i) $\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(x^2-(x+2\cos x)^2\right)\mathrm{d}x$ ***A1A1A1***

Note: Award ***A1*** for $x^2-(x+2\cos x)^2$, ***A1*** for correct limits and ***A1*** for π.

(ii) $6\pi^2(=59.2)$ ***A2***

Notes: Do not award **ft** from (b)(i).

[5 marks]

Total [8 marks]

6. (a) **METHOD 1**

sketch showing where the lines cross or zeros of $y = x(x+2)^6 - x$ ***(M1)***

$x = 0$ ***(A1)***

$x = -1$ and $x = -3$ ***(A1)***

the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

METHOD 2

separating into two cases $x > 0$ and $x < 0$ ***(M1)***

if $x > 0$ then $(x+2)^6 > 1 \Rightarrow$ always true ***(M1)***

if $x < 0$ then $(x+2)^6 < 1 \Rightarrow -3 < x < -1$ ***(M1)***

so the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

METHOD 3

$f(x) = x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x$ ***(A1)***

solutions to $x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 63x = 0$ are ***(M1)***

$x = 0$, $x = -1$ and $x = -3$ ***(A1)***

so the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

METHOD 4

$f(x) = x$ when $x(x+2)^6 = x$

either $x = 0$ or $(x+2)^6 = 1$ ***(A1)***

if $(x+2)^6 = 1$ then $x+2 = \pm 1$ so $x = -1$ or $x = -3$ ***(M1)(A1)***

the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

[5 marks]

continued …

Question 6 continued

(b) **METHOD 1** (by substitution)

substituting $u = x + 2$ ***(M1)***

$du = dx$

$\int (u-2)u^6 du$ ***M1A1***

$= \frac{1}{8}u^8 - \frac{2}{7}u^7 \ (+c)$ ***(A1)***

$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 \ (+c)$ ***A1***

METHOD 2 (by parts)

$u = x \Rightarrow \frac{du}{dx} = 1, \ \frac{dv}{dx} = (x+2)^6 \Rightarrow v = \frac{1}{7}(x+2)^7$ ***(M1)(A1)***

$\int x(x+2)^6 dx = \frac{1}{7}x(x+2)^7 - \frac{1}{7}\int (x+2)^7 dx$ ***M1***

$= \frac{1}{7}x(x+2)^7 - \frac{1}{56}(x+2)^8 \ (+c)$ ***A1A1***

METHOD 3 (by expansion)

$\int f(x) dx = \int \left(x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x\right) dx$ ***M1A1***

$= \frac{1}{8}x^8 + \frac{12}{7}x^7 + 10x^6 + 32x^5 + 60x^4 + 64x^3 + 32x^2 \ (+c)$ are ***M1A2***

Note: Award ***M1A1*** if at least four terms are correct.

[5 marks]

Total [10 marks]

7. if $n = 0$

$7^3 + 2 = 345$ which is divisible by 5, hence true for $n = 0$ ***A1***

> **Note:** Award ***A0*** for using $n = 1$ but do not penalize further in question.

assume true for $n = k$ ***M1***

> **Note:** Only award the ***M1*** if truth is assumed.

so $7^{8k+3} + 2 = 5p,\ p \in \bullet$ ***A1***

if $n = k + 1$

$7^{8(k+1)+3} + 2$ ***M1***

$= 7^8 7^{8k+3} + 2$ ***M1***

$= 7^8(5p - 2) + 2$ ***A1***

$= 7^8.5p - 2.7^8 + 2$

$= 7^8.5p - 11529600$

$= 5\left(7^8 p - 2305920\right)$ ***A1***

hence if true for $n = k$, then also true for $n = k + 1$. Since true for $n = 0$, then true for all $n \in \bullet$ ***R1***

[8 marks]

> **Note:** Only award the ***R1*** if the first two ***M1***s have been awarded.

8. (a) $\left(A\binom{6}{5}2^5 B+3\binom{6}{4}2^4 B^2\right)x^5$ ***M1A1A1***

$=\left(192AB+720B^2\right)x^5$ ***A1***

[4 marks]

(b) **METHOD 1**

$x=\frac{1}{6},\ A=\frac{3}{6}\left(=\frac{1}{2}\right),\ B=\frac{4}{6}\left(=\frac{2}{3}\right)$ ***A1A1A1***

probability is $\frac{4}{81}$ $(=0.0494)$ ***A1***

METHOD 2

$\text{P}(5\text{ eaten})=\text{P}(\text{M eats 1})\,\text{P}(\text{N eats 4})+\text{P}(\text{M eats 0})\,\text{P}(\text{N eats 5})$ ***(M1)***

$=\frac{1}{2}\binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2+\frac{1}{2}\binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$ ***(A1)(A1)***

$=\frac{4}{81}\ (=0.0494)$ ***A1***

[4 marks]

Total [8 marks]

9. (a) mean for week is 40.88 ***(A1)***

$P(S > 40) = 1 - P(S \leq 40) = 0.513$ ***A1***

[2 marks]

(b) $\dfrac{\text{probability there were more than 10 on Monday AND more than 40 over the week}}{\text{probability there were more than 10 on Monday}}$

M1

possibilities for the numerator are:

there were more than 40 birds on the power line on Monday ***R1***
11 on Monday and more than 29 over the course of the next 6 days ***R1***
12 on Monday and more than 28 over the course of the next 6 days … until
40 on Monday and more than 0 over the course of the next 6 days ***R1***
hence if X is the number on the power line on Monday and Y, the number on the power line Tuesday – Sunday then the numerator is ***M1***

$$P(X > 40) + P(X = 11) \times P(Y > 29) + P(X = 12) \times P(Y > 28) + \ldots$$
$$+ P(X = 40) \times P(Y > 0)$$
$$= P(X > 40) + \sum_{r=11}^{40} P(X = r) P(Y > 40 - r)$$

hence solution is $\dfrac{P(X > 40) + \sum_{r=11}^{40} P(X = r) P(Y > 40 - r)}{P(X > 10)}$ ***AG***

[5 marks]

Total [7 marks]

SECTION B

10. (a) $x \to -\infty \Rightarrow y \to -\frac{1}{2}$ so $y = -\frac{1}{2}$ is an asymptote *(M1)A1*

$e^x - 2 = 0 \Rightarrow x = \ln 2$ so $x = \ln 2 (= 0.693)$ is an asymptote *(M1)A1*

[4 marks]

(b) (i) $f'(x) = \frac{2(e^x - 2)e^{2x} - (e^{2x} + 1)e^x}{(e^x - 2)^2}$ *M1A1*

$= \frac{e^{3x} - 4e^{2x} - e^x}{(e^x - 2)^2}$

(ii) $f'(x) = 0$ when $e^{3x} - 4e^{2x} - e^x = 0$ *M1*

$e^x(e^{2x} - 4e^x - 1) = 0$

$e^x = 0,\ e^x = -0.236,\ e^x = 4.24\ \left(\text{or } e^x = 2 \pm \sqrt{5}\right)$ *A1A1*

> **Note:** Award ***A1*** for zero, ***A1*** for other two solutions.
> Accept any answers which show a zero, a negative and a positive.

as $e^x > 0$ exactly one solution *R1*

> **Note:** Do not award marks for purely graphical solution.

(iii) $(1.44, 8.47)$ *A1A1*

[8 marks]

(c) $f'(0) = -4$ *(A1)*

so gradient of normal is $\frac{1}{4}$ *(M1)*

$f(0) = -2$ *(A1)*

so equation of L_1 is $y = \frac{1}{4}x - 2$ *A1*

[4 marks]

continued …

Question 10 continued

(d) $f'(x)=\frac{1}{4}$ ***M1***

so $x=1.46$ ***(M1)A1***

$f(1.46)=8.47$ ***(A1)***

equation of L_2 is $y-8.47=\frac{1}{4}(x-1.46)$ ***A1***

(or $y=\frac{1}{4}x+8.11$)

[5 marks]

Total [21 marks]

11. (a) $\int_2^3 (ax+b)\,\mathrm{d}x\,(=1)$ ***M1A1***

$\left[\frac{1}{2}ax^2+bx\right]_2^3 (=1)$ ***A1***

$\frac{5}{2}a+b=1$ ***M1***

$5a+2b=2$ ***AG***

[4 marks]

(b) (i) $\int_2^3 \left(ax^2+bx\right)\mathrm{d}x\,(=\mu)$ ***M1A1***

$\left[\frac{1}{3}ax^3+\frac{1}{2}bx^2\right]_2^3 (=\mu)$ ***A1***

$\frac{19}{3}a+\frac{5}{2}b=\mu$ ***A1***

eliminating b ***M1***

eg

$\frac{19}{3}a+\frac{5}{2}\left(1-\frac{5}{2}a\right)=\mu$ ***A1***

$\frac{1}{12}a+\frac{5}{2}=\mu$

$a=12\mu-30$ ***AG***

Note: Elimination of *b* could be at different stages.

(ii) $b=1-\frac{5}{2}(12\mu-30)$

$=76-30\mu$ ***A1***

Note: This solution may be seen in part (i).

[7 marks]

(c) (i) $\int_2^{2.3} (ax+b)\,\mathrm{d}x\,(=0.5)$ ***(M1)(A1)***

$\left[\frac{1}{2}ax^2+bx\right]_2^{2.3} (=0.5)$

$0.645a+0.3b\,(=0.5)$ ***(A1)***

$0.645(12\mu-30)+0.3(76-30\mu)=0.5$ ***M1***

$\mu=2.34\left(=\frac{295}{126}\right)$ ***A1***

continued …

Question 11 continued

(ii) $\mathrm{E}\left(X^2\right)=\int_2^3 x^2\,(ax+b)\,\mathrm{d}x$ ***(M1)***

$a=12\mu-30=-\dfrac{40}{21},\ b=76-30\mu=\dfrac{121}{21}$ ***(A1)***

$\mathrm{E}\left(X^2\right)=\int_2^3 x^2\left(-\dfrac{40}{21}x+\dfrac{121}{21}\right)\mathrm{d}x=5.539\ldots\left(=\dfrac{349}{63}\right)$ ***(A1)***

$\mathrm{Var}\,(X)=5.539\mathrm{K}\ -(2.341\mathrm{K}\,)^2=0.05813\ldots$ ***(M1)***

$\sigma=0.241$ ***A1***

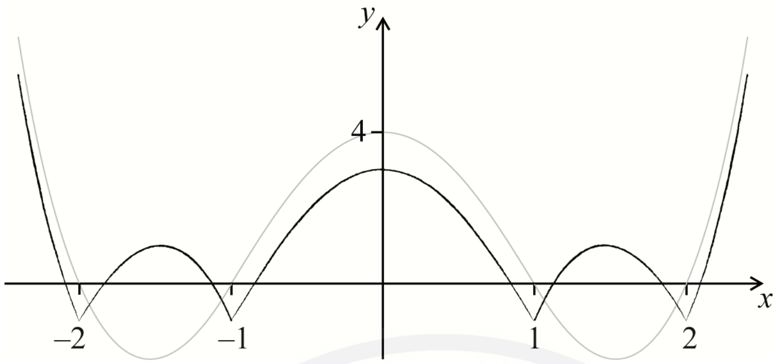
[10 marks]

Total [21 marks]

12. (a) (i) $f(0) = -1$ ***(M1)A1***

(ii) $(f \circ g)(0) = f(4) = 3$ ***A1***

(iii)

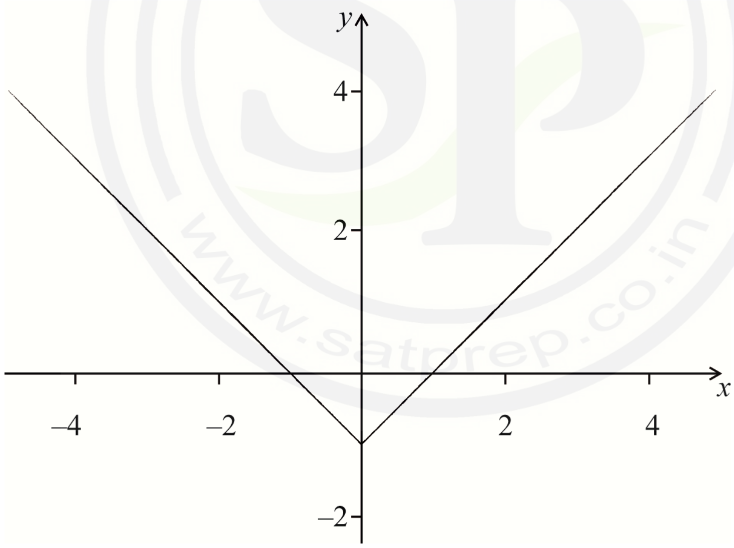


(M1)A1

Note:	Award ***M1*** for evidence that the lower part of the graph has been reflected and ***A1*** correct shape with *y*-intercept below 4.

[5 marks]

(b) (i)



(M1)A1

Note:	Award ***M1*** for any translation of $y = \lvert x \rvert$.

(ii) ± 1 ***A1***

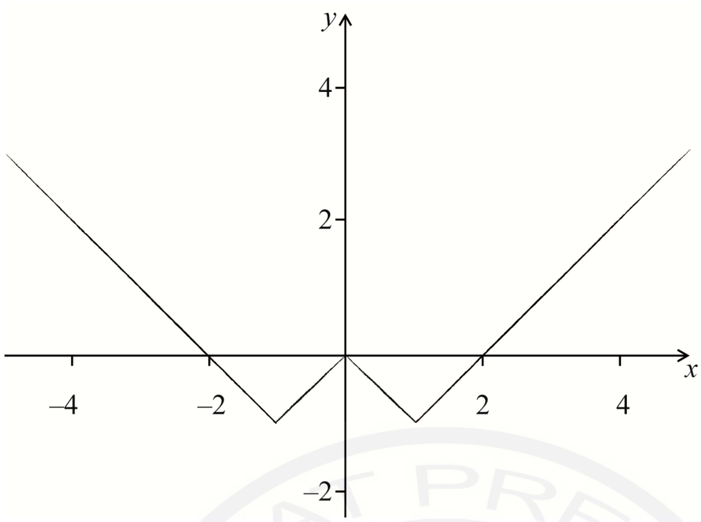
Note:	Do not award the ***A1*** if coordinates given, but do not penalise in the rest of the question

[3 marks]

continued …

Question 12 continued

(c) (i)



(M1)A1

Note: Award ***M1*** for evidence that lower part of (b) has been reflected in the *x*-axis and translated.

(ii) $0, \pm 2$ ***A1***

[3 marks]

(d) (i) $\pm 1, \pm 3$ ***A1***

(ii) $0, \pm 2, \pm 4$ ***A1***

(iii) $0, \pm 2, \pm 4, \pm 6, \pm 8$ ***A1***

[3 marks]

(e) (i) $(1, 3), (2, 5), \ldots$ ***(M1)***

$N = 2n + 1$ ***A1***

(ii) Using the formula of the sum of an arithmetic series ***(M1)***

EITHER

$$4(1 + 2 + 3 + \ldots + n) = \frac{4}{2}n(n+1)$$

$= 2n(n+1)$ ***A1***

OR

$$2(2 + 4 + 6 + \ldots + 2n) = \frac{2}{2}n(2n+2)$$

$= 2n(n+1)$ ***A1***

[4 marks]

Total [18 marks]

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MARKSCHEME

May 2014

MATHEMATICS

Higher Level

Paper 2

Instructions to Examiners

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1 General

Mark according to Scoris instructions and the document **"Mathematics HL: Guidance for e-marking May 2014"**. It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 ***N* marks**

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 **Implied marks**

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 **Follow through marks**

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 **Mis-read**

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 **Discretionary marks *(d)***

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \ \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. (a) (i) $n = 27$ *(A1)*

METHOD 1

$S_{27} = \frac{14+196}{2} \times 27$ *(M1)*

$= 2835$ *A1*

METHOD 2

$S_{27} = \frac{27}{2}(2 \times 14 + 26 \times 7)$ *(M1)*

$= 2835$ *A1*

METHOD 3

$S_{27} = \sum_{n=1}^{27} 7 + 7n$ *(M1)*

$= 2835$ *A1*

(ii) $\sum_{n=1}^{27}(7+7n)$ or equivalent *A1*

Note: Accept $\sum_{n=2}^{28} 7n$

[4 marks]

(b) $\frac{n}{2}\left(2000 - 6(n-1)\right) < 0$ *(M1)*

$n > 334.333$

$n = 335$ *A1*

Note: Accept working with equalities.

[2 marks]

Total [6 marks]

2. (a) **METHOD 1**

$\mu = \frac{1}{2} \times (17.1 + 21.3)$ ***(M1)***

$\mu = 19.2$ (kg) ***A1***

finding z value for the upper quartile $= 0.674489\text{K}$

$0.674489\text{K} = \frac{21.3 - 19.2}{\sigma}$ or $-0.674489\text{K} = \frac{17.1 - 19.2}{\sigma}$ ***M1***

$\sigma = 3.11$ (kg) ***A1***

METHOD 2

finding z value for the upper quartile $= 0.674489\text{K}$

from symmetry the z value for a lower quartile is -0.674489K ***M1***

forming two simultaneous equations:

$-0.674489\text{K} = \frac{17.1 - \mu}{\sigma}$

$0.674489\text{K} = \frac{21.3 - \mu}{\sigma}$ ***M1***

solving gives:

$\mu = 19.2$ (kg) ***A1***

$\sigma = 3.11$ (kg) ***A1***

[4 marks]

(b) using $100 \times \text{P}(X > 22) = 100 \times 0.184241\text{K}$

$= 18$ ***A1***

Note: Accept 18.4

[1 mark]

Total [5 marks]

3. (a) $x_A = 2.87$ ***A1***

$x_B = 6.78$ ***A1***

[2 marks]

(b) $\int_{2.87172\ldots}^{6.77681\ldots} 1 - 2\sin x - x^2 e^{-x} \, dx$ ***(M1)(A1)***

$= 6.76$ ***A1***

> **Note:** Award ***(M1)*** for definite integral and ***(A1)*** for a correct definite integral.

[3 marks]

Total [5 marks]

4. (a) **METHOD 1**

$2\arcsin\left(\frac{1.5}{4}\right)$ ***M1***

$\alpha = 0.769^c \ (44.0^\circ)$ ***A1***

METHOD 2

using the cosine rule:

$3^2 = 4^2 + 4^2 - 2(4)(4)\cos\alpha$ ***M1***

$\alpha = 0.769^c \ (44.0^\circ)$ ***A1***

[2 marks]

(b) one segment

$A_1 = \frac{1}{2} \times 4^2 \times 0.76879 - \frac{1}{2} \times 4^2 \times \sin(0.76879)$ ***M1A1***

$= 0.58819\ldots$ ***(A1)***

$2A_1 = 1.18 \ (\text{cm}^2)$ ***A1***

> **Note:** Award ***M1*** only if both sector and triangle are considered.

[4 marks]

Total [6 marks]

5. expanding $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ ***A1***

expanding $\left(\frac{1}{x} + 2x\right)^6$ gives

$64x^6 + 192x^4 + 240x^2 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6} + 160$ ***(M1)A1A1***

Note:	Award ***(M1)*** for an attempt at expanding using binomial. Award ***A1*** for $\frac{60}{x^2}$. Award ***A1*** for $\frac{12}{x^4}$.

$\frac{60}{x^2} \times -1 + \frac{12}{x^4} \times -3x^2$ ***(M1)***

Note:	Award ***(M1)*** only if both terms are considered.

therefore coefficient x^{-2} is -96 ***A1***

Note:	Accept $-96x^{-2}$

Note:	Award full marks if working with the required terms only without giving the entire expansion.

[6 marks]

6. (a) (i) $0.6^3 \times 0.4^3$ ***(M1)***

Note:	Award ***(M1)*** for use of the product of probabilities.

$= 0.0138$ ***A1***

(ii) binomial distribution $X : \mathrm{B}(6, 0.6)$ ***(M1)***

Note:	Award ***(M1)*** for recognizing the binomial distribution.

$\mathrm{P}(X = 3) = {}^6C_3\,(0.6)^3\,(0.4)^3$
$= 0.276$ ***A1***

Note:	Award ***(M1)A1*** for ${}^6C_3 \times 0.0138 = 0.276$.

[4 marks]

continued…

Question 6 continued

(b) $Y \sim B(n, 0.4)$
$P(Y \geq 1) > 0.995$
$1 - P(Y = 0) > 0.995$
$P(Y = 0) < 0.005$ ***(M1)***

Note: Award ***(M1)*** for any of the last three lines. Accept equalities.

$0.6^n < 0.005$ ***(M1)***

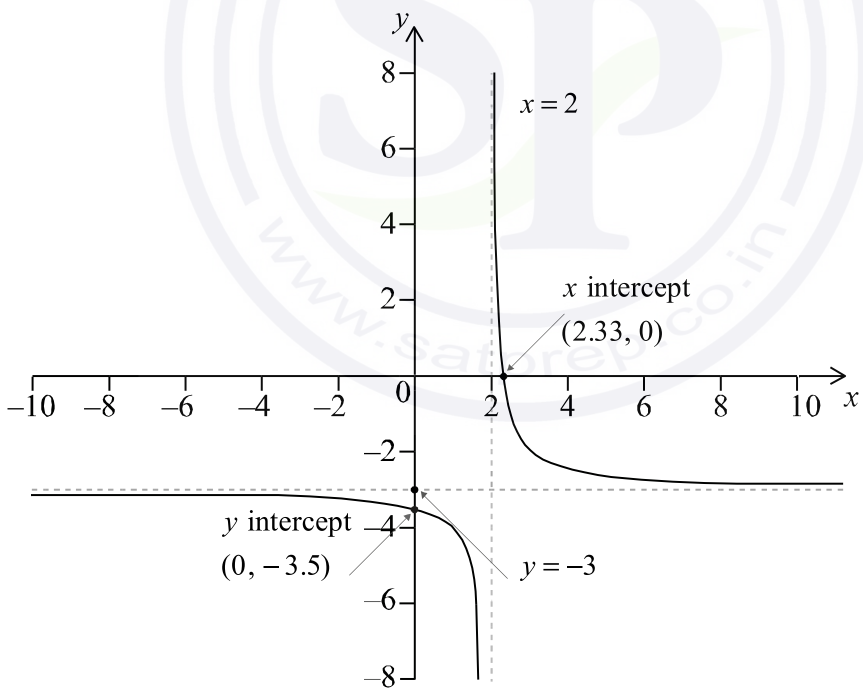
Note: Award ***(M1)*** for attempting to solve $0.6^n < 0.005$ using any method, *eg*, logs, graphically, use of solver. Accept an equality.

$n > 10.4$
$\therefore n = 11$ ***A1***

[3 marks]

Total [7 marks]

7. (a)



A1A1A1

Note: Award ***A1*** for correct shape, ***A1*** for $x = 2$ clearly stated and ***A1*** for $y = -3$ clearly stated.

x intercept $(2.33, 0)$ and y intercept $(0, -3.5)$ ***A1***

Note: Accept –3.5 and 2.33 (7/3) marked on the correct axes.

[4 marks]
continued…

Question 7 continued

(b) $x = -3 + \dfrac{1}{y-2}$ ***M1***

Note: Award ***M1*** for interchanging *x* and *y* (can be done at a later stage).

$x + 3 = \dfrac{1}{y-2}$

$y - 2 = \dfrac{1}{x+3}$ ***M1***

Note: Award ***M1*** for attempting to make *y* the subject.

$f^{-1}(x) = 2 + \dfrac{1}{x+3}\left(= \dfrac{2x+7}{x+3}\right),\ x \neq -3$ ***A1A1***

Note: Award ***A1*** only if $f^{-1}(x)$ is seen. Award ***A1*** for the domain.

[4 marks]

Total [8 marks]

8. (a) $\dfrac{\mu^2 e^{-\mu}}{2!} + \dfrac{\mu^3 e^{-\mu}}{3!} = \dfrac{\mu^5 e^{-\mu}}{5!}$ ***(M1)***

$\dfrac{\mu^2}{2} + \dfrac{\mu^3}{6} - \dfrac{\mu^5}{120} = 0$

$\mu = 5.55$ ***A1***

[2 marks]

(b) $\sigma = \sqrt{5.55...} = 2.35598...$ ***(M1)***

$\mathrm{P}(3.19 \le X \le 7.9)$

$\mathrm{P}(4 \le X \le 7)$

$= 0.607$ ***A1***

[2 marks]

Total [4 marks]

9. **METHOD 1**

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{dV}{dh} = \pi h^2$ ***(A1)***

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ (using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$) ***M1A1***

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3 \, \text{min}^{-1})$ ***A1***

METHOD 2

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{dV}{dt} = \frac{1}{3}\pi \times 3h^2 \times \frac{dh}{dt}$ ***A1***

when $h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ ***M1A1***

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3 \, \text{min}^{-1})$ ***A1***

METHOD 3

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$ ***M1A1***

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when $h = 4$, $r = 4$, $\frac{dV}{dt} = \frac{1}{3}\pi\left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5\right)$ ***M1A1***

$\frac{dV}{dt} = 8\pi \ (= 25.1) \ (\text{cm}^3 \, \text{min}^{-1})$ ***A1***

[5 marks]

10. (a) **METHOD 1**

expanding the brackets first:

$x^4 + 2x^2y^2 + y^4 = 4xy^2$ ***M1***

$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 4y^2 + 8xy\frac{dy}{dx}$ ***M1A1A1***

Note: Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent ***A1***

METHOD 2

$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 4y^2 + 8xy\frac{dy}{dx}$ ***M1A1A1***

Note: Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$

$x^3 + x^2y\frac{dy}{dx} + y^2x + y^3\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent ***A1***

[5 marks]

(b) **METHOD 1**

at $(1, 1)$, $\frac{dy}{dx}$ is undefined ***M1A1***

$y = 1$ ***A1***

METHOD 2

gradient of normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{(yx^2 - 2xy + y^3)}{(-x^3 - xy^2 + y^2)}$ ***M1***

at $(1, 1)$ gradient $= 0$ ***A1***

$y = 1$ ***A1***

[3 marks]

Total [8 marks]

SECTION B

11. (a) $a\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = 1$ *(M1)*

integrating by parts:

$u = x \qquad v' = \cos x$

$u' = 1 \qquad v = \sin x$ **M1**

$\int x\cos x\,\mathrm{d}x = x\sin x + \cos x$ **A1**

$[x\sin x + \cos x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$ **A1**

$a = \frac{1}{\frac{\pi}{2} - 1}$ **A1**

$= \frac{2}{\pi - 2}$ **AG**

[5 marks]

(b) $\mathrm{P}\left(X < \frac{\pi}{4}\right) = \frac{2}{\pi - 2}\int_0^{\frac{\pi}{4}} x\cos x\,\mathrm{d}x = 0.460$ *(M1)***A1**

> **Note:** Accept $\frac{2}{\pi - 2}\left(= \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2} - 1\right)$ or equivalent

[2 marks]

(c) (i) mode $=$ 0.860 **A1**

(x-value of a maximum on the graph over the given domain)

(ii) $\frac{2}{\pi - 2}\int_0^m x\cos x\,\mathrm{d}x = 0.5$ *(M1)*

$\int_0^m x\cos x\,\mathrm{d}x = \frac{\pi - 2}{4}$

$m\sin m + \cos m - 1 = \frac{\pi - 2}{4}$ *(M1)*

median $= 0.826$ **A1**

> **Note:** Do not accept answers containing additional solutions.

[4 marks]

continued…

Question 11 continued

(d) $\mathrm{P}\left(X < \frac{\pi}{8} \mid X < \frac{\pi}{4}\right) = \frac{\mathrm{P}\left(X < \frac{\pi}{8}\right)}{\mathrm{P}\left(X < \frac{\pi}{4}\right)}$ ***M1***

$= \frac{0.129912}{0.459826}$

$= 0.283$ ***A1***

[2 marks]

Total [13 marks]

12. (a) $C = \mathrm{AX} \times 5k + \mathrm{XB} \times k$ ***(M1)***

> **Note:** Award ***(M1)*** for attempting to express the cost in terms of AX, XB and *k*.

$= 5k\sqrt{450^2 + x^2} + (1000 - x)k$ ***A1***

$= 5k\sqrt{202500 + x^2} + (1000 - x)k$ ***AG***

[2 marks]

(b) (i) $\frac{\mathrm{d}C}{\mathrm{d}x} = k\left[\frac{5 \times 2x}{2\sqrt{202500 + x^2}} - 1\right] = k\left(\frac{5x}{\sqrt{202500 + x^2}} - 1\right)$ ***M1A1***

> **Note:** Award ***M1*** for an attempt to differentiate and ***A1*** for the correct derivative.

continued…

Question 12 continued

(ii) attempting to solve $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$ ***M1***

$\frac{5x}{\sqrt{202500 + x^2}} = 1$ ***(A1)***

$x = 91.9\,(\text{m}) \left(= \frac{75\sqrt{6}}{2}\,(\text{m}) \right)$ ***A1***

METHOD 1
for example,

at $x = 91$ $\frac{\mathrm{d}C}{\mathrm{d}x} = -0.00895k < 0$ ***M1***

at $x = 92$ $\frac{\mathrm{d}C}{\mathrm{d}x} = 0.001506k > 0$ ***A1***

Note: Award ***M1*** for attempting to find the gradient either side of $x = 91.9$ and ***A1*** for two correct values.

thus $x = 91.9$ gives a minimum ***AG***

METHOD 2

$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = \frac{1012500k}{\left(x^2 + 202500\right)^{\frac{3}{2}}}$

at $x = 91.9$ $\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = 0.010451k > 0$ ***(M1)A1***

Note: Award ***M1*** for attempting to find the second derivative and ***A1*** for the correct value.

Note: If $\frac{\mathrm{d}^2C}{\mathrm{d}x^2}$ is obtained and its value at $x = 91.9$ is not calculated, award ***(M1)A1*** for correct reasoning *eg*, both numerator and denominator are positive at $x = 91.9$.

thus $x = 91.9$ gives a minimum ***AG***

METHOD 3

Sketching the graph of either C versus x or $\frac{\mathrm{d}C}{\mathrm{d}x}$ versus x. ***M1***

Clearly indicating that $x = 91.9$ gives the minimum on their graph. ***A1***

[7 marks]

continued…

Question 12 continued

(c) $C_{\min} = 3205k$ **A1**

Note:	Accept 3200*k*. Accept 3204 *k*.

[1 mark]

(d) $\arctan\left(\dfrac{450}{91.855865\text{K}}\right) = 78.463\text{K}^{\text{o}}$ **M1**

$180 - 78.463\text{K} = 101.537\text{K}$

$= 102^{\text{o}}$ **A1**

[2 marks]

(e) (i) when $\theta = 120^{\text{o}},\ x = 260\,(\text{m})\ \left(\dfrac{450}{\sqrt{3}}(\text{m})\right)$ **A1**

(ii) $\dfrac{133.728\text{K}}{3204.5407685\text{K}} \times 100\%$ **M1**

$= 4.17\,(\%)$ **A1**

[3 marks]

Total [15 marks]

13. (a) let $\mathrm{P}(n)$ be the proposition $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, $n \in ¢^+$

let $n = 1 \Rightarrow$

$\text{LHS} = r(\cos\theta + \mathrm{i}\sin\theta)$

$\text{RHS} = r(\cos\theta + \mathrm{i}\sin\theta)$, $\therefore$ $\mathrm{P}(1)$ is true ***R1***

assume true for $n = k \Rightarrow r^k(\cos\theta + \mathrm{i}\sin\theta)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$ ***M1***

Note: Only award the ***M1*** if truth is assumed.

now show $n = k$ true implies $n = k+1$ also true

$r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ ***M1***

$= r^{k+1}\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$

$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$ ***A1***

$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$ ***A1***

$= r^{k+1}\left(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\right) \Rightarrow n = k+1$ is true ***A1***

$\mathrm{P}(k)$ true implies $\mathrm{P}(k+1)$ true and $\mathrm{P}(1)$ is true, therefore by mathematical induction statement is true for $n \geq 1$ ***R1***

Note: Only award the final ***R1*** if the first 4 marks have been awarded.

[7 marks]

(b) (i) $u = 2\,\mathrm{cis}\left(\frac{\pi}{3}\right)$ ***A1***

$v = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$ ***A1***

Notes: Accept 3 sf answers only. Accept equivalent forms. Accept $2e^{\frac{\pi}{3}i}$ and $\sqrt{2}\,e^{-\frac{\pi}{4}i}$.

(ii) $u^3 = 2^3\,\mathrm{cis}(\pi) = -8$

$v^4 = 4\,\mathrm{cis}(-\pi) = -4$ ***(M1)***

$u^3 v^4 = 32$ ***A1***

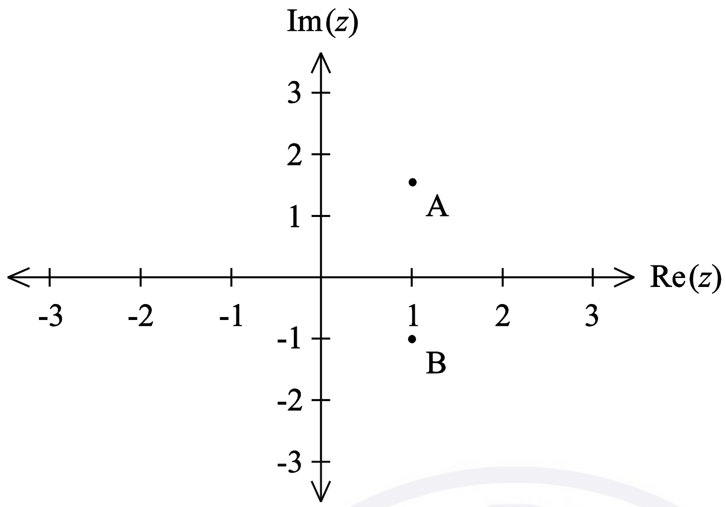
Notes: Award ***(M1)*** for an attempt to find u^3 and v^4. Accept equivalent forms.

[4 marks]

continued…

Question 13 continued

(c)



A1

Note: Award **A1** if A or $1+\sqrt{3}i$ and B or $1-i$ are in their correct quadrants, are aligned vertically and it is clear that $|u|>|v|$.

[1 mark]

(d) $\text{Area} = \frac{1}{2}\times 2\times\sqrt{2}\times\sin\left(\frac{5\pi}{12}\right)$ ***M1A1***

$=1.37\left(=\frac{\sqrt{2}}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)$ ***A1***

Notes: Award ***M1A0A0*** for using $\frac{7\pi}{12}$.

[3 marks]

(e) $(z-1+\mathrm{i})(z-1-\mathrm{i})=z^2-2z+2$ ***M1A1***

Note: Award ***M1*** for recognition that a complex conjugate is also a root.

$\left(z-1-\sqrt{3}\mathrm{i}\right)\left(z-1+\sqrt{3}\mathrm{i}\right)=z^2-2z+4$ ***A1***

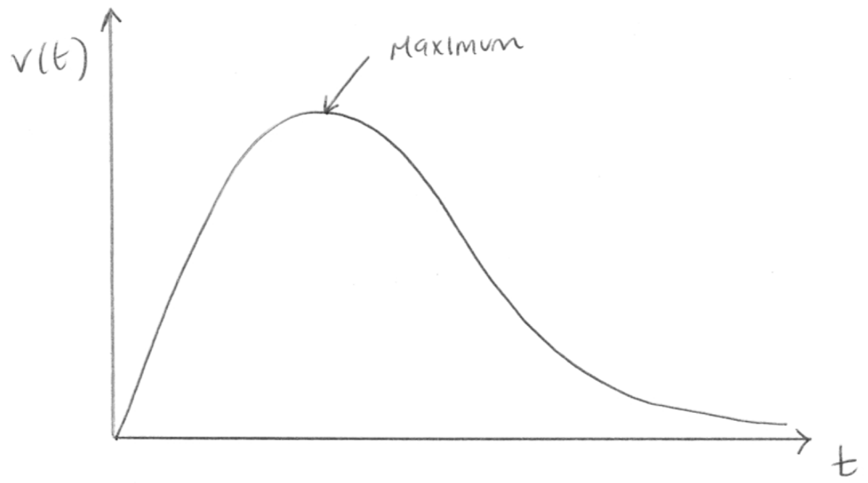
$\left(z^2-2z+2\right)\left(z^2-2z+4\right)=z^4-4z^3+10z^2-12z+8$ ***M1A1***

Note: Award ***M1*** for an attempt to expand two quadratics.

[5 marks]

Total [20 marks]

14. (a)



A1

A1 for correct shape and correct domain

(1.41, 0.0884) $\left(\sqrt{2}, \frac{\sqrt{2}}{16}\right)$ *A1*

[2 marks]

(b) **EITHER**

$u = t^2$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ *A1*

OR

$t = u^{\frac{1}{2}}$

$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$ *A1*

THEN

$\int \frac{t}{12+t^4}\,\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}u}{12+u^2}$ *M1*

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ *M1*

$= \frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent *A1*

[4 marks]

continued…

Question 14 continued

(c) $\int_0^6 \frac{t}{12+t^4}\,\mathrm{d}t$ ***(M1)***

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)$(m) ***A1***

> **Note:** Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

[3 marks]

(d) $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2\sqrt{s(1-s)}}$ ***(A1)***

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$

$a = \arcsin\left(\sqrt{s}\right)\times\frac{1}{2\sqrt{s(1-s)}}$ ***(M1)***

$a = \arcsin\left(\sqrt{0.1}\right)\times\frac{1}{2\sqrt{0.1\times 0.9}}$

$a = 0.536\ \left(\mathrm{ms}^{-2}\right)$ ***A1***

[3 marks]

Total [12 marks]

N13/5/MATHL/HP2/ENG/TZ0/XX/M

International Baccalaureate®
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MARKSCHEME

November 2013

MATHEMATICS

Higher Level

Paper 2

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document **"Mathematics HL: Guidance for e-marking November 2013"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. $\boldsymbol{AX} = \boldsymbol{B}$

EITHER

$\Rightarrow \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$ ***(M1)***

OR

attempting row reduction:

eg $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -6 \\ 0 & -2 & 0 & -1 \end{array}\right)$ ***(M1)***

THEN

$\Rightarrow \boldsymbol{X} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \\ 5 \end{pmatrix}$ ***A1A1A1***

Total [4 marks]

2. (a) **METHOD 1**

$34 = a + 3d$ and $76 = a + 9d$ ***(M1)***
$d = 7$ ***A1***
$a = 13$ ***A1***

METHOD 2

$76 = 34 + 6d$ ***(M1)***
$d = 7$ ***A1***
$34 = a + 21$
$a = 13$ ***A1***

[3 marks]

(b) $\frac{n}{2}\left(26 + 7(n-1)\right) > 5000$ ***(M1)(A1)***
$n > 36.463...$ ***(A1)***

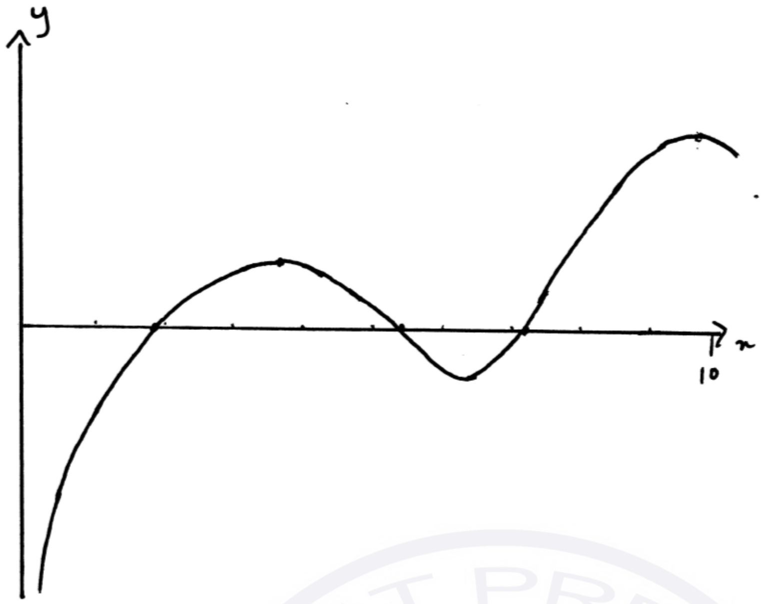
Note: Award ***M1A1A1*** for using either an equation, a graphical approach or a numerical approach.

$n = 37$ ***A1*** ***N3***

[4 marks]

Total [7 marks]

3. (a)



A correct graph shape for $0 < x \le 10$. *A1*

maxima $(3.78, 0.882)$ and $(9.70, 1.89)$ *A1*

minimum $(6.22, -0.885)$ *A1*

x-axis intercepts $(1.97, 0)$, $(5.24, 0)$ and $(7.11, 0)$ *A2*

Note: Award ***A1*** if two *x*-axis intercepts are correct.

[5 marks]

(b) $0 < x \le 1.97$ *A1*

$5.24 \le x \le 7.11$ *A1*

[2 marks]

Total [7 marks]

4. $\mathrm{P}\left(Z < \frac{780-\mu}{\sigma}\right) = 0.92$ and $\mathrm{P}\left(Z < \frac{755-\mu}{\sigma}\right) = 0.12$ ***(M1)***

use of inverse normal ***(M1)***

$\Rightarrow \frac{780-\mu}{\sigma} = 1.405\ldots$ and $\frac{755-\mu}{\sigma} = -1.174\ldots$ ***(A1)***

solving simultaneously ***(M1)***

Note:	Award ***M1*** for attempting to solve an incorrect pair of equations *eg*, inverse normal not used.

$\mu = 766.385$
$\sigma = 9.6897$
$\mu = 12$ hrs 46 mins ($= 766$ mins) ***A1***
$\sigma = 10$ mins ***A1***

Total [6 marks]

5. (a) $\mathrm{P}(F) = \left(\frac{1}{7}\times\frac{7}{9}\right) + \left(\frac{6}{7}\times\frac{4}{9}\right)$ ***(M1)(A1)***

Note:	Award ***M1*** for the sum of two products.

$= \frac{31}{63}$ $(= 0.4920\ldots)$ ***A1***

[3 marks]

(b) Use of $\mathrm{P}(S \mid F) = \frac{\mathrm{P}(S \cap F)}{\mathrm{P}(F)}$ to obtain $\mathrm{P}(S \mid F) = \frac{\frac{1}{7}\times\frac{7}{9}}{\frac{31}{63}}$. ***M1***

Note:	Award ***M1*** only if the numerator results from the product of two probabilities.

$= \frac{7}{31}$ $(= 0.2258\ldots)$ ***A1***

[2 marks]

Total [5 marks]

6. (a) $\dfrac{a+\mathrm{i}}{a-\mathrm{i}} \times \dfrac{a+\mathrm{i}}{a+\mathrm{i}}$ ***M1***

$= \dfrac{a^2-1+2a\mathrm{i}}{a^2+1} \left(= \dfrac{a^2-1}{a^2+1} + \dfrac{2a}{a^2+1}\mathrm{i}\right)$ ***A1***

(i) z is real when $a=0$ ***A1***

(ii) z is purely imaginary when $a=\pm 1$ ***A1***

Note: Award ***M1A0A1A0*** for $\dfrac{a^2-1+2a\mathrm{i}}{a^2-1} \left(=1+\dfrac{2a}{a^2-1}\mathrm{i}\right)$ leading to $a=0$ in (i).

[4 marks]

(b) **METHOD 1**

attempting to find either $|z|$ or $|z|^2$ by expanding and simplifying

eg $|z|^2 = \dfrac{\left(a^2-1\right)^2+4a^2}{\left(a^2+1\right)^2} = \dfrac{a^4+2a^2+1}{\left(a^2+1\right)^2}$ ***M1***

$= \dfrac{\left(a^2+1\right)^2}{\left(a^2+1\right)^2}$

$|z|^2 = 1 \Rightarrow |z| = 1$ ***A1***

METHOD 2

$|z| = \dfrac{|a+\mathrm{i}|}{|a-\mathrm{i}|}$ ***M1***

$|z| = \dfrac{\sqrt{a^2+1}}{\sqrt{a^2+1}} \Rightarrow |z| = 1$ ***A1***

[2 marks]

Total [6 marks]

7. (a) attempting to form $(3\cos\theta+6)(\cos\theta-2)+7(1+\sin\theta)=0$ ***M1***

$3\cos^2\theta-12+7\sin\theta+7=0$ ***A1***

$3\left(1-\sin^2\theta\right)+7\sin\theta-5=0$ ***M1***

$3\sin^2\theta-7\sin\theta+2=0$ ***AG***

[3 marks]

(b) attempting to solve algebraically (including substitution) or graphically for $\sin\theta$ ***(M1)***

$\sin\theta=\frac{1}{3}$ ***(A1)***

$\theta=0.340\quad(=19.5^\circ)$ ***A1***

[3 marks]

Total [6 marks]

8. (a) $A=\frac{1}{2}\times10^2\times\theta-\frac{1}{2}\times10^2\times\sin\theta$ ***M1A1***

> **Note:** Award ***M1*** for use of area of segment = area of sector − area of triangle.

$=50\theta-50\sin\theta$ ***AG***

[2 marks]

(b) **METHOD 1**

unshaded area $=\frac{\pi\times10^2}{2}-50(\theta-\sin\theta)$

(or equivalent *eg* $50\pi-50\theta+50\sin\theta$) ***(M1)***

$50\theta-50\sin\theta=\frac{1}{2}(50\pi-50\theta+50\sin\theta)$ ***(A1)***

$3\theta-3\sin\theta-\pi=0$

$\Rightarrow\theta=1.969$ (rad) ***A1***

METHOD 2

$50\theta-50\sin\theta=\frac{1}{3}\left(\frac{\pi\times10^2}{2}\right)$ ***(M1)(A1)***

$3\theta-3\sin\theta-\pi=0$

$\Rightarrow\theta=1.969$ (rad) ***A1***

[3 marks]

Total [5 marks]

9. (a) **METHOD 1**

for P on L_1, $\vec{\text{OP}} = \begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix}$

require $\begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ ***M1***

$5+\lambda-6+4\lambda+4+4\lambda=0$ (or equivalent) ***A1***

$\lambda = -\frac{1}{3}$ ***A1***

$\therefore \vec{\text{OP}} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen.

[5 marks]

METHOD 2

Calculating either $\left|\vec{\text{OP}}\right|$ or $\left|\vec{\text{OP}}\right|^2$ *eg*

$\left|\vec{\text{OP}}\right| = \sqrt{(-5-\lambda)^2 + (-3+2\lambda)^2 + (2+2\lambda)^2}$

$= \sqrt{9\lambda^2 + 6\lambda + 38}$ ***A1***

Solving either $\frac{\text{d}}{\text{d}\lambda}\left(\left|\vec{\text{OP}}\right|\right) = 0$ or $\frac{\text{d}}{\text{d}\lambda}\left(\left|\vec{\text{OP}}\right|^2\right) = 0$ for λ. ***M1***

$\lambda = -\frac{1}{3}$ ***A1***

$\vec{\text{OP}} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen. ***[5 marks]***

continued…

Question 9 continued

(b) **METHOD 1**

$\left|\overrightarrow{OP}\right| = \sqrt{\left(-\frac{14}{3}\right)^2 + \left(-\frac{11}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$ ***(M1)***

$= 6.08 \quad \left(= \sqrt{37}\right)$ ***A1***

METHOD 2

$\text{shortest distance} = \frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix} \times \begin{pmatrix}-5\\-3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\right|}$ ***(M1)***

$= \frac{|10\boldsymbol{i} + 8\boldsymbol{j} + 13\boldsymbol{k}|}{\sqrt{1+4+4}}$

$= 6.08 \left(= \sqrt{37}\right)$ ***A1***

[2 marks]

Total [7 marks]

10. **EITHER**

$\frac{\mathrm{d}x}{\mathrm{d}u} = 2\sec^2 u$ ***A1***

$\int \frac{2\sec^2 u \, \mathrm{d}u}{4\tan^2 u\sqrt{4+4\tan^2 u}}$ ***(M1)***

$= \int \frac{2\sec^2 u \, \mathrm{d}u}{4\tan^2 u \times 2\sec u}$ $\left(= \int \frac{\mathrm{d}u}{4\sin^2 u\sqrt{\tan^2 u + 1}} \text{ or } = \int \frac{2\sec^2 u \, \mathrm{d}u}{4\tan^2 u\sqrt{4\sec^2 u}}\right)$ ***A1***

OR

$u = \arctan \frac{x}{2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x^2+4}$ ***A1***

$\int \frac{\sqrt{4\tan^2 u + 4}\,\mathrm{d}u}{2 \times 4\tan^2 u}$ ***(M1)***

$\int \frac{2\sec u \, \mathrm{d}u}{2 \times 4\tan^2 u}$ ***A1***

THEN

$= \frac{1}{4}\int \frac{\sec u \, \mathrm{d}u}{\tan^2 u}$

$= \frac{1}{4}\int \operatorname{cosec} u \cot u \, \mathrm{d}u$ $\left(= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\mathrm{d}u\right)$ ***A1***

$= -\frac{1}{4}\operatorname{cosec} u \ (+C)$ $\left(= -\frac{1}{4\sin u} \ (+C)\right)$ ***A1***

use of either $\tan u = \frac{x}{2}$ or an appropriate trigonometric identity ***M1***

either $\sin u = \frac{x}{\sqrt{x^2+4}}$ or $\operatorname{cosec} u = \frac{\sqrt{x^2+4}}{x}$ (or equivalent) ***A1***

$= \frac{-\sqrt{x^2+4}}{4x} (+C)$ ***AG***

Total [7 marks]

SECTION B

11. (a) (i) $X \sim \text{Po}(0.6)$

$\text{P}(X = 0) = 0.549 \ \left(= \text{e}^{-0.6}\right)$ ***A1***

(ii) $\text{P}(X \geq 3) = 1 - \text{P}(X \leq 2)$ ***(M1)(A1)***

$= 1 - \left(\text{e}^{-0.6} + \text{e}^{-0.6} \times 0.6 + \text{e}^{-0.6} \times \dfrac{0.6^2}{2}\right)$

$= 0.0231$ ***A1***

(iii) $Y \sim \text{Po}(2.4)$ ***(M1)***

$\text{P}(Y \leq 5) = 0.964$ ***A1***

(iv) $Z \sim \text{B}(12,\ 0.451...)$ ***(M1)(A1)***

Note: Award ***M1*** for recognising binomial and ***A1*** for using correct parameters.

$\text{P}(Z = 4) = 0.169$ ***A1***

[9 marks]

(b) (i) $k\int_1^3 \ln x \,\text{d}x = 1$ ***(M1)***

$(k \times 1.2958... = 1)$

$k = 0.771702$ ***A1***

(ii) $\text{E}(X) = \int_1^3 kx \ln x \,\text{d}x$ ***(A1)***

attempting to evaluate their integral ***(M1)***

$= 2.27$ ***A1***

(iii) $x = 3$ ***A1***

(iv) $\int_1^m k \ln x \,\text{d}x = 0.5$ ***(M1)***

$k[x \ln x - x]_1^m = 0.5$

attempting to solve for m ***(M1)***

$m = 2.34$ ***A1***

[9 marks]

Total [18 marks]

12. (a) (i) **METHOD 1**

$v = \int 3\cos\frac{t}{4}\mathrm{d}t$ ***M1***

$= 12\sin\frac{t}{4} + c$ ***A1***

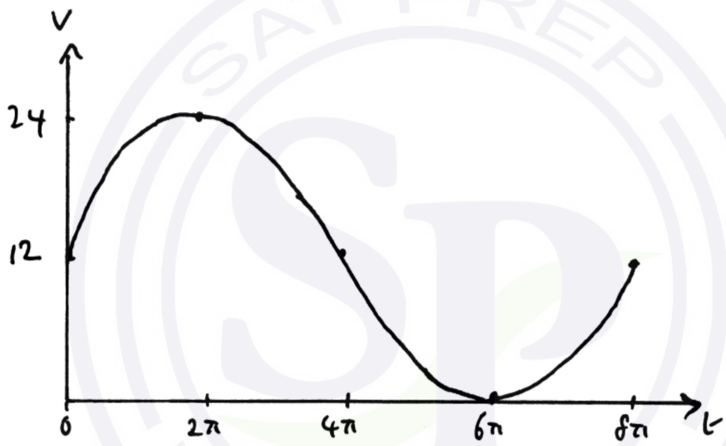
$t = 0,\ v = 12 \Rightarrow v = 12\sin\frac{t}{4} + 12$ ***A1***

METHOD 2

$v - 12 = \int_0^t 3\cos\frac{t}{4}\mathrm{d}t$ ***M1A1***

$v = 12\sin\frac{t}{4} + 12$ ***A1***

(ii)



A1A1A1

> **Note:** Award ***A1*** for shape and domain $0 \le t \le 8\pi$.
> Award ***A1*** for $(0, 12)$ and $(6\pi, 0)$ $\left((18.8, 0)\right)$.
> Award ***A1*** for $(2\pi, 24)$ $\left((6.28, 24)\right)$.

(iii) **METHOD 1**

$\int_0^{6\pi}\left(12\sin\frac{t}{4} + 12\right)\mathrm{d}t$ ***M1***

$= 274\ \text{(m)}\ \left(= 72\pi + 48\ \text{(m)}\right)$ ***A1***

METHOD 2

$s = \int 12\sin\frac{t}{4} + 12\ \mathrm{d}t$

$= -48\cos\frac{t}{4} + 12t + c$ ***M1***

When $t = 0$, $s = 0$ and so $c = 48$.

When $t = 6\pi$, $s = 274\ \text{(m)}\ \left(= 72\pi + 48\ \text{(m)}\right)$. ***A1***

[8 marks]

continued …

Question 12 continued

(b) (i) **METHOD 1**

$$\frac{dv}{dt} = -\left(v^2 + 4\right)$$ ***(A1)***

$$\int \frac{dv}{v^2+4} = -\int dt$$ ***M1***

$$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = -t + c$$ ***A1***

EITHER

$$t = 0,\ v = 2 \Rightarrow c = \frac{\pi}{8}$$ ***M1***

$$\arctan\left(\frac{v}{2}\right) = \frac{\pi}{4} - 2t$$ ***A1***

OR

$$v = 2\tan\left(2c - 2t\right)$$ ***A1***

$$t = 0,\ v = 2 \Rightarrow \tan 2c = 1 \text{ and so } c = \frac{\pi}{8}$$ ***M1***

THEN

$$v = 2\tan\left(\frac{\pi}{4} - 2t\right)$$ ***A1***

$$v = 2\tan\left(\frac{\pi - 8t}{4}\right)$$ ***AG***

METHOD 2

$$\frac{dv}{dt} = -4\sec^2\left(\frac{\pi - 8t}{4}\right)$$ ***M1A1***

Substituting $v = 2\tan\left(\frac{\pi - 8t}{4}\right)$ into $\frac{dv}{dt} = -\left(v^2 + 4\right)$:

$$\frac{dv}{dt} = -\left(4\tan^2\left(\frac{\pi - 8t}{4}\right) + 4\right)$$ ***M1***

$$= -4\left(\tan^2\left(\frac{\pi - 8t}{4}\right) + 1\right)$$ ***(A1)***

$$= -4\sec^2\left(\frac{\pi - 8t}{4}\right)$$ ***A1***

Verifying that $v = 2$ when $t = 0$. ***A1***

continued …

(ii) **METHOD 1**

$v\frac{\mathrm{d}v}{\mathrm{d}s} = -\left(v^2+4\right)$ ***A1***

$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\left(v^2+4\right)}{v}$ ***AG***

METHOD 2

$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}v}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}s}$ ***A1***

$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\left(v^2+4\right)}{v}$ ***AG***

(iii) **METHOD 1**

When $v=0$, $t=\frac{\pi}{8}$ $(t=0.392...)$. ***(M1)A1***

$s=\int_0^{\frac{\pi}{8}} 2\tan\left(\frac{\pi-8t}{4}\right)\mathrm{d}t$ ***(M1)***

$s=0.347\ \text{(m)}\ \left(s=\frac{1}{2}\ln 2\ \text{(m)}\right)$ ***A2***

METHOD 2

$\int\frac{v}{4+v^2}\mathrm{d}v = -\int\mathrm{d}s$ ***M1***

EITHER

$\frac{1}{2}\ln\left(v^2+4\right) = -s+c$ (or equivalent) ***A1***

$v=2,\ s=0 \Rightarrow c=\frac{1}{2}\ln 8$ ***M1***

$s=-\frac{1}{2}\ln\left(v^2+4\right)+\frac{1}{2}\ln 8\ \left(s=\frac{1}{2}\ln\left(\frac{8}{v^2+4}\right)\right)$ ***(A1)***

$v=0 \Rightarrow s=\frac{1}{2}\ln 2\ \text{(m)}\ \left(s=0.347\ \text{(m)}\right)$ ***A1***

OR

$-\int_2^0\frac{v}{4+v^2}\mathrm{d}v = s$ (or equivalent) ***M1A1***

Note: Award ***M1*** for setting up a definite integral and award ***A1*** for stating correct limits.

$s=0.347\ \text{(m)}\ \left(s=\frac{1}{2}\ln 2\ \text{(m)}\right)$ ***A2***

[12 marks]

Total [20 marks]

13. (a) (i) either counterexample or sketch or
recognising that $y = k$ $(k > 1)$ intersects the graph of $y = f(x)$ twice ***M1***
function is not 1–1 (does not obey horizontal line test) ***R1***
so f^{-1} does not exist ***AG***

(ii) $f'(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ ***(A1)***

$f'(\ln 3) = \frac{4}{3}$ $(= 1.33)$ ***(A1)***

$m = -\frac{3}{4}$ ***M1***

$f(\ln 3) = \frac{5}{3}$ $(= 1.67)$ ***A1***

EITHER

$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$ ***M1***

$4y - \frac{20}{3} = -3x + 3\ln 3$ ***A1***

OR

$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$ ***M1***

$c = \frac{5}{3} + \frac{3}{4}\ln 3$

$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$ ***A1***

$12y = -9x + 20 + 9\ln 3$

THEN

$9x + 12y - 9\ln 3 - 20 = 0$ ***AG***

(iii) The tangent at $(a,\ f(a))$ has equation $y - f(a) = f'(a)(x - a)$. ***(M1)***

$f'(a) = \frac{f(a)}{a}$ (or equivalent) ***(A1)***

$e^{a} - e^{-a} = \frac{e^{a} + e^{-a}}{a}$ (or equivalent) ***A1***

attempting to solve for a ***(M1)***
$a = \pm 1.20$ ***A1A1***

[14 marks]

continued …

Question 13 continued

(b) (i) $2y = e^x + e^{-x}$

$e^{2x} - 2ye^x + 1 = 0$ ***M1A1***

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$e^x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^x = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi\int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \, dy$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use $V = \pi\int_c^d x^2 dy$.

$= 37.1 \left(\text{units}^3\right)$ ***A1***

[8 marks]

Total [22 marks]

International Baccalaureate®
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MARKSCHEME

May 2013

MATHEMATICS

Higher Level

Paper 2

15 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty **(AP)** no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

*The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award **A0**. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. $\dfrac{5\times 6+6k+7\times 3+8\times 1+9\times 2+10\times 1}{13+k}=6.5$ (or equivalent) ***(M1)(A1)(A1)***

Note: Award ***(M1)(A1)*** for correct numerator, and ***(A1)*** for correct denominator.

$0.5k=2.5\Rightarrow k=5$ ***A1***

[4 marks]

2. **METHOD 1**

determinant $=0$ ***M1***
$k(-2-16)-(0-12)+2(0+3)=0$ ***(M1)(A1)***
$-18k+18=0$ ***(A1)***
$k=1$ ***A1***

METHOD 2

writes in the form

$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ (or attempts to solve simultaneous equations) ***(M1)***

Having two 0's in first column (obtaining two equations in the same two variables) ***M1***

$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 0 & 0 & 18k-18 & 21k-27 \end{pmatrix}$ (or isolating one variable) ***A1***

Note: The ***A1*** is to be awarded for the $18k-18$. The final column may not be seen.

$k=1$ ***(M1)A1***

[5 marks]

3. Let X represent the length of time a journey takes on a particular day.

(a) $P(X>15)=0.0912112819\ldots=0.0912$ ***(M1)A1***

(b) Use of correct Binomial distribution ***(M1)***
$N\sim B(5,0.091...)$
$1-0.0912112819\ldots=0.9087887181\ldots$
$1-(0.9087887181\ldots)^5=0.380109935\ldots=0.380$ ***(M1)A1***

Note: Allow answers to be given as percentages.

[5 marks]

4. volume $=\pi\int x^2\text{d}y$ ***(M1)***

$x=\arcsin y+1$ ***(M1)(A1)***

volume $=\pi\int_0^1(\arcsin y+1)^2\text{d}y$ ***A1***

Note: ***A1*** is for the limits, provided a correct integration of y.

$=2.608993\ldots\pi=8.20$ ***A2*** ***N5***

[6 marks]

5. $\frac{1}{2}r^2\times1=7$ ***M1***

$r=3.7\ldots\left(=\sqrt{14}\right)$ (or 37… mm) ***(A1)***

height $=2r\cos\left(\frac{\pi-1}{2}\right)$ $\left(\text{or } 2r\sin\frac{1}{2}\right)$ ***(M1)(A1)***

3.59 or anything that rounds to 3.6 ***A1***

so the dimensions are 3.7 by 3.6 (cm or 37 by 36 mm) ***A1***

[6 marks]

6. other root is $2-i$ ***(A1)***

a quadratic factor is therefore $(x-2+i)(x-2-i)$ ***(M1)***

$=x^2-4x+5$ ***A1***

$x+1$ is a factor ***A1***

$(x-2)^2$ is a factor ***A1***

$p(x)=a(x+1)(x-2)^2(x^2-4x+5)$ ***(M1)***

$p(0)=4\Rightarrow a=\frac{1}{5}$ ***A1***

$p(x)=\frac{1}{5}(x+1)(x-2)^2(x^2-4x+5)$

[7 marks]

7. (a) let the distance the cable is laid along the seabed be y

$y^2 = x^2 + 200^2 - 2 \times x \times 200 \cos 60^0$ ***(M1)***

(or equivalent method)

$y^2 = x^2 - 200x + 40000$ ***(A1)***

$\text{cost} = C = 80y + 20x$ ***(M1)***

$C = 80(x^2 - 200x + 40000)^{\frac{1}{2}} + 20x$ ***A1***

[4 marks]

(b) $x = 55.2786\ldots = 55$ (m to the nearest metre) ***(A1)A1***

$\left(x = 100 - \sqrt{2000}\right)$

[2 marks]

Total [6 marks]

8. (a) the three girls can sit together in $3! = 6$ ways ***(A1)***

this leaves 4 'objects' to arrange so the number of ways this can be done is $4!$ ***(M1)***

so the number of arrangements is $6 \times 4! = 144$ ***A1***

[3 marks]

(b) Finding more than one position that the girls can sit ***(M1)***

Counting exactly four positions ***(A1)***

number of ways = $4 \times 3! \times 3! = 144$ ***M1A1*** ***N2***

[4 marks]

Total [7 marks]

9. (a) $\Delta = b^2 - 4ac = 4k^2 - 4 \times 3 \times (k-1) = 4k^2 - 12k + 12$ ***M1A1***

Note: Award ***M1A1*** if expression seen within quadratic formula.

EITHER

$144 - 4 \times 4 \times 12 < 0$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

(and cannot be always negative as $a > 0$)

OR

sketch of $y = 4k^2 - 12k + 12$ or $y = k^2 - 3k + 3$ not crossing the x-axis ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

OR

write Δ as $4(k - 1.5)^2 + 3$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

[4 marks]

continued …

Question 9 continued

(b) closest together when Δ is least **(M1)**
minimum value occurs when $k = 1.5$ **(M1)A1**

[3 marks]

Total [7 marks]

10. (a) $X \sim \text{Po}(0.25\text{T})$ **(A1)**
Attempt to solve $\text{P}(X \le 3) = 0.6$ **(M1)**
$T = 12.8453\ldots = 13$ (minutes) **A1**

Note: Award ***A1M1A0*** if T found correctly but not stated to the nearest minute.

[3 marks]

(b) let X_1 be the number of cars that arrive during the first interval and X_2 be the number arriving during the second.
X_1 and X_2 are $\text{Po}(2.5)$ **(A1)**
P (all get on) $= \text{P}(X_1 \le 3) \times \text{P}(X_2 \le 3) + \text{P}(X_1 = 4) \times \text{P}(X_2 \le 2)$
$+ \text{P}(X_1 = 5) \times \text{P}(X_2 \le 1) + \text{P}(X_1 = 6) \times \text{P}(X_2 = 0)$ **(M1)**
$= 0.573922\ldots + 0.072654\ldots + 0.019192\ldots + 0.002285\ldots$ **(M1)**
$= 0.668\ (053\ldots)$ **A1**

[4 marks]

Total [7 marks]

SECTION B

11. (a) $\overrightarrow{PQ} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ ***(A1)***

equation of line: $\boldsymbol{r} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ (or equivalent) ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ is omitted.

[3 marks]

(b) **METHOD 1**

$x: \quad -4 + 5s = -3 + 8t$
$y: \quad 2s = -1 + 6t$
$z: \quad 4 = 2 + 4t$ ***M1***
solving any two simultaneously ***M1***
$t = 0.5,\ s = 1$ (or equivalent) ***A1***
verification that these values give R when substituted into **both** equations
(or that the three equations are consistent and that one gives R) ***R1***

METHOD 2

$(1, 2, 4)$ is given by $t = 0.5$ for L_1 and $s = 1$ for L_2 ***M1A1A1***
because $(1, 2, 4)$ is on both lines it is the point of intersection of the two lines ***R1***

[4 marks]

(c) $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 26 = \sqrt{29} \times \sqrt{29} \cos\theta$ ***M1***

$\cos\theta = \frac{26}{29}$ ***(A1)***

$\theta = 0.459$ or 26.3° ***A1***

[3 marks]

continued …

Question 11 continued

(d) $\vec{RP} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}, |\vec{RP}| = \sqrt{29}$ ***(M1)A1***

Note: This could also be obtained from $\left\| 0.5\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix} \right\|$

EITHER

$\vec{RS_1} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}, \ |\vec{RS_1}| = \sqrt{29}$ ***A1***

$\therefore \vec{OS_2} = \vec{OS_1} + 2\vec{S_1R} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$ ***M1A1***

$\left(\text{or } \vec{OS_2} = \vec{OR} + \vec{S_1R} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \right)$

$= \begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix}$

S_2 is $(6, 4, 4)$ ***A1***

OR

$\begin{pmatrix} -4+5s \\ 2s \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5s-5 \\ 2s-2 \\ 0 \end{pmatrix}$ ***M1***

$(5s-5)^2 + (2s-2)^2 = 29$ ***M1A1***

$29s^2 - 58s + 29 = 29$

$s(s-2) = 0, \ s = 0, 2$

$(6, 4, 4)$ (and $(-4, 0, 4)$) ***A1***

Note: There are several geometrical arguments possible using information obtained in previous parts, depending on what forms the previous answers had been given.

[6 marks]

(e) **EITHER**

midpoint of $[PS_1]$ is $M(-3.5, -0.5, 3)$ ***M1A1***

$$\vec{RM} = \begin{pmatrix} -4.5 \\ -2.5 \\ -1 \end{pmatrix}$$ ***A1***

OR

$$\vec{RS_1} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}$$ ***M1***

the direction of the line is $\vec{RS_1} + \vec{RP}$

$$\begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ -5 \\ -2 \end{pmatrix}$$ ***M1A1***

THEN

the equation of the line is:

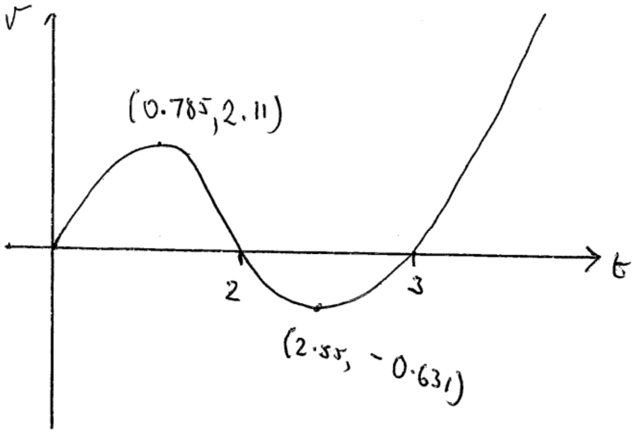
$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note:	Marks cannot be awarded for methods involving halving the angle, unless it is clear that the candidate considers also the equation of the plane of L_1 and L_2 to reduce the number of parameters involved to one (to obtain the vector equation of the required line).

[4 marks]

Total [20 marks]

12. (a)



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maximum and minimum, ***A1*** for intercepts.

Note: Follow through applies to (b) and (c).

[3 marks]

(b) $0 \le t < 0.785, \left(\text{or } 0 \le t < \dfrac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

and $t > 2.55 \left(\text{or } t > \dfrac{5+\sqrt{7}}{3}\right)$ ***A1***

[2 marks]

(c) $0 \le t < 0.785, \left(\text{or } 0 \le t < \dfrac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

$2 < t < 2.55, \left(\text{or } 2 < t < \dfrac{5+\sqrt{7}}{3}\right)$ ***A1***

$t > 3$ ***A1***

[3 marks]

(d) position of A: $x_A = \int t^3 - 5t^2 + 6t \, \mathrm{d}t$ ***(M1)***

$x_A = \dfrac{1}{4}t^4 - \dfrac{5}{3}t^3 + 3t^2 \quad (+c)$ ***A1***

when $t = 0$, $x_A = 0$ so $c = 0$ ***R1***

[3 marks]

continued …

Question 12 continued

(e) $\frac{dv_B}{dt} = -2v_B \Rightarrow \int \frac{1}{v_B} dv_B = \int -2dt$ ***(M1)***

$\ln|v_B| = -2t + c$ ***(A1)***

$v_B = Ae^{-2t}$ ***(M1)***

$v_B = -20$ when $t = 0$ so $v_B = -20e^{-2t}$ ***A1***

[4 marks]

(f) $x_B = 10e^{-2t}(+c)$ ***(M1)(A1)***

$x_B = 20$ when $t = 0$ so $x_B = 10e^{-2t} + 10$ ***(M1)A1***

meet when $\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 = 10e^{-2t} + 10$ ***(M1)***

$t = 4.41(290\ldots)$ ***A1***

[6 marks]

Total: [21 marks]

13. (a) $f(2) = 9$ ***(A1)***

$f^{-1}(x) = (x-1)^{\frac{1}{3}}$ ***A1***

$(f^{-1})'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$ ***(M1)***

$(f^{-1})'(9) = \frac{1}{12}$ ***A1***

$f'(x) = 3x^2$ ***(M1)***

$\frac{1}{f'(2)} = \frac{1}{3\times 4} = \frac{1}{12}$ ***A1***

Note: The last ***M1*** and ***A1*** are independent of previous marks.

[6 marks]

(b) $g'(x) = e^{x^2} + 2x^2e^{x^2}$ ***M1A1***

$g'(x) > 0$ as each part is positive ***R1***

[3 marks]

continued …

Question 13 continued

(c) to find the x-coordinate on $y = g(x)$ solve

$2 = xe^{x^2}$ ***(M1)***

$x = 0.89605022078\ldots$ ***(A1)***

gradient $= (g^{-1})'(2) = \dfrac{1}{g'(0.896\ldots)}$ ***(M1)***

$= \dfrac{1}{e^{(0.896\ldots)^2}\left(1 + 2 \times (0.896\ldots)^2\right)} = 0.172$ to 3sf ***A1***

(using the $\dfrac{dy}{dx}$ function on gdc $g'(0.896\ldots) = 5.7716028\ldots$

$\dfrac{1}{g'(0.896\ldots)} = 0.173$)

[4 marks]

(d) (i) $(x^3 + 1)e^{(x^3+1)^2} = 2$ ***A1***

$x = -0.470191\ldots$ ***A1***

(ii) **METHOD 1**

$(g \circ f)'(x) = 3x^2 e^{(x^3+1)^2}\left(2(x^3 + 1)^2 + 1\right)$ ***(M1)(A1)***

$(g \circ f)'(-0.470191\ldots) = 3.85755\ldots$ ***(A1)***

$h'(2) = \dfrac{1}{3.85755\ldots} = 0.259\ (232\ldots)$ ***A1***

Note: The solution can be found without the student obtaining the explicit form of the composite function.

METHOD 2

$h(x) = (f^{-1} \circ g^{-1})(x)$ ***A1***

$h'(x) = (f^{-1})'\left(g^{-1}(x)\right) \times (g^{-1})'(x)$ ***M1***

$= \dfrac{1}{3}\left(g^{-1}(x) - 1\right)^{-\frac{2}{3}} \times (g^{-1})'(x)$ ***M1***

$h'(2) = \dfrac{1}{3}\left(g^{-1}(2) - 1\right)^{-\frac{2}{3}} \times (g^{-1})'(2)$

$= \dfrac{1}{3}(0.89605\ldots - 1)^{-\frac{2}{3}} \times 0.171933\ldots$

$= 0.259\ (232\ldots)$ ***A1*** ***N4***

[6 marks]

Total [19 marks]

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MARKSCHEME

May 2013

MATHEMATICS

Higher Level

Paper 2

Instructions to Examiners

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R Marks awarded for clear **Reasoning**.

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AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document **"Mathematics HL: Guidance for e-marking May 2013"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. (a) **EITHER**

$\text{A}\hat{\text{O}}\text{B} = 2\arcsin\left(\frac{3}{4}\right)$ or equivalent (*eg* $\text{A}\hat{\text{O}}\text{B} = 2\arctan\left(\frac{3}{\sqrt{7}}\right)$, $\text{A}\hat{\text{O}}\text{B} = 2\arccos\left(\frac{\sqrt{7}}{4}\right)$)***(M1)***

OR

$\cos \text{A}\hat{\text{O}}\text{B} = \frac{4^2 + 4^2 - 6^2}{2 \times 4 \times 4}$ $\left(= -\frac{1}{8}\right)$ ***(M1)***

THEN

$= 1.696$ (correct to 4sf) ***A1***

[2 marks]

(b) use of area of segment $=$ area of sector – area of triangle ***(M1)***

$= \frac{1}{2} \times 4^2 \times 1.696 - \frac{1}{2} \times 4^2 \times \sin 1.696$ ***(A1)***

$= 5.63\ (\text{cm}^2)$ ***A1***

[3 marks]

Total [5 marks]

2. (a) attempting to express the system in matrix form ***M1***

$$\begin{pmatrix} 0.1 & -1.7 & 0.9 \\ -2.4 & 0.3 & 3.2 \\ 2.5 & 0.6 & -3.7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4.4 \\ 1.2 \\ 0.8 \end{pmatrix}$$ ***A1***

Note: Award ***M1A1*** for a correct augmented matrix.

[2 marks]

(b) either direct GDC use, attempting elimination or using an inverse matrix. ***(M1)***

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2.4 \\ 1.6 \\ -1.6 \end{pmatrix}$$ (correct to 2sf) or $$\begin{pmatrix} -2.40 \\ 1.61 \\ -1.57 \end{pmatrix}$$ (correct to 3sf) or $$\begin{pmatrix} -\frac{932}{389} \\ \frac{628}{389} \\ -\frac{612}{389} \end{pmatrix}$$ (exact) ***A2***

[3 marks]

Total [5 marks]

3. (a) $X \sim N(13.5, 9.5)$

$13.5 - \sqrt{9.5} < X < 13.5 + \sqrt{9.5}$ ***(M1)***

$10.4 < X < 16.6$ ***A1***

Note: Accept 6.16.

[2 marks]

(b) $\mathrm{P}(X < 10) = 0.12807\ldots$ ***(M1)(A1)***

estimate is 1281 (correct to the nearest whole number). ***A1***

Note: Accept 1280.

[3 marks]

Total [5 marks]

4. (a) $\int x\sec^2 x\,\mathrm{d}x = x\tan x - \int 1 \times \tan x\,\mathrm{d}x$ ***M1A1***

$= x\tan x + \ln\left|\cos x\right|\ (+c)\ \left(= x\tan x - \ln\left|\sec x\right|(+c)\right)$ ***M1A1***

[4 marks]

(b) attempting to solve an appropriate equation *eg* $m\tan m + \ln(\cos m) = 0.5$ ***(M1)***

$m = 0.822$ ***A1***

Note: Award ***A1*** if $m = 0.822$ is specified with other positive solutions.

[2 marks]

Total [6 marks]

5. (a) $u_n - v_n = 1.6 + (n-1)\times 1.5 - 3\times 1.2^{n-1}$ $(=1.5n + 0.1 - 3\times 1.2^{n-1})$ ***A1A1***

[2 marks]

(b) attempting to solve $u_n > v_n$ numerically or graphically. ***(M1)***

$n = 2.621\ldots, 9.695\ldots$ ***(A1)***

So $3 \le n \le 9$ ***A1***

[3 marks]

(c) The greatest value of $u_n - v_n$ is 1.642. ***A1***

Note: Do not accept 1.64.

[1 mark]

Total [6 marks]

6. (a) attempting to solve for $\cos x$ or for u where $u = \cos x$ or for x graphically. ***(M1)***

EITHER

$\cos x = \frac{2}{3}$ (and 2) ***(A1)***

OR

$x = 48.1897\ldots^\circ$ ***(A1)***

THEN

$x = 48^\circ$ ***A1***

Note: Award ***(M1)(A1)A0*** for $x = 48^\circ, 132^\circ$.

Note: Award ***(M1)(A1)A0*** for 0.841 radians.

[3 marks]

(b) attempting to solve for $\sec x$ or for v where $v = \sec x$. ***(M1)***

$\sec x = \pm\sqrt{2}\left(\text{and } \pm\sqrt{\frac{2}{3}}\right)$ ***(A1)***

$\sec x = \pm\sqrt{2}$ ***A1***

[3 marks]

Total [6 marks]

7. (a) $\int_0^{0.5} ax^2 \mathrm{d}x + \int_{0.5}^{1} 0.5a(1-x)\mathrm{d}x = 1$ ***M1A1***

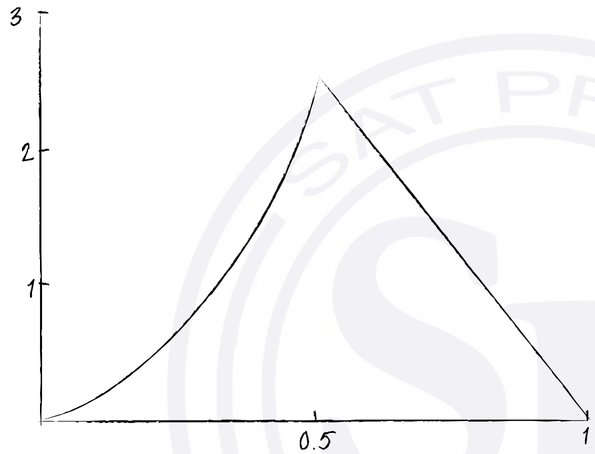
$\frac{5a}{48}$ (or equivalent) or $a \times 0.104... = 1$ ***A1***

Note:	Award ***M1*** for considering two definite integrals. Award ***A1*** for equating to 1. Award ***A1*** for a correct equation. The ***A1A1*** can be awarded in any order.

$a = 9.6$ ***AG***

[3 marks]

(b)



correct shape for $0 \le x \le 0.5$ and $f(0.5) \approx 2.4$ ***A1***

correct shape for $0.5 \le x \le 1$ and $f(1) = 0$ ***A1***

[2 marks]

(c) attempting to find $\mathrm{P}(X < 0.6)$ ***(M1)***

direct GDC use or *eg* $\mathrm{P}(0 \le X \le 0.5) + \mathrm{P}(0.5 \le X \le 0.6)$ or $1 - \mathrm{P}(0.6 \le X \le 1)$

$\mathrm{P}(X < 0.6) = 0.616 \left(= \frac{77}{125}\right)$ ***A1***

[2 marks]

Total [7 marks]

8. $\mathrm{P}(n): f(n) = 5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$

for $n = 1$, $f(1) = 5^2 - 24 - 1 = 0$

Zero is divisible by 576, (as every non-zero number divides zero), and so $\mathrm{P}(1)$ is true. ***R1***

Note: Award ***R0*** for $\mathrm{P}(1) = 0$ shown and zero is divisible by 576 not specified.

Note: Ignore $\mathrm{P}(2) = 576$ if $\mathrm{P}(1) = 0$ is shown and zero is divisible by 576 is specified.

Assume $\mathrm{P}(k)$ is true for some $k (\Rightarrow f(k) = N \times 576)$. ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1): f(k+1) = 5^{2(k+1)} - 24(k+1) - 1$ ***M1***

$= 25 \times 5^{2k} - 24k - 25$ ***A1***

EITHER

$= 25 \times (24k + 1 + N \times 576) - 24k - 25$ ***A1***

$= 576k + 25 \times 576N$ which is a multiple of 576 ***A1***

OR

$= 25 \times 5^{2k} - 600k - 25 + 600k - 24k$ ***A1***

$= 25\left(5^{2k} - 24k - 1\right) + 576k$ (or equivalent) which is a multiple of 576 ***A1***

THEN

$\mathrm{P}(1)$ is true and $\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k+1)$ true, so $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award ***R1*** only if at least four prior marks have been awarded.

Total [7 marks]

9. (a) $X \sim \text{Po}(1.2)$

$\text{P}(X=3)\times\text{P}(X=0)$ ***(M1)***

$=0.0867\ldots\times 0.3011\ldots$

$=0.0261$ ***A1***

[2 marks]

(b) Three requests over two days can occur as $(3, 0), (0, 3), (2, 1)$ or $(1, 2)$. ***R1***

using conditional probability, for example

$$\frac{\text{P}(3,0)}{\text{P}(3 \text{ requests}, m = 2.4)} = 0.125 \text{ or } \frac{\text{P}(2,1)}{\text{P}(3 \text{ requests}, m = 2.4)} = 0.375$$ ***M1A1***

expected income is

$2\times 0.125\times \text{US\$}120 + 2\times 0.375\times \text{US\$}180$ ***M1***

Note:	Award ***M1*** for attempting to find the expected income including both $(3, 0)$ and $(2, 1)$ cases.

$= \text{US\$}30 + \text{US\$}135$

$= \text{US\$}165$ ***A1***

[5 marks]

Total [7 marks]

10. **METHOD 1**

$\frac{dv}{dt} = \frac{1}{40}(60 - v)$ ***(M1)***

attempting to separate variables $\int \frac{dv}{60 - v} = \int \frac{dt}{40}$ ***M1***

$-\ln(60 - v) = \frac{t}{40} + c$ ***A1***

$c = -\ln 60$ (or equivalent) ***A1***

attempting to solve for v when $t = 30$ ***(M1)***

$v = 60 - 60e^{-\frac{3}{4}}$

$v = 31.7\ (\text{m s}^{-1})$ ***A1***

METHOD 2

$\frac{dv}{dt} = \frac{1}{40}(60 - v)$ ***(M1)***

$\frac{dt}{dv} = \frac{40}{60 - v}$ (or equivalent) ***M1***

$\int_0^{v_f} \frac{40}{60 - v}\, dv = 30$ where v_f is the velocity of the car after 30 seconds. ***A1A1***

attempting to solve $\int_0^{v_f} \frac{40}{60 - v}\, dv = 30$ for v_f ***(M1)***

$v = 31.7\ (\text{m s}^{-1})$ ***A1***

Total [6 marks]

SECTION B

11. (a) (i) $\sum_{k=1}^{n}(2k-1)$ (or equivalent) ***A1***

> **Note:** Award ***A0*** for $\sum_{n=1}^{n}(2n-1)$ or equivalent.

(ii) **EITHER**

$2\times\frac{n(n+1)}{2}-n$ ***M1A1***

OR

$\frac{n}{2}\left(2+(n-1)2\right)$ (using $S_n=\frac{n}{2}\left(2u_1+(n-1)d\right)$) ***M1A1***

OR

$\frac{n}{2}(1+2n-1)$ (using $S_n=\frac{n}{2}(u_1+u_n)$) ***M1A1***

THEN

$=n^2$ ***AG***

(iii) $47^2-14^2=2013$ ***A1***

[4 marks]

(b) (i) **EITHER**

a pentagon and five diagonals ***A1***

OR

five diagonals (circle optional) ***A1***

(ii) Each point joins to $n-3$ other points. ***A1***
a correct argument for $n(n-3)$ ***R1***
a correct argument for $\frac{n(n-3)}{2}$ ***R1***

(iii) attempting to solve $\frac{1}{2}n(n-3)>1000000$ for *n*. ***(M1)***
$n>1415.7$ ***(A1)***
$n=1416$ ***A1***

[7 marks]

continued …

Question 11 continued

(c) (i) $np = 4$ and $npq = 3$ ***(A1)***
attempting to solve for n and p ***(M1)***
$n = 16$ and $p = \frac{1}{4}$ ***A1***

(ii) $X \sim \mathrm{B}(16, 0.25)$ ***(A1)***
$\mathrm{P}(X = 1) = 0.0534538\ldots \left(= \binom{16}{1}(0.25)(0.75)^{15}\right)$ ***(A1)***
$\mathrm{P}(X = 3) = 0.207876\ldots \left(= \binom{16}{3}(0.25)^3(0.75)^{13}\right)$ ***(A1)***
$\mathrm{P}(X = 1) + \mathrm{P}(X = 3)$ ***(M1)***
$= 0.261$ ***A1***

[8 marks]

Total [19 marks]

12. (a) (i) **METHOD 1**

$\frac{dy}{dx} = -\sin x + \cos x$ ***A1***

$y\frac{dy}{dx} = (\cos x + \sin x)(-\sin x + \cos x)$ ***M1***

$= \cos^2 x - \sin^2 x$ ***A1***

$= \cos 2x$ ***AG***

METHOD 2

$y^2 = (\sin x + \cos x)^2$ ***A1***

$2y\frac{dy}{dx} = 2(\cos x + \sin x)(\cos x - \sin x)$ ***M1***

$y\frac{dy}{dx} = \cos^2 x - \sin^2 x$ ***A1***

$= \cos 2x$ ***AG***

(ii) attempting to separate variables $\int y\,dy = \int \cos 2x\,dx$ ***M1***

$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$ ***A1A1***

Note: Award ***A1*** for a correct LHS and ***A1*** for a correct RHS.

$y = \pm(\sin 2x + A)^{\frac{1}{2}}$ ***A1***

(iii) $\sin 2x + A \equiv (\cos x + \sin x)^2$ ***(M1)***

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x$

use of $\sin 2x \equiv 2\sin x\cos x$. ***(M1)***

$A = 1$ ***A1***

[10 marks]

continued …

Question 12 continued

(b) (i) substituting $x = \frac{\pi}{4}$ and $y = 2$ into $y = (\sin 2x + A)^{\frac{1}{2}}$ ***M1***

so $g(x) = (\sin 2x + 3)^{\frac{1}{2}}$. ***A1***

range g is $\left[\sqrt{2}, 2\right]$ ***A1A1A1***

Note:	Accept $[1.41, 2]$. Award ***A1*** for each correct endpoint and ***A1*** for the correct closed interval.

(ii) $\int_0^{\frac{\pi}{2}} (\sin 2x + 3)^{\frac{1}{2}}\,dx$ ***(M1)(A1)***

$= 2.99$ ***A1***

(iii) $\pi\int_0^{\frac{\pi}{2}} (\sin 2x + 3)\,dx - \pi(1)\left(\frac{\pi}{2}\right)$ (or equivalent) ***(M1)(A1)(A1)***

Note:	Award ***(M1)(A1)(A1)*** for $\pi\int_0^{\frac{\pi}{2}} (\sin 2x + 2)\,dx$

$= 17.946 - 4.935$ $\left(= \frac{\pi}{2}(3\pi + 2) - \pi\left(\frac{\pi}{2}\right)\right)$

$= 13.0$ ***A1***

Note:	Award ***A1*** for $\pi(\pi + 1)$.

[12 marks]

Total [22 marks]

13. (a) **EITHER**

$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent) ***M1A1***

> **Note:** Accept $\theta = 180° - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent).

OR

$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$ (or equivalent) ***M1A1***

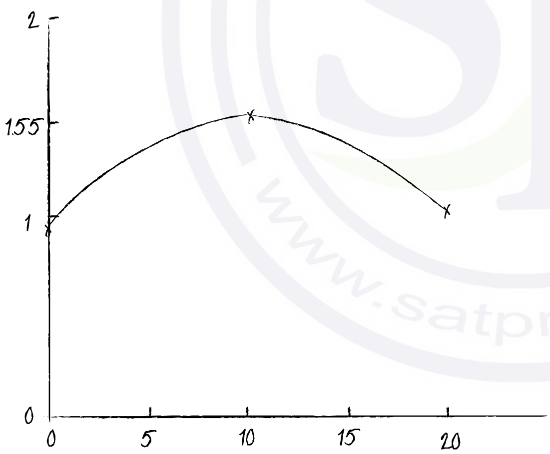
[2 marks]

(b) (i) $\theta = 0.994 \ (= \arctan\frac{20}{13})$ ***A1***

(ii) $\theta = 1.19 \ (= \arctan\frac{5}{2})$ ***A1***

[2 marks]

(c) correct shape. ***A1***
correct domain indicated. ***A1***



[2 marks]

continued …

Question 13 continued

(d) attempting to differentiate one $\arctan\left(f(x)\right)$ term **M1**

EITHER

$$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$$

$$\frac{d\theta}{dx} = \frac{8}{x^2} \times \frac{1}{1+\left(\frac{8}{x}\right)^2} - \frac{13}{(20-x)^2} \times \frac{1}{1+\left(\frac{13}{20-x}\right)^2}$$ **A1A1**

OR

$$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$$

$$\frac{d\theta}{dx} = \frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2} + \frac{-\frac{1}{13}}{1+\left(\frac{20-x}{13}\right)^2}$$ **A1A1**

THEN

$$= \frac{8}{x^2+64} - \frac{13}{569-40x+x^2}$$ **A1**

$$= \frac{8(569-40x+x^2) - 13(x^2+64)}{(x^2+64)(x^2-40x+569)}$$ **M1A1**

$$= \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$$ **AG**

[6 marks]

(e) Maximum light intensity at P occurs when $\frac{d\theta}{dx} = 0$. ***(M1)***

either attempting to solve $\frac{d\theta}{dx} = 0$ for x or using the graph of either θ or $\frac{d\theta}{dx}$ ***(M1)***

$x = 10.05$ (m) **A1**

[3 marks]

continued …

Question 13 continued

(f) $\frac{dx}{dt} = 0.5$ ***(A1)***

At $x = 10$, $\frac{d\theta}{dx} = 0.000453$ $\left(= \frac{5}{11\,029}\right)$. ***(A1)***

use of $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$ ***M1***

$\frac{d\theta}{dt} = 0.000227$ $\left(= \frac{5}{22058}\right)$ (rad s^{-1}) ***A1***

Note: Award ***(A1)*** for $\frac{dx}{dt} = -0.5$ and ***A1*** for $\frac{d\theta}{dt} = -0.000227$ $\left(= -\frac{5}{22058}\right)$.

Note: Implicit differentiation can be used to find $\frac{d\theta}{dt}$. Award as above.

[4 marks]

Total [19 marks]

N12/5/MATHL/HP2/ENG/TZ0/XX/M

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

November 2012

MATHEMATICS

Higher Level

Paper 2

19 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2012**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

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- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. **METHOD 1**

$102+105+\ldots+498$ ***(M1)***
so number of terms $=133$ ***(A1)***

EITHER

$=\frac{133}{2}(2\times102+132\times3)$ ***(M1)***

$=39900$ ***A1***

OR

$=(102+498)\times\frac{133}{2}$ ***(M1)***

$=39900$ ***A1***

OR

$\sum_{n=34}^{166}3n$ ***(M1)***

$=39900$ ***A1***

METHOD 2

$500\div3=166.666...$ and $100\div3=33.333...$

$102+105+\ldots+498=\sum_{n=1}^{166}3n-\sum_{n=1}^{33}3n$ ***(M1)***

$\sum_{n=1}^{166}3n=41583$ ***(A1)***

$\sum_{n=1}^{33}3n=1683$ ***(A1)***

the sum is 39 900 ***A1***

[4 marks]

2. $\Delta = (5-k)^2 + 4(k+2)$ *M1A1*

$= k^2 - 6k + 33$ *(A1)*

$= (k-3)^2 + 24$ which is positive for all k *R1*

Note: Accept analytical, graphical or other correct methods. In all cases only award ***R1*** if a reason is given in words or graphically. Award ***M1A1A0R1*** if mistakes are made in the simplification but the argument given is correct.

[4 marks]

3. $\det \boldsymbol{A} = 3\ln x - 2\ln(5-x)$ *(M1)(A1)*

$\boldsymbol{A}$ singular $\Rightarrow \det \boldsymbol{A} = 0$ *(M1)*

attempt to solve $3\ln x - 2\ln(5-x) = 0$ (*eg* graph sketch) *(M1)*

$x = 2.0547...$ *A1*

$x = 2.05$ (3sf)

Note: Award the last ***M1*** just in the cases where there is evidence that a correct method has been attempted.

[5 marks]

4. $\dfrac{\sum_{i=1}^{15} x_i}{15} = 11.5 \Rightarrow \sum_{i=1}^{15} x_i = 172.5$ *(A1)*

new mean $= \dfrac{172.5 - 22.1}{14}$ *(M1)*

$= 10.7428\ldots = 10.7$ (3sf) *A1*

$\dfrac{\sum_{i=1}^{15} x_i^2}{15} - 11.5^2 = 9.3$ *(M1)*

$\Rightarrow \sum_{i=1}^{15} x_i^2 = 2123.25$

new variance $= \dfrac{2123.25 - 22.1^2}{14} - (10.7428\ldots)^2$ *(M1)*

$= 1.37$ (3sf) *A1*

[6 marks]

5. the pieces have lengths $a, ar, \ldots, ar^9$ ***(M1)***

$8a = ar^9$ (or $8 = r^9$) ***A1***

$r = \sqrt[9]{8} = 1.259922\ldots$ ***A1***

$a\dfrac{r^{10}-1}{r-1} = 1 \quad \left(\text{or } a\dfrac{r^{10}-1}{r-1} = 1000\right)$ ***M1***

$a = \dfrac{r-1}{r^{10}-1} = 0.0286\ldots \quad \left(\text{or } a = \dfrac{r-1}{r^{10}-1} = 28.6\ldots\right)$ ***(A1)***

$a = 29\,\text{mm}$ (accept 0.029 m or any correct answer regardless the units) ***A1***

[6 marks]

6. $2s\dfrac{ds}{dt} + \dfrac{ds}{dt} - 2 = 0$ ***M1A1***

$v = \dfrac{ds}{dt} = \dfrac{2}{2s+1}$ ***A1***

EITHER

$a = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt}$ ***(M1)***

$\dfrac{dv}{ds} = \dfrac{-4}{(2s+1)^2}$ ***(A1)***

$a = \dfrac{-4}{(2s+1)^2}\dfrac{ds}{dt}$

OR

$2\left(\dfrac{ds}{dt}\right)^2 + 2s\dfrac{d^2s}{dt^2} + \dfrac{d^2s}{dt^2} = 0$ ***(M1)***

$\underbrace{\dfrac{d^2s}{dt^2}}_{a} = \dfrac{-2\left(\dfrac{ds}{dt}\right)^2}{2s+1}$ ***(A1)***

THEN

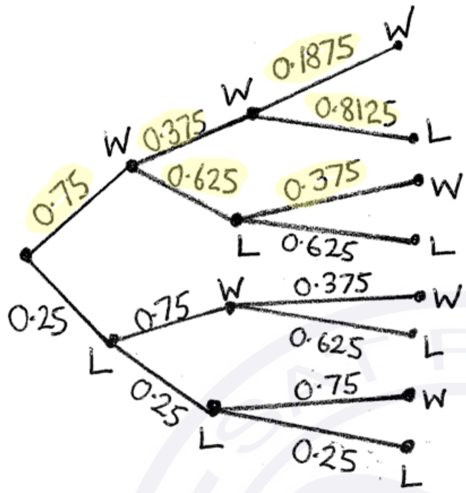
$a = \dfrac{-8}{(2s+1)^3}$ ***A1***

[6 marks]

7. (a) $\text{P}(\text{WWW}) = 0.75 \times 0.375 \times 0.1875 = 0.0527 \text{ (3sf)} \left(\frac{3}{4} \times \frac{3}{8} \times \frac{3}{16} = \frac{27}{512}\right)$ ***(M1)A1***

[2 marks]

(b)



(M1)(A1)

Note:	Award ***M1*** for any reasonable attempt to use a tree diagram showing that three games were played (do not award ***M1*** for tree diagrams that only show the first two games) and ***A1*** for the highlighted probabilities.

$\text{P}(\text{wins 2 games} \mid \text{wins first game}) = \frac{\text{P}(\text{WWL, WLW})}{\text{P}(\text{wins first game})}$ ***(M1)***

$= \frac{0.75 \times 0.375 \times 0.8125 + 0.75 \times 0.625 \times 0.375}{0.75}$ ***(A1)(A1)***

$= 0.539 \text{ (3sf)} \left(\text{or } \frac{69}{128}\right)$ ***A1***

Note:	Candidates may use the tree diagram to obtain the answer without using the conditional probability formula, *ie*, $\text{P}(\text{wins 2 games} \mid \text{wins first game}) = 0.375 \times 0.8125 + 0.625 \times 0.375 = 0.539$.

[6 marks]

Total [8 marks]

8. $x = \sin t$, $dx = \cos t \, dt$

$\int \frac{x^3}{\sqrt{1-x^2}} dx = \int \frac{\sin^3 t}{\sqrt{1-\sin^2 t}} \cos t \, dt$ ***M1***

$= \int \sin^3 t \, dt$ ***(A1)***

$= \int \sin^2 t \, \sin t \, dt$

$= \int (1 - \cos^2 t) \sin t \, dt$ ***M1A1***

$= \int \sin t \, dt - \int \cos^2 t \sin t \, dt$

$= -\cos t + \frac{\cos^3 t}{3} + C$ ***A1A1***

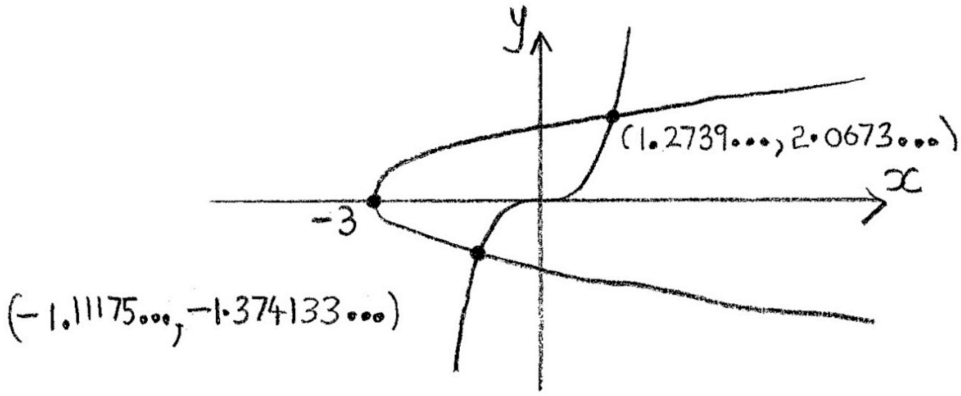
$= -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$ ***A1***

$\left(= -\sqrt{1-x^2}\left(1 - \frac{1}{3}(1-x^2)\right) + C \right)$

$\left(= -\frac{1}{3}\sqrt{1-x^2}\,(2+x^2) + C \right)$

[7 marks]

9.



intersection points *A1A1*

Note: Only either the x-coordinate or the y-coordinate is needed.

EITHER

$x = y^2 - 3 \Rightarrow y = \pm\sqrt{x+3}$ $\left(\text{accept } y = \sqrt{x+3}\right)$ *(M1)*

$A = \int_{-3}^{-1.111...} 2\sqrt{x+3}\,dx + \int_{-1.111...}^{1.2739...} \sqrt{x+3} - x^3\,dx$ *(M1)A1A1*

$= 3.4595... + 3.8841...$

$= 7.34$ (3sf) *A1*

OR

$y = x^3 \Rightarrow x = \sqrt[3]{y}$ *(M1)*

$A = \int_{-1.374...}^{2.067...} \sqrt[3]{y} - \left(y^2 - 3\right)dy$ *(M1)A1*

$= 7.34$ (3sf) *A2*

[7 marks]

10. **METHOD 1**

$(1-\omega^2)^* = (1-\text{cis } 2\theta)^* = \left((1-\cos 2\theta) - \text{i}\sin 2\theta\right)^*$ *M1A1*

$= (1-\cos 2\theta) + \text{i}\sin 2\theta$ *A1*

$\left|(1-\omega^2)^*\right| = \sqrt{(1-\cos 2\theta)^2 + \sin^2 2\theta} \left(= \sqrt{(2\sin^2\theta)^2 + \left(2\sin\theta\cos\theta\right)^2}\right)$ *M1*

$= |2\sin\theta|$ *A1*

$\arg\left((1-\omega^2)^*\right) = \alpha \Rightarrow \tan\alpha = \cot(\theta)$ *M1*

$\alpha = \frac{\pi}{2} - \theta$ *A1*

therefore:

modulus is $2|\sin\theta|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$

METHOD 2

EITHER

$(1-\omega^2)^* = (1-\text{cis } 2\theta)^* = \left((1-\cos 2\theta) - \text{i}\sin 2\theta\right)^*$ *M1A1*

$= (1-\cos 2\theta) + \text{i}\sin 2\theta$ *A1*

$= \left(1-1+2\sin^2\theta\right) + 2\text{i}\sin\theta\cos\theta$ *M1*

OR

$(1-\omega^2)^* = \left(1-(\cos\theta + i\sin\theta)^2\right)^*$ *M1A1*

$= \left(1-\cos^2\theta + \sin^2\theta - 2\text{i}\sin\theta\cos\theta\right)^*$ *A1*

$= 2\sin^2\theta + 2\text{i}\sin\theta\cos\theta$ *M1*

THEN

$= 2\sin\theta(\sin\theta + \text{i}\cos\theta)$ *(M1)*

$= 2\sin\theta\left(\cos\left(\frac{\pi}{2} - \theta\right) + \text{i}\sin\left(\frac{\pi}{2} - \theta\right)\right)$ *A1A1*

$= 2\sin\theta\,\text{cis}\left(\frac{\pi}{2} - \theta\right)$

therefore:

modulus is $2|\sin\theta|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$.

[7 marks]

SECTION B

11. (a) $2.2 \times 6 \times 60 = 792$ *(M1)A1*

[2 marks]

(b) $V \sim \text{Po}(2.2 \times 60)$ *(M1)*

$\text{P}(V > 100) = 0.998$ *(M1)A1*

[3 marks]

(c) $(0.997801\ldots)^6 = 0.987$ *(M1)A1*

[2 marks]

(d) $A \sim \text{N}(\mu, \sigma^2)$

$\text{P}(A < 35) = 0.29$ and $\text{P}(A > 55) = 0.23 \Rightarrow \text{P}(A < 55) = 0.77$

$\text{P}\left(Z < \frac{35-\mu}{\sigma}\right) = 0.29$ and $\text{P}\left(Z < \frac{55-\mu}{\sigma}\right) = 0.77$ *(M1)*

use of inverse normal *(M1)*

$\frac{35-\mu}{\sigma} = -0.55338\ldots$ and $\frac{55-\mu}{\sigma} = 0.738846\ldots$ *(A1)*

solving simultaneously *(M1)*

$\mu = 43.564...$ and $\sigma = 15.477...$ *A1A1*

$\mu = 43.6$ and $\sigma = 15.5\,(3\text{sf})$

[6 marks]

(e) $0.29n = 100 \Rightarrow n = 344.82\ldots$ *(M1)(A1)*

$\text{P}(A < 50) = 0.66121\ldots$ *(A1)*

expected number of visitors under $50 = 228$ *(M1)A1*

[5 marks]

Total [18 marks]

12. (a) $L = \mathrm{CA} + \mathrm{AD}$ ***M1***

$\sin\alpha = \dfrac{a}{\mathrm{CA}} \Rightarrow \mathrm{CA} = \dfrac{a}{\sin\alpha}$ ***A1***

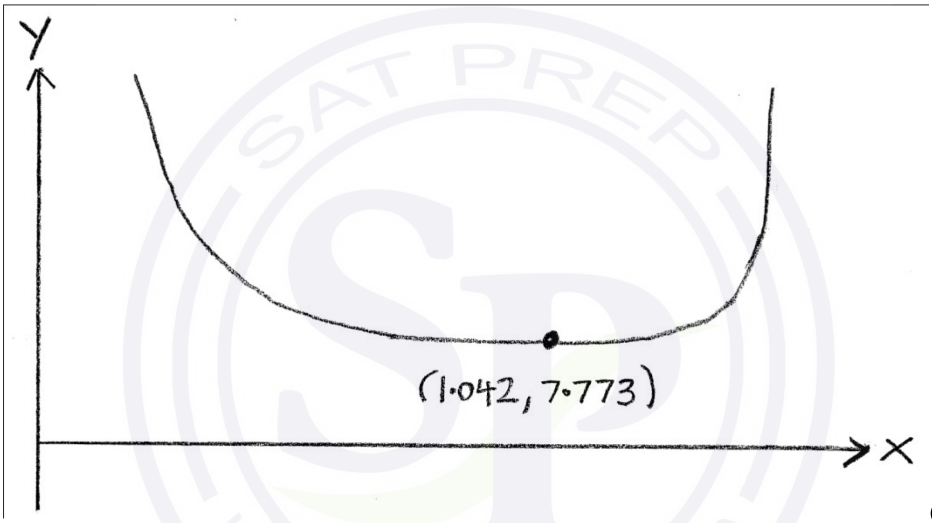
$\cos\alpha = \dfrac{b}{\mathrm{AD}} \Rightarrow \mathrm{AD} = \dfrac{b}{\cos\alpha}$ ***A1***

$L = \dfrac{a}{\sin\alpha} + \dfrac{b}{\cos\alpha}$ ***AG***

[3 marks]

(b) $a = 5$ and $b = 1 \Rightarrow L = \dfrac{5}{\sin\alpha} + \dfrac{1}{\cos\alpha}$

METHOD 1



(M1)

minimum from graph $\Rightarrow$ $L = 7.77$ ***(M1)A1***

minimum of L gives the max length of the painting ***R1***

[4 marks]

METHOD 2

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{-5\cos\alpha}{\sin^2\alpha} + \dfrac{\sin\alpha}{\cos^2\alpha}$ ***(M1)***

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \dfrac{\sin^3\alpha}{\cos^3\alpha} = 5 \Rightarrow \tan\alpha = \sqrt[3]{5}$ $(\alpha = 1.0416\ldots)$ ***(M1)***

minimum of L gives the max length of the painting ***R1***

maximum length $= 7.77$ ***A1***

[4 marks]

(c) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{-3k\cos\alpha}{\sin^2\alpha} + \dfrac{k\sin\alpha}{\cos^2\alpha}$ (or equivalent) ***M1A1A1***

[3 marks]

continued …

Question 12 continued

(d) $\frac{dL}{d\alpha} = \frac{-3k\cos^3\alpha + k\sin^3\alpha}{\sin^2\alpha\cos^2\alpha}$ ***(A1)***

$\frac{dL}{d\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = \frac{3k}{k} \Rightarrow \tan\alpha = \sqrt[3]{3} \quad (\alpha = 0.96454\ldots)$ ***M1A1***

$\tan\alpha = \sqrt[3]{3} \Rightarrow \frac{1}{\cos\alpha} = \sqrt{1+\sqrt[3]{9}} \qquad (1.755\ldots)$ ***(A1)***

and $\frac{1}{\sin\alpha} = \frac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}} \qquad (1.216\ldots)$ ***(A1)***

$L = 3k\left(\frac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}}\right) + k\sqrt{1+\sqrt[3]{9}} \qquad (L = 5.405598\ldots k)$ ***A1*** ***N4***

[6 marks]

(e) $L \le 8 \Rightarrow k \ge 1.48$ ***M1A1***

the minimum value is 1.48

[2 marks]

Total [18 marks]

13. **Note:** Accept alternative notation for vectors (*eg* $\langle a, b, c\rangle$ or (a, b, c)).

(a) $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ and $\boldsymbol{m} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ ***(A1)***

$\cos\theta = \dfrac{\boldsymbol{n}\bullet\boldsymbol{m}}{|\boldsymbol{n}||\boldsymbol{m}|}$ ***(M1)***

$\cos\theta = \dfrac{2+2+3}{\sqrt{1+4+9}\sqrt{4+1+1}} = \dfrac{7}{\sqrt{14}\sqrt{6}}$ ***A1***

$\theta = 40.2^{\circ}$ (0.702 rad) ***A1***

[4 marks]

(b) **METHOD 1**

eliminate *z* from $x-2y-3z=2$ and $2x-y-z=k$

$5x - y = 3k-2 \Rightarrow x = \dfrac{y-(2-3k)}{5}$ ***M1A1***

eliminate *y* from $x-2y-3z=2$ and $2x-y-z=k$

$3x+z=2k-2 \Rightarrow x = \dfrac{z-(2k-2)}{-3}$ ***A1***

$x=t$, $y=(2-3k)+5t$ and $z=(2k-2)-3t$ ***A1A1***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***AG***

[5 marks]

METHOD 2

$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} \Rightarrow$ direction is $\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***M1A1***

Let $x=0$

$0-2y-3z=2$ and $2\times 0 - y - z = k$ ***(M1)***

solve simultaneously ***(M1)***

$y = 2-3k$ and $z = 2k-2$ ***A1***

therefore $\mathbf{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***AG***

[5 marks]

continued …

Question 13 continued

METHOD 3

substitute $x = t$, $y = (2-3k)+5t$ and $z = (2k-2)-3t$ into π_1 and π_2 ***M1***

for $\pi_1 : t - 2(2-3k+5t) - 3(2k-2-3t) = 2$ ***A1***

for $\pi_2 : 2t - (2-3k+5t) - (2k-2-3t) = k$ ***A1***

the planes have a unique line of intersection ***R2***

therefore the line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ ***AG***

[5 marks]

(c) $5 - t = (2-3k+5t)+3 = 2-2(2k-2-3t)$ ***M1A1***

Note: Award ***M1A1*** if candidates use vector or parametric equations of L_2 *eg* $\begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$ or $\Rightarrow \begin{cases} t = 5-2s \\ 2-3k+5t = -3+2s \\ 2k-2-3t = 1+s \end{cases}$

solve simultaneously ***M1***

$k = 2, t = 1$ ($s = 2$) ***A1***

intersection point $(1, 1, -1)$ ***A1***

[5 marks]

(d) $\vec{l_2} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***A1***

$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 5 & -3 \\ 2 & -2 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -7 \\ -12 \end{pmatrix}$ ***(M1)A1***

$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix}$ ***(M1)***

$x + 7y + 12z = -4$ ***A1***

[5 marks]

continued …

Question 13 continued

(e) Let θ be the angle between the lines $\vec{l}_1 = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ and $\vec{l}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

$\cos\theta = \dfrac{|2-10-3|}{\sqrt{35}\sqrt{9}} \Rightarrow \theta = 0.902334\ldots\ (51.699\ldots^\circ)$ ***(M1)***

as the triangle XYZ has a right angle at Y,

$\text{XZ} = a \Rightarrow \text{YZ} = a\sin\theta$ and $\text{XY} = a\cos\theta$ ***(M1)***

$\text{area} = 3 \Rightarrow \dfrac{a^2\sin\theta\cos\theta}{2} = 3$ ***(M1)***

$a = 3.5122\ldots$ ***(A1)***

$\text{perimeter} = a + a\sin\theta + a\cos\theta = 8.44537\ldots = 8.45$ ***A1***

Note: If candidates attempt to find coordinates of Y and Z award ***M1*** for expression of vector YZ in terms of two parameters, ***M1*** for attempt to use perpendicular condition to determine relation between parameters, ***M1*** for attempt to use the area to find the parameters and ***A2*** for final answer.

[5 marks]

Total [24 marks]

M12/5/MATHL/HP2/ENG/TZ1/XX/M

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

May 2012

MATHEMATICS

Higher Level

Paper 2

15 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2012**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
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13 More than one solution

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SECTION A

1. $\frac{dy}{dx} = 3x^2 - 12x + k$ ***M1A1***

For use of discriminant $b^2 - 4ac = 0$ or completing the square $3(x-2)^2 + k - 12$ ***(M1)***

$144 - 12k = 0$ ***(A1)***

Note:	Accept trial and error, sketches of parabolas with vertex (2,0) or use of second derivative.

$k = 12$ ***A1***

[5 marks]

2. $k\int_1^2 2^{\frac{1}{x}}\,dx = 1 \Rightarrow k = \frac{1}{\int_1^2 2^{\frac{1}{x}}\,dx}$ $(= 0.61556\ldots)$ ***(M1)(A1)***

$E(X) = k\int_1^2 x2^{\frac{1}{x}}\,dx = 2.39....k$ or 1.47 ***M1A1***

Note:	Condone missing d*x* in any part of the question.

[4 marks]

3. (a) $\binom{10}{6} = 210$ ***(M1)A1***

[2 marks]

(b) $2 \times \binom{8}{5} = 112$ ***(M1)A1A1***

Note:	Accept 210 – 28 – 70 = 112

[3 marks]

(c) $\frac{112}{210}$ $\left(= \frac{8}{15} = 0.533\right)$ ***(M1)A1***

[2 marks]

Total [7 marks]

4. point on line is $x = \frac{-1-5\lambda}{5}$, $y = \frac{9+5\lambda}{5}$, $z = \lambda$ or similar ***M1A1***

Note:	Accept use of point on the line or elimination of one of the variables using the equations of the planes

$$\frac{-1-5\lambda}{5} - \frac{9+5\lambda}{5} + 2\lambda = k$$ ***M1A1***

Note:	Award ***M1A1*** if coordinates of point and equation of a plane is used to obtain linear equation in *k* or equations of the line are used in combination with equation obtained by elimination to get linear equation in *k*.

$k = -2$ ***A1***

[5 marks]

5. (a) 50 ***A1***

[1 mark]

(b) Lower quartile is 4 so at least 26 obtained a 4 ***R1***
Lower bound is 26 ***A1***

Minimum is 2 but the rest could be 4 ***R1***

So upper bound is 49 ***A1***

Note:	Do not allow follow through for ***A*** marks.

Note:	If answers are incorrect award ***R0A0***; if argument is correct but no clear lower/upper bound is stated award ***R1A0;*** award ***R0A1*** for correct answer without explanation or incorrect explanation.

[4 marks]

Total [5 marks]

6. $h(x) = f(x-3) - 2 = \ln(x-3) - 2$ ***(M1)(A1)***
$g(x) = -h(x) = 2 - \ln(x-3)$ ***M1***

Note:	Award **M1** only if it is clear the effect of the reflection in the *x*-axis: the expression is correct ***OR*** there is a change of signs of the previous expression ***OR*** there's a graph or an explanation making it explicit

$= \ln e^2 - \ln(x-3)$ ***M1***

$= \ln\left(\frac{e^2}{x-3}\right)$ ***A1***

[5 marks]

7.

$X \sim \mathrm{Po}(m)$

$\mathrm{P}(X = 2) = \mathrm{P}(X < 2)$ ***(M1)***

$\frac{1}{2}m^2 e^{-m} = e^{-m}(1+m)$ ***(A1)(A1)***

$m = 2.73 \quad \left(1+\sqrt{3}\right)$ ***A1***

in four hours the expected value is 10.9 $\left(4+4\sqrt{3}\right)$ ***A1***

Note: Value of *m* does not need to be rounded.

[5 marks]

8. $x = r - \frac{r}{h}y$ or $x = \frac{r}{h}(h-y)$ (or equivalent) ***(A1)***

$\int \pi x^2 \,\mathrm{d}y$

$= \pi\int_0^h \left(r - \frac{r}{h}y\right)^2 \mathrm{d}y$ ***M1A1***

Note: Award ***M1*** for $\int x^2 \,\mathrm{d}y$ and ***A1*** for correct expression. Accept $\pi\int_0^h \left(\frac{r}{h}y - r\right)^2 \mathrm{d}y$ and $\pi\int_0^h \left(\pm\left(r - \frac{r}{h}x\right)\right)^2 \mathrm{d}x$

$= \pi\int_0^h \left(r^2 - \frac{2r^2}{h}y + \frac{r^2}{h^2}y^2\right)\mathrm{d}y$ ***A1***

Note: Accept substitution method and apply markscheme to corresponding steps.

$= \pi\left[r^2 y - \frac{r^2y^2}{h} + \frac{r^2y^3}{3h^2}\right]_0^h$ ***M1A1***

Note: Award ***M1*** for attempted integration of any quadratic trinomial.

$= \pi\left(r^2h - r^2h + \frac{1}{3}r^2h\right)$ ***M1A1***

Note: Award ***M1*** for attempted substitution of limits in a trinomial.

$= \frac{1}{3}\pi r^2 h$ ***A1***

Note: Throughout the question do not penalize missing d*x*/d*y* as long as the integrations are done with respect to correct variable

[9 marks]

9. $\left(x-\frac{2}{x}\right)^4 = x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ ***(M1)(A1)***

$\left(x^2+\frac{2}{x}\right)^3 = x^6 + 6x^3 + 12 + \frac{8}{x^3}$ ***(M1)(A1)***

Note:	Accept unsimplified or uncalculated coefficients in the constant term

$= 24 \times 12$ ***(M1)(A1)***

$= 288$ ***A1***

[7 marks]

10. attempt to find intersections ***M1***

intersections are $\left(\frac{10}{m+2}, \frac{10m}{m+2}\right)$ and $\left(\frac{10m}{2m-1}, -\frac{10}{2m-1}\right)$ ***A1A1***

area of triangle $= \frac{1}{2} \times \frac{\sqrt{100+100m^2}}{(m+2)} \times \frac{\sqrt{100+100m^2}}{(2m-1)}$ ***M1A1A1***

$= \frac{50(1+m^2)}{(m+2)(2m-1)}$

minimum when $m = 3$ ***(M1)A1***

[8 marks]

SECTION B

11. (a) $(3.79, -5)$ ***A1***

[1 mark]

(b) $p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$ ***A1A1***

[2 marks]

(c) $f'(x) = 3\cos x - 4\sin x$ ***(M1)(A1)***

$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43\ldots$ ***(A1)***

$(y = -4)$ ***A1***

Coordinates are $(4.43, -4)$

[4 marks]

(d) $m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}$ ***(M1)***

gradient at P is -4 so gradient of normal at P is $\frac{1}{4}$ ***(A1)***

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$ ***(A1)***

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570\ldots)$ $\left(\text{or } y = 0.25x + 2.60...\right)$ ***(M1)***

equation of normal at Q is $y - 3 = -\frac{1}{4}\left(x - 5.999\ldots\right)$ $\left(\text{or } y = -0.25x + \underbrace{4.499...}\right)$ ***(M1)***

Note:	Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where (a, b) are coordinates of P and Q (or in $y = mx + c$ with c determined using coordinates of Pand Q.

intersect at $(3.79, 3.55)$ ***A1A1***

Note:	Award ***N2*** for 3.79 without other working.

[7 marks]

Total [14 marks]

12. (a) (i) $X \sim \text{Po}(11)$ ***(M1)***

$\text{P}(X \le 11) = 0.579$ ***(M1)A1***

(ii) $\text{P}(X > 8 | X < 12) =$ ***(M1)***

$= \frac{\text{P}(8 < X < 12)}{\text{P}(X < 12)} \left(\text{or } \frac{\text{P}(X \le 11) - \text{P}(X \le 8)}{\text{P}(X \le 11)} \text{ or } \frac{0.3472...}{0.5792...} \right)$ ***A1***

$= 0.600$ ***A1*** ***N2***

[6 marks]

(b) (i) $Y \sim \text{Po}(m)$

$\text{P}(Y > 3) = 0.24$ ***(M1)***

$\text{P}(Y \le 3) = 0.76$ ***(M1)***

$e^{-m}\left(1 + m + \frac{1}{2}m^2 + \frac{1}{6}m^3\right) = 0.76$ ***(A1)***

Note: At most two of the above lines can be implied.

Attempt to solve equation with GDC ***(M1)***

$m = 2.49$ ***A1***

(ii) $A \sim \text{Po}(4.98)$

$\text{P}(A > 5) = 1 - \text{P}(A \le 5) = 0.380\ldots$ ***M1A1***

$W \sim \text{B}(4, 0.380\ldots)$ ***(M1)***

$\text{P}(W \ge 2) = 1 - \text{P}(W \le 1) = 0.490$ ***M1A1***

[10 marks]

(c) $\text{P}(A < 25) = 0.8$, $\text{P}(A < 18) = 0.4$

$\frac{25 - \mu}{\sigma} = 0.8416\ldots$ ***(M1)(A1)***

$\frac{18 - \mu}{\sigma} = -0.2533\ldots$ (or -0.2534 from tables) ***(M1)(A1)***

solving these equations ***(M1)***

$\mu = 19.6$ ***A1***

Note: Accept just 19.6, 19 or 20; award A0 to any other final answer.

[6 marks]

Total [22 marks]

13. (a) $\vec{AB} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \Rightarrow AB = \sqrt{72}$ ***A1***

$\vec{AC} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \Rightarrow AC = \sqrt{72}$ ***A1***

so they are the same ***AG***

$\vec{AB} \cdot \vec{AC} = 36 = \left(\sqrt{72}\right)\left(\sqrt{72}\right)\cos\theta$ ***(M1)***

$\cos\theta = \dfrac{36}{\left(\sqrt{72}\right)\left(\sqrt{72}\right)} = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$ ***A1AG***

Note: Award ***M1A1*** if candidates find BC and claim that triangle ABC is equilateral.

[4 marks]

(b) **METHOD 1**

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 6 & -6 \\ -6 & 0 & -6 \end{vmatrix} = -36\boldsymbol{i} + 36\boldsymbol{j} + 36\boldsymbol{k}$ ***(M1)A1***

equation of plane is $x - y - z = k$ ***(M1)***

goes through A, B or C $\Rightarrow x - y - z = 2$ ***A1***

[4 marks]

METHOD 2

$x + by + cz = d$ (or similar) ***M1***

$5 - 2b + 5c = d$

$5 + 4b - c = d$ ***A1***

$-1 - 2b - c = d$

solving simultaneously ***M1***

$b = -1,\ c = -1,\ d = 2$

so $x - y - z = 2$ ***A1***

[4 marks]

(c) (i) midpoint is $(5, 1, 2)$, so equation of Π_1 is $y - z = -1$ ***A1A1***

(ii) midpoint is $(2, -2, 2)$, so equation of Π_2 is $x + z = 4$ ***A1A1***

Note: In each part, award ***A1*** for midpoint and ***A1*** for the equation of the plane.

[4 marks]

continued …

Question 12 continued …

(d) **EITHER**

solving the two equations above *M1*

$L:\boldsymbol{r}=\begin{pmatrix}4\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ *A1*

OR

L has the direction of the vector product of the normal vectors to the planes Π_1 and Π_2 *(M1)*

$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\0 & 1 & -1\\1 & 0 & 1\end{vmatrix}=\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$ (or its opposite) *A1*

THEN

direction is $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ as required *R1*

[3 marks]

(e) D is of the form $(4-\lambda, -1+\lambda, \lambda)$ *M1*

$(1+\lambda)^2+(-1-\lambda)^2+(5-\lambda)^2=72$ *M1*

$3\lambda^2-6\lambda-45=0$

$\lambda=5$ or $\lambda=-3$ *A1*

D(–1, 4, 5) *AG*

Note:	Award ***M0M0A0*** if candidates just show that D(–1, 4, 5) satisfies AB=AD; Award ***M1M1A0*** if candidates also show that D is of the form $(4-\lambda, -1+\lambda, \lambda)$

[3 marks]

continued …

Question 12 continued …

(f) **EITHER**

G is of the form $(4-\lambda, -1+\lambda, \lambda)$ and $\text{DG} = \text{AG}, \text{BG}$ or CG ***M1***

e.g. $(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = (5-\lambda)^2 + (5-\lambda)^2 + (5-\lambda)^2$ ***M1***

$(1+\lambda)^2 = (5-\lambda)^2$

$\lambda = 2$ ***A1***

$\text{G}(2, 1, 2)$ ***AG***

OR

G is the centre of mass (barycentre) of the regular tetrahedron ABCD ***(M1)***

$$\text{G}\left(\frac{5+5+(-1)+(-1)}{4}, \frac{-2+4+(-2)+4}{4}, \frac{5+(-1)+(-1)+5}{4}\right)$$ ***M1A1***

THEN

Note: the following part is independent of previous work and candidates may use ***AG*** to answer it (here it is possible to award ***M0M0A0A1M1A1)***

$$\vec{\text{GD}} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \text{ and } \vec{\text{GA}} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$ ***A1***

$$\cos\theta = \frac{-9}{\left(3\sqrt{3}\right)\left(3\sqrt{3}\right)} = -\frac{1}{3} \Rightarrow \theta = 109^\circ \text{ (or 1.91 radians)}$$ ***M1A1***

[6 marks]

Total [24 marks]

M12/5/MATHL/HP2/ENG/TZ2/XX/M

International Baccalaureate®
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MARKSCHEME

May 2012

MATHEMATICS

Higher Level

Paper 2

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document **"Mathematics HL: Guidance for e-marking May 2012"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

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- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
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SECTION A

1. (a) $S_n = \frac{n}{2}[2a+(n-1)d]$

$212 = \frac{16}{2}(2a+15d) \quad (=16a+120d)$ ***A1***

n^{th} term is $a+(n-1)d$

$8 = a+4d$ ***A1***

solving simultaneously: ***(M1)***

$d = 1.5,\ a = 2$ ***A1***

[4 marks]

(b) $\frac{n}{2}[4+1.5(n-1)] > 600$ ***(M1)***

$\Rightarrow 3n^2 + 5n - 2400 > 0$ ***(A1)***

$\Rightarrow n > 27.4\ldots,\ (n < -29.1\ldots)$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ ***A1***

[3 marks]

Total [7 marks]

2. (a) $\mathrm{E}(X) = np$

$\Rightarrow 10 = 30p$

$\Rightarrow p = \frac{1}{3}$ ***A1***

[1 mark]

(b) $\mathrm{P}(X=10) = \binom{30}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{20} = 0.153$ ***(M1)A1***

[2 marks]

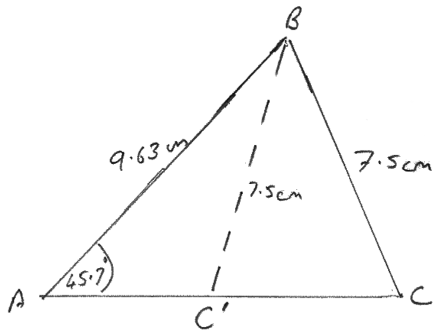
(c) $\mathrm{P}(X \geq 15) = 1 - \mathrm{P}(X \leq 14)$ ***(M1)***

$= 1 - 0.9565\ldots = 0.0435$ ***A1***

[2 marks]

Total [5 marks]

3. (a)



A2

Note:	Accept 2 separate triangles. The diagram(s) should show that one triangle has an acute angle and the other triangle has an obtuse angle. The values 9.63, 7.5 and 45.7 and/or the letters, A, B C′ and C should be correctly marked on the diagram(s).

[2 marks]

(b) **METHOD 1**

$\frac{\sin 45.7}{7.5} = \frac{\sin C}{9.63}$ *M1*

$\Rightarrow \hat{C} = 66.77\ldots^\circ, 113.2\ldots^\circ$ *(A1)(A1)*

$\Rightarrow \hat{B} = 67.52\ldots^\circ, 21.07\ldots^\circ$ *(A1)*

$\frac{b}{\sin B} = \frac{7.5}{\sin 45.7} \Rightarrow b = 9.68\text{(cm)}, b = 3.77\text{(cm)}$ *A1A1*

Note:	If only the acute value of $\hat{C}$ is found, award ***M1(A1)(A0)(A0)A1A0***.

METHOD 2

$7.5^2 = 9.63^2 + b^2 - 2 \times 9.63 \times b \cos 45.7^\circ$ *M1A1*

$b^2 - 13.45\ldots b + 36.48\ldots = 0$

$b = \frac{13.45\ldots \pm \sqrt{13.45\ldots^2 - 4 \times 36.48\ldots}}{2}$ *(M1)(A1)*

$\text{AC} = 9.68\text{(cm)}, \text{AC} = 3.77\text{(cm)}$ *A1A1*

[6 marks]

Total [8 marks]

4. (a) number of arrangements of boys is $15!$ and number of arrangements of girls is $10!$ ***(A1)***

total number of arrangements is $15!\times 10!\times 2 (= 9.49\times 10^{18})$ ***M1A1***

Note: If 2 is omitted, award ***(A1)M1A0***.

[3 marks]

(b) number of ways of choosing two boys is $\binom{15}{2}$ and the number of ways of choosing three girls is $\binom{10}{3}$ ***(A1)***

number of ways of choosing two boys and three girls is $\binom{15}{2}\times\binom{10}{3} = 12600$ ***M1A1***

[3 marks]

Total [6 marks]

5. (a) $\mathrm{P}(X = 5) = \mathrm{P}(X = 3) + \mathrm{P}(X = 4)$

$\frac{\mathrm{e}^{-m}m^5}{5!} = \frac{\mathrm{e}^{-m}m^3}{3!} + \frac{\mathrm{e}^{-m}m^4}{4!}$ ***M1(A1)***

$m^2 - 5m - 20 = 0$

$\Rightarrow m = \frac{5+\sqrt{105}}{2} = (7.62)$ ***A1***

[3 marks]

(b) $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \le 2)$ ***(M1)***

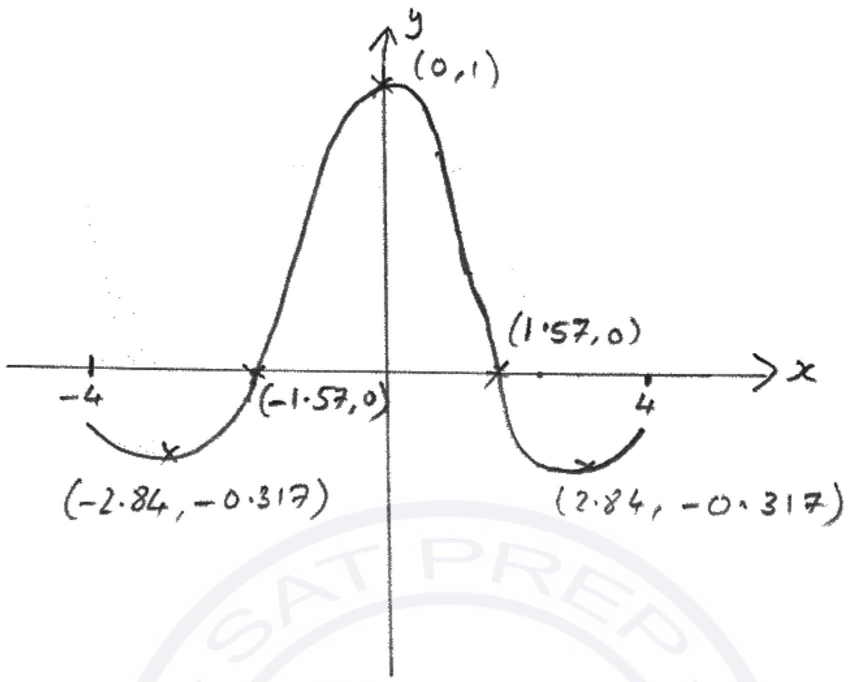
$= 1 - 0.018\ldots$

$= 0.982$ ***A1***

[2 marks]

Total [5 marks]

6. (a)



A1A1A1A1

Note:	Award ***A1*** for correct shape. Do not penalise if too large a domain is used, ***A1*** for correct *x*-intercepts, ***A1*** for correct coordinates of two minimum points, ***A1*** for correct coordinates of maximum point. Accept answers which correctly indicate the position of the intercepts, maximum point and minimum points.

[4 marks]

(b) gradient at $x = 1$ is -0.786 ***A1***

[1 mark]

(c) gradient of normal is $\frac{-1}{-0.786}(=1.272\ldots)$ ***(A1)***

when $x = 1,\ y = 0.3820...$ ***(A1)***

Equation of normal is $y - 0.382 = 1.27(x-1)$ ***A1***

$(\Rightarrow y = 1.27x - 0.890)$

[3 marks]

Total [8 marks]

continued …

7. (a) $\int_0^a \frac{1}{1+x^4}\,dx = 1$ ***M2***

$a = 1.40$ ***A1***

[3 marks]

(b) $E(X) = \int_0^a \frac{x}{1+x^4}\,dx$ ***M1***

$\left(= \frac{1}{2}\arctan(a^2)\right)$

$= 0.548$ ***A1***

[2 marks]

Total [5 marks]

8. (a) height $= 4 \times 0.95^4$ *(A1)*

$= 3.26$ (metres) ***A1***

[2 marks]

(b) $4 \times 0.95^n < 1$ *(M1)*

$0.95^n < 0.25$

$\Rightarrow n > \frac{\ln 0.25}{\ln 0.95}$ *(A1)*

$\Rightarrow n > 27.0$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ ***A1***

Note: If candidates have used $n-1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

[3 marks]

(c) **METHOD 1**

recognition of geometric series with sum to infinity, first term of 4×0.95 and common ratio 0.95 ***M1***

recognition of the need to double this series and to add 4 ***M1***

total distance travelled is $2\left(\frac{4 \times 0.95}{1-0.95}\right) + 4 = 156$ (metres) ***A1***

[3 marks]

Note: If candidates have used $n-1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

METHOD 2

recognition of a geometric series with sum to infinity, first term of 4 and common ratio 0.95 **M1**

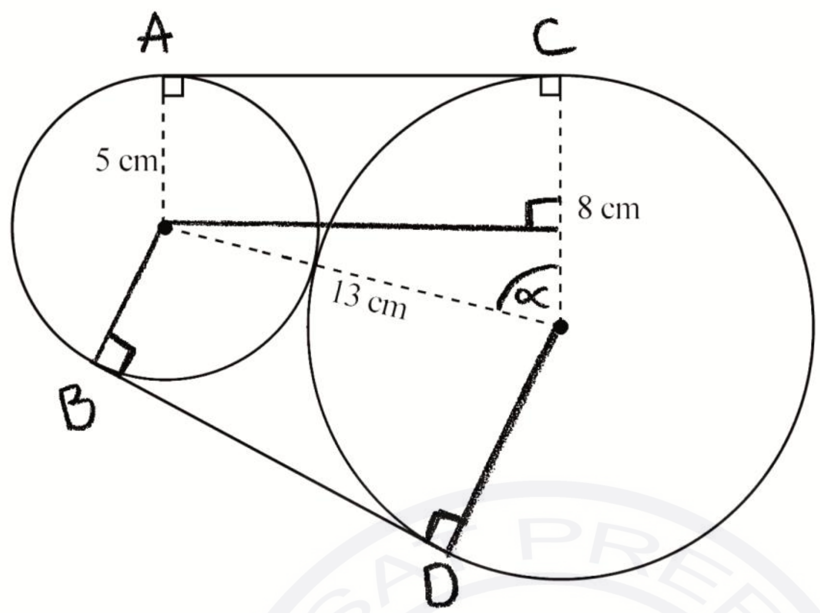
recognition of the need to double this series and to subtract 4 **M1**

total distance travelled is $2\left(\frac{4}{1-0.95}\right) - 4 = 156$ (metres) **A1**

[3 marks]

Total [8 marks]

9.



$AC = BD = \sqrt{13^2 - 3^2} = 12.64\ldots$ ***(A1)***

$\cos\alpha = \frac{3}{13} \Rightarrow \alpha = 1.337\ldots\left(76.65\ldots^\circ\right)$ ***(M1)(A1)***

attempt to find either arc length AB or arc length CD ***(M1)***

arc length $AB = 5(\pi - 2\times 0.232\ldots)\ (=13.37\ldots)$ ***(A1)***

arc length $CD = 8(\pi + 2\times 0.232\ldots)\ (=28.85\ldots)$ ***(A1)***

length of string $= 13.37\ldots + 28.85\ldots + 2(12.64\ldots)$ ***(M1)***

$= 67.5\,(\text{cm})$ ***A1***

[8 marks]

SECTION B

10. (a) (i) $P(X > 225) = 0.158\ldots$ ***(M1)(A1)***

expected number $= 450 \times 0.158\ldots = 71.4$ ***A1***

(ii) $P(X < m) = 0.7$ ***(M1)***

$\Rightarrow m = 213$ (grams) ***A1***

[5 marks]

(b) $\frac{270 - \mu}{\sigma} = 1.40\ldots$ ***(M1)A1***

$\frac{250 - \mu}{\sigma} = -1.03\ldots$ ***A1***

Note: These could be seen in graphical form.

solving simultaneously ***(M1)***

$\mu = 258,\ \sigma = 8.19$ ***A1A1***

[6 marks]

(c)

$X \sim N(80, 4^2)$

$P(X > 82) = 0.3085\ldots$ ***A1***

recognition of the use of binomial distribution. ***(M1)***

$X \sim B(5, 0.3085\ldots)$

$P(X = 3) = 0.140$ ***A1***

[3 marks]

.

Total [14 marks]

11. (a) in augmented matrix form $\left|\begin{matrix} 1 & -3 & 1 & 3 \\ 1 & 5 & -2 & 1 \\ 0 & 16 & -6 & k \end{matrix}\right|$

attempt to find a line of zeros ***(M1)***

$r_2 - r_1 \left|\begin{matrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 16 & -6 & k \end{matrix}\right|$ ***(A1)***

$r_3 - 2r_2 \left|\begin{matrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{matrix}\right|$ ***(A1)***

there is an infinite number of solutions when $k = -4$ ***R1***

there is no solution when

$k \neq -4, (k \in \mathbb{R})$ ***R1***

Note: Approaches other than using the augmented matrix are acceptable.

[5 marks]

(b) using $\left|\begin{matrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{matrix}\right|$ and letting $z = \lambda$ ***(M1)***

$8y - 3\lambda = -2$

$\Rightarrow y = \dfrac{3\lambda - 2}{8}$ ***(A1)***

$x - 3y + z = 3$

$\Rightarrow x - \left(\dfrac{9\lambda - 6}{8}\right) + \lambda = 3$ ***(M1)***

$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$

$\Rightarrow x = \dfrac{18 + \lambda}{8}$ ***(A1)***

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{18}{8} \\ -\frac{2}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{pmatrix}$ ***(M1)(A1)***

$\boldsymbol{r} = \begin{pmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$ ***A1***

Note: Accept equivalent answers.

[7 marks]

continued...

Question 11 continued

(c) recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane ***(A1)***

direction normal of the plane is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$ ***(M1)***

$= 16\boldsymbol{i} + 24\boldsymbol{j} - 11\boldsymbol{k}$ ***A1***

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is $(1, 2, 0)$ ***(M1)***

$\Rightarrow 16 + 48 = d$

$d = 64$ ***A1***

hence Cartesian equation of plane is $16x + 24y - 11z = 64$ ***AG***

Note:	Accept alternative methods using dot product.

[5 marks]

(d) the plane crosses the *z*-axis when $x = y = 0$ ***(M1)***

coordinates of P are $\left(0, 0, -\dfrac{64}{11}\right)$ ***A1***

Note:	Award ***A1*** for stating $z = -\dfrac{64}{11}$.

Note:	Accept. $\begin{pmatrix} 0 \\ 0 \\ -\dfrac{64}{11} \end{pmatrix}$

[2 marks]

(e) recognition that the angle between the line and the direction normal is given by:

$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta$ where θ is the angle between the line and the normal vector **_M1A1_**

$\Rightarrow 122 = \sqrt{29}\sqrt{953}\cos\theta$ **_(A1)_**

$\Rightarrow \theta = 42.8^\circ$ (0.747 radians) **_(A1)_**

hence the angle between the line and the plane is $90^\circ - 42.8^\circ = 47.2^\circ$ (0.824 radians) **_A1_**

[5 marks]

Note: Accept use of the formula $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

Total [24 marks]

12. (a) $\frac{dv}{dt} = -v^2 - 1$

attempt to separate the variables **_M1_**

$\int \frac{1}{1+v^2} dv = \int -1 dt$ **_A1_**

$\arctan v = -t + k$ **_A1A1_**

Note: Do not penalize the lack of constant at this stage.

when $t = 0, v = 1$ **_M1_**

$\Rightarrow k = \arctan 1 = \left(\frac{\pi}{4}\right) = \left(45^\circ\right)$ **_A1_**

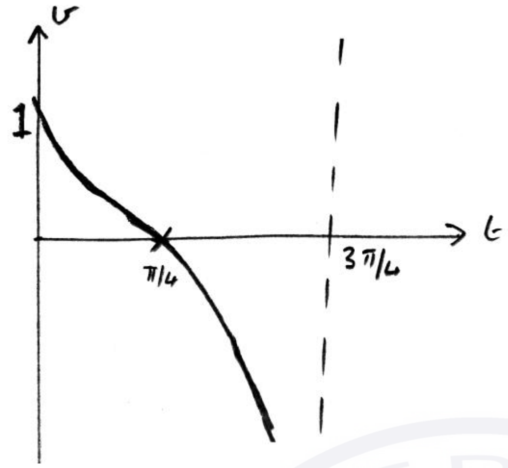
$\Rightarrow v = \tan\left(\frac{\pi}{4} - t\right)$ **_A1_**

[7 marks]

continued...

Question 12 continued

(b)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for asymptote, ***A1*** for correct t and v intercept.

Note:	Do not penalise if a larger domain is used.

[3 marks]

(c) (i) $T = \frac{\pi}{4}$ ***A1***

(ii) area under curve $= \int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - t\right) \mathrm{d}t$ ***(M1)***

$= 0.347 \left(= \frac{1}{2}\ln 2\right)$ ***A1***

[3 marks]

continued...

Question 12 continued

(d) $v = \tan\left(\frac{\pi}{4} - t\right)$

$s = \int \tan\left(\frac{\pi}{4} - t\right)\mathrm{d}t$ ***M1***

$\int \frac{\sin\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)}\mathrm{d}t$ ***(M1)***

$= \ln\cos\left(\frac{\pi}{4} - t\right) + k$ ***A1***

when $t = 0$, $s = 0$

$k = -\ln\cos\frac{\pi}{4}$ ***A1***

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}\ \left(= \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - t\right)\right]\right)$ ***A1***

[5 marks]

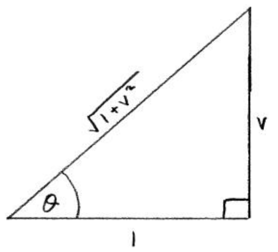
(e) **METHOD 1**

$\frac{\pi}{4} - t = \arctan v$ ***M1***

$t = \frac{\pi}{4} - \arctan v$

$s = \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{\pi}{4} + \arctan v\right)\right]$

$s = \ln\left[\sqrt{2}\cos(\arctan v)\right]$ ***M1A1***



$s = \ln\left[\sqrt{2}\cos\left(\arccos\frac{1}{\sqrt{1+v^2}}\right)\right]$ ***A1***

$= \ln\frac{\sqrt{2}}{\sqrt{1+v^2}}$

$= \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

continued...

Question 12 continued

METHOD 2

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$

$= -\ln\sec\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + \tan^2\left(\frac{\pi}{4} - t\right)} - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + v^2} - \ln\cos\frac{\pi}{4}$ ***A1***

$= \ln\frac{1}{\sqrt{1 + v^2}} + \ln\sqrt{2}$ ***A1***

$= \frac{1}{2}\ln\frac{2}{1 + v^2}$ ***AG***

METHOD 3

$v\frac{dv}{ds} = -v^2 - 1$ ***M1***

$\int\frac{v}{v^2 + 1}dv = -\int 1 ds$ ***M1***

$\frac{1}{2}\ln(v^2 + 1) = -s + k$ ***A1***

when $s = 0$, $t = 0 \Rightarrow v = 1$

$\Rightarrow k = \frac{1}{2}\ln 2$ ***A1***

$\Rightarrow s = \frac{1}{2}\ln\frac{2}{1 + v^2}$ ***AG***

[4 marks]

Total [22 marks]

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MARKSCHEME

November 2011

MATHEMATICS

Higher Level

Paper 2

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty **(AP)** no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

*The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award **A0**. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

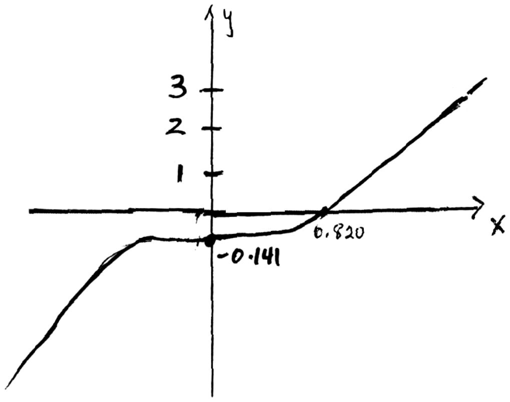
If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. (a)



A1A1A1

Note: Award *A1* for shape, *A1* for x-intercept is 0.820, accept sin(–3) or –sin(3) *A1* for y-intercept is -0.141.

(b) $A=\int_0^{0.8202}\left|x+\sin(x-3)\right|\mathrm{d}x\approx 0.0816$ sq units *(M1)A1*

[5 marks]

2. $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}$ *(M1)*

$=a\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix}-\begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix}+\begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix}$

$=a(a^2-1)-(a-1)+(1-a)$ *(A1)*

$=a^3-3a+2$ *A1*

set $a^3-3a+2=0$ *M1*

$\Rightarrow a=-2;\ a=1$ *A1A1*

hence the system has a unique solution for all reals such that

$a\neq -2;\ a\neq 1$ *R1*

Note: Award ***R1*** for their values of a.

[7 marks]

3. (a) $m=\dfrac{300}{60}=5$ *(A1)*

$\mathrm{P}(X=0)=0.00674$ *A1*

or e^{-5}

(b) $\mathrm{E}(X)=5\times 2=10$ *A1*

(c) $\mathrm{P}(X>10)=1-\mathrm{P}(X\le 10)$ *(M1)*

$=0.417$ *A1*

[5 marks]

4. (a) $\tan\left(\arctan\frac{1}{2} - \arctan\frac{1}{3}\right) = \tan(\arctan a)$ ***(M1)***

$a = 0.14285\ldots = \frac{1}{7}$ ***(A1)A1***

(b) $\arctan\left(\frac{1}{7}\right) = \arcsin(x) \Rightarrow x = \sin\left(\arctan\frac{1}{7}\right) \approx 0.141$ ***(M1)A1***

> **Note:** Accept exact value of $\left(\frac{1}{\sqrt{50}}\right)$.

[5 marks]

5. (a) $X \sim \mathrm{B}(5,\, 0.1)$ ***(M1)***

$\mathrm{P}(X = 2) = 0.0729$ ***A1***

(b) $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0)$ ***(M1)***

$0.9 < 1 - \left(\frac{9}{10}\right)^n$ ***(M1)***

$n > \frac{\ln 0.1}{\ln 0.9}$

$n = 22$ days ***A1***

[5 marks]

6. METHOD 1

$\arg(z_1 z_2) = \frac{5\pi}{6}$ (150°) ***(A1)***

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ (90°) ***(A1)***

$\Rightarrow \arg(z_1) + \arg(z_2) = \frac{5\pi}{6}$; $\arg(z_1) - \arg(z_2) = \frac{\pi}{2}$ ***M1***

solving simultaneously

$\arg(z_1) = \frac{2\pi}{3}$ (120°) and $\arg(z_2) = \frac{\pi}{6}$ (30°) ***A1A1***

Note: Accept decimal approximations of the radian measures.

$|z_1 z_2| = 2 \Rightarrow |z_1||z_2| = 2$; $\left|\frac{z_1}{z_2}\right| = 2 \Rightarrow \frac{|z_1|}{|z_2|} = 2$ ***M1***

solving simultaneously

$|z_1| = 2$; $|z_2| = 1$ ***A1***

[7 marks]

METHOD 2

$z_1 = 2iz_2$ $\quad 2iz_2{}^2 = -\sqrt{3} + i$ ***(M1)***

$z_2{}^2 = \frac{-\sqrt{3}+i}{2i}$ ***A1***

$z_2 = \sqrt{\frac{-\sqrt{3}+i}{2i}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $e^{\frac{\pi}{6}i}$ ***(M1)(A1)***

(allow $0.866 + 0.5i$ or $e^{0.524i}$)

$z_1 = -1 + \sqrt{3}i$ or $2e^{\frac{2\pi}{3}i}$ – (allow $-1+1.73i$ or $2e^{2.09i}$) ***(A1)***

z_1 modulus = 2, argument = $\frac{2\pi}{3}$ ***A1***

z_2 modulus = 1, argument = $\frac{\pi}{6}$ ***A1***

Note: Accept degrees and decimal approximations to radian measure.

[7 marks]

7. (a) for the series to have a finite sum, $\left|\frac{2x}{x+1}\right|<1$ ***R1***

(sketch from gdc or algebraic method) ***M1***

S_∞ exists when $-\frac{1}{3}<x<1$ ***A1A1***

Note: Award ***A1*** for bounds and ***A1*** for strict inequalities.

(b) $S_\infty=\dfrac{\frac{2x}{x+1}}{1-\frac{2x}{x+1}}=\dfrac{2x}{1-x}$ ***M1A1***

[6 marks]

8. (a) $y=\dfrac{1}{1+\mathrm{e}^{-x}}$

$y(1+\mathrm{e}^{-x})=1$ ***M1***

$1+\mathrm{e}^{-x}=\dfrac{1}{y}\Rightarrow \mathrm{e}^{-x}=\dfrac{1}{y}-1$ ***A1***

$\Rightarrow x=-\ln\left(\dfrac{1}{y}-1\right)$ ***A1***

$f^{-1}(x)=-\ln\left(\dfrac{1}{x}-1\right)\quad\left(=\ln\left(\dfrac{x}{1-x}\right)\right)$ ***A1***

domain: $0<x<1$ ***A1A1***

Note: Award ***A1*** for endpoints and ***A1*** for strict inequalities.

(b) 0.659 ***A1***

[7 marks]

9. $V=\dfrac{\pi}{3}r^2h$

$\dfrac{\mathrm{d}V}{\mathrm{d}t}=\dfrac{\pi}{3}\left[2rh\dfrac{\mathrm{d}r}{\mathrm{d}t}+r^2\dfrac{\mathrm{d}h}{\mathrm{d}t}\right]$ ***M1A1A1***

at the given instant

$\dfrac{\mathrm{d}V}{\mathrm{d}t}=\dfrac{\pi}{3}\left[2(40)(200)\left(-\dfrac{1}{2}\right)+40^2(3)\right]$ ***M1***

$=\dfrac{-3200\pi}{3}=-3351.03\ldots\approx-3350$ ***A1***

hence, the volume is decreasing (at approximately 3350 mm^3 per century) ***R1***

[6 marks]

10. **METHOD 1**

$\frac{2-\text{i}}{1+\text{i}}=\frac{1-3\text{i}}{2}$ *A1*

$\frac{6+8\text{i}}{u+\text{i}}\times\frac{u-\text{i}}{u-\text{i}}=\frac{6u+8+(8u-6)\text{i}}{u^2+1}$ *M1A1*

$\Rightarrow\frac{2-\text{i}}{1+\text{i}}-\frac{6+8u}{u+\text{i}}=\frac{1}{2}-\frac{6u+8}{u^2+1}-\left(\frac{3}{2}+\frac{8u-6}{u^2+1}\right)\text{i}$

$\text{Im}\,z=\text{Re}\,z$

$\Rightarrow\frac{1}{2}-\frac{6u+8}{u^2+1}=-\frac{3}{2}-\frac{8u-6}{u^2+1}$ *A1*

(sketch from gdc, or algebraic method) *(M1)*

$u=-3;\ u=2$ *A1A1* *N2*

[7 marks]

METHOD 2

$\frac{2-\text{i}}{1+\text{i}}-\frac{6+8\text{i}}{u+\text{i}}=\frac{(2-\text{i})(u+\text{i})-(1+\text{i})(6+8\text{i})}{(u-1)+\text{i}(u+1)}$ *M1A1*

$=\frac{(2-\text{i})(u+\text{i})-(1+\text{i})(6+8\text{i})}{(u-1)+\text{i}(u+1)}\cdot\frac{(u-1)-\text{i}(u+1)}{(u-1)-\text{i}(u+1)}$ *M1*

$=\frac{u^2-12u-15+\text{i}(-3u^2-16u+9)}{2(u^2+1)}$ *A1*

$\text{Re}\,z=\text{Im}\,z\Rightarrow u^2-12u-15=-3u^2-16u+9$ *M1*

$u=-3;\ u=2$ *A1A1* *N2*

[7 marks]

SECTION B

11. (a) $X \sim \mathrm{N}(60.33, 1.95^2)$

$\mathrm{P}(X < x) = 0.2 \Rightarrow x = 58.69$ m ***(M1)A1***

[2 marks]

(b) $z = -0.8416\ldots$ ***(A1)***

$-0.8416 = \dfrac{56.52 - 59.39}{\sigma}$ ***(M1)***

$\sigma \approx 3.41$ ***A1***

[3 marks]

(c) Jan $X \sim \mathrm{N}(60.33, 1.95^2)$; Sia $X \sim \mathrm{N}(59.50, 3.00^2)$

(i) Jan: $\mathrm{P}(X > 65) \approx 0.00831$ ***(M1)A1***

Sia: $\mathrm{P}(Y > 65) \approx 0.0334$ ***A1***

Sia is more likely to qualify ***R1***

Note: Only award ***R1*** if ***(M1)*** has been awarded.

(ii) Jan: $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0)$ ***(M1)***

$= 1 - (1 - 0.00831\ldots)^3 \approx 0.0247$ ***(M1)A1***

Sia: $\mathrm{P}(Y \geq 1) = 1 - \mathrm{P}(Y = 0) = 1 - (1 - 0.0334\ldots)^3 \approx 0.0968$ ***A1***

Note: Accept 0.0240 and 0.0969.

hence, $\mathrm{P}(X \geq 1 \text{ and } Y \geq 1) = 0.0247 \times 0.0968 = 0.00239$ ***(M1)A1***

[10 marks]

Total [15 marks]

12. (a) $S_{2n} = \frac{2n}{2}\left(2(8) + (2n-1)\frac{1}{4}\right)$ ***(M1)***

$= n\left(16 + \frac{2n-1}{4}\right)$ ***A1***

$S_{3n} = \frac{3n}{2}\left(2 \times 8 + (3n-1)\frac{1}{4}\right)$ ***(M1)***

$= \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ ***A1***

$S_{2n} = S_{3n} - S_{2n} \Rightarrow 2S_{2n} = S_{3n}$ ***M1***

solve $2S_{2n} = S_{3n}$

$\Rightarrow 2n\left(16 + \frac{2n-1}{4}\right) = \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ ***A1***

$\left(\Rightarrow 2\left(16 + \frac{2n-1}{4}\right) = \frac{3}{2}\left(16 + \frac{3n-1}{4}\right)\right)$

(gdc or algebraic solution) ***(M1)***

$n = 63$ ***A2***

[9 marks]

(b) $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \ldots$

$= (a_1 - a_1 r)^2 + (a_1 r - a_1 r^2)^2 + (a_1 r^2 - a_1 r^3) + \ldots$ ***M1A1***

$= \left[a_1(1-r)\right]^2 + \left[a_1 r(1-r)\right]^2 + \left[a_1 r^2(1-r)\right]^2 + \ldots + \left[a_1 r^{n-1}(1-r)\right]^2$ ***(A1)***

Note: This ***A1*** is for the expression for the last term.

$= a_1^{\,2}(1-r)^2 + a_1^{\,2} r^2(1-r)^2 + a_1^{\,2} r^4(1-r)^2 + \ldots + a_1^{\,2} r^{2n-2}(1-r)^2$ ***A1***

$= a_1^{\,2}(1-r)^2(1 + r^2 + r^4 + \ldots + r^{2n-2})$ ***A1***

$= a_1^{\,2}(1-r)^2\left(\frac{1-r^{2n}}{1-r^2}\right)$ ***M1A1***

$= \frac{a_1^{\,2}(1-r)(1-r^{2n})}{1+r}$ ***AG***

[7 marks]

Total [16 marks]

13. (a) **METHOD 1**

solving simultaneously (gdc) *(M1)*

$x = 1 + 2z$; $y = -1 - 5z$ *A1A1*

$$L: \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$ *A1A1A1*

Note: 1st ***A1*** is for $\boldsymbol{r} =$.

[6 marks]

METHOD 2

direction of line $= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$ (last two rows swapped) ***M1***

$= 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ ***A1***

putting $z = 0$, a point on the line satisfies $2x + y = 1$, $3x + y = 2$ ***M1***

i.e. (1, −1, 0) ***A1***

the equation of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$ ***A1A1***

Note: Award ***A0A1*** if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is missing.

[6 marks]

(b) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ ***M1***

$= 6\boldsymbol{i} - 12\boldsymbol{k}$ ***A1***

hence, $\boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{k}$

$\boldsymbol{n} \cdot \boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1$ ***M1A1***

therefore $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} \Rightarrow x - 2z = 1$ ***AG***

[4 marks]

continued …

Question 13 continued

(c) **METHOD 1**

$\text{P} = (-2, 4, 1)$, $\text{Q} = (x, y, z)$

$\overrightarrow{\text{PQ}} = \begin{pmatrix} x+2 \\ y-4 \\ z-1 \end{pmatrix}$ ***A1***

$\overrightarrow{\text{PQ}}$ is perpendicular to $3x + y - z = 2$

$\Rightarrow \overrightarrow{\text{PQ}}$ is parallel to $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ ***R1***

$\Rightarrow x + 2 = 3t$; $y - 4 = t$; $z - 1 = -t$ ***A1***

$1 - z = t \Rightarrow x + 2 = 3 - 3z \Rightarrow x + 3z = 1$ ***A1***

solving simultaneously $x + 3z = 1$; $x - 2z = 1$ ***M1***

$5z = 0 \Rightarrow z = 0$; $x = 1$, $y = 5$ ***A1***

hence, $\text{Q} = (1, 5, 0)$

[6 marks]

METHOD 2

Line passing through PQ has equation

$\mathbf{r} = \begin{matrix} -2 \\ 4 \\ 1 \end{matrix} + t \begin{matrix} 3 \\ 1 \\ -1 \end{matrix}$ ***M1A1***

Meets π_3 when:

$-2 + 3t - 2(1 - t) = 1$ ***M1A1***

$t = 1$ ***A1***

Q has coordinates (1, 5, 0) ***A1***

[6 marks]

Total [16 marks]

14. (a) $\left|e^{i\theta}\right| \ \left(=\left|\cos\theta + i\sin\theta\right|\right) = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ ***M1AG***

[1 mark]

(b) $z = \frac{1}{3}e^{i\theta}$ ***A1***

$|z| = \left|\frac{1}{3}e^{i\theta}\right| = \frac{1}{3}$ ***A1AG***

[2 marks]

(c) $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}e^{i\theta}}$ ***(M1)A1***

[2 marks]

(d) **EITHER**

$S_\infty = \frac{1}{1-\frac{1}{3}\cos\theta - \frac{1}{3}i\sin\theta}$ ***A1***

$= \frac{1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta}{\left(1-\frac{1}{3}\cos\theta - \frac{1}{3}i\sin\theta\right)\left(1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta\right)}$ ***M1A1***

$= \frac{1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta}{\left(1-\frac{1}{3}\cos\theta\right)^2 + \frac{1}{9}\sin^2\theta}$ ***A1***

$= \frac{1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta}{1-\frac{2}{3}\cos\theta + \frac{1}{9}}$ ***A1***

continued …

Question 14 continued

OR

$$S_\infty = \frac{1}{1-\frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}}$$

$$= \frac{1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{\left(1-\frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}\right)\left(1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}\right)}$$ ***M1A1***

$$= \frac{1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{1-\frac{1}{3}(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta})+\frac{1}{9}}$$ ***A1***

$$= \frac{1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$ ***A1***

$$= \frac{1-\frac{1}{3}(\cos\theta - \mathrm{i}\sin\theta)}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$ ***A1***

THEN

taking imaginary parts on both sides

$$\frac{1}{3}\sin\theta+\frac{1}{9}\sin 2\theta+\ldots = \frac{\frac{1}{3}\sin\theta}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$ ***M1A1A1***

$$= \frac{\sin\theta}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$

$$\Rightarrow \sin\theta+\frac{1}{3}\sin 2\theta+\ldots = \frac{9\sin\theta}{10-6\cos\theta}$$ ***AG***

[8 marks]

Total ***[13 marks]***