Diploma Programme
Programme du diplôme
Programa del Diploma

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Thursday 21 November 2019 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

8819 – 7208

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1. [Maximum mark: 7]

The function f is defined by $f(x)=\begin{cases}\dfrac{x-3}{x-5}, & x<3\\ \ln(x-2), & x\geq 3\end{cases}$.

(a) Show that f is continuous at $x=3$. [3]

(b) Show that f is not differentiable at $x=3$. [4]

2. [Maximum mark: 10]

Determine whether each of the following infinite series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{3n}{2n^2+5}$ [4]

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{3^n(n!)^2}$ [6]

3. [Maximum mark: 11]

The function f is defined by $f(x)=\arcsin(2x)$, where $-\frac{1}{2}\leq x\leq\frac{1}{2}$.

(a) By finding a suitable number of derivatives of f, find the first two non-zero terms in the Maclaurin series for f. [8]

(b) Hence or otherwise, find $\displaystyle\lim_{x\to 0}\frac{\arcsin(2x)-2x}{(2x)^3}$. [3]

4. [Maximum mark: 22]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^2 + y^2 - xy}{x^2}$, with $y = 2$ when $x = 1$.

(a) Use Euler's method, with step length $h = 0.1$, to find an approximate value of y when $x = 1.4$. [5]

(b) Sketch the isoclines for $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$. [3]

(c) (i) Express $m^2 - 2m + 4$ in the form $(m - a)^2 + b$, where $a, b \in \mathbb{Z}$.

(ii) Solve the differential equation, for $x > 0$, giving your answer in the form $y = f(x)$.

(iii) Sketch the graph of $y = f(x)$ for $1 \le x \le 1.4$.

(iv) With reference to the curvature of your sketch in part (c)(iii), and without further calculation, explain whether you conjecture $f(1.4)$ will be less than, equal to, or greater than your answer in part (a). [14]

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Mathematics
Higher level
Paper 3 – calculus

Wednesday 15 May 2019 (morning)

1 hour

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2219 – 7208

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1. [Maximum mark: 7]

A simple model to predict the population of the world is set up as follows. At time t years the population of the world is x, which can be assumed to be a continuous variable. The rate of increase of x due to births is $0.056x$ and the rate of decrease of x due to deaths is $0.035x$.

(a) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.021x$. [1]

(b) Find a prediction for the number of years it will take for the population of the world to double. [6]

2. [Maximum mark: 9]

(a) Show that $1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1+x^2}$, where $|x| < 1$. [1]

(b) Hence write down the first four non-zero terms of the power series for $f(x) = \frac{1}{1+4x^2}$. [2]

(c) Using the result in (b), find the first four non-zero terms of the power series for $g(x) = \arctan 2x$. [6]

3. [Maximum mark: 9]

Consider the series $\sum_{n=1}^{\infty} \frac{A \times 8^n}{3^{2n+1}}$.

(a) Given that $A = \frac{1}{n}$, use the comparison test to show that the series converges. [4]

(b) Given that $A = n$, determine whether the series diverges or converges. [5]

4. [Maximum mark: 9]

Using L'Hôpital's rule, find $\lim_{x \to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)$. [9]

5. [Maximum mark: 16]

Consider the differential equation $2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2$, where $x > 0$.

(a) Solve the differential equation and show that a general solution is $x^2 + y^2 = cx$ where c is a positive constant. [11]

(b) Prove that there are two horizontal tangents to the general solution curve and state their equations, in terms of c. [5]

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Mathematics
Higher level
Paper 3 – calculus

Thursday 15 November 2018 (afternoon)

1 hour

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8818 – 7208

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1. [Maximum mark: 10]

(a) Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2}$ converges or diverges. [5]

(b) Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{n!}(x-1)^n$ converges for all $x \in \mathbb{R}$. [5]

2. [Maximum mark: 8]

(a) Use L'Hôpital's rule to determine the value of

$$\lim_{x \to 0}\left(\frac{e^{-3x^2} + 3\cos(2x) - 4}{3x^2}\right).$$ [5]

(b) Hence find $\lim_{x \to 0}\left(\frac{\int_0^x \left(e^{-3t^2} + 3\cos(2t) - 4\right)dt}{\int_0^x 3t^2\,dt}\right)$. [3]

3. [Maximum mark: 14]

Consider the differential equation

$$(x+2)^2\frac{\mathrm{d}y}{\mathrm{d}x}=(x+1)y\text{ , where } x\neq -2$$

with initial condition $y=2$ when $x=1$.

(a) Show that $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\dfrac{3x+7}{(x+2)^2}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [5]

Taylor polynomials, about $x=1$, are used to approximate $y(x)$.

(b) Find the Taylor polynomial of

(i) degree 2;

(ii) degree 3. [7]

(c) Find the difference between the approximated values of $y(1.05)$ that is obtained using the two answers to part (b). [2]

4. [Maximum mark: 18]

Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+\dfrac{y}{x}$, where $x\neq 0$.

(a) Given that $y(1)=1$, use Euler's method with step length $h=0.25$ to find an approximation for $y(2)$. Give your answer to two significant figures. [4]

(b) Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+\dfrac{y}{x}$ for $y(1)=1$. [6]

(c) Find the percentage error when $y(2)$ is approximated by the final rounded value found in part (a). Give your answer to two significant figures. [3]

Consider the family of curves which satisfy the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+\dfrac{y}{x}$, where $x\neq 0$.

(d) (i) Find the equation of the isocline corresponding to $\dfrac{\mathrm{d}y}{\mathrm{d}x}=k$, where $k\neq 0,\ k\in\mathbb{R}$.

(ii) Show that such an isocline can never be a normal to any of the family of curves that satisfy the differential equation. [5]

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Mathematics
Higher level
Paper 3 – calculus

Wednesday 9 May 2018 (afternoon)

1 hour

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2218 – 7208

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1. [Maximum mark: 10]

(a) Given that $n > \ln n$ for $n > 0$, use the comparison test to show that the series $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ is divergent. [3]

(b) Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(3x)^n}{\ln(n+2)}$. [7]

2. [Maximum mark: 6]

The function f is defined by

$$f(x) = \begin{cases} |x-2|+1 & x < 2 \\ ax^2 + bx & x \ge 2 \end{cases}$$

where a and b are real constants.

Given that both f and its derivative are continuous at $x = 2$, find the value of a and the value of b.

3. [Maximum mark: 11]

(a) Find the value of $\int_4^{\infty} \frac{1}{x^3}\,dx$. [3]

(b) Illustrate graphically the inequality $\sum_{n=5}^{\infty} \frac{1}{n^3} < \int_4^{\infty} \frac{1}{x^3}\,dx < \sum_{n=4}^{\infty} \frac{1}{n^3}$. [4]

(c) Hence write down a lower bound for $\sum_{n=4}^{\infty} \frac{1}{n^3}$. [1]

(d) Find an upper bound for $\sum_{n=4}^{\infty} \frac{1}{n^3}$. [3]

4. [Maximum mark: 11]

The function f is defined by $f(x) = (\arcsin x)^2$, $-1 \le x \le 1$.

(a) Show that $f'(0) = 0$. [2]

The function f satisfies the equation $\left(1 - x^2\right) f''(x) - x f'(x) - 2 = 0$.

(b) By differentiating the above equation twice, show that

$$\left(1 - x^2\right) f^{(4)}(x) - 5x f^{(3)}(x) - 4 f''(x) = 0$$

where $f^{(3)}(x)$ and $f^{(4)}(x)$ denote the 3rd and 4th derivative of $f(x)$ respectively. [4]

(c) Hence show that the Maclaurin series for $f(x)$ up to and including the term in x^4 is $x^2 + \frac{1}{3}x^4$. [3]

(d) Use this series approximation for $f(x)$ with $x = \frac{1}{2}$ to find an approximate value for π^2. [2]

5. [Maximum mark: 12]

Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^p + 1$ where $x \in \mathbb{R}, x \ne 0$ and p is a positive integer, $p > 1$.

(a) Solve the differential equation given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$. [8]

(b) (i) Show that the x-coordinate(s) of the points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ satisfy the equation $x^{p-1} = \frac{1}{p}$.

(ii) Deduce the set of values for p such that there are two points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Give a reason for your answer. [4]

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Mathematics
Higher level
Paper 3 – calculus

Thursday 16 November 2017 (afternoon)

1 hour

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8817 – 7208

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1. [Maximum mark: 5]

The function f is defined by

$$f(x) = \begin{cases} x^2 - 2, & x < 1 \\ ax + b, & x \geq 1 \end{cases}$$

where a and b are real constants.

Given that both f and its derivative are continuous at $x = 1$, find the value of a and the value of b.

2. [Maximum mark: 10]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

(a) Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. [4]

(b) Solve the differential equation giving your answer in the form $y = f(x)$. [6]

3. [Maximum mark: 12]

(a) Use the limit comparison test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2}$ is convergent. [3]

Let $S = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2+2}$.

(b) Find the interval of convergence for S. [9]

4. [Maximum mark: 10]

The mean value theorem states that if f is a continuous function on $[a\,,b]$ and differentiable on $]a\,,b[$ then $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ for some $c\in\,]a\,,b[$.

The function g, defined by $g(x)=x\cos\left(\sqrt{x}\right)$, satisfies the conditions of the mean value theorem on the interval $[0\,,5\pi]$.

(a) For $a=0$ and $b=5\pi$, use the mean value theorem to find all possible values of c for the function g. [6]

(b) Sketch the graph of $y=g(x)$ on the interval $[0\,,5\pi]$ and hence illustrate the mean value theorem for the function g. [4]

5. [Maximum mark: 13]

Consider the function $f(x)=\sin(p\arcsin x)$, $-1<x<1$ and $p\in\mathbb{R}$.

(a) Show that $f'(0)=p$. [2]

The function f and its derivatives satisfy

$$\left(1-x^2\right)f^{(n+2)}(x)-(2n+1)\,xf^{(n+1)}(x)+\left(p^2-n^2\right)f^{(n)}(x)=0\,,\ n\in\mathbb{N}$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ and $f^{(0)}(x)$ is $f(x)$.

(b) Show that $f^{(n+2)}(0)=\left(n^2-p^2\right)f^{(n)}(0)$. [1]

(c) For $p\in\mathbb{R}\setminus\{\pm1\,,\pm3\}$, show that the Maclaurin series for $f(x)$, up to and including the x^5 term, is

$$px+\frac{p\left(1-p^2\right)}{3!}x^3+\frac{p\left(9-p^2\right)\left(1-p^2\right)}{5!}x^5\,.$$ [4]

(d) Hence or otherwise, find $\displaystyle\lim_{x\to0}\frac{\sin(p\arcsin x)}{x}$. [2]

(e) If p is an odd integer, prove that the Maclaurin series for $f(x)$ is a polynomial of degree p. [4]

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Mathematics
Higher level
Paper 3 – calculus

Monday 8 May 2017 (afternoon)

1 hour

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2217 – 7208

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1. [Maximum mark: 7]

Use l'Hôpital's rule to determine the value of

$$\lim_{x \to 0} \frac{\sin^2 x}{x \ln(1+x)}.$$

2. [Maximum mark: 6]

Let the Maclaurin series for $\tan x$ be

$$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \dots$$

where a_1, a_3 and a_5 are constants.

(a) Find series for $\sec^2 x$, in terms of a_1, a_3 and a_5, up to and including the x^4 term

(i) by differentiating the above series for $\tan x$;

(ii) by using the relationship $\sec^2 x = 1 + \tan^2 x$. [3]

(b) Hence, by comparing your two series, determine the values of a_1, a_3 and a_5. [3]

3. [Maximum mark: 9]

Use the integral test to determine whether the infinite series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is convergent or divergent.

4. [Maximum mark: 13]

(a) Consider the differential equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), x > 0 .$$

Use the substitution $y = vx$ to show that the general solution of this differential equation is

$$\int \frac{dv}{f(v) - v} = \ln x + \text{Constant} .$$

[3]

(b) Hence, or otherwise, solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}, x > 0 ,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = g(x)$. [10]

5. [Maximum mark: 15]

Consider the curve $y = \frac{1}{x}, x > 0$.

(a) By drawing a diagram and considering the area of a suitable region under the curve, show that for $r > 0$,

$$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}.$$ [4]

(b) Hence, given that n is a positive integer greater than one, show that

(i) $\sum_{r=1}^{n} \frac{1}{r} > \ln(1+n)$;

(ii) $\sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n$. [6]

Let $U_n = \sum_{r=1}^{n} \frac{1}{r} - \ln n$.

(c) Hence, given that n is a positive integer greater than one, show that

(i) $U_n > 0$;

(ii) $U_{n+1} < U_n$. [4]

(d) Explain why these two results prove that $\{U_n\}$ is a convergent sequence. [1]

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Mathematics
Higher level
Paper 3 – calculus

Friday 18 November 2016 (morning)

1 hour

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8816 – 7208

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1. [Maximum mark: 11]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right)y = x^2$, given that $y = 2$ when $x = 0$.

(a) Show that $1 + x^2$ is an integrating factor for this differential equation. [5]

(b) Hence solve this differential equation. Give the answer in the form $y = f(x)$. [6]

2. [Maximum mark: 18]

(a) By successive differentiation find the first four non-zero terms in the Maclaurin series for $f(x) = (x+1)\ln(1+x) - x$. [11]

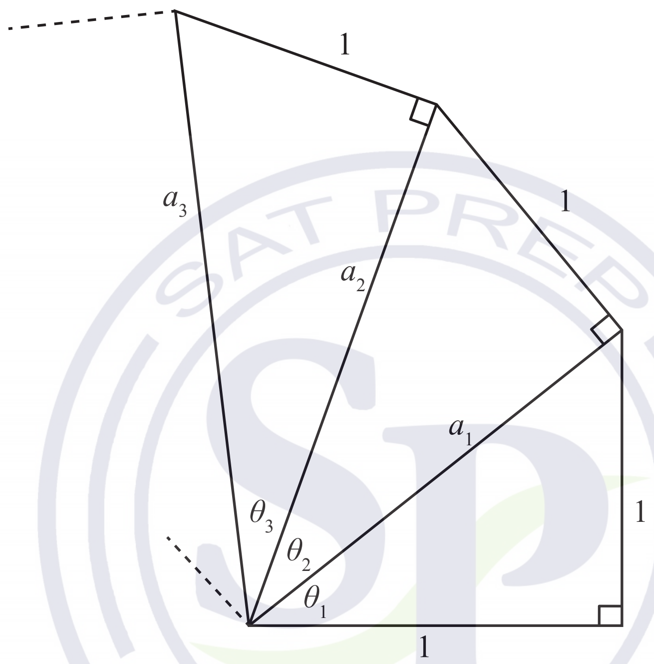
(b) Deduce that, for $n \geq 2$, the coefficient of x^n in this series is $(-1)^n \frac{1}{n(n-1)}$. [1]

(c) By applying the ratio test, find the radius of convergence for this Maclaurin series. [6]

3. [Maximum mark: 15]

(a) Using l'Hôpital's rule, find $\lim\limits_{x \to \infty}\left(\dfrac{\arcsin\left(\dfrac{1}{\sqrt{(x+1)}}\right)}{\dfrac{1}{\sqrt{x}}}\right)$. [6]

Consider the infinite spiral of right angle triangles as shown in the following diagram.



The nth triangle in the spiral has central angle θ_n, hypotenuse of length a_n and opposite side of length 1, as shown in the diagram. The first right angle triangle is isosceles with the two equal sides being of length 1.

(b) (i) Find a_1 and a_2 and hence write down an expression for a_n.

(ii) Show that $\theta_n = \arcsin\dfrac{1}{\sqrt{(n+1)}}$. [3]

Consider the series $\sum\limits_{n=1}^{\infty}\theta_n$.

(c) Using a suitable test, determine whether this series converges or diverges. [6]

4. [Maximum mark: 16]

(a) State the mean value theorem for a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. [2]

Let $f(x)$ be a function whose first and second derivatives both exist on the closed interval $[0, h]$.

Let $g(x) = f(h) - f(x) - (h - x)f'(x) - \frac{(h-x)^2}{h^2}\left(f(h) - f(0) - hf'(0)\right)$.

(b) (i) Find $g(0)$.

(ii) Find $g(h)$.

(iii) Apply the mean value theorem to the function $g(x)$ on the closed interval $[0, h]$ to show that there exists c in the open interval $]0, h[$ such that $g'(c) = 0$.

(iv) Find $g'(x)$.

(v) Hence show that $-(h - c)f''(c) + \frac{2(h-c)}{h^2}\left(f(h) - f(0) - hf'(0)\right) = 0$.

(vi) Deduce that $f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(c)$. [9]

(c) Hence show that, for $h > 0$

$$1 - \cos(h) \leq \frac{h^2}{2}.$$

[5]

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Wednesday 18 May 2016 (morning)

1 hour

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2216 – 7208

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1. [Maximum mark: 17]

The function f is defined by $f(x) = e^x \sin x$, $x \in \mathbb{R}$.

(a) By finding a suitable number of derivatives of f, determine the Maclaurin series for $f(x)$ as far as the term in x^3. [7]

(b) Hence, or otherwise, determine the exact value of $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^3}$. [3]

(c) The Maclaurin series is to be used to find an approximate value for $f(0.5)$.

(i) Use the Lagrange form of the error term to find an upper bound for the absolute value of the error in this approximation.

(ii) Deduce from the Lagrange error term whether the approximation will be greater than or less than the actual value of $f(0.5)$. [7]

2. [Maximum mark: 7]

A function f is given by $f(x) = \int_0^x \ln(2 + \sin t)\,dt$.

(a) Write down $f'(x)$. [1]

(b) By differentiating $f(x^2)$, obtain an expression for the derivative of $\int_0^{x^2} \ln(2 + \sin t)\,dt$ with respect to x. [3]

(c) Hence obtain an expression for the derivative of $\int_x^{x^2} \ln(2 + \sin t)\,dt$ with respect to x. [3]

3. [Maximum mark: 9]

(a) Given that $f(x) = \ln x$, use the mean value theorem to show that, for $0 < a < b$,

$$\frac{b-a}{b} < \ln\frac{b}{a} < \frac{b-a}{a}.$$ [7]

(b) Hence show that $\ln(1.2)$ lies between $\frac{1}{m}$ and $\frac{1}{n}$, where m, n are consecutive positive integers to be determined. [2]

4. [Maximum mark: 13]

Consider the differential equation $\frac{dy}{dx} = \frac{x}{y} - xy$ where $y > 0$ and $y = 2$ when $x = 0$.

(a) Show that putting $z = y^2$ transforms the differential equation into $\frac{dz}{dx} + 2xz = 2x$. [4]

(b) By solving this differential equation in z, obtain an expression for y in terms of x. [9]

5. [Maximum mark: 14]

Consider the infinite series $S = \sum_{n=0}^{\infty} u_n$ where $u_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t} dt$.

(a) Explain why the series is alternating. [1]

(b) (i) Use the substitution $T = t - \pi$ in the expression for u_{n+1} to show that $|u_{n+1}| < |u_n|$.

(ii) Show that the series is convergent. [9]

(c) Show that $S < 1.65$. [4]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Wednesday 18 November 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

8815 – 7208

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1. [Maximum mark: 5]

The function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f : x \to \begin{cases} 1 & , x < 0 \\ 1 - x, & x \geq 0 \end{cases}$.

By considering limits, prove that f is

(a) continuous at $x = 0$; [2]

(b) not differentiable at $x = 0$. [3]

2. [Maximum mark: 10]

Let $f(x) = e^x \sin x$.

(a) Show that $f''(x) = 2(f'(x) - f(x))$. [4]

(b) By further differentiation of the result in part (a), find the Maclaurin expansion of $f(x)$, as far as the term in x^5. [6]

3. [Maximum mark: 11]

(a) Prove by induction that $n! > 3^n$, for $n \geq 7$, $n \in \mathbb{Z}$. [5]

(b) Hence use the comparison test to prove that the series $\sum_{r=1}^{\infty} \frac{2^r}{r!}$ converges. [6]

4. [Maximum mark: 14]

Consider the function $f(x) = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$.

(a) Illustrate graphically the inequality, $\dfrac{1}{5}\displaystyle\sum_{r=1}^{5} f\left(\dfrac{r}{5}\right) < \int_0^1 f(x)\,\mathrm{d}x < \dfrac{1}{5}\sum_{r=0}^{4} f\left(\dfrac{r}{5}\right)$. [3]

(b) Use the inequality in part (a) to find a lower and upper bound for π. [5]

(c) Show that $\displaystyle\sum_{r=0}^{n-1} (-1)^r x^{2r} = \dfrac{1+(-1)^{n-1}x^{2n}}{1+x^2}$. [2]

(d) Hence show that $\pi = 4\left(\displaystyle\sum_{r=0}^{n-1} \dfrac{(-1)^r}{2r+1} - (-1)^{n-1}\int_0^1 \dfrac{x^{2n}}{1+x^2}\,\mathrm{d}x\right)$. [4]

5. [Maximum mark: 20]

The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1, 0)$ and are defined by the differential equations $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$ respectively.

(a) Show that the tangent to the curve $y = f(x)$ at the point $(1, 0)$ is normal to the curve $y = g(x)$ at the point $(1, 0)$. [2]

(b) Find $g(x)$. [6]

(c) Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places. [5]

(d) Explain why $y = f(x)$ cannot cross the isocline $x - y^2 = 0$, for $x > 1$. [3]

(e) (i) Sketch the isoclines $x - y^2 = -2, 0, 1$.

(ii) On the same set of axes, sketch the graph of f. [4]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Thursday 21 May 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
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- The maximum mark for this examination paper is **[60 marks]**.

2215 – 7208

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1. [Maximum mark: 7]

The function f is defined by $f(x) = \mathrm{e}^{-x}\cos x + x - 1$.

By finding a suitable number of derivatives of f, determine the first non-zero term in its Maclaurin series.

2. [Maximum mark: 8]

(a) Show that $y = \frac{1}{x}\int f(x)\,\mathrm{d}x$ is a solution of the differential equation

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x)$, $x > 0$. [3]

(b) Hence solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^{-\frac{1}{2}}$, $x > 0$, given that $y = 2$ when $x = 4$. [5]

3. [Maximum mark: 17]

(a) Show that the series $\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$ converges. [3]

(b) (i) Show that $\ln(n) + \ln\left(1 + \frac{1}{n}\right) = \ln(n+1)$.

(ii) Using this result, show that an application of the ratio test fails to determine whether or not $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ converges. [6]

(c) (i) State why the integral test can be used to determine the convergence or divergence of $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

(ii) Hence determine the convergence or divergence of $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$. [8]

4. [Maximum mark: 12]

(a) Use l'Hôpital's rule to find $\lim_{x\to\infty} x^2 e^{-x}$. [4]

(b) Show that the improper integral $\int_0^\infty x^2 e^{-x}\,dx$ converges, and state its value. [8]

5. [Maximum mark: 16]

(a) The mean value theorem states that if f is a continuous function on $[a, b]$ and differentiable on $]a, b[$ then $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some $c \in]a, b[$.

(i) Find the two possible values of c for the function defined by $f(x) = x^3 + 3x^2 - 2$ on the interval $[-3, 1]$.

(ii) Illustrate this result graphically. [7]

(b) (i) The function f is continuous on $[a, b]$, differentiable on $]a, b[$ and $f'(x) = 0$ for all $x \in]a, b[$. Show that $f(x)$ is constant on $[a, b]$.

(ii) Hence, prove that for $x \in [0, 1]$, $2\arccos x + \arccos\left(1 - 2x^2\right) = \pi$. [9]

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – CALCULUS

Thursday 13 November 2014 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
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1. *[Maximum mark: 14]*

(a) Use the integral test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}.$$ *[3]*

(b) Let $S = \sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n \times n^{0.5}}$.

(i) Use the ratio test to show that S is convergent for $-3 < x < 1$.

(ii) Hence find the interval of convergence for S. *[11]*

2. *[Maximum mark: 14]*

(a) Use an integrating factor to show that the general solution for $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{x}{t} = -\frac{2}{t},\ t > 0$ is $x = 2 + ct$, where c is a constant. *[4]*

The weight in kilograms of a dog, t weeks after being bought from a pet shop, can be modelled by the following function:

$$w(t) = \begin{cases} 2 + ct & 0 \le t \le 5 \\ 16 - \dfrac{35}{t} & t > 5 \end{cases}.$$

(b) Given that $w(t)$ is continuous, find the value of c. *[2]*

(c) Write down

(i) the weight of the dog when bought from the pet shop;

(ii) an upper bound for the weight of the dog. *[2]*

(d) Prove from first principles that $w(t)$ is differentiable at $t = 5$. *[6]*

3. *[Maximum mark: 10]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ where $f(x, y) = y - 2x$.

(a) Sketch, on one diagram, the four isoclines corresponding to $f(x, y) = k$ where k takes the values $-1, -0.5, 0$ and 1. Indicate clearly where each isocline crosses the y axis. *[2]*

A curve, C, passes through the point $(0, 1)$ and satisfies the differential equation above.

(b) Sketch C on your diagram. *[3]*

(c) State a particular relationship between the isocline $f(x, y) = -0.5$ and the curve C, at their point of intersection. *[1]*

(d) Use Euler's method with a step interval of 0.1 to find an approximate value for y on C, when $x = 0.5$. *[4]*

4. *[Maximum mark: 22]*

In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

(a) Consider the infinite geometric series

$$1 - x^2 + x^4 - x^6 + \ldots \qquad |x| < 1.$$

Show that the sum of the series is $\dfrac{1}{1+x^2}$. *[1]*

(b) Hence show that an expansion of $\arctan x$ is $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ *[4]*

(c) f is a continuous function defined on $[a, b]$ and differentiable on $]a, b[$ with $f'(x) > 0$ on $]a, b[$.

Use the mean value theorem to prove that for any $x, y \in [a, b]$, if $y > x$ then $f(y) > f(x)$. *[4]*

(d) (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$.

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$. *[4]*

(e) Use the result from part (c) to prove that $\arctan x > x - \dfrac{x^3}{3}$, for $x > 0$. *[5]*

(f) Hence show that $\dfrac{16}{3\sqrt{3}} < \pi < \dfrac{6}{\sqrt{3}}$. *[4]*

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – CALCULUS

Thursday 15 May 2014 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

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1. *[Maximum mark: 16]*

Consider the functions f and g given by $f(x)=\frac{e^x+e^{-x}}{2}$ and $g(x)=\frac{e^x-e^{-x}}{2}$.

(a) Show that $f'(x)=g(x)$ and $g'(x)=f(x)$. *[2]*

(b) Find the first three non-zero terms in the Maclaurin expansion of $f(x)$. *[5]*

(c) Hence find the value of $\lim_{x\to 0}\frac{1-f(x)}{x^2}$. *[3]*

(d) Find the value of the improper integral $\int_0^{\infty}\frac{g(x)}{\left[f(x)\right]^2}\,dx$. *[6]*

2. *[Maximum mark: 17]*

(a) Consider the functions $f(x)=(\ln x)^2$, $x>1$ and $g(x)=\ln\left(f(x)\right)$, $x>1$.

(i) Find $f'(x)$.

(ii) Find $g'(x)$.

(iii) Hence, show that $g(x)$ is increasing on $]1,\infty[$. *[5]*

(b) Consider the differential equation

$$(\ln x)\frac{dy}{dx}+\frac{2}{x}y=\frac{2x-1}{(\ln x)}, \; x>1.$$

(i) Find the general solution of the differential equation in the form $y=h(x)$.

(ii) Show that the particular solution passing through the point with coordinates $\left(e,\, e^2\right)$ is given by $y=\frac{x^2-x+e}{(\ln x)^2}$.

(iii) Sketch the graph of your solution for $x>1$, clearly indicating any asymptotes and any maximum or minimum points. *[12]*

3. *[Maximum mark: 12]*

Each term of the power series $\frac{1}{1\times 2}+\frac{1}{4\times 5}x+\frac{1}{7\times 8}x^2+\frac{1}{10\times 11}x^3+\ldots$ has the form $\frac{1}{b(n)\times c(n)}x^n$, where $b(n)$ and $c(n)$ are linear functions of n.

(a) Find the functions $b(n)$ and $c(n)$. *[2]*

(b) Find the radius of convergence. *[4]*

(c) Find the interval of convergence. *[6]*

4. *[Maximum mark: 15]*

The function f is defined by $f(x)=\begin{cases} e^{-x^2}\left(-x^3+2x^2+x\right), & x\le 1 \\ ax+b, & x>1 \end{cases}$, where a and b are constants.

(a) Find the exact values of a and b if f is continuous and differentiable at $x=1$. *[8]*

(b) (i) Use Rolle's theorem, applied to f, to prove that $2x^4-4x^3-5x^2+4x+1=0$ has a root in the interval $]-1, 1[$.

(ii) Hence prove that $2x^4-4x^3-5x^2+4x+1=0$ has at least two roots in the interval $]-1, 1[$. *[7]*

N13/5/MATHL/HP3/ENG/TZ0/SE

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Tuesday 19 November 2013 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

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1. *[Maximum mark: 10]*

Consider the infinite series $\sum_{n=1}^{\infty} \frac{2}{n^2+3n}$.

(a) Use a comparison test to show that the series converges. *[2]*

(b) (i) Express $\frac{2}{n^2+3n}$ in partial fractions.

(ii) Hence find the value of $\sum_{n=1}^{\infty} \frac{2}{n^2+3n}$. *[8]*

2. *[Maximum mark: 9]*

The general term of a sequence $\{a_n\}$ is given by the formula $a_n = \frac{e^n + 2^n}{2e^n}$, $n \in \mathbb{Z}^+$.

(a) Determine whether the sequence $\{a_n\}$ is decreasing or increasing. *[3]*

(b) Show that the sequence $\{a_n\}$ is convergent and find the limit L. *[2]*

(c) Find the smallest value of $N \in \mathbb{Z}^+$ such that $|a_n - L| < 0.001$, for all $n \geq N$. *[4]*

3. *[Maximum mark: 19]*

Consider the differential equation $\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$, for $x,\ y > 0$.

(a) Use Euler's method starting at the point $(x,\ y) = (1,\ 2)$, with interval $h = 0.2$, to find an approximate value of y when $x = 1.6$. *[7]*

(b) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = \frac{v}{1+\sqrt{v}} - v$. *[3]*

(c) (i) Hence find the solution of the differential equation in the form $f(x,\ y) = 0$, given that $y = 2$ when $x = 1$.

(ii) Find the value of y when $x = 1.6$. *[9]*

4. *[Maximum mark: 13]*

Let $g(x) = \sin x^2$, where $x \in \mathbb{R}$.

(a) Using the result $\lim_{t \to 0} \frac{\sin t}{t} = 1$, or otherwise, calculate $\lim_{x \to 0} \frac{g(2x) - g(3x)}{4x^2}$. *[4]*

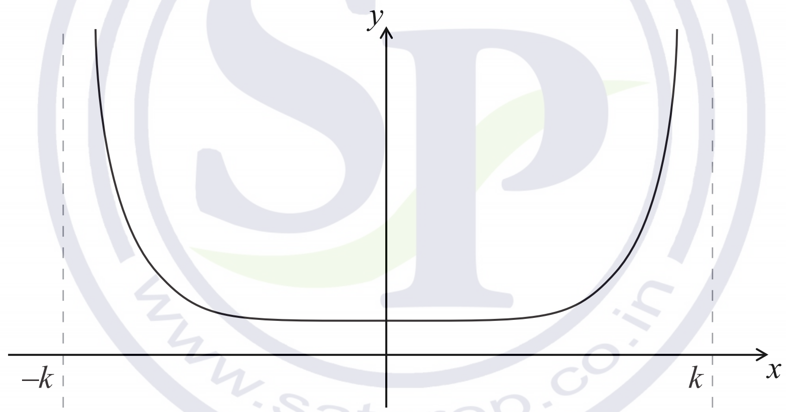
(b) Use the Maclaurin series of $\sin x$ to show that $g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$. *[2]*

(c) Hence determine the minimum number of terms of the expansion of $g(x)$ required to approximate the value of $\int_0^1 g(x)\,\mathrm{d}x$ to four decimal places. *[7]*

5. *[Maximum mark: 9]*

A function f is defined in the interval $]-k, k[$, where $k > 0$. The gradient function f' exists at each point of the domain of f.

The following diagram shows the graph of $y = f(x)$, its asymptotes and its vertical symmetry axis.



(a) Sketch the graph of $y = f'(x)$. *[2]*

Let $p(x) = a + bx + cx^2 + dx^3 + \ldots$ be the Maclaurin expansion of $f(x)$.

(b) (i) Justify that $a > 0$.

(ii) Write down a condition for the largest set of possible values for each of the parameters b, c and d. *[5]*

(c) State, with a reason, an upper bound for the radius of convergence. *[2]*

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Tuesday 21 May 2013 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

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1. *[Maximum mark: 9]*

The Taylor series of $\sqrt{x}$ about $x = 1$ is given by

$$a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \dots$$

(a) Find the values of a_0, a_1, a_2 and a_3. *[6 marks]*

(b) Hence, or otherwise, find the value of $\lim\limits_{x \to 1} \dfrac{\sqrt{x}-1}{x-1}$. *[3 marks]*

2. *[Maximum mark: 15]*

Consider the differential equation $\dfrac{dy}{dx} + y\tan x = \cos^2 x$, given that $y = 2$ when $x = 0$.

(a) Use Euler's method with a step length of 0.1 to find an approximation to the value of y when $x = 0.3$. *[5 marks]*

(b) (i) Show that the integrating factor for solving the differential equation is $\sec x$.

(ii) Hence solve the differential equation, giving your answer in the form $y = f(x)$. *[10 marks]*

3. *[Maximum mark: 11]*

Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$.

(a) Find the radius of convergence. *[4 marks]*

(b) Find the interval of convergence. *[3 marks]*

(c) Given that $x = -0.1$, find the sum of the series correct to three significant figures. *[4 marks]*

4. *[Maximum mark: 11]*

(a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. *[3 marks]*

(b) Let $S_n = \displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)}$.

(i) Show that $S_n = \dfrac{an^2 + bn}{4(n+1)(n+2)}$, where a and b are positive integers whose values should be determined.

(ii) Write down the value of $\lim\limits_{n\to\infty} S_n$. *[8 marks]*

5. *[Maximum mark: 14]*

(a)

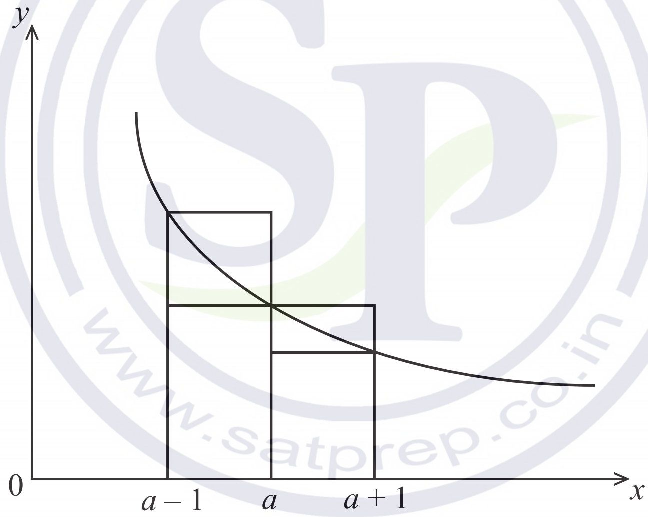

Figure 1

Figure 1 shows part of the graph of $y = \dfrac{1}{x}$ together with line segments parallel to the coordinate axes.

(i) By considering the areas of appropriate rectangles, show that

$$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)}.$$

(ii) Hence find lower and upper bounds for $\ln(1.2)$. *[9 marks]*

(This question continues on the following page)

(Question 5 continued)

(b)

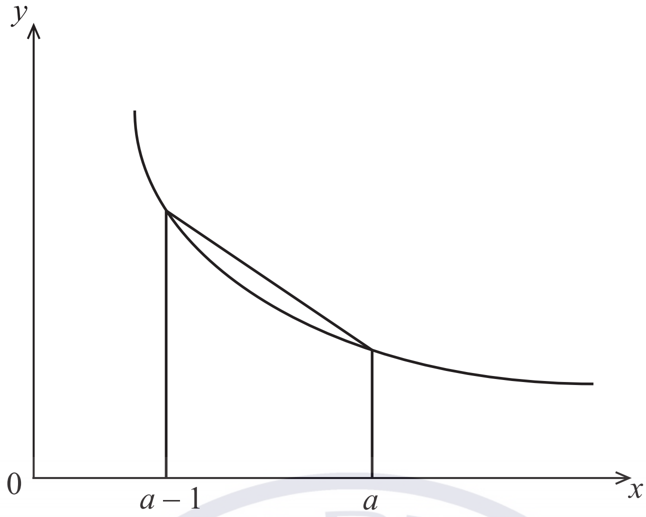

Figure 2

An improved upper bound can be found by considering Figure 2 which again shows part of the graph of $y=\dfrac{1}{x}$.

(i) By considering the areas of appropriate regions, show that

$$\ln\left(\frac{a}{a-1}\right)<\frac{2a-1}{2a(a-1)}.$$

(ii) Hence find an upper bound for $\ln(1.2)$. *[5 marks]*

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Thursday 8 November 2012 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

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- Answer all the questions.
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1. *[Maximum mark: 13]*

A differential equation is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$, where $x > 0$ and $y > 0$.

(a) Solve this differential equation by separating the variables, giving your answer in the form $y = f(x)$. *[3 marks]*

(b) Solve the same differential equation by using the standard homogeneous substitution $y = vx$. *[4 marks]*

(c) Solve the same differential equation by the use of an integrating factor. *[5 marks]*

(d) If $y = 20$ when $x = 2$, find y when $x = 5$. *[1 mark]*

2. *[Maximum mark: 12]*

Let the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x+y}$, $(x + y \geq 0)$ satisfying the initial conditions $y = 1$ when $x = 1$. Also let $y = c$ when $x = 2$.

(a) Use Euler's method to find an approximation for the value of c, using a step length of $h = 0.1$. Give your answer to four decimal places. *[6 marks]*

You are told that if Euler's method is used with $h = 0.05$ then $c \simeq 2.7921$, if it is used with $h = 0.01$ then $c \simeq 2.8099$ and if it is used with $h = 0.005$ then $c \simeq 2.8121$.

(b) Plot on graph paper, with h on the horizontal axis and the approximation for c on the vertical axis, the four points (one of which you have calculated and three of which have been given). Use a scale of 1 cm = 0.01 on both axes. Take the horizontal axis from 0 to 0.12 and the vertical axis from 2.76 to 2.82. *[3 marks]*

(c) Draw, by eye, the straight line that best fits these four points, using a ruler. *[1 mark]*

(d) Use your graph to give the best possible estimate for c, giving your answer to three decimal places. *[2 marks]*

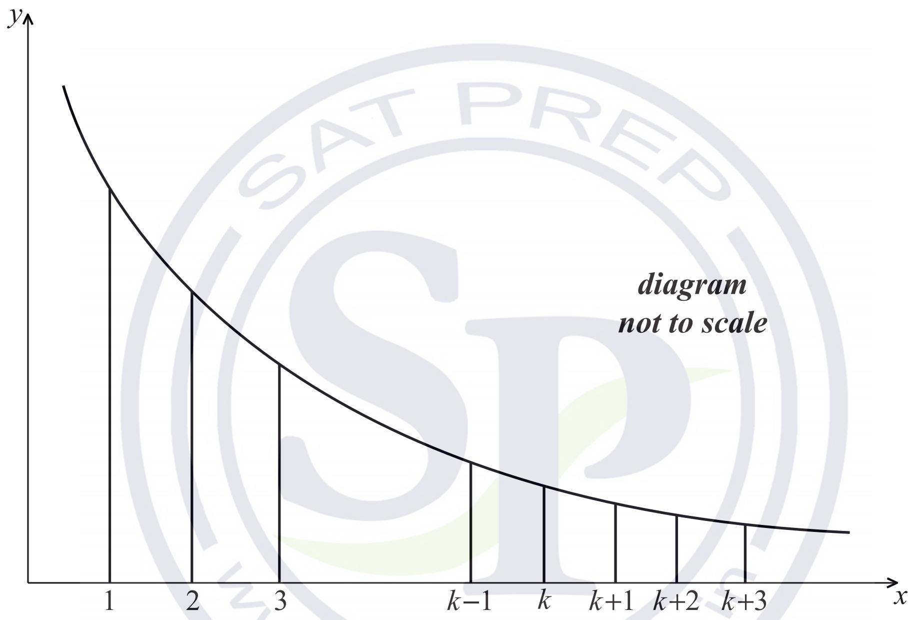
3. *[Maximum mark: 17]*

(a) Prove that $\lim_{H \to \infty} \int_a^H \frac{1}{x^2} \mathrm{d}x$ exists and find its value in terms of a (where $a \in \mathbb{R}^+$). *[3 marks]*

(b) Use the integral test to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. *[3 marks]*

Let $\sum_{n=1}^{\infty} \frac{1}{n^2} = L$.

(c) The diagram below shows the graph of $y = \frac{1}{x^2}$.



(i) Shade suitable regions on a copy of the diagram above and show that $\sum_{n=1}^{k} \frac{1}{n^2} + \int_{k+1}^{\infty} \frac{1}{x^2} \mathrm{d}x < L$.

(ii) Similarly shade suitable regions on another copy of the diagram above and show that $L < \sum_{n=1}^{k} \frac{1}{n^2} + \int_{k}^{\infty} \frac{1}{x^2} \mathrm{d}x$. *[6 marks]*

(d) Hence show that $\sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k+1} < L < \sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k}$. *[2 marks]*

You are given that $L = \frac{\pi^2}{6}$.

(e) By taking $k = 4$, use the upper bound and lower bound for L to find an upper bound and lower bound for π. Give your bounds to three significant figures. *[3 marks]*

4. *[Maximum mark: 18]*

(a) Use the limit comparison test to prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. *[5 marks]*

(b) Express $\frac{1}{n(n+1)}$ in partial fractions and hence find the value of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. *[4 marks]*

(c) Using the Maclaurin series for $\ln(1+x)$, show that the Maclaurin series for $(1+x)\ln(1+x)$ is $x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)}$. *[3 marks]*

(d) Hence find $\lim_{x \to -1} (1+x)\ln(1+x)$. *[2 marks]*

(e) Write down $\lim_{x \to 0} x \ln(x)$. *[1 mark]*

(f) Hence find $\lim_{x \to 0} x^x$. *[3 marks]*

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Monday 7 May 2012 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 6]*

Use L'Hôpital's Rule to find $\lim\limits_{x\to 0}\dfrac{e^x-1-x\cos x}{\sin^2 x}$.

2. *[Maximum mark: 21]*

Consider the differential equation $\dfrac{dy}{dx}=\dfrac{y^2}{1+x}$, where $x>-1$ and $y=1$ when $x=0$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x=0.5$. *[7 marks]*

(b) (i) Show that $\dfrac{d^2y}{dx^2}=\dfrac{2y^3-y^2}{(1+x)^2}$.

(ii) Hence find the Maclaurin series for y, up to and including the term in x^2. *[8 marks]*

(c) (i) Solve the differential equation.

(ii) Find the value of a for which $y\to\infty$ as $x\to a$. *[6 marks]*

3. *[Maximum mark: 7]*

Find the general solution of the differential equation $t\dfrac{dy}{dt}=\cos t-2y$, for $t>0$.

4. *[Maximum mark: 15]*

The sequence $\{u_n\}$ is defined by $u_n = \dfrac{3n+2}{2n-1}$, for $n \in \mathbb{Z}^+$.

(a) Show that the sequence converges to a limit L, the value of which should be stated. *[3 marks]*

(b) Find the least value of the integer N such that $|u_n - L| < \varepsilon$, for all $n > N$ where

(i) $\varepsilon = 0.1$;

(ii) $\varepsilon = 0.00001$. *[4 marks]*

(c) For each of the sequences $\left\{\dfrac{u_n}{n}\right\}, \left\{\dfrac{1}{2u_n - 2}\right\}$ and $\left\{(-1)^n u_n\right\}$, determine whether or not it converges. *[6 marks]*

(d) Prove that the series $\sum_{n=1}^{\infty}(u_n - L)$ diverges. *[2 marks]*

5. *[Maximum mark: 11]*

(a) Find the set of values of k for which the improper integral $\int_2^{\infty} \dfrac{\mathrm{d}x}{x(\ln x)^k}$ converges. *[6 marks]*

(b) Show that the series $\sum_{r=2}^{\infty} \dfrac{(-1)^r}{r \ln r}$ is convergent but not absolutely convergent. *[5 marks]*

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Friday 4 November 2011 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 5]*

Find $\lim_{x\to\frac{1}{2}}\left(\frac{\left(\frac{1}{4}-x^2\right)}{\cot\pi x}\right)$.

2. *[Maximum mark: 5]*

(a) Show that $n!\geq 2^{n-1}$, for $n\geq 1$. *[2 marks]*

(b) Hence use the comparison test to determine whether the series $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges or diverges. *[3 marks]*

3. *[Maximum mark: 11]*

Consider the series $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n\times 2^n}$.

(a) Find the radius of convergence of the series. *[7 marks]*

(b) Hence deduce the interval of convergence. *[4 marks]*

4. *[Maximum mark: 8]*

(a) Using the integral test, show that $\sum_{n=1}^{\infty}\frac{1}{4n^2+1}$ is convergent. *[4 marks]*

(b) (i) Show, by means of a diagram, that $\sum_{n=1}^{\infty}\frac{1}{4n^2+1}<\frac{1}{4\times 1^2+1}+\int_1^{\infty}\frac{1}{4x^2+1}\,\mathrm{d}x$.

(ii) Hence find an upper bound for $\sum_{n=1}^{\infty}\frac{1}{4n^2+1}$. *[4 marks]*

5. *[Maximum mark: 16]*

(a) Given that $y = \ln\left(\frac{1+e^{-x}}{2}\right)$, show that $\frac{dy}{dx} = \frac{e^{-y}}{2} - 1$. *[5 marks]*

(b) Hence, by repeated differentiation of the above differential equation, find the Maclaurin series for y as far as the term in x^3, showing that two of the terms are zero. *[11 marks]*

6. *[Maximum mark: 15]*

The real and imaginary parts of a complex number $x + iy$ are related by the differential equation $(x + y)\frac{dy}{dx} + (x - y) = 0$.

By solving the differential equation, given that $y = \sqrt{3}$ when $x = 1$, show that the relationship between the modulus r and the argument θ of the complex number is $r = 2e^{\frac{\pi}{3} - \theta}$.