N19/5/MATHL/HP3/ENG/TZ0/SE/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2019

Calculus

Higher level

Paper 3

14 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets** eg **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
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*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

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*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

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*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

7 Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

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12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
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Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) attempt to substitute $x = 3$ in both parts of f ***(M1)***

$\lim_{x \to 3^-} f(x) \left(= \frac{3-3}{3-5} \right) = 0$ ***A1***

$\lim_{x \to 3^+} f(x) \left(= \ln(3-2) \right) = 0$ ***A1***

(since $\lim_{x \to 3^-} f(x) = 0 = \lim_{x \to 3^+} f(x)$), f is continuous at $x = 3$ ***AG***

[3 marks]

(b) **METHOD 1**

for $x < 3$, $f'(x) = \frac{-2}{(x-5)^2}$ ***M1***

$\Rightarrow \lim_{x \to 3^-} f'(x) = -\frac{1}{2}$ (or equivalent) ***A1***

Note: Award ***A0*** for $f'(3) = -\frac{1}{2}$.

for $x > 3$, $f'(x) = \frac{1}{x-2}$ ***M1***

Note: Condone $x \geq 3$.

$\Rightarrow \lim_{x \to 3^+} f'(x) = 1$ (or equivalent) ***A1***

Note: Award ***A0*** for $f'(3) = 1$.

(since $-\frac{1}{2} \neq 1$), f is not differentiable at $x = 3$ ***AG***

METHOD 2

$\lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h}$ ***M1***

$= \lim_{h \to 0^-} \frac{\frac{h}{h-2} - 0}{h} = \lim_{h \to 0^-} \frac{1}{h-2}$

$= -\frac{1}{2}$ ***A1***

$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h}$ ***M1***

$= \lim_{h \to 0^+} \frac{\ln(1+h)}{h} = \lim_{h \to 0^+} \frac{1}{1+h}$ (by L'Hôpital)

$= 1$ ***A1***

(since $-\frac{1}{2} \neq 1$), f is not differentiable at $x = 3$ ***AG***

[4 marks]

Total [7 marks]

2. (a) **METHOD 1**

attempt to use limit comparison test and choosing an appropriate b_n ***M1***

let $a_n = \frac{3n}{2n^2+5}$ and $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{3n}{2n^2+5}}{\frac{1}{n}}$$ ***(A1)***

$$= \lim_{n\to\infty}\frac{3}{2+\frac{5}{n^2}}$$

$$= \frac{3}{2}\ (>0)$$ ***A1***

since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}\frac{3n}{2n^2+5}$ also diverges (by the limit comparison test) ***R1***

NOTE: Do not award ***R1*** if candidates omit sigma.

METHOD 2

attempt to find $\int_1^{\infty}\frac{3x}{2x^2+5}\mathrm{d}x$ ***M1***

$$= \lim_{R\to\infty}\int_1^{R}\frac{3x}{2x^2+5}\mathrm{d}x$$

$$= \lim_{R\to\infty}\left[\frac{3}{4}\ln(2x^2+5)\right]_1^R$$ ***A1***

NOTE: Condone use of ∞ as the upper limit.

$$= \lim_{R\to\infty}\frac{3}{4}\ln(2R^2+5) - \frac{3}{4}\ln 7$$

$= \infty$ (accept limit DNE) ***A1***

since $\int_1^{\infty}\frac{3x}{2x^2+5}\mathrm{d}x$ diverges, $\sum_{n=1}^{\infty}\frac{3n}{2n^2+5}$ also diverges (by the integral test.) ***R1***

continued…

Question 2 continued

METHOD 3

attempt to use comparison test and choosing an appropriate b_n ***M1***

EITHER

for $n > 2$ ***A1***

$\frac{3n}{2n^2+5} > \frac{1}{n}$ ***A1***

OR

for $n \geq 1$ ***(A1)***

$\frac{3n}{2n^2+5} > \frac{1}{3n}$ ***A1***

THEN

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{3n}{2n^2+5}$ also diverges (by the comparison test.) ***R1***

Note:	In both cases accept valid alternative inequalities. Do not award ***R1*** if candidates omit sigma.

[4 marks]

(b) attempt to use ratio test

$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{3^{n+1}\left((n+1)!\right)^2} \times \frac{3^n (n!)^2}{(2n)!}$ (***M1***)

attempt to simplify factorials (***M1***)

$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{3(n+1)^2} \quad \left(= \frac{4n^2+6n+2}{3n^2+6n+3} \right)$ ***A1***

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{3\left(1+\frac{1}{n}\right)^2}$ ***(M1)***

$= \frac{4}{3}$ ***A1***

since $\frac{4}{3} > 1$, $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n (n!)^2}$ diverges (by the ratio test) ***R1***

Note:	Award ***R1*** for correct reasoning consistent with their limit.

[6 marks]

Total [10 marks]

3. (a) $f(x) = \arcsin(2x)$

$$f'(x) = \frac{2}{\sqrt{1-4x^2}}$$ ***M1A1***

> **Note:** Award ***M1A0*** for $f'(x) = \frac{1}{\sqrt{1-4x^2}}$.

$$f''(x) = \frac{8x}{\left(1-4x^2\right)^{\frac{3}{2}}}$$ ***A1***

EITHER

$$f'''(x) = \frac{8\left(1-4x^2\right)^{\frac{3}{2}} - 8x\left(\frac{3}{2}(-8x)\left(1-4x^2\right)^{\frac{1}{2}}\right)}{\left(1-4x^2\right)^3} \left(= \frac{8\left(1-4x^2\right)^{\frac{3}{2}} + 96x^2\left(1-4x^2\right)^{\frac{1}{2}}}{\left(1-4x^2\right)^3}\right)$$ ***A1***

OR

$$f'''(x) = 8\left(1-4x^2\right)^{-\frac{3}{2}} + 8x\left(-\frac{3}{2}\left(1-4x^2\right)^{-\frac{5}{2}}\right)(-8x) \left(= 8\left(1-4x^2\right)^{-\frac{3}{2}} + 96x^2\left(1-4x^2\right)^{-\frac{5}{2}}\right)$$ ***A1***

THEN

substitute $x = 0$ into f or any of its derivatives ***(M1)***

$f(0) = 0$, $f'(0) = 2$ and $f''(0) = 0$ ***A1***

$f'''(0) = 8$

the Maclaurin series is

$$f(x) = 2x + \frac{8x^3}{6} + \ldots \left(= 2x + \frac{4x^3}{3} + \ldots\right)$$ ***(M1)A1***

[8 marks]

continued…

Question 3 continued

(b) **METHOD 1**

$$\lim_{x\to 0}\frac{\arcsin(2x)-2x}{(2x)^3}=\lim_{x\to 0}\frac{2x+\frac{4x^3}{3}+\ldots-2x}{8x^3}$$ ***M1***

$$=\lim_{x\to 0}\frac{\frac{4}{3}+\ldots\text{ terms with }x}{8}$$ ***(M1)***

$$=\frac{1}{6}$$ ***A1***

Note: Condone the omission of +… in their working.

METHOD 2

$$\lim_{x\to 0}\frac{\arcsin(2x)-2x}{(2x)^3}=\frac{0}{0}$$ indeterminate form, using L'Hôpital's rule

$$=\lim_{x\to 0}\frac{\frac{2}{\sqrt{1-4x^2}}-2}{24x^2}$$ ***M1***

$$=\frac{0}{0}$$ indeterminate form, using L'Hôpital's rule again

$$=\lim_{x\to 0}\frac{\frac{8x}{\left(1-4x^2\right)^{\frac{3}{2}}}}{48x}\left(=\lim_{x\to 0}\frac{1}{6\left(1-4x^2\right)^{\frac{3}{2}}}\right)$$ ***M1***

Note: Award ***M1*** only if their previous expression is in indeterminate form.

$$=\frac{1}{6}$$ ***A1***

Note: Award ***FT*** for use of their derivatives from part (a).

[3 marks]

Total [11 marks]

4. (a)

x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$	
1	2	6	***(M1)***
1.1	2.6	7.22	***(A1)***
1.2	3.32	8.89652	***(A1)***
1.3	4.21	11.26	***(A1)***
1.4	5.34		***A1***

$y(1.4) \approx 5.34$

Note: Award ***A1*** for each correct y value.
For the intermediate y values, accept answers that are accurate to 2 significant figures.
The final y value must be accurate to 3 significant figures or better.

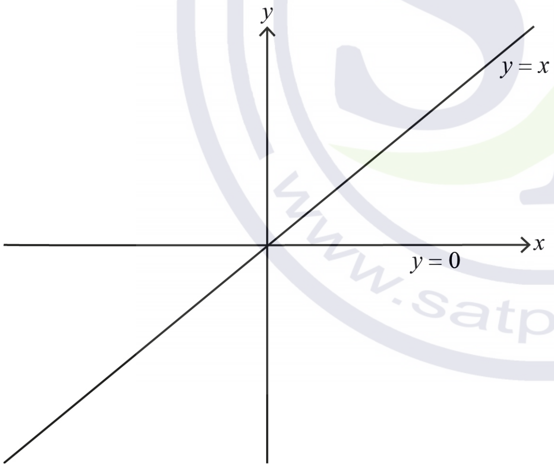
[5 marks]

(b) attempt to solve $\frac{4x^2 + y^2 - xy}{x^2} = 4$ ***(M1)***

$\Rightarrow y^2 - xy = 0$

$y(y - x) = 0$

$y = 0$ or $y = x$



A1A1

[3 marks]

(c) (i) $m^2 - 2m + 4 = (m-1)^2 + 3 \quad (a = 1, b = 3)$ ***A1***

continued…

Question 4 continued

(ii) recognition of homogeneous equation,
let $y = vx$ **M1**

the equation can be written as

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 4 + v^2 - v$$ ***(A1)***

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - 2v + 4$$

$$\int \frac{1}{v^2 - 2v + 4}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$$ ***M1***

Note: Award ***M1*** for attempt to separate the variables.

$$\int \frac{1}{(v-1)^2 + 3}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$$ from part (c)(i) ***M1***

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{v-1}{\sqrt{3}}\right) = \ln x \ (+c)$$ ***A1A1***

$$x = 1, y = 2 \Rightarrow v = 2$$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) = \ln 1 + c$$ ***M1***

Note: Award ***M1*** for using initial conditions to find c.

$$\Rightarrow c = \frac{\pi}{6\sqrt{3}} \quad (= 0.302)$$ ***A1***

$$\arctan\left(\frac{v-1}{\sqrt{3}}\right) = \sqrt{3}\ln x + \frac{\pi}{6}$$

substituting $v = \frac{y}{x}$ ***M1***

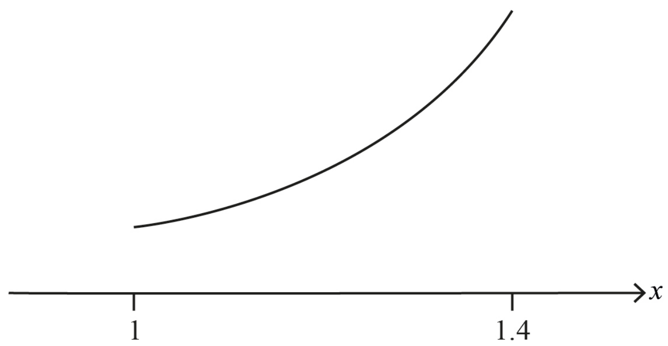
Note: This ***M1*** may be awarded earlier.

$$y = x\left(\sqrt{3}\tan\left(\sqrt{3}\ln x + \frac{\pi}{6}\right) + 1\right)$$ ***A1***

continued…

Question 4 continued

(iii)



curve drawn over correct domain ***A1***

(iv) the sketch shows that f is concave up ***A1***

Note: Accept f' is increasing.

this means the tangent drawn using Euler's method will give an underestimate of the real value, so $f(1.4) >$ estimate in part (a) ***R1***

Note: The ***R1*** is dependent on the ***A1***.

[14 marks]

Total [22 marks]

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1. (a) $\frac{dx}{dt} = 0.056x - 0.035x$ ***A1***

$\frac{dx}{dt} = 0.021x$ ***AG***

[1 mark]

(b) **METHOD 1**

$\frac{dx}{dt} = 0.021x$

attempt to separate variables ***M1***

$\int \frac{1}{x} dx = \int 0.021 dt$ ***A1***

$\ln x = 0.021t (+c)$ ***A1***

EITHER

$x = Ae^{0.021t}$

$\Rightarrow 2A = Ae^{0.021t}$ ***A1***

Note: This ***A1*** is independent of the following marks.

OR

$t = 0, x = x_0 \Rightarrow c = \ln x_0$

$\Rightarrow \ln 2x_0 = 0.021t + \ln x_0$ ***A1***

Note: This ***A1*** is independent of the following marks.

THEN

$\Rightarrow \ln 2 = 0.021t$ ***(M1)***

$\Rightarrow t = 33$ years ***A1***

Note: If a candidate writes $t = 33.007$, so $t = 34$ then award the final ***A1***.

[6 marks]

continued…

Question 1 continued

METHOD 2

$\frac{dx}{dt} = 0.021x$

attempt to separate variables **M1**

$\int_A^{2A} \frac{1}{x} dx = \int_0^t 0.021 du$ **A1A1**

Note: Award ***A1*** for correct integrals and ***A1*** for correct limits seen anywhere. Do not penalize use of t in place of u.

$[\ln x]_A^{2A} = [0.021u]_0^t$ **A1**

$\Rightarrow \ln 2 = 0.021t$ ***(M1)***

$\Rightarrow t = 33$ **A1**

[6 marks]

Total [7 marks]

2. (a) $S = 1 - x^2 + x^4 - x^6 + \dots$

recognition of GP $u_1 = 1,\ r = -x^2$ **M1**

$S_\infty = \frac{1}{1+x^2}$ **AG**

Note: Accept a correct algebraic method such as $(1+x^2)(1-x^2+x^4-x^6+...) = 1+x^2-x^2-x^4+x^4+... = 1$.

Note: Accept finding the Maclaurin series for $\frac{1}{(1+x^2)}$ only if the first four derivatives and their values at $x = 0$ are shown.

Note: Accept a correct argument based on using the Maclaurin series for $\arctan x$.

[1 mark]

(b) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$

attempt to substitute $2x$ ***(M1)***

$f(x) = \frac{1}{1+4x^2} = 1 - 4x^2 + 16x^4 - 64x^6 + \dots$ **A1**

Note: Accept use of a GP with $r = -4x^2$.

[2 marks]

continued…

Question 2 continued

(c) **EITHER**

$\int \frac{1}{1+4x^2}\,\mathrm{d}x = \frac{1}{2}\arctan 2x\,(+c)$ ***M1A1***

OR

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan 2x) = \frac{2}{1+4x^2}$ ***M1A1***

THEN

$\frac{1}{2}\arctan 2x\,(+c) = \int \left(1 - 4x^2 + 16x^4 - 64x^6 + \ldots\right)\mathrm{d}x$ ***(M1)***

$= x - \frac{4x^3}{3} + \frac{16x^5}{5} - \frac{64x^7}{7} + \ldots$ ***(A1)***

when $x = 0$ $\arctan 2x = 0 \Rightarrow c = 0$ ***R1***

$(\arctan 2x) = 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7}$ ***A1***

Note: No accuracy marks should be lost due to absence of c.

[6 marks]

Total [9 marks]

3. (a) attempt to use the comparison test with any convergent series ***M1***

$\frac{8^n}{n3^{2n+1}} < \frac{8^n}{3^{2n}}$ ***(A1)***

Note: Award ***A0*** for comparing series, *eg*, $\sum_{n=1}^{\infty}\frac{8^n}{n3^{2n+1}} < \sum_{n=1}^{\infty}\frac{8^n}{3^{2n}}$. However, subsequent marks may still be awarded.

$< \left(\frac{8}{9}\right)^n$ ***A1***

Note: Award ***A1*** for recognition of a geometric series with $r = \frac{8}{9}$.

$\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n$ converges (as it is a geometric series with common ratio $\lvert r\rvert < 1$) ***R1***

Note: Award ***R0*** for a statement such as "$\left(\frac{8}{9}\right)^n$ converges".

series converges by comparison test ***AG***

Note: Award a maximum of ***M0A0A0R1***, if the limit comparison test is used instead of the comparison test.

[4 marks]

(b) attempt to use the ratio test ***M1***

$\frac{a_{n+1}}{a_n} = \frac{(n+1)8^{n+1}}{3^{2n+3}} \times \frac{3^{2n+1}}{n8^n}$ ***A1***

$= \frac{8(n+1)}{9n}\left(= \frac{8}{9}\left(1+\frac{1}{n}\right)\right)$ ***(M1)***

$\to \frac{8}{9}$ (as $n \to \infty$) ***A1***

since $\frac{8}{9} < 1$ series converges (by the ratio test) ***R1***

Note: Award ***R1*** for comparing their limit to 1 and stating a consistent conclusion. Award ***R0*** if their limit equals 1.

[5 marks]

Total [9 marks]

4. $\lim_{x\to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)$

$\lim_{x\to 0}\left(\frac{3\sec^2 3x - 3\sec^2 x}{3\cos 3x - 3\cos x}\right) \quad \left(= \lim_{x\to 0}\left(\frac{\sec^2 3x - \sec^2 x}{\cos 3x - \cos x}\right)\right)$ ***M1A1A1***

Note:	Award ***M1*** for attempt at differentiation using l'Hopital's rule, ***A1*** for numerator, ***A1*** for denominator.

METHOD 1

using l'Hopital's rule again

$= \lim_{x\to 0}\left(\frac{18\sec^2 3x\tan 3x - 6\sec^2 x\tan x}{-9\sin 3x + 3\sin x}\right)\left(= \lim_{x\to 0}\left(\frac{6\sec^2 3x\tan 3x - 2\sec^2 x\tan x}{-3\sin 3x + \sin x}\right)\right)$ ***A1A1***

EITHER

$= \lim_{x\to 0}\left(\frac{108\sec^2 3x\tan^2 3x + 54\sec^4 3x - 12\sec^2 x\tan^2 x - 6\sec^4 x}{-27\cos 3x + 3\cos x}\right)$ ***A1A1***

Note:	Not all terms in numerator need to be written in final fraction. Award ***A1*** for $54\sec^4 3x + ... - 6\sec^4 x - ...$. However, if the terms are written, they must be correct to award ***A1***.

attempt to substitute $x = 0$ ***M1***

$= \frac{48}{-24}$

OR

$\frac{d}{dx}\left(18\sec^2 3x\tan 3x - 6\sec^2 x\tan x\right)\Big|_{x=0} = 48$ ***(M1)A1***

$\frac{d}{dx}\left(-9\sin 3x + 3\sin x\right)\Big|_{x=0} = -24$ ***A1***

THEN

$\left(\lim_{x\to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)\right) = -2$ ***A1***

continued…

Question 4 continued

METHOD 2

$$= \lim_{x \to 0} \left(\frac{\frac{3}{\cos^2 3x} - \frac{3}{\cos^2 x}}{3\cos 3x - 3\cos x} \right)$$ ***M1***

$$= \lim_{x \to 0} \left(\frac{\cos^2 x - \cos^2 3x}{\cos^2 3x \cos^2 x \left(\cos 3x - \cos x\right)} \right)$$ ***A1***

$$= \lim_{x \to 0} \left(\frac{\cos x + \cos 3x}{-\cos^2 3x \cos^2 x} \right)$$ ***M1A1***

attempt to substitute $x = 0$ ***M1***

$$= \frac{2}{-1}$$

$$= -2$$ ***A1***

Total [9 marks]

5. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - x^2}{2xy}$

let $y = vx$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2 - x^2}{2vx^2}$ ***(M1)***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2 - 1}{2v} \quad \left(= \frac{v}{2} - \frac{1}{2v}\right)$ ***(A1)***

Note: Or equivalent attempt at simplification.

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-v^2 - 1}{2v} \quad \left(= -\frac{v}{2} - \frac{1}{2v}\right)$ ***A1***

$\frac{2v}{1+v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{x}$ ***(M1)***

$\int \frac{2v}{1+v^2}\mathrm{d}v = \int -\frac{1}{x}\mathrm{d}x$ ***(A1)***

$\ln\left(1 + v^2\right) = -\ln x + \ln c$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS and a constant.

$\ln\left(1 + \left(\frac{y}{x}\right)^2\right) = -\ln x + \ln c$ ***M1***

Note: Award ***M1*** for substituting $v = \frac{y}{x}$. May be seen at a later stage.

$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}$ ***A1***

Note: Award ***A1*** for any correct equivalent equation without logarithms.

$x^2 + y^2 = cx$ ***AG***

[11 marks]

continued…

Question 5 continued

(b) **METHOD 1**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - x^2}{2xy}$$

(for horizontal tangents) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$\left(\Rightarrow y^2 = x^2\right) \Rightarrow y = \pm x$

EITHER

using $x^2 + y^2 = cx \Rightarrow 2x^2 = cx$ ***M1***

$2x^2 - cx = 0 \Rightarrow x = \frac{c}{2}$ ***A1***

Note: Award ***M1A1*** for $2y^2 = \pm cy$.

OR

using implicit differentiation of $x^2 + y^2 = cx$

$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = c$ ***M1***

Note: Accept differentiation of $y = \sqrt{cx - x^2}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = \frac{c}{2}$ ***A1***

THEN

tangents at $y = \frac{c}{2},\ y = -\frac{c}{2}$ ***A1A1***

hence there are two tangents ***AG***

METHOD 2

$x^2 + y^2 = cx$

$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$ ***M1A1***

this is a circle radius $\frac{c}{2}$ centre $\left(\frac{c}{2}, 0\right)$ ***A1***

hence there are two tangents ***AG***

tangents at $y = \frac{c}{2}, y = -\frac{c}{2}$ ***A1A1***

[5 marks]

Total [16 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2018

Calculus

Higher level

Paper 3

12 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2018**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets** eg **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) compare with $\sum_{n=1}^{\infty}\frac{1}{n}$ ***R1***

> **Note:** Accept comparison with $\sum_{n=1}^{\infty}\frac{1}{3n}$ or similar.

$\lim_{n\to\infty}\left(\frac{2n+1}{3n^2}\div\frac{1}{n}\right)$ ***M1***

$=\lim_{n\to\infty}\frac{2n^2+n}{3n^2}$

$=\frac{2}{3}$ ***A1***

$\frac{2}{3}$ (is finite and) not equal to 0 (so both series do the same) ***R1***

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

so $\sum_{n=1}^{\infty}\frac{2n+1}{3n^2}$ diverges ***A1***

> **Note:** Award ***R0M1A0R1A0*** if candidates apply compare with a convergent series; award the last ***R1*** only if the reason is consistent with the limit value.

[5 marks]

(b) $u_n=\frac{n^2}{n!}(x-1)^n \Rightarrow u_{n+1}=\frac{(n+1)^2}{(n+1)!}(x-1)^{n+1}$

$\left|\frac{u_{n+1}}{u_n}\right|=\frac{(n+1)^2}{(n+1)!}\frac{n!}{n^2}|x-1|=\frac{n+1}{n^2}|x-1|$ ***M1A1***

> **Note:** Award ***A1*** for any correct expression without the factorials that allows calculation of the limit below.

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=0, \forall x\in\mathbb{R}$ ***M1A1***

as $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1, \forall x\in\mathbb{R}$, ***R1***

the series converges $\forall x\in\mathbb{R}$ ***AG***

[5 marks]

Total [10 marks]

2. (a) $\lim\limits_{x\to 0}\dfrac{e^{-3x^2}+3\cos 2x-4}{3x^2}=\left(\dfrac{0}{0}\right)$

$=\lim\limits_{x\to 0}\dfrac{-6xe^{-3x^2}-6\sin 2x}{6x}=\left(\dfrac{0}{0}\right)$ ***M1A1A1***

$=\lim\limits_{x\to 0}\dfrac{-6e^{-3x^2}+36x^2e^{-3x^2}-12\cos 2x}{6}$ ***A1***

$=-3$ ***A1***

[5 marks]

(b) $\lim\limits_{x\to 0}\left(\dfrac{\int_0^x\left(e^{-3t^2}+3\cos 2t-4\right)dt}{\int_0^x 3t^2\,dt}\right)$ is of the form $\dfrac{0}{0}$

applying l'Hôpital´s rule ***(M1)***

$=\lim\limits_{x\to 0}\dfrac{e^{-3x^2}+3\cos 2x-4}{3x^2}$ ***(A1)***

$=-3$ ***A1***

[3 marks]

Total [8 marks]

3. (a) **METHOD 1**

attempt at differentiation ***M1***

$2(x+2)\dfrac{dy}{dx}+(x+2)^2\dfrac{d^2y}{dx^2}=(x+1)\dfrac{dy}{dx}+y$ ***A1A1***

Note: Award ***A1*** for LHS, ***A1*** for RHS.

$2\dfrac{dy}{dx}+2(x+2)\dfrac{d^2y}{dx^2}+2(x+2)\dfrac{d^2y}{dx^2}+(x+2)^2\dfrac{d^3y}{dx^3}=(x+1)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+\dfrac{dy}{dx}$ ***M1***

$4(x+2)\dfrac{d^2y}{dx^2}+(x+2)^2\dfrac{d^3y}{dx^3}=(x+1)\dfrac{d^2y}{dx^2}$

$(x+2)^2\dfrac{d^3y}{dx^3}=\left((x+1)-4(x+2)\right)\dfrac{d^2y}{dx^2}$ ***A1***

$\dfrac{d^3y}{dx^3}=-\dfrac{3x+7}{(x+2)^2}\dfrac{d^2y}{dx^2}$ ***AG***

[5 marks]

continued…

Question 3 continued

METHOD 2

$$\frac{dy}{dx}=\frac{y(x+1)}{(x+2)^2}$$

$$\frac{d^2y}{dx^2}=\frac{(x+2)^2\left(\frac{dy}{dx}(x+1)+y\right)-y(x+1)\times 2(x+2)}{(x+2)^4}$$ ***M1A1***

$$=\frac{y\left((x+1)^2+(x+2)^2-2(x+1)(x+2)\right)}{(x+2)^4}$$

$$=\frac{y\left((x+2)-(x+1)\right)^2}{(x+2)^4}$$

$$\frac{d^2y}{dx^2}=\frac{y}{(x+2)^4}\ \left(\text{or } \frac{d^2y}{dx^2}=\frac{1}{(x+2)^2(x+1)}\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3}=\frac{(x+2)^4\frac{dy}{dx}-y\times 4(x+2)^3}{(x+2)^8}$$ ***A1***

$$=\frac{(x+2)^4\frac{(x+1)y}{(x+2)^2}-4y(x+2)^3}{(x+2)^8}$$ ***M1***

$$=\frac{y}{(x+2)^4}\times\left(\frac{x+1}{(x+2)^2}-\frac{4}{x+2}\right)$$

$$=\frac{d^2y}{dx^2}\left(\frac{x+1}{(x+2)^2}-\frac{4(x+2)}{(x+2)^2}\right)$$ ***A1***

$$\frac{d^3y}{dx^3}=-\frac{3x+7}{(x+2)^2}\frac{d^2y}{dx^2}$$ ***AG***

(b) (i) $y(1)=2\Rightarrow\frac{dy}{dx}=\frac{4}{9}\ (=0.4444\ldots)$ ***A1***

$$(x+2)^2\frac{d^2y}{dx^2}=-\frac{dy}{dx}(x+3)+y$$ ***(M1)***

$$\left.\frac{d^2y}{dx^2}\right|_{(1,2)}=\frac{2}{81}\ (=0.02469\ldots)$$ ***A1***

$$y(x)=y(1)+y'(1)(x-1)+y''(1)\frac{(x-1)^2}{2}$$ ***(M1)***

$$=2+\frac{4}{9}(x-1)+\frac{1}{81}(x-1)^2$$ ***A1***

$$\left(=2+0.444(x-1)+0.0123(x-1)^2\right)$$

Note: Allow coefficients rounded to two correct significant figures.

continued…

Question 3 continued

(ii) $\left.\frac{d^3y}{dx^3}\right|_{(1,2)} = -\frac{20}{729}(= -0.02743...)$ ***A1***

$y(x) = y(1) + y'(1)(x-1) + y''(1)\frac{(x-1)^2}{2} + y'''(1)\frac{(x-1)^3}{6}$

$= 2 + \frac{4}{9}(x-1) + \frac{1}{81}(x-1)^2 - \frac{10}{2187}(x-1)^3$ ***A1***

$\left(= 2 + 0.444(x-1) + 0.0123(x-1)^2 - 0.00457(x-1)^3\right)$

Note: Allow coefficients rounded to two correct significant figures.

[7 marks]

(c) difference is $\frac{10}{2187}(0.05)^3\left(= 5.72\times10^{-7}\right)$ ***(M1)A1***

Note: Accept any answer that rounds to 5.7×10^{-7}. Accept $\pm5.72\times10^{-7}$.

Note: Allow ***FT*** only if the answer is obtained from the degree 3 term of the polynomial in b(ii)

[2 marks]

Total [14 marks]

4. (a) attempt to apply Euler's method ***(M1)***

$x_{n+1} = x_n + 0.25;\ y_{n+1} = y_n + 0.25\times\left(1+\frac{y_n}{x_n}\right)$

x	y	$\frac{dy}{dx}$
1.00	1.00000	2.00000
1.25	1.50000	2.20000
1.50	2.05000	2.36667
1.75	2.64167	2.50952
2.00	3.26905	

(A1)(A1)

Note: Award ***A1*** for correct x values, ***A1*** for first three correct y values.

$y = 3.3$ ***A1***

[4 marks]

continued…

Question 4 continued

(b) **METHOD 1**

$I(x) = e^{\int -\frac{1}{x}dx}$ ***(M1)***

$= e^{-\ln x}$

$= \frac{1}{x}$ ***(A1)***

$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}$ ***(M1)***

$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$

$\frac{y}{x} = \ln|x| + C$ ***A1***

$y(1) = 1 \Rightarrow C = 1$ ***M1***

$y = x\ln|x| + x$ ***A1***

METHOD 2

$v = \frac{y}{x}$ ***M1***

$\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y$ ***(A1)***

$v + x\frac{dv}{dx} = 1 + v$ ***M1***

$\int 1 dv = \int \frac{1}{x}dx$

$v = \ln|x| + C$

$\frac{y}{x} = \ln|x| + C$ ***A1***

$y(1) = 1 \Rightarrow C = 1$ ***M1***

$y = x\ln|x| + x$ ***A1***

Note: Modulus sign need only be seen in the final answer.

[6 marks]

(c) $y(2) = 2\ln 2 + 2 = 3.38629\ldots$

percentage error $= \frac{3.38629\ldots - 3.3}{3.38629\ldots} \times 100\%$ ***(M1)(A1)***

$= 2.5\%$ ***A1***

[3 marks]

continued…

Question 4 continued

(d) (i) $\frac{dy}{dx} = k \Rightarrow 1 + \frac{y}{x} = k$ ***A1***

$y = (k-1)x$

(ii) gradient of isocline equals gradient of normal ***(M1)***

$k - 1 = -\frac{1}{k}$ or $k(k-1) = -1$ ***A1***

$k^2 - k + 1 = 0$ ***A1***

$\Delta = 1 - 4 < 0$ ***R1***

$\therefore$ no solution ***AG***

Note: Accept alternative reasons for no solutions.

[5 marks]

Total [18 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2018

Calculus

Higher level

Paper 3

15 pages

Instructions to Examiners

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

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  Examples

	Correct answer seen	**Further working seen**	**Action**
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1. (a) **METHOD 1**

$\ln(n+2) < n+2$ ***(A1)***

$\Rightarrow \frac{1}{\ln(n+2)} > \frac{1}{n+2}$ (for $n \geq 0$) ***A1***

Note: Award ***A0*** for statements such as $\sum_{n=0}^{\infty}\frac{1}{\ln(n+2)} > \sum_{n=0}^{\infty}\frac{1}{n+2}$.
However condone such a statement if the above ***A1*** has already been awarded.

$\sum_{n=0}^{\infty}\frac{1}{n+2}$ (is a harmonic series which) diverges ***R1***

Note: The ***R1*** is independent of the ***A1***s.
Award ***R0*** for statements such as "$\frac{1}{n+2}$ diverges".

so $\sum_{n=0}^{\infty}\frac{1}{\ln(n+2)}$ diverges by the comparison test ***AG***

METHOD 2

$\frac{1}{\ln n} > \frac{1}{n}$ (for $n \geq 2$) ***A1***

Note: Award ***A0*** for statements such as $\sum_{n=2}^{\infty}\frac{1}{\ln n} > \sum_{n=2}^{\infty}\frac{1}{n}$.
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a correct statement linking n and $n+2$ *eg*,
$\sum_{n=0}^{\infty}\frac{1}{\ln(n+2)} = \sum_{n=2}^{\infty}\frac{1}{\ln n}$ or $\sum_{n=0}^{\infty}\frac{1}{n+2} = \sum_{n=2}^{\infty}\frac{1}{n}$ ***A1***

Note: Award ***A0*** for $\sum_{n=0}^{\infty}\frac{1}{n}$.

$\sum_{n=2}^{\infty}\frac{1}{n}$ (is a harmonic series which) diverges

(which implies that $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by the comparison test) ***R1***

Note: The ***R1*** is independent of the ***A1***s.
Award ***R0*** for statements such as $\sum_{n=0}^{\infty}\frac{1}{n}$ diverges and "$\frac{1}{n}$ diverges".
Award ***A1A0R1*** for arguments based on $\sum_{n=1}^{\infty}\frac{1}{n}$.

so $\sum_{n=0}^{\infty}\frac{1}{\ln(n+2)}$ diverges by the comparison test ***AG***

continued…

Question 1 continued

(b) applying the ratio test $\lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{\ln(n+3)}\times\frac{\ln(n+2)}{(3x)^n}\right|$ ***M1***

$=|3x|$ (as $\lim_{n\to\infty}\left|\frac{\ln(n+2)}{\ln(n+3)}\right|=1$) ***A1***

Note: Condone the absence of limits and modulus signs.

Note: Award ***M1A0*** for $3x^n$. Subsequent marks can be awarded.

series converges for $-\frac{1}{3}<x<\frac{1}{3}$

considering $x=-\frac{1}{3}$ and $x=\frac{1}{3}$ ***M1***

Note: Award ***M1*** to candidates who consider one endpoint.

when $x=\frac{1}{3}$, series is $\sum_{n=0}^{\infty}\frac{1}{\ln(n+2)}$ which is divergent (from (a)) ***A1***

Note: Award this ***A1*** if $\sum_{n=0}^{\infty}\frac{1}{\ln(n+2)}$ is not stated but reference to part (a) is.

when $x=-\frac{1}{3}$, series is $\sum_{n=0}^{\infty}\frac{(-1)^n}{\ln(n+2)}$ ***A1***

$\sum_{n=0}^{\infty}\frac{(-1)^n}{\ln(n+2)}$ converges (conditionally) by the alternating series test ***R1***

(strictly alternating, $|u_n|>|u_{n+1}|$ for $n\ge 0$ and $\lim_{n\to\infty}(u_n)=0$)

so the interval of convergence of S is $-\frac{1}{3}\le x<\frac{1}{3}$ ***A1***

Note: The final ***A1*** is dependent on previous ***A1***s – *ie*, considering correct series when $x=-\frac{1}{3}$ and $x=\frac{1}{3}$ and on the final ***R1***.

Award as above to candidates who firstly consider $x=-\frac{1}{3}$ and then state conditional convergence implies divergence at $x=\frac{1}{3}$.

[7 marks]

Total [10 marks]

2. considering continuity at $x = 2$

$\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 2^+} f(x) = 4a + 2b$ ***(M1)***

$4a + 2b = 1$ ***A1***

considering differentiability at $x = 2$

$f'(x) = \begin{cases} -1 & x < 2 \\ 2ax + b & x \geq 2 \end{cases}$ ***(M1)***

$\lim_{x \to 2^-} f'(x) = -1$ and $\lim_{x \to 2^+} f'(x) = 4a + b$ ***(M1)***

Note: The above ***M1*** is for attempting to find the left and right limit of their derived piecewise function at $x = 2$.

$4a + b = -1$ ***A1***

$a = -\frac{3}{4}$ and $b = 2$ ***A1***

[6 marks]

3. (a) $\int_4^\infty \frac{1}{x^3}\,dx = \lim_{R \to \infty} \int_4^R \frac{1}{x^3}\,dx$ ***(A1)***

Note: The above ***A1*** for using a limit can be awarded at any stage. Condone the use of $\lim_{x \to \infty}$. Do not award this mark to candidates who use ∞ as the upper limit throughout.

$= \lim_{R \to \infty} \left[-\frac{1}{2}x^{-2}\right]_4^R \left(= \left[-\frac{1}{2}x^{-2}\right]_4^\infty\right)$ ***M1***

$= \lim_{R \to \infty} \left(-\frac{1}{2}\left(R^{-2} - 4^{-2}\right)\right)$

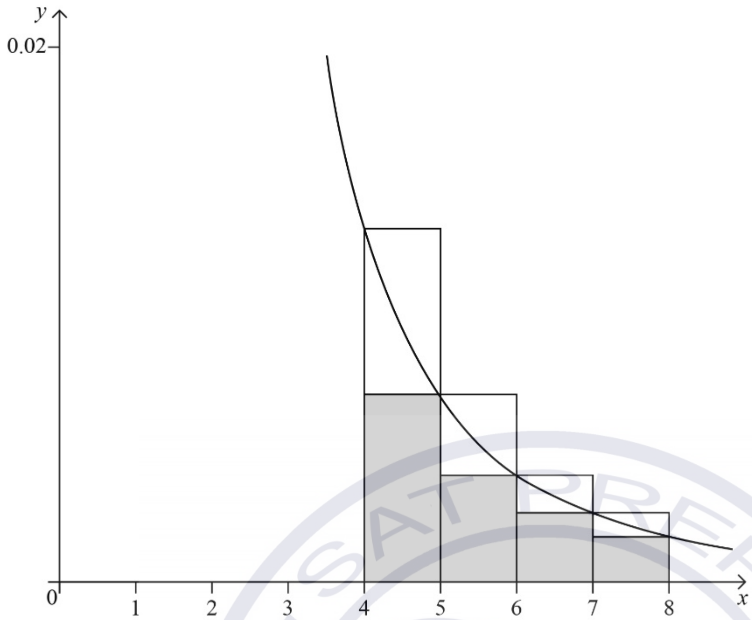
$= \frac{1}{32}$ ***A1***

[3 marks]

continued…

Question 3 continued

(b)



A1A1A1A1

A1 for the curve
A1 for rectangles starting at $x = 4$
A1 for at least three upper rectangles
A1 for at least three lower rectangles

Note: Award ***A0A1*** for two upper rectangles and two lower rectangles.

sum of areas of the lower rectangles $<$ the area under the curve $<$ the sum of the areas of the upper rectangles so

$$\sum_{n=5}^{\infty}\frac{1}{n^3} < \int_4^{\infty}\frac{1}{x^3}\,\mathrm{d}x < \sum_{n=4}^{\infty}\frac{1}{n^3}$$ ***AG***

[4 marks]

(c) a lower bound is $\frac{1}{32}$ ***A1***

Note: Allow ***FT*** from part (a).

[1 mark]

(d) **METHOD 1**

$$\sum_{n=5}^{\infty}\frac{1}{n^3} < \frac{1}{32}$$ ***(M1)***

$$\frac{1}{64} + \sum_{n=5}^{\infty}\frac{1}{n^3} < \frac{1}{32} + \frac{1}{64}$$ ***(M1)***

$$\sum_{n=4}^{\infty}\frac{1}{n^3} < \frac{3}{64}$$, an upper bound ***A1***

Note: Allow ***FT*** from part (a).

continued…

Question 3 continued

METHOD 2

changing the lower limit in the inequality in part (b) gives

$$\sum_{n=4}^{\infty}\frac{1}{n^3}<\int_3^{\infty}\frac{1}{x^3}\,\mathrm{d}x\left(<\sum_{n=3}^{\infty}\frac{1}{n^3}\right)$$ ***(A1)***

$$\sum_{n=4}^{\infty}\frac{1}{n^3}<\lim_{R\to\infty}\left[-\frac{1}{2}x^{-2}\right]_3^R$$ ***(M1)***

$\sum_{n=4}^{\infty}\frac{1}{n^3}<\frac{1}{18}$, an upper bound ***A1***

Note: Condone candidates who do not use a limit.

[3 marks]

Total [11 marks]

4. (a) $f'(x)=\frac{2\arcsin(x)}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for an attempt at chain rule differentiation.
Award ***M0A0*** for $f'(x)=2\arcsin(x)$.

$f'(0)=0$ ***AG***

[2 marks]

(b) differentiating gives $\left(1-x^2\right)f^{(3)}(x)-2xf''(x)-f'(x)-xf''(x)(=0)$ ***M1A1***

differentiating again gives $\left(1-x^2\right)f^{(4)}(x)-2xf^{(3)}(x)-3f''(x)-3xf^{(3)}(x)-f''(x)(=0)$

M1A1

Note: Award ***M1*** for an attempt at product rule differentiation of at least one product in each of the above two lines.
Do not penalise candidates who use poor notation.

$\left(1-x^2\right)f^{(4)}(x)-5xf^{(3)}(x)-4f''(x)=0$ ***AG***

[4 marks]

continued…

Question 4 continued

(c) attempting to find **one of** $f''(0)$, $f^{(3)}(0)$ or $f^{(4)}(0)$ by substituting $x=0$ into relevant differential equation(s) ***(M1)***

Note: Condone $f''(0)$ found by calculating $\frac{d}{dx}\left(\frac{2\arcsin(x)}{\sqrt{1-x^2}}\right)$ at $x=0$.

$\left(f(0)=0, f'(0)=0\right)$

$f''(0)=2$ and $f^{(4)}(0)-4f''(0)=0 \Rightarrow f^{(4)}(0)=8$ ***A1***

$f^{(3)}(0)=0$ and so $\frac{2}{2!}x^2+\frac{8}{4!}x^4$ ***A1***

Note: Only award the above ***A1***, for correct first differentiation in part (b) leading to $f^{(3)}(0)=0$ stated or $f^{(3)}(0)=0$ seen from use of the general Maclaurin series. **Special Case:** Award ***(M1)A0A1*** if $f^{(4)}(0)=8$ is stated without justification or found by working backwards from the general Maclaurin series.

so the Maclaurin series for $f(x)$ up to and including the term in x^4 is $x^2+\frac{1}{3}x^4$ ***AG***

[3 marks]

(d) substituting $x=\frac{1}{2}$ into $x^2+\frac{1}{3}x^4$ ***M1***

the series approximation gives a value of $\frac{13}{48}$

so $\pi^2 \simeq \frac{13}{48}\times 36$

$\simeq 9.75\left(\simeq\frac{39}{4}\right)$ ***A1***

Note: Accept 9.76.

[2 marks]

Total [11 marks]

5. (a) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x^{p-1} + \frac{1}{x}$ ***(M1)***

integrating factor $= \mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}$ ***M1***

$= \mathrm{e}^{-\ln x}$ ***(A1)***

$= \frac{1}{x}$ ***A1***

$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = x^{p-2} + \frac{1}{x^2}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = x^{p-2} + \frac{1}{x^2}$

$\frac{y}{x} = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of C.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1$, $y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = \frac{1}{p-1}\left(x^p - x\right) - 1$ ***A1***

[8 marks]

continued…

Question 5 continued

METHOD 2

put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1(A1)***

substituting, ***M1***

$x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - vx = x^p + 1$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-1} + \frac{1}{x}$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-2} + \frac{1}{x^2}$

$v = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

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[8 marks]

(b) (i) **METHOD 1**

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p-1}\left(px^{p-1} - 1\right)$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow px^{p-1} - 1 = 0$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

continued…

Question 5 continued

METHOD 2

substitute $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and their y into the differential equation and solve for x

$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow -\left(\frac{x^p - x}{p-1}\right)+1 = x^p + 1$ ***M1***

$x^p - x = x^p - px^p$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

(ii) there are two solutions for x when p is odd (and $p > 1$) ***A1***

if $p-1$ is even there are two solutions (to $x^{p-1} = \frac{1}{p}$)

and if $p-1$ is odd there is only one solution (to $x^{p-1} = \frac{1}{p}$) ***R1***

Note: Only award the ***R1*** if both cases are considered.

[4 marks]

Total [12 marks]

Markscheme

November 2017

Calculus

Higher level

Paper 3

Instructions to Examiners

Abbreviations

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1. considering continuity $\lim_{x \to 1^-}\left(x^2 - 2\right) = -1$ ***(M1)***

$a + b = -1$ ***(A1)***

considering differentiability $2x = a$ when $x = 1$ ***(M1)***

$\Rightarrow a = 2$ ***A1***

$b = -3$ ***A1***

[5 marks]

2. (a) **METHOD 1**

integrating factor $= \mathrm{e}^{\int \frac{x}{x^2+1}\mathrm{d}x}$ ***(M1)***

$\int \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\ln\left(x^2+1\right)$ ***(M1)***

Note: Award ***M1*** for use of $u = x^2 + 1$ for example or $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x)$.

integrating factor $= \mathrm{e}^{\frac{1}{2}\ln\left(x^2+1\right)}$ ***A1***

$= \mathrm{e}^{\ln\left(\sqrt{x^2+1}\right)}$ ***A1***

Note: Award ***A1*** for $\mathrm{e}^{\ln\sqrt{u}}$ where $u = x^2 + 1$.

$= \sqrt{x^2+1}$ ***AG***

METHOD 2

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{x^2+1}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{x^2+1} + \frac{x}{\sqrt{x^2+1}}y$ ***M1A1***

$\sqrt{x^2+1}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y\right)$ ***M1A1***

Note: Award ***M1*** for attempting to express in the form $\sqrt{x^2+1} \times$(LHS of de).

so $\sqrt{x^2+1}$ is an integrating factor for this differential equation ***AG***

[4 marks]

continued…

Question 2 continued

(b) $\sqrt{x^2+1}\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{x}{\sqrt{x^2+1}}y=x\sqrt{x^2+1}$ (or equivalent) ***(M1)***

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{x^2+1}\right)=x\sqrt{x^2+1}$

$y\sqrt{x^2+1}=\int x\sqrt{x^2+1}\,\mathrm{d}x \quad \left(y=\dfrac{1}{\sqrt{x^2+1}}\int x\sqrt{x^2+1}\,\mathrm{d}x\right)$ ***A1***

$=\dfrac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}+C$ ***(M1)A1***

Note: Award ***M1*** for using an appropriate substitution.

Note: Condone the absence of C.

substituting $x=0,\ y=1\Rightarrow C=\dfrac{2}{3}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y=\dfrac{1}{3}\left(x^2+1\right)+\dfrac{2}{3\sqrt{x^2+1}}\quad \left(y=\dfrac{\left(x^2+1\right)^{\frac{3}{2}}+2}{3\sqrt{x^2+1}}\right)$ ***A1***

[6 marks]

Total [10 marks]

3. (a) $\lim\limits_{n\to\infty}\dfrac{\frac{1}{n^2+2}}{\frac{1}{n^2}}=\lim\limits_{n\to\infty}\dfrac{n^2}{n^2+2}=\left(\lim\limits_{n\to\infty}\left(1-\dfrac{2}{n^2+2}\right)\right)$ ***M1***

$=1$ ***A1***

since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ converges (a p-series with $p=2$) ***R1***

by limit comparison test, $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+2}$ also converges ***AG***

Notes: The ***R1*** is independent of the ***A1***.

[3 marks]

continued…

Question 3 continued

(b) applying the ratio test $\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)^2+2}\times\frac{n^2+2}{(x-3)^n}\right|$ ***M1A1***

$=|x-3|$ (as $\lim_{n\to\infty}\frac{(n^2+2)}{(n+1)^2+2}=1$) ***A1***

converges if $|x-3|<1$ (converges for $2<x<4$) ***M1***

considering endpoints $x=2$ and $x=4$ ***M1***

when $x=4$, series is $\sum_{n=1}^{\infty}\frac{1}{n^2+2}$, convergent from (a) ***A1***

when $x=2$, series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+2}$ ***A1***

EITHER

$\sum_{n=1}^{\infty}\frac{1}{n^2+2}$ is convergent therefore $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+2}$ is (absolutely) convergent ***R1***

OR

$\frac{1}{n^2+2}$ is a decreasing sequence and $\lim_{n\to\infty}\frac{1}{n^2+2}=0$ so series converges by the alternating series test ***R1***

THEN

interval of convergence is $2\le x\le 4$ ***A1***

Note:	The final ***A1*** is dependent on previous ***A1***s – *ie*, considering correct series when $x=2$ and $x=4$ and on the final ***R1***.

[9 marks]

Total [12 marks]

4. (a) $\dfrac{g(5\pi) - g(0)}{5\pi - 0} = -0.6809\ldots\left(= \cos\sqrt{5\pi}\right)$ (gradient of chord) ***(A1)***

$g'(x) = \cos\left(\sqrt{x}\right) - \dfrac{\sqrt{x}\sin\left(\sqrt{x}\right)}{2}$ (or equivalent) ***(M1)(A1)***

Note: Award ***M1*** to candidates who attempt to use the product and chain rules.

attempting to solve $\cos\left(\sqrt{c}\right) - \dfrac{\sqrt{c}\sin\left(\sqrt{c}\right)}{2} = -0.6809\ldots$ for c ***(M1)***

Notes: Award ***M1*** to candidates who attempt to solve their $g'(c) =$ gradient of chord.
Do not award ***M1*** to candidates who just attempt to rearrange their equation.

$c = 2.26,\ 11.1$ ***A1A1***

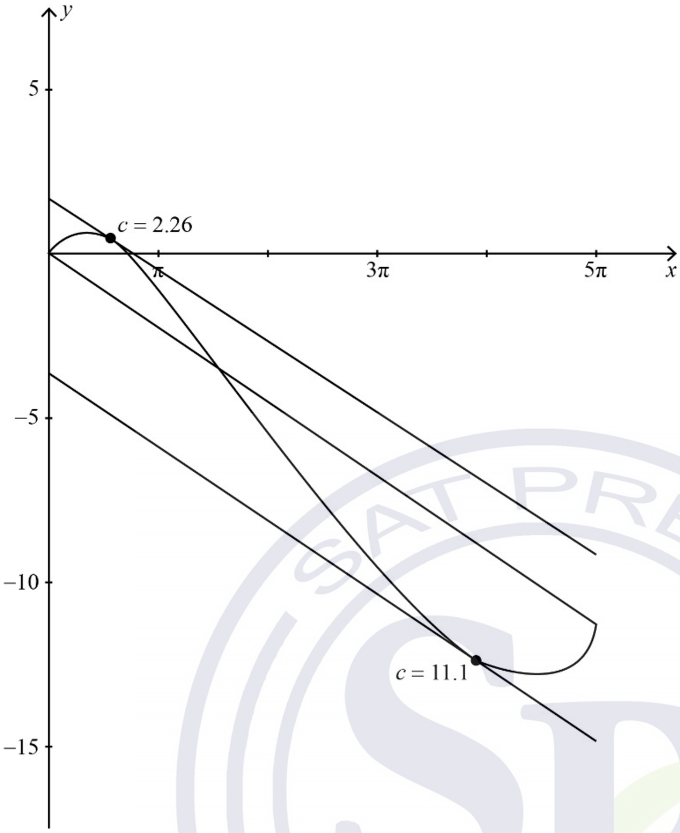
Note: Condone candidates working in terms of x.

[6 marks]

continued…

Question 4 continued

(b)



correct graph: 2 turning points close to the endpoints, endpoints indicated and correct endpoint behaviour ***A1***

Notes:	Endpoint coordinates are not required. Candidates do not need to indicate axes scales.

correct chord ***A1***

tangents drawn at their values of c which are approximately parallel to the chord ***A1A1***

Notes:	Award ***A1A0A1A0*** to candidates who draw a correct graph, do not draw a chord but draw 2 tangents at their values of c. Condone the absence of their c – values stated on their sketch. However do not award marks for tangents if no c – values were found in (a).

[4 marks]

Total [10 marks]

5. (a) $f'(x) = \dfrac{p\cos(p\arcsin x)}{\sqrt{1 - x^2}}$ ***(M1)A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f'(0) = p$ ***AG***

[2 marks]

(b) **EITHER**

$f^{(n+2)}(0) + \left(p^2 - n^2\right) f^{(n)}(0) = 0$ (or equivalent) ***A1***

OR

for *eg*, $\left(1 - x^2\right) f^{(n+2)}(x) = (2n+1)xf^{(n+1)}(x) - \left(p^2 - n^2\right) f^{(n)}(x)$ ***A1***

Note: Award ***A1*** for *eg*, $\left(1 - x^2\right) f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) = -\left(p^2 - n^2\right) f^{(n)}(x)$.

THEN

$f^{(n+2)}(0) = \left(n^2 - p^2\right) f^{(n)}(0)$ ***AG***

[1 mark]

(c) considering f and its derivatives at $x = 0$ ***(M1)***

$f(0) = 0$ and $f'(0) = p$ from (a) ***A1***

$f''(0) = 0$, $f^{(4)}(0) = 0$ ***A1***

$f^{(3)}(0) = \left(1 - p^2\right) f^{(1)}(0) = \left(1 - p^2\right) p$,

$f^{(5)}(0) = \left(9 - p^2\right) f^{(3)}(0) = \left(9 - p^2\right)\left(1 - p^2\right) p$ ***A1***

Note: Only award the last ***A1*** if either $f^{(3)}(0) = \left(1 - p^2\right) f^{(1)}(0)$ and $f^{(5)}(0) = \left(9 - p^2\right) f^{(3)}(0)$ have been stated or the general Maclaurin series has been stated and used.

$px + \dfrac{p\left(1 - p^2\right)}{3!}x^3 + \dfrac{p\left(9 - p^2\right)\left(1 - p^2\right)}{5!}x^5$ ***AG***

[4 marks]

continued…

Question 5 continued

(d) **METHOD 1**

$$\lim_{x\to 0}\frac{\sin(p\arcsin x)}{x} = \lim_{x\to 0}\frac{px + \frac{p\left(1-p^2\right)}{3!}x^3 + \ldots}{x}$$ ***M1***

$= p$ ***A1***

METHOD 2

by l'Hôpital's rule $\lim_{x\to 0}\frac{\sin(p\arcsin x)}{x} = \lim_{x\to 0}\frac{p\cos(p\arcsin x)}{\sqrt{1-x^2}}$ ***M1***

$= p$ ***A1***

[2 marks]

(e) the coefficients of all even powers of x are zero ***A1***

the coefficient of x^p for (p odd) is non-zero (or equivalent *eg*, the coefficients of all odd powers of x up to p are non-zero) ***A1***

$f^{(p+2)}(0) = \left(p^2 - p^2\right) f^{(p)}(0) = 0$ and so the coefficient of x^{p+2} is zero ***A1***

the coefficients of all odd powers of x greater than $p+2$ are zero (or equivalent) ***A1***

so the Maclaurin series for $f(x)$ is a polynomial of degree p ***AG***

[4 marks]

Total [13 marks]

Markscheme

May 2017

Calculus

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2017**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets** eg **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. attempt to use l'Hôpital's rule, ***M1***

limit $= \lim_{x\to 0} \dfrac{2\sin x\cos x}{\ln(1+x) + \dfrac{x}{1+x}}$ or $\dfrac{\sin 2x}{\ln(1+x)+\dfrac{x}{1+x}}$ ***A1A1***

Note: Award ***A1*** for numerator ***A1*** for denominator.

this gives $0/0$ so use the rule again ***(M1)***

$= \lim_{x \to 0} \dfrac{2\cos^2 x - 2\sin^2 x}{\dfrac{1}{1+x} + \dfrac{1+x-x}{(1+x)^2}}$ or $\dfrac{2\cos 2x}{\dfrac{2+x}{(1+x)^2}}$ ***A1A1***

Note: Award ***A1*** for numerator ***A1*** for denominator.

$= 1$ ***A1***

Note: This ***A1*** is dependent on all previous marks being awarded, except when the first application of L'Hopital's does not lead to $0/0$, when it should be awarded for the correct limit of their derived function.

[7 marks]

2. (a) (i) $(\sec^2 x =)\ a_1 + 3a_3x^2 + 5a_5x^4 + \ldots$ ***A1***

(ii) $\sec^2 x = 1 + \left(a_1x + a_3x^3 + a_5x^5 + \ldots\right)^2$

$= 1 + a_1^2x^2 + 2a_1a_3x^4 + \ldots$ ***M1A1***

Note: Condone the presence of terms with powers greater than four.

[3 marks]

(b) equating constant terms: $a_1 = 1$ ***A1***

equating x^2 terms: $3a_3 = a_1^2 = 1 \Rightarrow a_3 = \dfrac{1}{3}$ ***A1***

equating x^4 terms: $5a_5 = 2a_1a_3 = \dfrac{2}{3} \Rightarrow a_5 = \dfrac{2}{15}$ ***A1***

[3 marks]

Total [6 marks]

3. consider $I = \int_2^N \frac{\mathrm{d}x}{x\sqrt{\ln x}}$ ***M1A1***

Note:	Do not award ***A1*** if n is used as the variable or if lower limit equal to 1, but some subsequent ***A*** marks can still be awarded. Allow ∞ as upper limit.

let $y = \ln x$ ***(M1)***

$\mathrm{d}y = \frac{\mathrm{d}x}{x}$, ***(A1)***

$[2, N] \Rightarrow [\ln 2, \ln N]$

$I = \int_{\ln 2}^{\ln N} \frac{\mathrm{d}y}{\sqrt{y}}$ ***(A1)***

Note:	Condone absence of limits, or wrong limits.

$= \left[2\sqrt{y}\right]_{\ln 2}^{\ln N}$ ***A1***

Note:	***A1*** is for the correct integral, irrespective of the limits used. Accept correct use of integration by parts.

$= 2\sqrt{\ln N} - 2\sqrt{\ln 2}$ ***(M1)***

Note:	***M1*** is for substituting their limits into their integral and subtracting.

$\to \infty$ as $N \to \infty$ ***A1***

Notes:	Allow "$=\infty$", "limit does not exist", "diverges" or equivalent. Do not award if wrong limits substituted into the integral but allow N or ∞ as an upper limit in place of $\ln N$.

(by the integral test) the series is divergent (because the integral is divergent) ***A1***

Notes:	Do not award this mark if ∞ used as upper limit throughout.

[9 marks]

4. (a) $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ***M1***

the differential equation becomes

$v + x\frac{dv}{dx} = f(v)$ ***A1***

$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$ ***A1***

integrating, $\int \frac{dv}{f(v) - v} = \ln x + \text{Constant}$ ***AG***

[3 marks]

(b) **EITHER**

$f(v) = 1 + 3v + v^2$ ***(A1)***

$\left(\int \frac{dv}{f(v) - v} = \right)\int \frac{dv}{1 + 3v + v^2 - v} = \ln x + C$ ***M1A1***

$\int \frac{dv}{(1+v)^2} = (\ln x + C)$ ***A1***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v}(= \ln x + C)$ ***A1***

OR

$v + x\frac{dv}{dx} = 1 + 3v + v^2$ ***A1***

$\int \frac{dv}{1 + 2v + v^2} = \int \frac{1}{x}dx$ ***M1***

$\int \frac{dv}{(1+v)^2}\left(= \int \frac{1}{x}dx\right)$ ***(A1)***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v} = \ln x(+C)$ ***A1A1***

continued…

Question 4 continued

THEN

substitute $y = 1$ or $v = 1$ when $x = 1$ ***(M1)***

therefore $C = -\frac{1}{2}$ ***A1***

Note: This ***A1*** can be awarded anywhere in their solution.

substituting for v,

$$-\frac{1}{\left(1+\frac{y}{x}\right)} = \ln x - \frac{1}{2}$$ ***M1***

Note: Award for correct substitution of $\frac{y}{x}$ into their expression.

$$1+\frac{y}{x} = \frac{1}{\frac{1}{2}-\ln x}$$ ***(A1)***

Note: Award for any rearrangement of a correct expression that has y in the numerator.

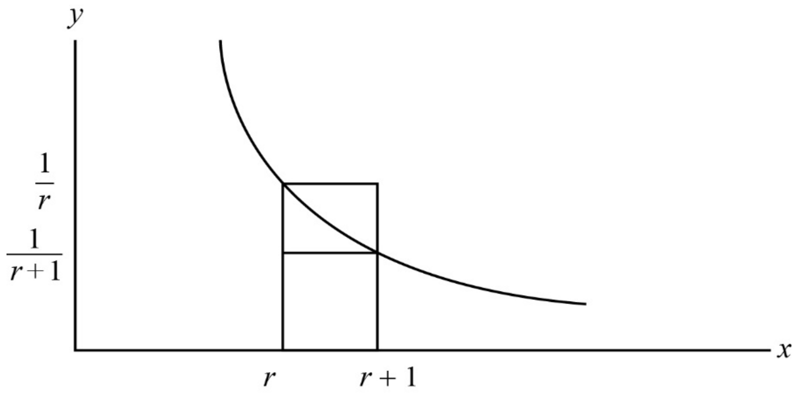
$$y = x\left(\frac{1}{\left(\frac{1}{2}-\ln x\right)} - 1\right)$$ (or equivalent) ***A1***

$$\left(= x\left(\frac{1+2\ln x}{1-2\ln x}\right)\right)$$

[10 marks]

Total [13 marks]

5. (a)



A1

Note: Curve, both rectangles and correct x- values required.

area of rectangles $\frac{1}{r}$ and $\frac{1}{1+r}$ *A1*

Note: Correct values on the y-axis are sufficient evidence for this mark if not otherwise indicated.

in the above diagram, the area below the curve between $x = r$ and $x = r + 1$ is between the areas of the larger and smaller rectangle

or $\frac{1}{r+1} < \int_r^{r+1} \frac{dx}{x} < \frac{1}{r}$ ***(R1)***

integrating, $\int_r^{r+1} \frac{dx}{x} = [\ln x]_r^{r+1} \left(= \ln(r+1) - \ln(r)\right)$ *A1*

$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}$ *AG*

[4 marks]

(b) (i) summing the right-hand part of the above inequality from $r = 1$ to $r = n$,

$\sum_{r=1}^{n} \frac{1}{r} > \sum_{r=1}^{n} \ln\left(\frac{r+1}{r}\right)$ *M1*

$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n+1}{n}\right)$ ***(A1)***

EITHER

$= \ln\left(\frac{2}{1} \times \frac{3}{2} \times \ldots \times \frac{n}{n-1} \times \frac{n+1}{n}\right)$ *A1*

OR

$\ln 2 - \ln 1 + \ln 3 - \ln 2 + \ldots + \ln(n+1) - \ln(n)$ *A1*

$= \ln(n+1)$ *AG*

continued…

Question 5 continued

(ii) $\sum_{r=1}^{n}\frac{1}{r}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<1+\ln\left(\frac{2}{1}\right)+\ln\left(\frac{3}{2}\right)+\ldots+\ln\left(\frac{n}{n-1}\right)$ ***M1A1A1***

$\left(1+\sum_{r=1}^{n-1}\frac{1}{r+1}<1+\sum_{r=1}^{n-1}\ln\left(\frac{r+1}{r}\right)\right)$

Note:	***M1*** is for using the correct inequality from (a), ***A1*** for both sides beginning with 1, ***A1*** for completely correct expression.

Note:	The 1 might be added after the sums have been calculated.

$=1+\ln n$ ***AG***

[6 marks]

(c) (i) from (b)(i) $U_n>\ln(1+n)-\ln n>0$ ***A1***

(ii) $U_{n+1}-U_n=\sum_{r=1}^{n+1}\frac{1}{r}-\ln(n+1)-\sum_{r=1}^{n}\frac{1}{r}+\ln n$ ***M1***

$=\frac{1}{n+1}-\ln\left(\frac{n+1}{n}\right)$ ***A1***

<0 (using the result proved in (a)) ***A1***

$U_{n+1}<U_n$ ***AG***

[4 marks]

(d) it follows from the two results that $\{U_n\}$ cannot be divergent either in the sense of tending to $-\infty$ or oscillating therefore it must be convergent ***R1***

[1 mark]

Note:	Accept the use of the result that a bounded (monotonically) decreasing sequence is convergent (allow "positive, decreasing sequence").

Total [15 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2016

Calculus

Higher level

Paper 3

12 pages

Instructions to Examiners

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(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

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- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

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  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

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If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

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A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

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1. (a) **METHOD 1**

attempting to find an integrating factor *(M1)*

$\int \frac{2x}{1+x^2}\,\mathrm{d}x = \ln\left(1+x^2\right)$ *(M1)A1*

IF is $e^{\ln\left(1+x^2\right)}$ *(M1)A1*

$= 1 + x^2$ *AG*

METHOD 2

multiply by the integrating factor

$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x^2\left(1+x^2\right)$ *M1A1*

left hand side is equal to the derivative of $\left(1+x^2\right)y$ *A3*

[5 marks]

(b) $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \left(1+x^2\right)x^2$ *(M1)*

$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1+x^2\right)y\right] = x^2 + x^4$

$\left(1+x^2\right)y = \left(\int x^2 + x^4\,\mathrm{d}x =\right) \frac{x^3}{3} + \frac{x^5}{5}\,(+\,c)$ *A1A1*

$y = \frac{1}{1+x^2}\left(\frac{x^3}{3} + \frac{x^5}{5} + c\right)$

$x = 0,\ y = 2 \Rightarrow c = 2$ *M1A1*

$y = \frac{1}{1+x^2}\left(\frac{x^3}{3} + \frac{x^5}{5} + 2\right)$ *A1*

[6 marks]

Total [11 marks]

2. (a) $f(x) = (x+1)\ln(1+x) - x$ $\quad f(0) = 0$ ***A1***

$f'(x) = \ln(1+x) + \dfrac{x+1}{1+x} - 1 \left(= \ln(1+x)\right)$ $\quad f'(0) = 0$ ***M1A1A1***

$f''(x) = (1+x)^{-1}$ $\quad f''(0) = 1$ ***A1A1***

$f'''(x) = -(1+x)^{-2}$ $\quad f'''(0) = -1$ ***A1***

$f^{(4)}(x) = 2(1+x)^{-3}$ $\quad f^{(4)}(0) = 2$ ***A1***

$f^{(5)}(x) = -3\times 2(1+x)^{-4}$ $\quad f^{(5)}(0) = -3\times 2$ ***A1***

$$f(x) = \frac{x^2}{2!} - \frac{1x^3}{3!} + \frac{2x^4}{4!} - \frac{6x^5}{5!}\ldots$$ ***M1A1***

$$f(x) = \frac{x^2}{1\times 2} - \frac{x^3}{2\times 3} + \frac{x^4}{3\times 4} - \frac{x^5}{4\times 5}\ldots$$

$$f(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20}\ldots$$

Note: Allow follow through from the first error in a derivative (provided future derivatives also include the chain rule), no follow through after a second error in a derivative.

[11 marks]

(b) $f^{(n)}(0) = (-1)^n(n-2)!$ So coefficient of $x^n = (-1)^n\dfrac{(n-2)!}{n!}$ ***A1***

coefficient of x^n is $(-1)^n\dfrac{1}{n(n-1)}$ ***AG***

[1 mark]

(c) applying the ratio test to the series of absolute terms

$$\lim_{n\to\infty}\frac{\dfrac{|x|^{n+1}}{(n+1)n}}{\dfrac{|x|^n}{n(n-1)}}$$ ***M1A1***

$$= \lim_{n\to\infty}|x|\frac{(n-1)}{(n+1)}$$ ***A1***

$$= |x|$$ ***A1***

so for convergence $|x| < 1$, giving radius of convergence as 1 ***(M1)A1***

[6 marks]

Total [18 marks]

3. (a) $\lim_{x\to\infty}\left(\dfrac{\arcsin\left(\dfrac{1}{\sqrt{(x+1)}}\right)}{\dfrac{1}{\sqrt{x}}}\right)$ is of the form $\dfrac{0}{0}$

and so will equal the limit of $\dfrac{\dfrac{\dfrac{-1}{2}(x+1)^{-\frac{3}{2}}}{\sqrt{1-\left(\dfrac{1}{x+1}\right)}}}{\dfrac{-1}{2}x^{-\frac{3}{2}}}$ ***M1M1A1A1***

Note: ***M1*** for attempting differentiation of the top and bottom, ***M1A1*** for derivative of top (only award ***M1*** if chain rule is used), ***A1*** for derivative of bottom.

$= \lim_{x\to\infty}\dfrac{\left(\dfrac{x}{(x+1)}\right)^{\frac{3}{2}}}{\sqrt{\dfrac{x}{x+1}}} = \lim_{x\to\infty}\left(\dfrac{x}{x+1}\right)$ ***M1***

Note: Accept any intermediate tidying up of correct derivative for the method mark.

$=1$ ***A1***

[6 marks]

(b) (i) $a_1=\sqrt{2}$, $a_2=\sqrt{3}$ ***A1***

$a_n=\sqrt{n+1}$ ***A1***

(ii) $\sin\theta_n=\dfrac{1}{a_n}=\dfrac{1}{\sqrt{n+1}}$ ***A1***

Note: Allow $\theta_n=\arcsin\left(\dfrac{1}{a_n}\right)$ if $a_n=\sqrt{n+1}$ in b(i).

so $\theta_n=\arcsin\dfrac{1}{\sqrt{(n+1)}}$ ***AG***

[3 marks]

continued…

Question 3 continued

(c) for $\sum_{n=1}^{\infty} \arcsin \frac{1}{\sqrt{(n+1)}}$ apply the limit comparison test (since both series of positive terms) ***M1***

with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ***A1***

from (a) $\lim_{n \to \infty} \frac{\arcsin \frac{1}{\sqrt{(n+1)}}}{\frac{1}{\sqrt{n}}} = 1$, so the two series either both converge or both diverge ***M1R1***

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as is a p-series with $p = \frac{1}{2}$) ***A1***

hence $\sum_{n=1}^{\infty} \theta_n$ diverges ***A1***

[6 marks]

Total [15 marks]

4. (a) there exists c in the open interval $]a, b[$ such that ***A1***

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$ ***A1***

Note: Open interval is required for the ***A1***.

[2 marks]

(b) (i) $g(0)=f(h)-f(0)-hf'(0)-\frac{h^2}{h^2}\left(f(h)-f(0)-hf'(0)\right)$

$=0$ ***A1***

(ii) $g(h)=f(h)-f(h)-0-0$

$=0$ ***A1***

(iii) ($g(x)$ is a differentiable function since it is a combination of other differentiable functions f, f' and polynomials.)

there exists c in the open interval $]0, h[$ such that

$$\frac{g(h)-g(0)}{h}=g'(c)$$ ***A1***

$$\frac{g(h)-g(0)}{h}=0$$ ***A1***

hence $g'(c)=0$ ***AG***

(iv) $g'(x)=-f'(x)+f'(x)-(h-x)f''(x)+\frac{2(h-x)}{h^2}\left(f(h)-f(0)-hf'(0)\right)$ ***A1A1***

Note: ***A1*** for the second and third terms and ***A1*** for the other terms (all terms must be seen).

$$=-(h-x)f''(x)+\frac{2(h-x)}{h^2}\left(f(h)-f(0)-hf'(0)\right)$$

(v) putting $x=c$ and equating to zero ***M1***

$$-(h-c)f''(c)+\frac{2(h-c)}{h^2}\left(f(h)-f(0)-hf'(0)\right)=g'(c)=0$$ ***AG***

(vi) $-f''(c)+\frac{2}{h^2}\left(f(h)-f(0)-hf'(0)\right)=0$ ***A1***

since $h-c\neq 0$ ***R1***

$$\frac{h^2}{2}f''(c)=f(h)-f(0)-hf'(0)$$

$$f(h)=f(0)+hf'(0)+\frac{h^2}{2}f''(c)$$ ***AG***

[9 marks]

continued…

Question 4 continued

(c) letting $f(x) = \cos(x)$ **M1**

$f'(x) = -\sin(x)$ $\quad$ $f''(x) = -\cos(x)$ **A1**

$\cos(h) = 1 + 0 - \frac{h^2}{2}\cos(c)$ **A1**

$1 - \cos(h) = \frac{h^2}{2}\cos(c)$ ***(A1)***

since $\cos(c) \le 1$ **R1**

$1 - \cos(h) \le \frac{h^2}{2}$ **AG**

[5 marks]

Note: Allow $f(x) = a \pm b\cos x$.

Total [16 marks]

Markscheme

May 2016

Calculus

Higher level

Paper 3

Instructions to Examiners

Abbreviations

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(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

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Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2016"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

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1. (a) attempt to use product rule *(M1)*

$f'(x) = e^x \sin x + e^x \cos x$ ***A1***

$f''(x) = 2e^x \cos x$ ***A1***

$f''(x) = 2e^x \cos x - 2e^x \sin x$ ***A1***

$f(0) = 0,\ f'(0) = 1$

$f''(0) = 2,\ f'''(0) = 2$ *(M1)*

$e^x \sin x = x + x^2 + \dfrac{x^3}{3} + \dots$ ***(M1)A1***

[7 marks]

(b) **METHOD 1**

$$\frac{e^x \sin x - x - x^2}{x^3} = \frac{x + x^2 + \frac{x^3}{3} + \dots - x - x^2}{x^3}$$ ***M1A1***

$\rightarrow \dfrac{1}{3}$ as $x \rightarrow 0$ ***A1***

METHOD 2

$$\lim_{x\to 0}\frac{e^x \sin x - x - x^2}{x^3} = \lim_{x\to 0}\frac{e^x \sin x + e^x \cos x - 1 - 2x}{3x^2}$$ ***A1***

$$= \lim_{x\to 0}\frac{2e^x \cos x - 2}{6x}$$ ***A1***

$$= \lim_{x\to 0}\frac{2e^x \cos x - 2e^x \sin x}{6} = \frac{1}{3}$$ ***A1***

[3 marks]

continued…

Question 1 continued

(c) (i) attempt to find 4th derivative from the 3rd derivative obtained in (a) ***M1***

$f''''(x) = -4e^x \sin x$ ***A1***

Lagrange error term = $\dfrac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}$ (where c lies between 0 and x)

$= -\dfrac{4e^c \sin c \times 0.5^4}{4!}$ ***(M1)***

the maximum absolute value of this expression occurs when $c = 0.5$ ***(A1)***

Note: This ***A1*** is independent of previous ***M*** marks.

therefore

upper bound = $\dfrac{4e^{0.5} \sin 0.5 \times 0.5^4}{4!}$ ***(M1)***

$= 0.00823$ ***A1***

(ii) the approximation is greater than the actual value because the Lagrange error term is negative ***R1***

[7 marks]

Total [17 marks]

2. (a) $\ln(2 + \sin x)$ ***A1***

Note: Do not accept $\ln(2 + \sin t)$.

[1 mark]

(b) attempt to use chain rule ***(M1)***

$\dfrac{d}{dx}\left(f\left(x^2\right)\right) = 2x\, f'\left(x^2\right)$ ***(A1)***

$= 2x \ln\left(2 + \sin\left(x^2\right)\right)$ ***A1***

[3 marks]

continued…

Question 2 continued

(c) $\int_x^{x^2} \ln(2+\sin t)\mathrm{d}t = \int_0^{x^2} \ln(2+\sin t)\mathrm{d}t - \int_0^{x} \ln(2+\sin t)\mathrm{d}t$ ***(M1)(A1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{x^2} \ln(2+\sin t)\mathrm{d}t\right) = 2x\ln\left(2+\sin\left(x^2\right)\right) - \ln(2+\sin x)$ ***A1***

[3 marks]

Total [7 marks]

3. (a) $f'(x) = \frac{1}{x}$ ***(A1)***

using the MVT $f'(c) = \frac{f(b)-f(a)}{b-a}$ (where c lies between a and b) ***(M1)***

$f'(c) = \frac{\ln b - \ln a}{b-a}$ ***A1***

$\ln\frac{b}{a} = \ln b - \ln a$ ***(M1)***

$f'(c) = \frac{\ln\frac{b}{a}}{b-a}$

since $f'(x)$ is a decreasing function or $a < c < b \Rightarrow \frac{1}{b} < \frac{1}{c} < \frac{1}{a}$ ***R1***

$f'(b) < f'(c) < f'(a)$ ***(M1)***

$\frac{1}{b} < \frac{\ln\frac{b}{a}}{b-a} < \frac{1}{a}$ ***A1***

$\frac{b-a}{b} < \ln\frac{b}{a} < \frac{b-a}{a}$ ***AG***

[7 marks]

(b) putting $b = 1.2$, $a = 1$, or equivalent ***M1***

$\frac{1}{6} < \ln 1.2 < \frac{1}{5}$ ***A1***

$(m = 6,\ n = 5)$

[2 marks]

Total [9 marks]

4. (a) **METHOD 1**

$z = y^2 \Rightarrow y = z^{1/2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2z^{1/2}}\frac{dz}{dx}$ ***M1A1***

substituting, $\frac{1}{2z^{1/2}}\frac{dz}{dx} = \frac{x}{z^{1/2}} - xz^{1/2}$ ***M1A1***

$\frac{dz}{dx} + 2xz = 2x$ ***AG***

METHOD 2

$z = y^2$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$ ***M1A1***

$\frac{dz}{dx} = 2x - 2xy^2$ ***M1A1***

$\frac{dz}{dx} + 2xz = 2x$ ***AG***

[4 marks]

(b) **METHOD 1**

integrating factor $= e^{\int 2x dx} = e^{x^2}$ ***(M1)A1***

$e^{x^2}\frac{dz}{dx} + 2xe^{x^2}z = 2xe^{x^2}$ ***(M1)***

$ze^{x^2} = \int 2xe^{x^2} dx$ ***A1***

$= e^{x^2} + C$ ***A1***

substitute $y = 2$ therefore $z = 4$ when $x = 0$ ***(M1)***

$4 = 1 + C$

$C = 3$ ***(A1)***

the solution is $z = 1 + 3e^{-x^2}$ ***(M1)***

Note: This line may be seen before determining the value of C.

so that $y = \sqrt{1 + 3e^{-x^2}}$ ***A1***

continued…

Question 4 continued

METHOD 2

$\frac{dz}{dx} = 2x(1-z)$

$\int \frac{1}{1-z}dz = \int 2x dx$ ***M1***

$-\ln(1-z) = x^2 + C$ ***A1A1***

$1-z = e^{-x^2-c}$ (or $1-z = Be^{-x^2}$) ***M1A1***

solving for z ***(M1)***

$z = 1 + Ae^{-x^2}$

$z = 4$ when $x = 0$ ***(M1)***

so $A = 3$ ***(A1)***

the solution is $z = 1 + 3e^{-x^2}$

so $y = \sqrt{1+3e^{-x^2}}$ ***A1***

[9 marks]

Total [13 marks]

5. (a) as t moves through the intervals $[0, \pi]$, $[\pi, 2\pi]$, $[2\pi, 3\pi]$, $[3\pi, 4\pi]$, *etc*, the sign of $\sin t$, (and therefore the sign of the integral) alternates $+, -, +, -$, *etc*, so that the series is alternating ***R1***

Note:	Award ***R1*** only if it includes a clear reason that justifies that the sign of the integrand alternates between – and + and this pattern is valid for all the terms. The change of signs can be justified by a labelled graph of $y = \sin(x)$ or $y = \frac{\sin x}{x}$ that shows the intervals $[0, \pi]$, $[\pi, 2\pi]$, $[2\pi, 3\pi]$, …

[1 mark]

continued…

Question 5 continued

(b) (i) $u_{n+1} = \int_{(n+1)\pi}^{(n+2)\pi} \frac{\sin t}{t}\,\mathrm{d}t$ ***(M1)***

put $T = t - \pi$ and $\mathrm{d}T = \mathrm{d}t$ ***(M1)***

the limits change to $n\pi$, $(n+1)\pi$

$\left|u_{n+1}\right| = \int_{n\pi}^{(n+1)\pi} \frac{\left|\sin(T+\pi)\right|}{T+\pi}\,\mathrm{d}T$ (or equivalent) ***A1***

$\left|\sin(T+\pi)\right| = \left|\sin(T)\right|$ or $\sin(T+\pi) = -\sin(T)$ ***(M1)***

$= \int_{n\pi}^{(n+1)\pi} \frac{\left|\sin T\right|}{T+\pi}\,\mathrm{d}T$

$< \int_{n\pi}^{(n+1)\pi} \frac{\left|\sin T\right|}{T}\,\mathrm{d}T = \left|u_n\right|$ ***A1AG***

(ii) $\left|u_n\right| = \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t}\,\mathrm{d}t$

$< \int_{n\pi}^{(n+1)\pi} \frac{1}{t}\,\mathrm{d}t$ ***M1***

$= [\ln t]_{n\pi}^{(n+1)\pi}$ ***A1***

$= \ln\left(\frac{n+1}{n}\right)$ ***A1***

$\to \ln 1 = 0$ as $n \to \infty$

from part (i) $\left|u_n\right|$ is a decreasing sequence and since $\lim_{n\to\infty}\left|u_n\right| = 0$, ***R1***

the series is convergent ***AG***

[9 marks]

continued…

Question 5 continued

(c) attempt to calculate the partial sums $\sum_{i=0}^{n-1} u_i = \int_0^{n\pi} \frac{\sin t}{t}\,dt$ ***(M1)***

the first partial sums are

n	$\sum_{i=0}^{n-1} u_i$
1	1.85 (or 1.8519…)
2	1.42 (or 1.4181…)
3	1.67 (or 1.6747…)
4	1.49 (or 1.4921…)
5	1.63 (or 1.6339…)

two consecutive partial sums for $n \geq 4$ ***A1A1***

(*eg* $S_4 = 1.49$ and $S_5 = 1.63$ or $S_{100} = 1.567\ldots$ and $S_{101} = 1.573\ldots$)

Note: These answers must be given to a minimum of 3 significant figures.

the sum to infinity lies between any two consecutive partial sums,
eg between 1.49 and 1.63 ***R1***

so that $S < 1.65$ ***AG***

Note: Award ***A1A1R1*** to candidates who calculate at least two partial sums for only odd values of n and state that the upper bound is less than these values.

[4 marks]

Total [14 marks]

Markscheme

November 2015

Calculus

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2015**". It is essential that you read this document before you start marking.
In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).
A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

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An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

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Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

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Unless the question specifies otherwise, ***accept*** *equivalent forms.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) consider upper or lower limits *M1*

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} 1 = 1 \left(= f(0)\right), \ \lim_{x\to0^+} f(x) = \lim_{x\to0^+} (1-x) = 1\left(= f(0)\right)$ *A1*

$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)$ so f is continuous *AG*

[2 marks]

(b) $\lim_{h\to0^-} \frac{1-1}{h} = 0$ *M1A1*

$\lim_{h\to0^+} \frac{1-h-1}{h} = \lim_{h\to0^+} (-1) = -1$ *A1*

Note: Award ***M1*** for an attempt to find limits in either case.

$\lim_{h\to0^+} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to0^-} \frac{f(0+h)-f(0)}{h}$ so f is not differentiable *AG*

Note: Award ***M1A1A0*** for correct differentiation of left and right sides, ONLY if limits then used to show that both functions have different limits.

[3 marks]

Total [5 marks]

2. (a) $f'(x) = e^x \sin x + e^x \cos x$ *M1A1*

$f''(x) = e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x = 2e^x \cos x$ *A1*

$= 2\left(e^x \sin x + e^x \cos x - e^x \sin x\right)$ *M1*

$= 2\,(f'(x) - f(x))$ *AG*

[4 marks]

(b) $f(0) = 0,\ f'(0) = 1,\ f''(0) = 2(1-0) = 2$ *(M1)A1*

Note: Award ***M1*** for attempt to find $f(0)$, $f'(0)$ and $f''(0)$.

$f'''(x) = 2(f''(x) - f'(x))$ *(M1)*

$f'''(0) = 2(2-1) = 2,\ f^{IV}(0) = 2(2-2) = 0,\ f^{V}(0) = 2(0-2) = -4$ *A1*

so $f(x) = x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 + \ldots$ *(M1)A1*

$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$

[6 marks]

Total [10 marks]

3. (a) if $n = 7$ then $7! > 3^7$ **A1**

so true for $n = 7$

assume true for $n = k$ **M1**

so $k! > 3^k$

consider $n = k + 1$

$(k+1)! = (k+1)k!$ **M1**

$> (k+1)3^k$

$> 3.3^k$ (as $k > 6$) **A1**

$= 3^{k+1}$

hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 7$, so true for all $n \geq 7$. **R1**

Note: Do not award the ***R1*** if the two ***M*** marks have not been awarded.

[5 marks]

(b) consider the series $\sum_{r=7}^{\infty} a_r$ where $a_r = \frac{2^r}{r!}$ **R1**

Note: Award the ***R1*** for starting at $r = 7$.

compare to the series $\sum_{r=7}^{\infty} b_r$ where $b_r = \frac{2^r}{3^r}$ **M1**

$\sum_{r=7}^{\infty} b_r$ is an infinite Geometric Series with $r = \frac{2}{3}$ and hence converges **A1**

Note: Award the ***A1*** even if series starts at $r = 1$.

as $r! > 3^r$ so $(0 <)\ a_r < b_r$ for all $r \geq 7$ **M1R1**

as $\sum_{r=7}^{\infty} b_r$ converges and $a_r < b_r$ so $\sum_{r=7}^{\infty} a_r$ must converge

Note: Award the ***A1*** even if series starts at $r = 1$.

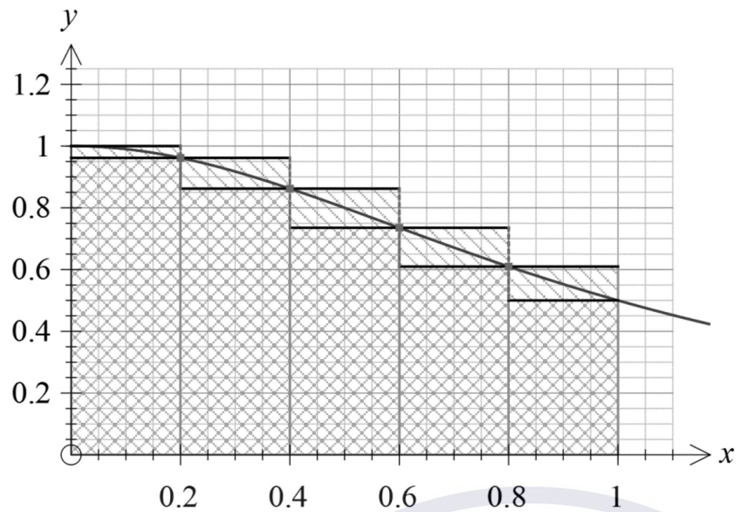
as $\sum_{r=1}^{6} a_r$ is finite, so $\sum_{r=1}^{\infty} a_r$ must converge **R1**

Note: If the limit comparison test is used award marks to a maximum of ***R1M1A1M0A0R1***.

[6 marks]

Total [11 marks]

4. (a)



A1A1A1

A1 for upper rectangles, ***A1*** for lower rectangles, ***A1*** for curve in between with $0 \le x \le 1$

hence $\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \int_0^1 f(x)\,dx < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$ ***AG***

[3 marks]

(b) attempting to integrate from 0 to 1 ***(M1)***

$\int_0^1 f(x)\,dx = [\arctan x]_0^1$

$= \frac{\pi}{4}$ ***A1***

attempt to evaluate either summation ***(M1)***

$\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \frac{\pi}{4} < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$

hence $\frac{4}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \pi < \frac{4}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$

so $2.93 < \pi < 3.33$ ***A1A1***

Note: Accept any answers that round to 2.9 and 3.3.

[5 marks]

continued…

Question 4 continued

(c) **EITHER**

recognise $\sum_{r=0}^{n-1}(-1)^r x^{2r}$ as a geometric series with $r = -x^2$ ***M1***

sum of n terms is $\frac{1-\left(-x^2\right)^n}{1--x^2} = \frac{1+\left(-1\right)^{n-1}x^{2n}}{1+x^2}$ ***M1AG***

OR

$\sum_{r=0}^{n-1}(-1)^r\left(1+x^2\right)x^{2r} = \left(1+x^2\right)x^0 - \left(1+x^2\right)x^2 + \left(1+x^2\right)x^4 + \ldots$

$+ (-1)^{n-1}\left(1+x^2\right)x^{2n-2}$ ***M1***

cancelling out middle terms ***M1***

$= 1 + (-1)^{n-1}x^{2n}$ ***AG***

[2 marks]

(d) $\sum_{r=0}^{n-1}(-1)^r x^{2r} = \frac{1}{1+x^2} + (-1)^{n-1}\frac{x^{2n}}{1+x^2}$

integrating from 0 to 1 ***M1***

$\left[\sum_{r=0}^{n-1}(-1)^r \frac{x^{2r+1}}{2r+1}\right]_0^1 = \int_0^1 f(x)\,\mathrm{d}x + (-1)^{n-1}\int_0^1 \frac{x^{2n}}{1+x^2}\mathrm{d}x$ ***A1A1***

$\int_0^1 f(x)\,\mathrm{d}x = \frac{\pi}{4}$ ***A1***

so $\pi = 4\left(\sum_{r=0}^{n-1}\frac{(-1)^r}{2r+1} - (-1)^{n-1}\int_0^1 \frac{x^{2n}}{1+x^2}\mathrm{d}x\right)$ ***AG***

[4 marks]

Total [14 marks]

5. (a) gradient of f at $(1, 0)$ is $1 - 0^2 = 1$ and the gradient of g at $(1, 0)$ is $0 - 1^2 = -1$ ***A1***

so gradient of normal is 1 ***A1***

= Gradient of the tangent of f at $(1,0)$ ***AG***

[2 marks]

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} - y = -x^2$

integrating factor is $\mathrm{e}^{\int -1\,\mathrm{d}x} = \mathrm{e}^{-x}$ ***M1***

$y\mathrm{e}^{-x} = \int -x^2\,\mathrm{e}^{-x}\,\mathrm{d}x$ ***A1***

$= x^2\,\mathrm{e}^{-x} - \int 2x\mathrm{e}^{-x}\,\mathrm{d}x$ ***M1***

$= x^2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x} - \int 2\mathrm{e}^{-x}\,\mathrm{d}x$

$= x^2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x} + 2\mathrm{e}^{-x} + c$ ***A1***

$\Rightarrow g(x) = x^2 + 2x + 2 + c\mathrm{e}^{x}$

$g(1) = 0 \Rightarrow c = -\frac{5}{\mathrm{e}}$ ***M1***

$\Rightarrow g(x) = x^2 + 2x + 2 - 5\mathrm{e}^{x-1}$ ***A1***

[6 marks]

(c) use of $y_{n+1} = y_n + hf'(x_n, y_n)$ ***(M1)***

$x_0 = 1,\ y_0 = 0$

$x_1 = 1.2,\ y_1 = 0.2$ ***A1***

$x_2 = 1.4,\ y_2 = 0.432$ ***(M1)(A1)***

$x_3 = 1.6,\ y_3 = 0.67467\ldots$

$x_4 = 1.8,\ y_4 = 0.90363\ldots$

$x_5 = 2,\ y_5 = 1.1003255...$

answer $= 1.10033$ ***A1*** ***N3***

Note: Award ***A0*** or ***N1*** if 1.10 given as answer.

[5 marks]

(d) at the point $(1, 0)$, the gradient of f is positive so the graph of f passes into the first quadrant for $x > 1$

in the first quadrant below the curve $x - y^2 = 0$ the gradient of f is positive ***R1***

the curve $x - y^2 = 0$ has positive gradient in the first quadrant ***R1***

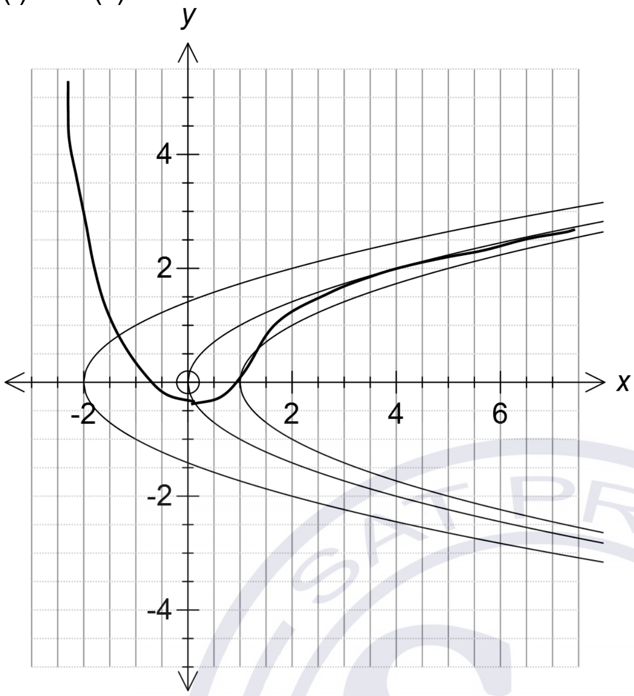
if f were to reach $x - y^2 = 0$ it would have gradient of zero, and therefore would not cross ***R1***

[3 marks]

continued…

Question 5 continued

(e) (i) and (ii)



A4

Note:	Award ***A1*** for 3 correct isoclines. Award ***A1*** for f not reaching $x - y^2 = 0$. Award ***A1*** for turning point of f on $x - y^2 = 0$. Award ***A1*** for negative gradient to the left of the turning point.

Note:	Award ***A1*** for correct shape and position if curve drawn without any isoclines.

[4 marks]

Total [20 marks]

Markscheme

May 2015

Calculus

Higher level

Paper 3

12 pages

Instructions to Examiners

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Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. $f(0)=0$ ***A1***

$f'(x) = -e^{-x}\cos x - e^{-x}\sin x + 1$ ***M1A1***

$f'(0)=0$ ***(M1)***

$f''(x) = 2e^{-x}\sin x$ ***A1***

$f''(0)=0$

$f^{(3)}(x) = -2e^{-x}\sin x + 2e^{-x}\cos x$ ***A1***

$f^{(3)}(0)=2$

the first non-zero term is $\frac{2x^3}{3!}\left(=\frac{x^3}{3}\right)$ ***A1***

Note: Award no marks for using known series.

[7 marks]

2. (a) **METHOD 1**

$\frac{dy}{dx} = -\frac{1}{x^2}\int f(x)\,dx + \frac{1}{x}f(x)$ ***M1M1A1***

$x\frac{dy}{dx} + y = f(x),\ x > 0$ ***AG***

Note: ***M1*** for use of product rule, ***M1*** for use of the fundamental theorem of calculus, ***A1*** for all correct.

METHOD 2

$x\frac{dy}{dx} + y = f(x)$

$\frac{d(xy)}{dx} = f(x)$ ***(M1)***

$xy = \int f(x)dx$ ***M1A1***

$y = \frac{1}{x}\int f(x)dx$ ***AG***

[3 marks]

continued…

Question 2 continued

(b) $y = \frac{1}{x}\left(2x^{\frac{1}{2}} + c\right)$ ***A1A1***

Note: ***A1*** for correct expression apart from the constant, ***A1*** for including the constant in the correct position.

attempt to use the boundary condition ***M1***

$c = 4$ ***A1***

$y = \frac{1}{x}\left(2x^{\frac{1}{2}} + 4\right)$ ***A1***

[5 marks]

Note: Condone use of integrating factor.

Total [8 marks]

3. (a) **METHOD 1**

$(0 <)\frac{1}{n^2 \ln(n)} < \frac{1}{n^2}$, (for $n \geq 3$) ***A1***

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the comparison test ($\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges implies) $\sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)}$ converges ***R1***

Note: Mention of using the comparison test may have come earlier. Only award ***R1*** if previous 2 ***A1***s have been awarded.

METHOD 2

$\lim_{n \to \infty}\left(\frac{\frac{1}{n^2 \ln n}}{\frac{1}{n^2}}\right) = \lim_{n \to \infty} \frac{1}{\ln n} = 0$ ***A1***

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the limit comparison test (if the limit is 0 and the series represented by the denominator converges, then so does the series represented by the

continued…

Question 3 continued

numerator, hence) $\sum_{n=2}^{\infty}\frac{1}{n^2(\ln n)}$ converges ***R1***

> **Note:** Mention of using the limit comparison test may come earlier.
> Do not award the ***R1*** if incorrect justifications are given, for example the series "converge or diverge together".
> Only award ***R1*** if previous 2 ***A1***s have been awarded.

[3 marks]

(b) (i) **EITHER**

$\ln(n)+\ln\left(1+\frac{1}{n}\right)=\ln\left(n\left(1+\frac{1}{n}\right)\right)$ ***A1***

OR

$\ln(n)+\ln\left(1+\frac{1}{n}\right)=\ln(n)+\ln\left(\frac{n+1}{n}\right)$

$=\ln(n)+\ln(n+1)-\ln(n)$ ***A1***

THEN

$=\ln(n+1)$ ***AG***

(ii) attempt to use the ratio test $\frac{n}{(n+1)}\frac{\ln(n)}{\ln(n+1)}$ ***M1***

$\frac{n}{n+1}\to 1$ as $n\to\infty$ ***(A1)***

$\frac{\ln(n)}{\ln(n+1)}=\frac{\ln(n)}{\ln(n)+\ln\left(1+\frac{1}{n}\right)}$ ***M1***

$\to 1$ (as $n\to\infty$) ***(A1)***

$\frac{n}{(n+1)}\frac{\ln(n)}{\ln(n+1)}\to 1$ (as $n\to\infty$) hence ratio test is inconclusive ***R1***

> **Note:** A link with the limit equalling 1 and the result being inconclusive needs to be given for ***R1***.

[6 marks]

(c) (i) consider $f(x)=\frac{1}{x\ln x}$ (for $x>1$) ***A1***

$f(x)$ is continuous and positive ***A1***

and is (monotonically) decreasing ***A1***

> **Note:** If a candidate uses n rather than x, award as follows
> $\frac{1}{n\ln n}$ is positive and decreasing ***A1A1***
> $\frac{1}{n\ln n}$ is continuous for $n\in\mathbb{R}$, $n>1$ ***A1*** (only award this mark if the domain has been explicitly changed).

continued…

Question 3 continued

(ii) consider $\int_2^R \frac{1}{x \ln x} \mathrm{d}x$ ***M1***

$= \left[\ln(\ln x)\right]_2^R$ ***(M1)A1***

$\to \infty$ as $R \to \infty$ ***R1***

hence series diverges ***A1***

Note:	Condone the use of ∞ in place of R.

Note:	If the lower limit is not equal to 2, but the expression is integrated correctly award ***M0M1A1R0A0***.

[8 marks]

Total [17 marks]

4. (a) $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$ ***M1A1***

$\lim_{x\to\infty} \frac{2}{e^x} = 0$ ***M1A1***

Note:	Award ***M1*** for an attempt at differentiating for a second time.

[4 marks]

(b) attempt to integrate by parts ***M1***

$\int x^2 e^{-x} \mathrm{d}x = -x^2 e^{-x} + \int 2x e^{-x} \mathrm{d}x$ ***(A1)***

$= -x^2 e^{-x} - 2x e^{-x} + \int 2e^{-x} \mathrm{d}x$ ***(A1)***

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} (+c)$ ***A1***

$\int_0^R x^2 e^{-x} \mathrm{d}x = -R^2 e^{-R} - 2R e^{-R} - 2e^{-R} + 2$ ***M1A1***

$\lim_{R\to\infty} \left(\int_0^R x^2 e^{-x} \mathrm{d}x\right) = 2$ ***M1A1***

Note:	Award ***M1*** for consideration of the limit and ***A1*** for correct limiting value.

hence the improper integral converges ***AG***

Note:	Do not award the final four marks to candidates who do not consider *R*.

[8 marks]

Total [12 marks]

5. (a) (i) $f'(x) = 3x^2 + 6x$ ***A1***

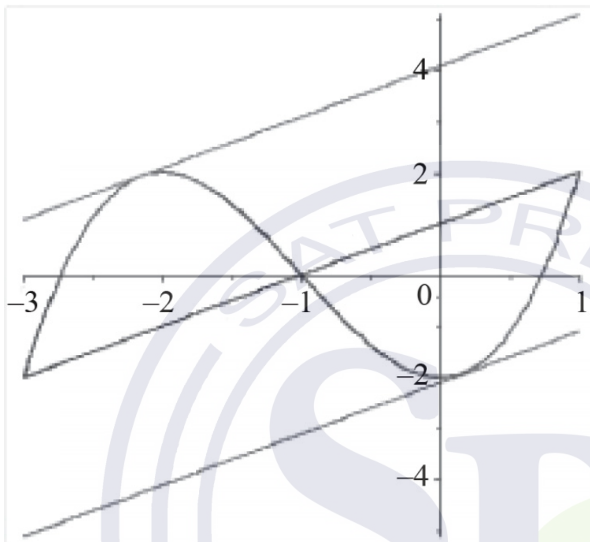
gradient of chord $= 1$ ***A1***

$3c^2 + 6c = 1$

$c = \dfrac{-3 \pm 2\sqrt{3}}{3}$ $(= -2.15,\ 0.155)$ ***A1A1***

Note: Accept any answers that round to the correct 2sf answers $(-2.2,\ 0.15)$.

(ii)



award ***A1*** for correct shape and clear indication of correct domain, ***A1*** for chord (from $x = -3$ to $x = 1$) and ***A1*** for two tangents drawn at their values of c ***A1A1A1***

[7 marks]

(b) (i) **METHOD 1**

(if a theorem is true for the interval $[a, b]$, it is also true for any interval $[x_1, x_2]$ which belongs to $[a, b]$)

suppose $x_1, x_2 \in [a, b]$ ***M1***

by the MVT, there exists c such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$ ***M1A1***

hence $f(x_1) = f(x_2)$ ***R1***

as x_1, x_2 are arbitrarily chosen, $f(x)$ is constant on $[a, b]$

Note: If the above is expressed in terms of a and b award ***M0M1A0R0***.

METHOD 2

(if a theorem is true for the interval $[a, b]$, it is also true for any interval $[x_1, x_2]$ which belongs to $[a, b]$)

suppose $x \in [a, b]$ ***M1***

continued…

Question 5 continued

by the MVT, there exists c such that $f'(c)=\dfrac{f(x)-f(a)}{x-a}=0$ ***M1A1***

hence $f(x)=f(a)=$ constant ***R1***

(ii) attempt to differentiate $(x)=2\arccos x+\arccos\left(1-2x^2\right)$ ***M1***

$$-2\frac{1}{\sqrt{1-x^2}}-\frac{-4x}{\sqrt{1-\left(1-2x^2\right)^2}}$$ ***A1A1***

$$=-2\frac{1}{\sqrt{1-x^2}}+\frac{4x}{\sqrt{4x^2-4x^4}}=0$$ ***A1***

Note:	Only award ***A1*** for 0 if a correct attempt to simplify the denominator is also seen.

$$f(x)=f(0)=2\times\frac{\pi}{2}+0=\pi$$ ***A1AG***

Note:	This ***A1*** is not dependent on previous marks.

Note:	Allow any value of $x\in[0,1]$.

[9 marks]

Total [16 marks]

International Baccalaureate®
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MARKSCHEME

November 2014

MATHEMATICS
CALCULUS

Higher Level

Paper 3

12 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2014**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $\int_1^{\infty} x^{-0.5}\,\mathrm{d}x$ ***M1***

$= \lim_{H \to \infty}\left[2x^{0.5}\right]_1^H$ ***A1***

> **Note:** Accept $\left[2x^{0.5}\right]_1^{\infty}$.

this is not finite so series is divergent ***R1***

[3 marks]

> **Note:** Accept equivalent eg $\to \infty$, or "limit does not exist".
> If lower limit is not equal to 1 award ***M0A0***, but the ***R1*** can still be awarded if the final reasoning is correct.

(b) (i) applying the ratio test ***M1***

$\lim_{n \to \infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}(n+1)^{0.5}} \times \frac{2^n n^{0.5}}{(x+1)^n}\right|$ ***A1***

$\lim_{n \to \infty}\left|\frac{(x+1)n^{0.5}}{2(n+1)^{0.5}}\right| = \left|\frac{(x+1)}{2}\right|$ ***A1***

> **Note:** Do not penalize the absence of limits and modulus signs.

converges if $\left|\frac{x+1}{2}\right| < 1 \Rightarrow -1 < \frac{(x+1)}{2} < 1$ ***M1***

$\Rightarrow -3 < x < 1$ ***A1***

> **Note:** Accept $-2 < x+1 < 2$.

(ii) considering end points ***M1***

when $x = -3$, series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$ ***A1***

$\frac{1}{n^{0.5}}$ is a decreasing sequence with limit zero, ***R1***

so series converges by alternating series test ***R1***

when $x = 1$, series is $\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$ which diverges by part (a) or *p*-series ***A1***

> **Note:** This ***A1*** is for both the reasoning and the statement it diverges.

interval of convergence is $-3 \le x < 1$ ***A1***

[11 marks]

Total [14 marks]

2. (a) integrating factor $e^{\int -\frac{1}{t}\mathrm{d}t} = e^{-\ln t} \left(= \frac{1}{t}\right)$ ***M1A1***

$\frac{x}{t} = \int -\frac{2}{t^2}\,\mathrm{d}t = \frac{2}{t} + c$ ***A1A1***

> **Note:** Award ***A1*** for $\frac{x}{t}$ and ***A1*** for $\frac{2}{t} + c$.

$x = 2 + ct$ ***AG***

[4 marks]

(b) given continuity at $x = 5$

$5c + 2 = 16 - \frac{35}{5} \Rightarrow c = \frac{7}{5}$ ***M1A1***

[2 marks]

(c) (i) 2 ***A1***

(ii) any value ≥ 16 ***A1***

> **Note:** Accept values less than 16 if fully justified by reference to the maximum age for a dog.

[2 marks]

continued …

Question 2 continued

(d) $\lim_{h\to 0-}\left(\dfrac{\frac{7}{5}(5+h)+2-\frac{7}{5}(5)-2}{h}\right)=\dfrac{7}{5}$ ***M1A1***

$\lim_{h\to 0+}\left(\dfrac{16-\frac{35}{5+h}-16+\frac{35}{5}}{h}\right)\left(=\lim_{h\to 0+}\left(\dfrac{\frac{-35}{5+h}+7}{h}\right)\right)$ ***M1***

$=\lim_{h\to 0+}\left(\dfrac{\frac{-35+35+7h}{(5+h)}}{h}\right)=\lim_{h\to 0+}\left(\dfrac{7}{5+h}\right)=\dfrac{7}{5}$ ***M1A1***

both limits equal so differentiable at $t=5$ ***R1AG***

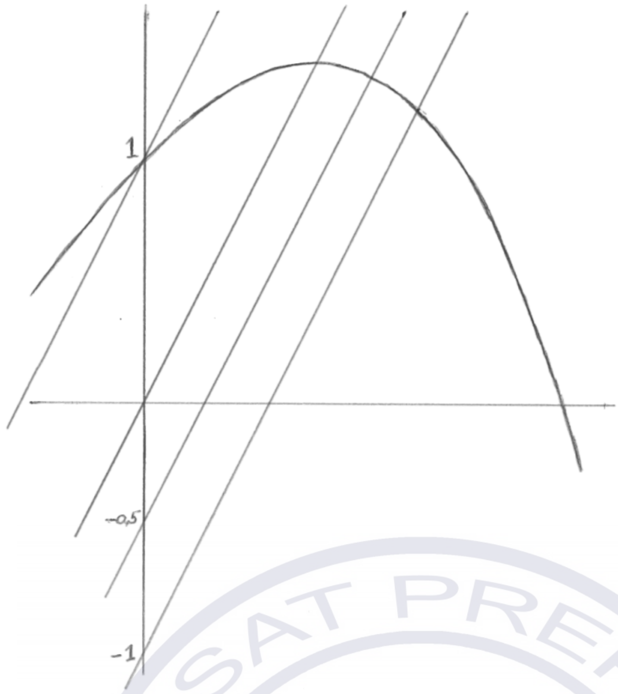
Note:	The limits $t\to 5$ could also be used. For each value of $\frac{7}{5}$ obtained by standard differentiation award ***A1***. To gain the other 4 marks a rigorous explanation must be given on how you can get from the left and right hand derivatives to the derivative.

Note:	If the candidate works with t and then substitutes $t = 5$ at the end award as follows First ***M1*** for using formula with t in the linear case, ***A1*** for $\frac{7}{5}$ Award next 2 method marks even if $t = 5$ not substituted, ***A1*** for $\frac{7}{5}$

[6 marks]

Total [14 marks]

3. (a) and (b)



(a) ***A1*** for 4 parallel straight lines with a positive gradient ***A1***
A1 for correct y intercepts ***A1***

[2 marks]

(b) ***A1*** for passing through $(0, 1)$ with positive gradient less than 2
A1 for stationary point on $y = 2x$
A1 for negative gradient on both of the other 2 isoclines ***A1A1A1***

[3 marks]

(c) the isocline is perpendicular to C ***R1***

[1mark]

(d) $y_{n+1} = y_n + 0.1(y_n - 2x_n)(= 1.1y_n - 0.2x_n)$ ***(M1)(A1)***

Note: Also award ***M1A1*** if no formula seen but y_2 is correct.

$y_0 = 1,\ y_1 = 1.1,\ y_2 = 1.19,\ y_3 = 1.269,\ y_4 = 1.3359$ ***(M1)***
$y_5 = 1.39$ to 3sf ***A1***

Note: ***M1*** is for repeated use of their formula, with steps of 0.1.

Note: Accept 1.39 or 1.4 only.

[4 marks]

Total [10 marks]

4. (a) $r = -x^2, \quad S = \dfrac{1}{1+x^2}$ ***A1AG***

[1 mark]

(b) $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

EITHER

$\int \dfrac{1}{1+x^2}\mathrm{d}x = \int 1 - x^2 + x^4 - x^6 + \ldots \mathrm{d}x$ ***M1***

$\arctan x = c + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ ***A1***

Note: Do not penalize the absence of *c* at this stage.

when $x = 0$ we have $\arctan 0 = c$ hence $c = 0$ ***M1A1***

OR

$\int_0^x \dfrac{1}{1+t^2}\mathrm{d}t = \int_0^x 1 - t^2 + t^4 - t^6 + \ldots \mathrm{d}t$ ***M1A1A1***

Note: Allow *x* as the variable as well as the limit. ***M1*** for knowing to integrate, ***A1*** for each of the limits.

$\left[\arctan t\right]_0^x = \left[t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \dfrac{t^7}{7} + \ldots\right]_0^x$ ***A1***

hence $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ ***AG***

[4 marks]

(c) applying the MVT to the function f on the interval $[x, y]$ ***M1***

$\dfrac{f(y) - f(x)}{y - x} = f'(c)$ (for some $c \in]x, y[$) ***A1***

$\dfrac{f(y) - f(x)}{y - x} > 0$ (as $f'(c) > 0$) ***R1***

$f(y) - f(x) > 0$ as $y > x$ ***R1***

$\Rightarrow f(y) > f(x)$ ***AG***

[4 marks]

Note: If they use *x* rather than c they should be awarded ***M1A0R0***, but could get the next ***R1***.

continued…

Question 4 continued

(d) (i) $g(x) = x - \arctan x \Rightarrow g'(x) = 1 - \dfrac{1}{1+x^2}$ ***A1***

this is greater than zero because $\dfrac{1}{1+x^2} < 1$ ***R1***

so $g'(x) > 0$ ***AG***

(ii) (g is a continuous function defined on $[0, b]$ and differentiable on $]0, b[$ with $g'(x) > 0$ on $]0, b[$ for all $b \in \mathbb{R}$)

(If $x \in [0, b]$ then) from part (c) $g(x) > g(0)$ ***M1***

$x - \arctan x > 0 \Rightarrow \arctan x < x$ ***M1***

(as b can take any positive value it is true for all $x > 0$) ***AG***

[4 marks]

(e) let $h(x) = \arctan x - \left(x - \dfrac{x^3}{3}\right)$ ***M1***

(h is a continuous function defined on $[0, b]$ and differentiable on $]0, b[$ with $h'(x) > 0$ on $]0, b[$)

$h'(x) = \dfrac{1}{1+x^2} - (1 - x^2)$ ***A1***

$= \dfrac{1-(1-x^2)(1+x^2)}{1+x^2} = \dfrac{x^4}{1+x^2}$ ***M1A1***

$h'(x) > 0$ hence (for $x \in [0, b]$) $h(x) > h(0) (= 0)$ ***R1***

$\Rightarrow \arctan x > x - \dfrac{x^3}{3}$ ***AG***

[5 marks]

Note: Allow correct working with $h(x) = x - \dfrac{x^3}{3} - \arctan x$.

continued …

Question 4 continued

(f) use of $x - \frac{x^3}{3} < \arctan x < x$ ***M1***

choice of $x = \frac{1}{\sqrt{3}}$ ***A1***

$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***M1***

$\frac{8}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***A1***

Note: Award final ***A1*** for a correct inequality with a single fraction on each side that leads to the final answer.

$\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$ ***AG***

[4 marks]

Total [22 marks]

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MARKSCHEME

May 2014

MATHEMATICS
CALCULUS

Higher Level

Paper 3

13 pages

Instructions to Examiners

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R Marks awarded for clear **Reasoning**.

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Using the markscheme

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- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

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*Award **N** marks for **correct** answers where there is **no** working.*

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Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

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Follow through ***(FT)*** *marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

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Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

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A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
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Students must always use correct mathematical notation, not calculator notation.

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13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) any correct step before the given answer *A1AG*

eg, $f'(x) = \frac{(e^x)' + (e^{-x})'}{2} = \frac{e^x - e^{-x}}{2} = g(x)$

any correct step before the given answer *A1AG*

eg, $g'(x) = \frac{(e^x)' - (e^{-x})'}{2} = \frac{e^x + e^{-x}}{2} = f(x)$

[2 marks]

(b) **METHOD 1**

statement and attempted use of the general Maclaurin expansion formula ***(M1)***

$f(0) = 1$; $g(0) = 0$ (or equivalent in terms of derivative values) ***A1A1***

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ or $f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ ***A1A1***

METHOD 2

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ ***A1***

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ ***A1***

adding and dividing by 2 ***M1***

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ or $f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ ***A1A1***

Notes: Accept $1, \frac{x^2}{2}$ and $\frac{x^4}{24}$ or $1, \frac{x^2}{2!}$ and $\frac{x^4}{4!}$.
Award ***A1*** if two correct terms are seen.

[5 marks]

continued…

Question 1 continued

(c) **METHOD 1**

attempted use of the Maclaurin expansion from (b) ***M1***

$$\lim_{x\to 0}\frac{1-f(x)}{x^2}=\lim_{x\to 0}\frac{1-\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\ldots\right)}{x^2}$$

$$\lim_{x\to 0}\left(-\frac{1}{2}-\frac{x^2}{24}-\ldots\right)$$ ***A1***

$$=-\frac{1}{2}$$ ***A1***

METHOD 2

attempted use of L'Hôpital and result from (a) ***M1***

$$\lim_{x\to 0}\frac{1-f(x)}{x^2}=\lim_{x\to 0}\frac{-g(x)}{2x}$$

$$\lim_{x\to 0}\frac{-f(x)}{2}$$ ***A1***

$$=-\frac{1}{2}$$ ***A1***

[3 marks]

(d) **METHOD 1**

use of the substitution $u=f(x)$ and $(\mathrm{d}u=g(x)\mathrm{d}x)$ ***(M1)(A1)***

attempt to integrate $\int_1^{\infty}\frac{\mathrm{d}u}{u^2}$ ***(M1)***

obtain $\left[-\frac{1}{u}\right]_1^{\infty}$ or $\left[-\frac{1}{f(x)}\right]_0^{\infty}$ ***A1***

recognition of an improper integral by use of a limit or statement saying the integral converges ***R1***

obtain 1 ***A1*** ***N0***

continued…

Question 1 continued

METHOD 2

$$\int_0^{\infty} \frac{\frac{e^x - e^{-x}}{2}}{\left(\frac{e^x + e^{-x}}{2}\right)^2} dx = \int_0^{\infty} \frac{2\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2} dx$$ ***(M1)***

use of the substitution $u = e^x + e^{-x}$ and $\left(du = e^x - e^{-x}\,dx\right)$ ***(M1)***

attempt to integrate $\int_2^{\infty} \frac{2du}{u^2}$ ***(M1)***

obtain $\left[-\frac{2}{u}\right]_2^{\infty}$ ***A1***

recognition of an improper integral by use of a limit or statement saying the integral converges ***R1***

obtain 1 ***A1*** ***N0***

[6 marks]

Total [16 marks]

2. (a) (i) attempt at chain rule *(M1)*

$f'(x) = \dfrac{2\ln x}{x}$ ***A1***

(ii) attempt at chain rule *(M1)*

$g'(x) = \dfrac{2}{x\ln x}$ ***A1***

(iii) $g'(x)$ is positive on $]1, \infty[$ ***A1***

so $g(x)$ is increasing on $]1, \infty[$ ***AG***

[5 marks]

(b) (i) rearrange in standard form:

$\dfrac{dy}{dx} + \dfrac{2}{x\ln x}y = \dfrac{2x-1}{(\ln x)^2},\ x > 1$ ***(A1)***

integrating factor:

$e^{\int \frac{2}{x\ln x}dx}$ ***(M1)***

$= e^{\ln((\ln x)^2)}$

$= (\ln x)^2$ ***(A1)***

multiply by integrating factor ***(M1)***

$(\ln x)^2 \dfrac{dy}{dx} + \dfrac{2\ln x}{x}y = 2x-1$

$\dfrac{d}{dx}\left(y(\ln x)^2\right) = 2x-1$ (or $y(\ln x)^2 = \int 2x-1\,dx$) ***M1***

attempt to integrate: ***M1***

$(\ln x)^2 y = x^2 - x + c$

$y = \dfrac{x^2-x+c}{(\ln x)^2}$ ***A1***

(ii) attempt to use the point (e, e^2) to determine c: ***M1***

eg, $(\ln e)^2 e^2 = e^2 - e + c$ or $e^2 = \dfrac{e^2-e+c}{(\ln e)^2}$ or $e^2 = e^2 - e + c$

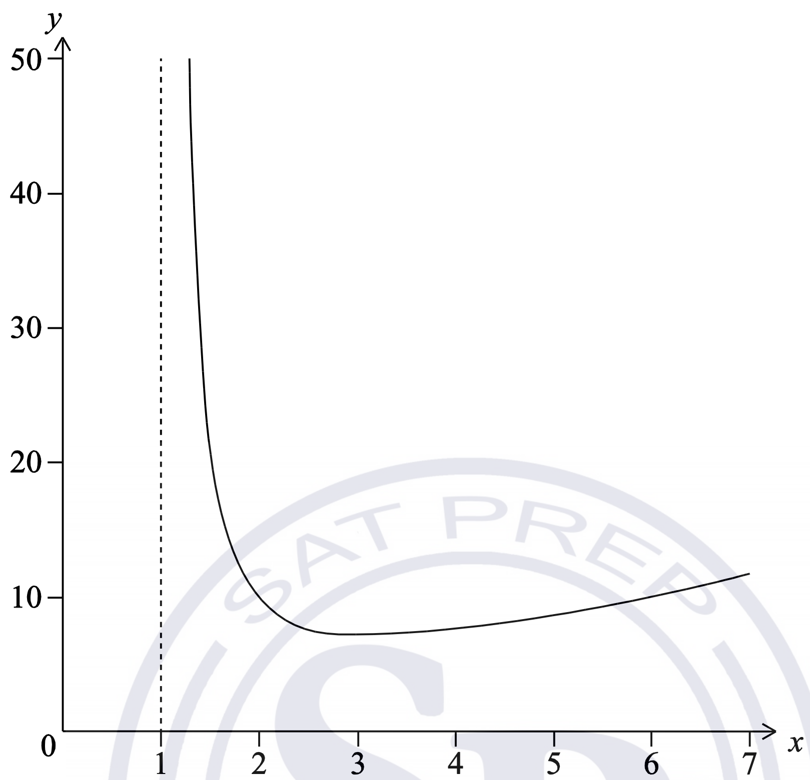
$c = e$ ***A1***

$y = \dfrac{x^2-x+e}{(\ln x)^2}$ ***AG***

continued…

Question 2 continued

(iii)



graph with correct shape ***A1***

minimum at $x = 3.1$ (accept answers to a minimum of 2 s.f) ***A1***

asymptote shown at $x = 1$ ***A1***

Note: y-coordinate of minimum not required for ***A1***;
Equation of asymptote not required for ***A1*** if VA appears on the sketch.
Award ***A0*** for asymptotes if more than one asymptote are shown

[12 marks]

Total [17 marks]

3. (a) $b(n) = 3n + 1$ ***A1***
$c(n) = 3n + 2$ ***A1***

> **Note:** $b(n)$ and $c(n)$ may be reversed.

[2 marks]

(b) consider the ratio of the $(n+1)^{\text{th}}$ and n^{th} terms: ***M1***

$$\frac{3n+1}{3n+4} \times \frac{3n+2}{3n+5} \times \frac{x^{n+1}}{x^n}$$ ***A1***

$$\lim_{n \to \infty} \frac{3n+1}{3n+4} \times \frac{3n+2}{3n+5} \times \frac{x^{n+1}}{x^n} = x$$ ***A1***

radius of convergence: $R = 1$ ***A1***

[4 marks]

(c) any attempt to study the series for $x = -1$ or $x = 1$ ***(M1)***

converges for $x = 1$ by comparing with p-series $\sum \frac{1}{n^2}$ ***R1***

attempt to use the alternating series test for $x = -1$ ***(M1)***

> **Note:** At least one of the conditions below needs to be attempted for ***M1***.

$|\text{terms}| \approx \frac{1}{9n^2} \to 0$ and terms decrease monotonically in absolute value ***A1***

series converges for $x = -1$ ***R1***

interval of convergence: $[-1, 1]$ ***A1***

> **Note:** Award the ***R1***s only if an attempt to corresponding correct test is made;
> award the final ***A1*** only if at least one of the ***R1***s is awarded;
> Accept study of absolute convergence at end points.

[6 marks]

Total [12 marks]

4. (a) $\lim_{x\to 1^-} e^{-x^2}\left(-x^3+2x^2+x\right)=\lim_{x\to 1^+}(ax+b)$ $(=a+b)$ ***M1***

$2e^{-1}=a+b$ ***A1***

differentiability: attempt to differentiate **both** expressions ***M1***

$f'(x)=-2xe^{-x^2}\left(-x^3+2x^2+x\right)+e^{-x^2}\left(-3x^2+4x+1\right)$ $(x<1)$ ***A1***

(or $f'(x)=e^{-x^2}\left(2x^4-4x^3-5x^2+4x+1\right)$)

$f'(x)=a$ $(x>1)$ ***A1***

substitute $x=1$ in **both** expressions and equate

$-2e^{-1}=a$ ***A1***

substitute value of a and find $b=4e^{-1}$ ***M1A1***

[8 marks]

(b) (i) $f'(x)=e^{-x^2}\left(2x^4-4x^3-5x^2+4x+1\right)$ (for $x\le 1$) ***M1***

$f(1)=f(-1)$ ***M1***

Rolle's theorem statement ***(A1)***

by Rolle's Theorem, $f'(x)$ has a zero in $]-1,1[$ ***R1***

hence quartic equation has a root in $]-1,1[$ ***AG***

continued…

Question 4 continued

(ii) let $g(x) = 2x^4 - 4x^3 - 5x^2 + 4x + 1$.

$g(-1) = g(1) < 0$ and $g(0) > 0$ ***M1***

as g is a polynomial function it is continuous in $[-1, 0]$ and $[0, 1]$. ***R1***

(or g is a polynomial function continuous in any interval of real numbers)

then the graph of g must cross the x-axis at least once in $]-1, 0[$ ***R1***

and at least once in $]0, 1[$.

[7 marks]

Total [15 marks]

N13/5/MATHL/HP3/ENG/TZ0/SE/M

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

November 2013

MATHEMATICS
SERIES AND DIFFERENTIAL EQUATIONS

Higher Level

Paper 3

12 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

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R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking November 2013**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

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- If a part is completely wrong, stamp ***A0*** by the final answer.
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1. (a) **EITHER**

$$\sum_{n=1}^{\infty}\frac{2}{n^2+3n}<\sum_{n=1}^{\infty}\frac{2}{n^2}$$ ***M1***

which is convergent ***A1***

the given series is therefore convergent using the comparison test ***AG***

OR

$$\lim_{n\to\infty}\frac{\frac{2}{n^2+3n}}{\frac{1}{n^2}}=2$$ ***M1A1***

the given series is therefore convergent using the limit comparison test ***AG***

[2 marks]

(b) (i) let $\frac{2}{n^2+3n}=\frac{A}{n}+\frac{B}{n+3}=\frac{A(n+3)+Bn}{n(n+3)}$

solve for A and B ***(M1)***

$A=\frac{2}{3}$ ***(A1)***

$B=-\frac{2}{3}$ ***(A1)***

$\frac{2}{n^2+3n}=\frac{2}{3n}-\frac{2}{3(n+3)}$ ***A1***

(ii) using partial fractions

$\sum_{n=1}^{\infty}\frac{2}{n^2+3n}=\frac{2}{3}\left(\frac{\mathbf{1}}{\mathbf{1}}-\frac{1}{4}+\frac{\mathbf{1}}{\mathbf{2}}-\frac{1}{5}+\frac{\mathbf{1}}{\mathbf{3}}-\frac{1}{6}+\frac{1}{4}\ldots\right)$ ***M1A1***

recognizing the cancellation (in the telescoping series) (*eg* crossing out) ***R1***

$\sum_{n=1}^{\infty}\frac{2}{n^2+3n}=\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right)=\frac{11}{9}\ \left(1\frac{2}{9}\right)$ ***A1***

[8 marks]

Total [10 marks]

2. (a) $a_n = \dfrac{e^n + 2^n}{2e^n} = \dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{2}{e}\right)^n > \dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{2}{e}\right)^{n+1} = a_{n+1}$ ***M1A1***

the sequence is decreasing (as terms are positive) ***A1***

Note: Accept reference to the sum of a constant and a decreasing geometric sequence.

Note: Accept use of derivative of $f(x) = \dfrac{e^x + 2^x}{2e^x}$ (and condone use of n) and graphical methods (graph of the sequence or graph of corresponding function f or graph of its derivative f'). Accept a list of consecutive terms of the sequence clearly decreasing (*eg* 0.8678…, 0.77067…, …).

[3 marks]

(b) $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{2}{e}\right)^n = \dfrac{1}{2} + \dfrac{1}{2} \times 0 = \dfrac{1}{2}$ ***M1A1***

[2 marks]

(c) $\left|a_n - \dfrac{1}{2}\right| = \left|\dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{2}{e}\right)^n - \dfrac{1}{2}\right| = \left|\dfrac{1}{2}\left(\dfrac{2}{e}\right)^n\right| < \dfrac{1}{1000}$ ***M1***

EITHER

$\Rightarrow \left(\dfrac{e}{2}\right)^n > 500$ ***(A1)***

$\Rightarrow n > 20.25\ldots$ ***(A1)***

OR

$\Rightarrow \left(\dfrac{2}{e}\right)^n < 500$

$\Rightarrow n > 20.25\ldots$ ***(A1)(A1)***

Note: ***A1*** for correct inequality; ***A1*** for correct value.

THEN

therefore $N = 21$ ***A1***

[4 marks]

Total [9 marks]

3. (a) let $f(x,\ y)=\dfrac{y}{x+\sqrt{xy}}$

$y(1.2)=y(1)+0.2f(1,\ 2)\quad(=2+0.1656...)$ ***(M2)(A1)***

$=2.1656\ldots$ ***A1***

$y(1.4)=2.1656\ldots+0.2f(1.2,\ 2.1256\ldots)\quad(=2.1656...+0.1540...)$ ***(M1)***

> **Note:** ***M1*** is for attempt to apply formula using point $(1.2,\ y(1.2))$.

$=2.3197\ldots$ ***A1***

$y(1.6)=2.3197\ldots+0.2f(1.4,\ 2.3197\ldots)\quad(=2.3297...+0.1448...)$

$=2.46\quad\text{(3sf)}$ ***A1*** ***N3***

[7 marks]

(b) $y=vx\Rightarrow\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ ***(M1)***

$\dfrac{dy}{dx}=\dfrac{y}{x+\sqrt{xy}}\Rightarrow v+x\dfrac{dv}{dx}=\dfrac{vx}{x+\sqrt{vx^2}}$ ***M1***

$\Rightarrow v+x\dfrac{dv}{dx}=\dfrac{vx}{x+x\sqrt{v}}$ (as $x>0$) ***A1***

$\Rightarrow x\dfrac{dv}{dx}=\dfrac{v}{1+\sqrt{v}}-v$ ***AG***

[3 marks]

(c) (i) $x\dfrac{dv}{dx}=\dfrac{v}{1+\sqrt{v}}-v$

$x\dfrac{dv}{dx}=\dfrac{-v\sqrt{v}}{1+\sqrt{v}}\Rightarrow\dfrac{1+\sqrt{v}}{-v\sqrt{v}}dv=\dfrac{1}{x}dx$ ***M1***

$\displaystyle\int\frac{1+\sqrt{v}}{-v\sqrt{v}}dv=\int\frac{1}{x}dx$ ***(M1)***

$\dfrac{2}{\sqrt{v}}-\ln v=\ln x+C$ ***A1A1***

> **Note:** Do not penalize absence of $+C$ at this stage; ignore use of absolute values on v and x (which are positive anyway).

continued ...

Question 3 continued

$$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} = \ln x + C \text{ as } y = vx \Rightarrow v = \frac{y}{x}$$ ***M1***

$y = 2$ when $x = 1 \Rightarrow \sqrt{2} - \ln 2 = 0 + C$ ***M1***

$$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} = \ln x + \sqrt{2} - \ln 2$$

$$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} - \ln x - \sqrt{2} + \ln 2 = 0 \quad \left(2\sqrt{\frac{x}{y}} - \ln y - \sqrt{2} + \ln 2 = 0\right)$$ ***A1***

(ii) $2\sqrt{\frac{1.6}{y}} - \ln\frac{y}{1.6} - \ln 1.6 - \sqrt{2} + \ln 2 = 0$ ***(M1)***

$y = 2.45$ ***A1***

[9 marks]

Total [19 marks]

4. (a) **METHOD 1**

$$\lim_{x\to 0}\frac{\sin 4x^2 - \sin 9x^2}{4x^2}$$ ***M1***

$$= \lim_{x\to 0}\frac{\sin 4x^2}{4x^2} - \frac{9}{4}\lim_{x\to 0}\frac{\sin 9x^2}{9x^2}$$ ***A1A1***

$$= 1 - \frac{9}{4}\times 1 = -\frac{5}{4}$$ ***A1***

METHOD 2

$$\lim_{x\to 0}\frac{\sin 4x^2 - \sin 9x^2}{4x^2}$$ ***M1***

$$= \lim_{x\to 0}\frac{8x\cos 4x^2 - 18x\cos 9x^2}{8x}$$ ***M1A1***

$$= \frac{8-18}{8} = -\frac{10}{8} = -\frac{5}{4}$$ ***A1***

[4 marks]

continued …

Question 4 continued

(b) since $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$ $\left(\text{or } \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$ ***(M1)***

$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!}$ $\left(\text{or } \sin x = \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right)$ ***A1***

$g(x) = \sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ ***AG***

[2 marks]

(c) let $I = \int_0^1 \sin x^2 \, dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^1 x^{4n+2} dx$ $\left(\int_0^1 \frac{x^2}{1!} dx - \int_0^1 \frac{x^6}{3!} dx + \int_0^1 \frac{x^{10}}{5!} dx - \ldots\right)$ ***M1***

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{[x^{4n+3}]_0^1}{(4n+3)}$ $\left(\left[\frac{x^3}{3\times 1!}\right]_0^1 - \left[\frac{x^7}{7\times 3!}\right]_0^1 + \left[\frac{x^{11}}{11\times 5!}\right]_0^1 - \ldots\right)$ ***M1***

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!(4n+3)}$ $\left(\frac{1}{3\times 1!} - \frac{1}{7\times 3!} + \frac{1}{11\times 5!} - \ldots\right)$ ***A1***

$= \sum_{n=0}^{\infty} (-1)^n a_n$ where $a_n = \frac{1}{(4n+3)(2n+1)!} > 0$ for all $n \in \mathbb{N}$

as $\{a_n\}$ is decreasing the sum of the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$

lies between $\sum_{n=0}^{N} (-1)^n a_n$ and $\sum_{n=0}^{N} (-1)^n a_n \pm a_{N+1}$ ***R1***

hence for four decimal place accuracy, we need $|a_{N+1}| < 0.00005$ ***M1***

N	$\lvert a_{N+1} \rvert$
1	$\frac{1}{11(5!)} = 0.0000757576$
2	$\frac{1}{15(7!)} = 0.0000132275$

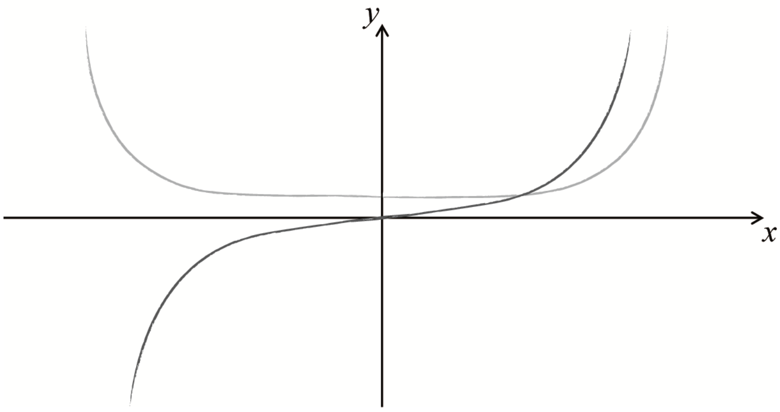
since $a_{2+1} < 0.00005$ ***R1***

so $N = 2$ (or 3 terms) ***A1***

[7 marks]

Total [13 marks]

5. (a)



A1 for shape, ***A1*** for passing through origin ***A1A1***

Note: Asymptotes not required.

[2 marks]

(b) $p(x) = \underbrace{f(0)}_{a} + \underbrace{f'(0)}_{b}x + \underbrace{\frac{f''(0)}{2!}}_{c}x^2 + \underbrace{\frac{f^{(3)}(0)}{3!}}_{d}x^3 + \ldots$

(i) because the y-intercept of f is positive ***R1***

(ii) $b = 0$ ***A1***
$c \geq 0$ ***A1A1***

Note: ***A1*** for $>$ and ***A1*** for $=$.

$d = 0$ ***A1***

[5 marks]

(c) as the graph has vertical asymptotes $x = \pm k$, $k > 0$, ***R1***
the radius of convergence has an upper bound of k ***A1***

Note: Accept $r < k$.

[2 marks]

Total [9 marks]

M13/5/MATHL/HP3/ENG/TZ0/SE/M

International Baccalaureate®
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Bachillerato Internacional

MARKSCHEME

May 2013

MATHEMATICS
SERIES AND DIFFERENTIAL EQUATIONS

Higher Level

Paper 3

12 pages

Instructions to Examiners

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Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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Students must always use correct mathematical notation, not calculator notation.

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Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) let $f(x)=\sqrt{x}$, $f(1)=1$ ***(A1)***

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$, $f'(1)=\frac{1}{2}$ ***(A1)***

$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$, $f''(1)=-\frac{1}{4}$ ***(A1)***

$f'''(x)=\frac{3}{8}x^{-\frac{5}{2}}$, $f'''(1)=\frac{3}{8}$ ***(A1)***

$a_1=\frac{1}{2}.\frac{1}{1!}$, $a_2=-\frac{1}{4}\cdot\frac{1}{2!}$, $a_3=\frac{3}{8}\cdot\frac{1}{3!}$ ***(M1)***

$a_0=1$, $a_1=\frac{1}{2}$, $a_2=-\frac{1}{8}$, $a_3=\frac{1}{16}$ ***A1***

Note: Accept $y=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3+\dots$

[6 marks]

(b) **METHOD 1**

$$\lim_{x\to1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to1}\frac{\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\dots}{x-1}$$ ***M1***

$$=\lim_{x\to1}\left(\frac{1}{2}-\frac{1}{8}(x-1)+\dots\right)$$ ***A1***

$$=\frac{1}{2}$$ ***A1***

METHOD 2

using l'Hôpital's rule, ***M1***

$$\lim_{x\to1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to1}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1}$$ ***A1***

$$=\frac{1}{2}$$ ***A1***

METHOD 3

$$\frac{\sqrt{x}-1}{x+1}=\frac{1}{\sqrt{x}+1}$$ ***M1A1***

$$\lim_{x\to1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}$$ ***A1***

[3 marks]

Total [9 marks]

2. (a) use of $y \to y + \frac{h\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{h\mathrm{d}y}{\mathrm{d}x}$
0	2	1	0.1
0.1	2.1	0.7793304775	0.07793304775
0.2	2.17793304775	0.5190416116	0.05190416116
0.3	2.229837209		

A1A1

Note: Award ***A1*** for $y(0.1)$ and ***A1*** for $y(0.2)$

$y(0.3) = 2.23$ ***A2***

[5 marks]

(b) (i) IF $= \mathrm{e}^{\left(\int \tan x \, \mathrm{d}x\right)}$ ***(M1)***

IF $= \mathrm{e}^{\left(\int \frac{\sin x}{\cos x} \mathrm{d}x\right)}$ ***(M1)***

Note: Only one of the two ***(M1)*** above may be implied.

$= \mathrm{e}^{(-\ln \cos x)}$ (or $\mathrm{e}^{(\ln \sec x)}$) ***A1***

$= \sec x$ ***AG***

(ii) multiplying by the IF ***(M1)***

$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sec x \tan x = \cos x$ ***(A1)***

$\frac{\mathrm{d}}{\mathrm{d}x}(y \sec x) = \cos x$ ***(A1)***

$y \sec x = \sin x + c$ ***A1A1***

putting $x = 0$, $y = 2 \Rightarrow c = 2$ ***M1***

$y = \cos x(\sin x + 2)$ ***A1***

[10 marks]

Total [15 marks]

3. (a) $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{\frac{(n+1)^2x^{n+1}}{2^{n+1}}}{\frac{n^2x^n}{2^n}}$ ***M1***

$=\lim\limits_{n\to\infty}\frac{(n+1)^2}{n^2}\times\frac{x}{2}$ ***A1***

$=\frac{x}{2}$ (since $\lim\to\frac{x}{2}$ as $n\to\infty$) ***A1***

the radius of convergence R is found by equating this limit to 1, giving
$R=2$ ***A1***

[4 marks]

(b) when $x=2$, the series is $\sum n^2$ which is divergent because the terms do not converge to 0 ***R1***

when $x=-2$, the series is $\sum(-1)^n n^2$ which is divergent because the terms do not converge to 0 ***R1***

the interval of convergence is $]-2, 2[$ ***A1***

[3 marks]

(c) putting $x=-0.1$, ***(M1)***

for any correct partial sum ***(A1)***

–0.05
–0.04
–0.041125
–0.041025
–0.0410328 ***(A1)***

the sum is –0.0410 correct to 3 significant figures ***A1***

[4 marks]

Total [11 marks]

4. (a) let $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2} = \frac{A(r+2)+Br}{r(r+2)}$ ***(M1)***

$A = \frac{1}{2},\ B = -\frac{1}{2}$ ***A1A1***

$\left(\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}\right)$

[3 marks]

(b) (i) attempt to sum using partial fractions ***(M1)***

$S_n = \frac{1}{2} - \frac{1}{6}$

$+\frac{1}{4} - \frac{1}{8}$

$+\frac{1}{6} - \frac{1}{10}$ ***(A1)***

……………..

$+\frac{1}{2(n-1)} - \frac{1}{2(n+1)}$

$+\frac{1}{2n} - \frac{1}{2(n+2)}$ ***(A1)***

$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$ ***(M1)A1***

$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$ ***A1***

Note: Award ***A1*** for alternative intermediate steps.

$S_n = \frac{3n^2 + 5n}{4(n+1)(n+2)}$ ***A1***

$(a = 3,\ b = 5)$

(ii) $\lim_{n \to \infty} S_n = \frac{3}{4}$ ***A1***

[8 marks]

Total [11 marks]

5. (a) (i) the area under the curve between $a-1$ and $a+1$

$= \int_{a-1}^{a+1} \frac{dx}{x}$ ***M1***

$= [\ln x]_{a-1}^{a+1}$ ***A1***

$= \ln\left(\frac{a+1}{a-1}\right)$ ***A1***

lower sum $= \frac{1}{a} + \frac{1}{a+1}$ ***M1A1***

$= \frac{2a+1}{a(a+1)}$ ***AG***

upper sum $= \frac{1}{a-1} + \frac{1}{a}$ ***A1***

$= \frac{2a-1}{a(a-1)}$ ***AG***

it follows that

$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)}$

because the area of the region under the curve lies between the areas of the regions defined by the lower and upper sums ***R1***

(ii) putting

$\left(\frac{a+1}{a-1} = 1.2\right) \Rightarrow a = 11$ ***A1***

therefore, $\text{UB} = \frac{21}{110} (= 0.191)$, $\text{LB} = \frac{23}{132} (= 0.174)$ ***A1***

[9 marks]

continued …

Question 5 continued

(b) (i) the area under the curve between $a-1$ and a

$= \int_{a-1}^{a} \frac{\mathrm{d}x}{x}$ ***A1***

$= [\ln x]_{a-1}^{a} = \ln\left(\frac{a}{a-1}\right)$

attempt to find area of trapezium ***M1***

area of trapezoidal "upper sum" $= \frac{1}{2}\left(\frac{1}{a-1} + \frac{1}{a}\right)$ or equivalent ***A1***

$= \frac{2a-1}{2a(a-1)}$

it follows that $\ln\left(\frac{a}{a-1}\right) < \frac{2a-1}{2a(a-1)}$ ***AG***

(ii) putting

$\left(\frac{a}{a-1} = 1.2\right) \Rightarrow a = 6$ ***A1***

therefore, $\text{UB} = \frac{11}{60} (= 0.183)$ ***A1***

[5 marks]

Total [14 marks]

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MARKSCHEME

November 2012

MATHEMATICS
SERIES AND DIFFERENTIAL EQUATIONS

Higher Level

Paper 3

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$\Rightarrow \ln y=\ln x+c$ ***A1***

$\Rightarrow \ln y=\ln x+\ln k=\ln kx$

$\Rightarrow y=kx$ ***A1***

[3 marks]

(b) $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$ ***(A1)***

so $v+x\frac{dv}{dx}=v$ ***M1***

$\Rightarrow x\frac{dv}{dx}=0\Rightarrow\frac{dv}{dx}=0$ (as $x\neq 0$) ***R1***

$\Rightarrow v=k$

$\Rightarrow \frac{y}{x}=k$ $(\Rightarrow y=kx)$ ***A1***

[4 marks]

(c) $\frac{dy}{dx}+\left(\frac{-1}{x}\right)y=0$ ***(M1)***

$\text{IF}=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\frac{1}{x}$ ***M1A1***

$x^{-1}\frac{dy}{dx}-x^{-2}y=0$

$\Rightarrow \frac{d[x^{-1}y]}{dx}=0$ ***(M1)***

$\Rightarrow x^{-1}y=k$ $(\Rightarrow y=kx)$ ***A1***

[5 marks]

(d) $20=2k\Rightarrow k=10$ so $y(5)=10\times 5=50$ ***A1***

[1 mark]

Total [13 marks]

2. (a) using $x_0 = 1,\ y_0 = 1$

$x_n = 1 + 0.1n\ ,\ y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$ ***(M1)(M1)(A1)***

Note:	If they have not written down formulae but have $x_1 = 1.1$ and $y_1 = 1.14142...$ award ***M1M1A1***.

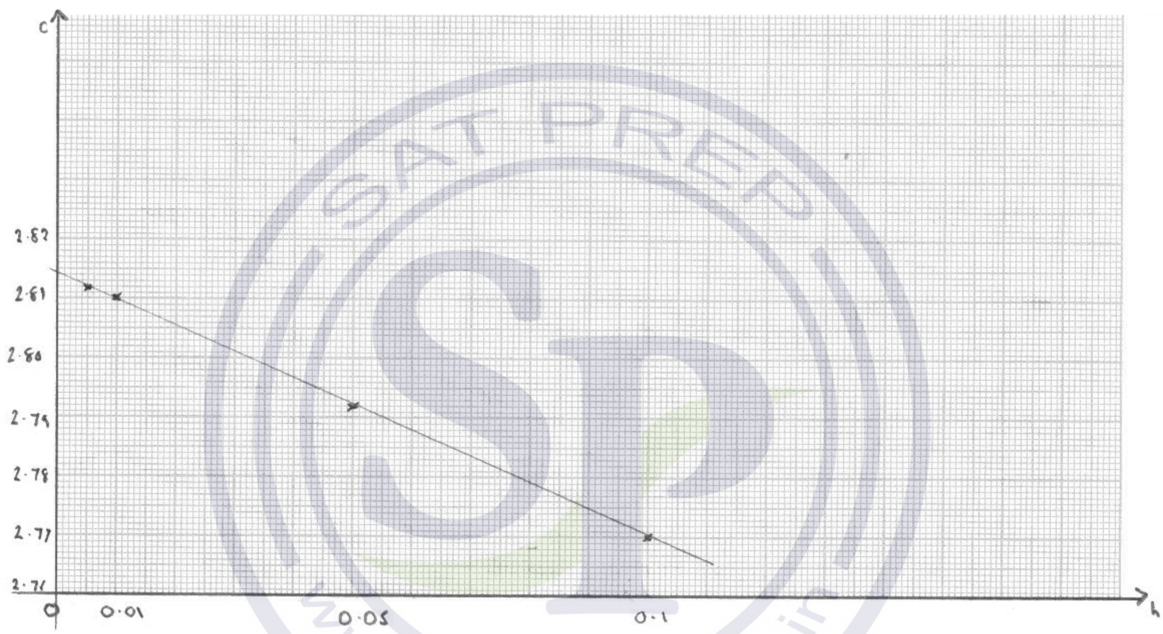
gives by GDC $x_{10} = 2,\ y_{10} = 2.770114792\ldots$ ***(M1)(A1)***

so $a \simeq 2.7701$ (4dp) ***A1*** ***N6***

Note:	Do not penalize over-accuracy.

[6 marks]

(b)



points drawn on graph above ***A1A1A1***

Note:	Award ***A1*** for scales, ***A1*** for 2 points correctly plotted, ***A1*** for other 2 points correctly plotted (second and third ***A1*** dependent on the first being correct).

[3 marks]

(c) suitable line of best fit placed on graph ***A1***

[1 mark]

(d) letting $h \to 0$ we approach the y intercept on the graph so ***(R1)***

$c \simeq 2.814$ (3dp) ***A1***

Note:	Accept 2.815.

[2 marks]

Total [12 marks]

3. (a) $\lim_{H\to\infty}\int_a^H \frac{1}{x^2}\mathrm{d}x = \lim_{H\to\infty}\left[\frac{-1}{x}\right]_a^H$ ***A1***

$= \lim_{H\to\infty}\left(\frac{-1}{H}+\frac{1}{a}\right)$ ***A1***

$= \frac{1}{a}$ ***A1***

[3 marks]

(b) as $\left\{\frac{1}{n^2}\right\}$ is a positive decreasing sequence we consider the function $\frac{1}{x^2}$

we look at $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$ ***M1***

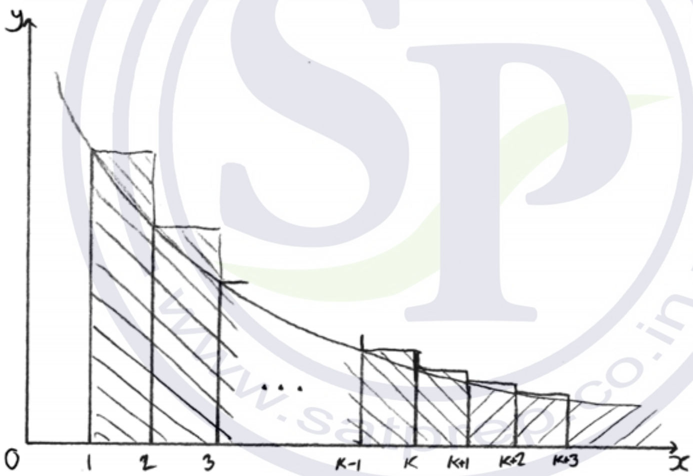
$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = 1$ ***A1***

since this is finite (allow "limit exists" or equivalent statement) ***R1***

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges ***AG***

[3 marks]

(c) (i)



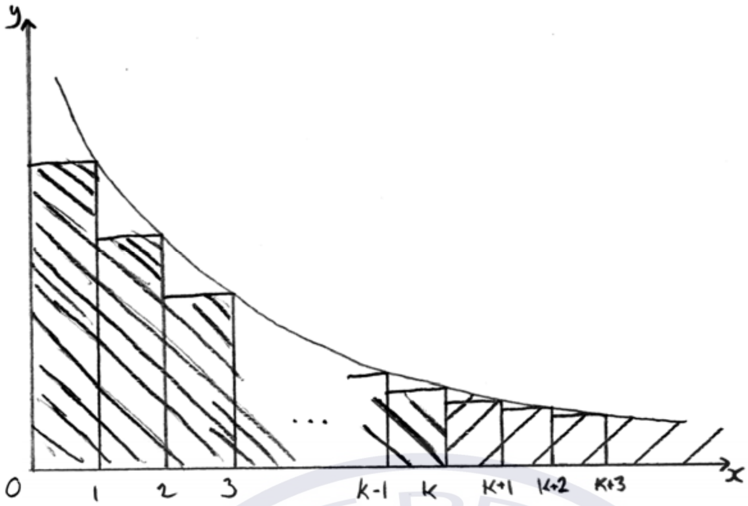
attempt to shade rectangles ***M1***
correct start and finish points for rectangles ***A1***
since the area shaded is less that the area of the required staircase we have ***R1***

$\sum_{n=1}^{k}\frac{1}{n^2}+\int_{k+1}^{\infty}\frac{1}{x^2}\mathrm{d}x < L$ ***AG***

continued …

Question 3 continued

(ii)



attempt to shade rectangles **M1**
correct start and finish points for rectangles **A1**
since the area shaded is greater that the area of the required staircase we have **R1**

$L < \sum_{n=1}^{k} \frac{1}{n^2} + \int_k^\infty \frac{1}{x^2}\,dx$ **AG**

Note: Alternative shading and rearranging of the inequality is acceptable.

[6 marks]

(d) $\int_{k+1}^\infty \frac{1}{x^2}\,dx = \frac{1}{k+1}$, $\int_k^\infty \frac{1}{x^2}\,dx = \frac{1}{k}$ **A1A1**

$\sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k+1} < L < \sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k}$ **AG**

[2 marks]

(e) $\frac{205}{144} + \frac{1}{5} < \frac{\pi^2}{6} < \frac{205}{144} + \frac{1}{4}$ $\left(1.6236\ldots < \frac{\pi^2}{6} < 1.6736\ldots\right)$ **A1**

$\sqrt{6\left(\frac{205}{144} + \frac{1}{5}\right)} < \pi < \sqrt{6\left(\frac{205}{144} + \frac{1}{4}\right)}$ ***(M1)***

$3.12 < \pi < 3.17$ **A1** ***N2***

[3 marks]

Total [17 marks]

4. (a) apply the limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ ***M1***

$$\lim_{n\to\infty}\frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n(n+1)}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1$$ ***M1A1***

(since the limit is finite and $\neq 0$) both series do the same ***R1***

we know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges and hence $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ also converges ***R1AG***

[5 marks]

(b) $\frac{1}{n(n+1)}\equiv\frac{A}{n}+\frac{B}{n+1}\Rightarrow 1\equiv A(n+1)+Bn$ ***(M1)***

putting $n=0\Rightarrow A=1$ and putting $n=-1\Rightarrow B=-1$

giving $\frac{1}{n(n+1)}\equiv\frac{1}{n}+\frac{-1}{n+1}$ ***A1***

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}\ldots$$ ***M1***

$=1$ ***A1***

[4 marks]

(c) $(1+x)\ln(1+x)=(1+x)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots\right)$ ***A1***

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots\right)+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}\ldots\right)$$

EITHER

$$=x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n}$$ ***A1***

$$=x+\sum_{n=1}^{\infty}(-1)^{n+1}x^{n+1}\left(\frac{-1}{n+1}+\frac{1}{n}\right)$$ ***M1***

OR

$$x+\left(1-\frac{1}{2}\right)x^2-\left(\frac{1}{2}-\frac{1}{3}\right)x^3+\left(\frac{1}{3}-\frac{1}{4}\right)x^4-\ldots$$ ***A1***

$$=x+\sum_{n=1}^{\infty}(-1)^{n+1}x^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$ ***M1***

$$=x+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$$ ***AG***

[3 marks]

continued …

Question 4 continued

(d) $\lim_{x \to -1} (1+x)\ln(1+x) = -1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = -1+1=0$ ***M1A1***

[2 marks]

(e) $\lim_{x \to 0} x \ln x = 0$ (replacing $1+x$ with x) ***A1***

[1 mark]

(f) $x^x = e^{x \ln x}$ ***M1***

therefore $\lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = e^0 = 1$ ***M1A1***

[3 marks]

Total [18 marks]

M12/5/MATHL/HP3/ENG/TZ0/SE/M

MARKSCHEME

May 2012

MATHEMATICS
SERIES AND DIFFERENTIAL EQUATIONS

Higher Level

Paper 3

10 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2012**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
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9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. apply l'Hôpital's Rule to a $0/0$ type limit

$\lim_{x\to 0}\frac{e^x - 1 - x\cos x}{\sin^2 x} = \lim_{x\to 0}\frac{e^x - \cos x + x\sin x}{2\sin x\cos x}$ ***M1A1***

noting this is also a $0/0$ type limit, apply l'Hôpital's Rule again ***(M1)***

obtain $\lim_{x\to 0}\frac{e^x + \sin x + x\cos x + \sin x}{2\cos 2x}$ ***A1***

substitution of $x = 0$ ***(M1)***

$= 0.5$ ***A1***

[6 marks]

2. (a) attempt the first step of

$y_{n+1} = y_n + (0.1)f(x_n, y_n)$ with $y_0 = 1,\ x_0 = 0$ ***(M1)***

$y_1 = 1.1$ ***A1***

$y_2 = 1.1 + (0.1)\frac{1.1^2}{1.1} = 1.21$ ***(M1)A1***

$y_3 = 1.332(0)$ ***(A1)***

$y_4 = 1.4685$ ***(A1)***

$y_5 = 1.62$ ***A1***

[7 marks]

(b) (i) recognition of both quotient rule and implicit differentiation ***M1***

$\frac{d^2y}{dx^2} = \frac{(1+x)2y\frac{dy}{dx} - y^2\times 1}{(1+x)^2}$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for everything else correct.

$= \frac{(1+x)2y\frac{y^2}{1+x} - y^2\times 1}{(1+x)^2}$ ***M1A1***

$= \frac{2y^3 - y^2}{(1+x)^2}$ ***AG***

(ii) attempt to use $y = y(0) + x\frac{dy}{dx}(0) + \frac{x^2}{2!}\frac{d^2y}{dx^2}(0) + \ldots$ ***(M1)***

$= 1 + x + \frac{x^2}{2}$ ***A1A1***

Note: Award ***A1*** for correct evaluation of $y(0), \frac{dy}{dx}(0), \frac{d^2y}{dx^2}(0)$, ***A1*** for correct series.

[8 marks]

continued …

Question 2 continued

(c) (i) separating the variables $\int \frac{1}{y^2}\mathrm{d}y = \int \frac{1}{1+x}\mathrm{d}x$ ***M1***

obtain $-\frac{1}{y} = \ln(1+x) + (c)$ ***A1***

impose initial condition $-1 = \ln 1 + c$ ***M1***

obtain $y = \frac{1}{1-\ln(1+x)}$ ***A1***

(ii) $y \to \infty$ if $\ln(1+x) \to 1$, so $a = \mathrm{e} - 1$ ***(M1)A1***

Note:	To award ***A1*** must see either $x \to e-1$ or $a = e-1$. Do not accept $x = e-1$.

[6 marks]

Total [21 marks]

3. recognise equation as first order linear and attempt to find the IF ***M1***

IF $= \mathrm{e}^{\int \frac{2}{t}\mathrm{d}t} = t^2$ ***A1***

solution $yt^2 = \int t\cos t\,\mathrm{d}t$ ***M1A1***

using integration by parts with the correct choice of u and v ***(M1)***

$\int t\cos t\,\mathrm{d}t = t\sin t + \cos t\,(+C)$ ***A1***

obtain $y = \frac{\sin t}{t} + \frac{\cos t + C}{t^2}$ ***A1***

[7 marks]

4. (a) $u_n = \dfrac{3+\frac{2}{n}}{2-\frac{1}{n}}$ or $\dfrac{3}{2}+\dfrac{A}{2n-1}$ ***M1***

using $\lim\limits_{n\to\infty}\dfrac{1}{n}=0$ ***(M1)***

obtain $\lim\limits_{n\to\infty}u_n = \dfrac{3}{2} = L$ ***A1*** ***N1***

[3 marks]

(b) $u_n - L = \dfrac{7}{2(2n-1)}$ ***(A1)***

$|u_n - L| < \varepsilon \Rightarrow n > \dfrac{1}{2}\left(1+\dfrac{7}{2\varepsilon}\right)$ ***(M1)***

(i) $\varepsilon = 0.1 \Rightarrow N = 18$ ***A1***

(ii) $\varepsilon = 0.00001 \Rightarrow N = 175000$ ***A1***

[4 marks]

(c) $u_n \to L$ and $\dfrac{1}{n}\to 0$ ***M1***

$\Rightarrow \dfrac{u_n}{n} \to (L\times 0) = 0$, hence converges ***A1***

$2u_n - 2 \to 2L - 2 = 1 \Rightarrow \dfrac{1}{2u_n - 2} \to 1$, hence converges ***M1A1***

Note:	To award ***A1*** the value of the limit and a statement of convergence must be clearly seen for each sequence.

$(-1)^n u_n$ does not converge ***A1***

The sequence alternates (or equivalent wording) between values close to $\pm L$ ***R1***

[6 marks]

(d) $u_n - L > \dfrac{7}{4n}$ (re: harmonic sequence) ***M1***

$\Rightarrow \sum\limits_{n=1}^{\infty}(u_n - L)$ diverges by the comparison theorem ***R1***

Note:	Accept alternative methods.

[2 marks]

Total [15 marks]

5. (a) consider the limit as $R \to \infty$ of the (proper) integral

$\int_2^R \frac{\mathrm{d}x}{x(\ln x)^k}$ ***(M1)***

substitute $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (***M1)***

obtain $\int_{\ln 2}^{\ln R} \frac{1}{u^k}\mathrm{d}u = \left[-\frac{1}{k-1}\frac{1}{u^{k-1}}\right]_{\ln 2}^{\ln R}$ ***A1***

Note: Ignore incorrect limits or omission of limits at this stage.

or $[\ln u]_{\ln 2}^{\ln R}$ if $k = 1$ ***A1***

Note: Ignore incorrect limits or omission of limits at this stage.

because $\ln R$ (and $\ln\ln R$) $\to \infty$ as $R \to \infty$ ***(M1)***

converges in the limit if $k > 1$ ***A1***

[6 marks]

(b) C: terms $\to 0$ as $r \to \infty$ ***A1***

$|u_{r+1}| < |u_r|$ for all r ***A1***

convergence by alternating series test ***R1***

AC: $(x\ln x)^{-1}$ is positive and decreasing on $[2, \infty)$ ***A1***

not absolutely convergent by integral test using part (a) for $k = 1$ ***R1***

[5 marks]

Total [11 marks]

International Baccalaureate®
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MARKSCHEME

November 2011

MATHEMATICS
SERIES AND DIFFERENTIAL EQUATIONS

Higher Level

Paper 3

11 pages

Instructions to Examiners

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AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

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1. **using l'Hôpital's Rule** *(M1)*

$$\lim_{x\to\frac{1}{2}}\left(\frac{\left(\frac{1}{4}-x^2\right)}{\cot \pi x}\right)=\lim_{x\to\frac{1}{2}}\left[\frac{-2x}{-\pi \operatorname{cosec}^2 \pi x}\right]$$ ***A1A1***

$$=\frac{-1}{-\pi \operatorname{cosec}^2 \frac{\pi}{2}}=\frac{1}{\pi}$$ ***(M1)A1***

[5 marks]

2. (a) for $n \ge 1$, $n! = n(n-1)(n-2)...3\times 2\times 1 \ge 2\times 2\times 2...2\times 2\times 1 = 2^{n-1}$ ***M1A1***

$\Rightarrow n! \ge 2^{n-1}$ for $n \ge 1$ ***AG***

[2 marks]

(b) $n! \ge 2^{n-1} \Rightarrow \frac{1}{n!} \le \frac{1}{2^{n-1}}$ for $n \ge 1$ ***A1***

$\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ is a positive converging geometric series ***R1***

hence $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges by the comparison test ***R1***

[3 marks]

Total [5 marks]

3. (a) using the ratio test (and absolute convergence implies convergence) ***(M1)***

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n x^n}{(n)2^n}}\right|$$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\times x^{n+1}\times n\times 2^n}{(-1)^n\times(n+1)\times 2^{n+1}\times x^n}\right|$$

$$= \lim_{n\to\infty}\frac{n}{2(n+1)}|x|$$ ***(A1)***

$$= \frac{|x|}{2}$$ ***A1***

for convergence we require $\frac{|x|}{2} < 1$ ***M1***

$\Rightarrow |x| < 2$

hence radius of convergence is 2 ***A1***

[7 marks]

(b) we now need to consider what happens when $x = \pm 2$ ***(M1)***

when $x = 2$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which is convergent (by the alternating series test) ***A1***

when $x = -2$ we have $\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent ***A1***

hence interval of convergence is $]-2, 2]$ ***A1***

[4 marks]

Total [11 marks]

4. (a) $\int \frac{1}{4x^2+1}\,\mathrm{d}x = \frac{1}{2}\arctan 2x + k$ ***(M1)(A1)***

Note: Do not penalize the absence of "$+k$".

$\int_1^{\infty} \frac{1}{4x^2+1}\,\mathrm{d}x = \frac{1}{2}\lim_{a\to\infty}\left[\arctan 2x\right]_1^a$ ***(M1)***

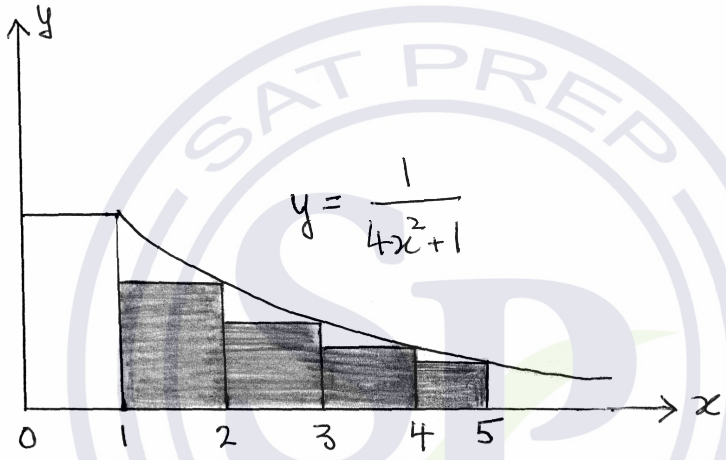
Note: Accept $\frac{1}{2}\left[\arctan 2x\right]_1^{\infty}$.

$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right)$ $(= 0.232)$ ***A1***

hence the series converges ***AG***

[4 marks]

(b) (i)



A2

The shaded rectangles lie within the area below the graph so that $\sum_{n=2}^{\infty}\frac{1}{4n^2+1} < \int_1^{\infty}\frac{1}{4x^2+1}\,\mathrm{d}x$. Adding the first term in the series, $\frac{1}{4\times1^2+1}$, gives $\sum_{n=1}^{\infty}\frac{1}{4n^2+1} < \frac{1}{4\times1^2+1} + \int_1^{\infty}\frac{1}{4x^2+1}\,\mathrm{d}x$. ***R1AG***

(ii) upper bound $= \frac{1}{5} + \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right)$ $(= 0.432)$ ***A1***

[4 marks]

Total [8 marks]

5. (a) METHOD 1

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\frac{dy}{dx} = \frac{-2e^{-x}}{2(1+e^{-x})} = \frac{-e^{-x}}{1+e^{-x}}$ ***M1A1***

now $\frac{1+e^{-x}}{2} = e^y$ ***M1***

$\Rightarrow 1+e^{-x} = 2e^y$

$\Rightarrow e^{-x} = 2e^y - 1$ ***(A1)***

$\Rightarrow \frac{dy}{dx} = \frac{-2e^y+1}{2e^y}$ ***(A1)***

Note: Only one of the two above ***A1*** marks may be implied.

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} - 1$ ***AG***

Note: Candidates may find $\frac{dy}{dx}$ as a function of x and then work backwards from the given answer. Award full marks if done correctly

METHOD 2

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\Rightarrow e^y = \frac{1+e^{-x}}{2}$ M1

$\Rightarrow e^{-x} = 2e^y - 1$

$\Rightarrow x = -\ln(2e^y - 1)$ A1

$\Rightarrow \frac{dx}{dy} = -\frac{1}{2e^y-1} \times 2e^y$ M1 A1

$\Rightarrow \frac{dy}{dx} = \frac{2e^y-1}{-2e^y}$ A1

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} - 1$

AG

[5 marks]

(b)

METHOD 1

when $x=0$, $y = \ln 1 = 0$ ***A1***

when $x=0$, $\frac{dy}{dx} = \frac{1}{2} - 1 = -\frac{1}{2}$ ***A1***

$\frac{d^2y}{dx^2} = -\frac{e^{-y}}{2}\frac{dy}{dx}$ ***M1A1***

when $x=0$, $\frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ ***A1***

$\frac{d^3y}{dx^3} = \frac{e^{-y}}{2}\left(\frac{dy}{dx}\right)^2 - \frac{e^{-y}}{2}\frac{d^2y}{dx^2}$ ***M1A1A1***

when $x = 0$, $\frac{d^3y}{dx^3} = \frac{1}{2} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{4} = 0$ ***A1***

$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$ ***(M1)A1***

two of the above terms are zero ***AG***

METHOD 2

when $x = 0$, $y = \ln 1 = 0$ ***A1***

when $x = 0$, $\frac{dy}{dx} = \frac{1}{2} - 1 = -\frac{1}{2}$ ***A1***

$\frac{d^2y}{dx^2} = \frac{-e^{-y}}{2}\frac{dy}{dx} = \frac{-e^{-y}}{2}\left(\frac{e^{-y}}{2} - 1\right) = \frac{-e^{2y}}{4} + \frac{e^{-y}}{2}$ ***M1A1***

when $x = 0$, $\frac{d^2y}{dx^2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$ ***A1***

$\frac{d^3y}{dx^3} = \left(\frac{e^{-2y}}{2} - \frac{e^{-y}}{2}\right)\frac{dy}{dx}$ ***M1A1A1***

when $x = 0$, $\frac{d^3y}{dx^3} = -\frac{1}{2} \times \left(\frac{1}{2} - \frac{1}{2}\right) = 0$ ***A1***

$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$ ***(M1)A1***

two of the above terms are zero ***AG***

[11 marks]

Total [16 marks]

6. $(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+(x-y)=0$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-x}{x+y}$

let $y=vx$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***A1***

$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{vx-x}{x+vx}$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v-1}{v+1}-v=\frac{v-1-v^2-v}{v+1}=\frac{-1-v^2}{1+v}$ ***A1***

$\int\frac{v+1}{1+v^2}\mathrm{d}v=-\int\frac{1}{x}\mathrm{d}x$ ***M1***

$\int\frac{v}{1+v^2}\mathrm{d}v+\int\frac{1}{1+v^2}\mathrm{d}v=-\int\frac{1}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \frac{1}{2}\ln\left|1+v^2\right|+\arctan v=-\ln\left|x\right|+k$ ***A1A1***

Notes: Award ***A1*** for $\frac{1}{2}\ln\left|1+v^2\right|$, ***A1*** for the other two terms.

Do not penalize missing k or missing modulus signs at this stage.

$\Rightarrow \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right|+\arctan\frac{y}{x}=-\ln\left|x\right|+k$ ***M1***

$\Rightarrow \frac{1}{2}\ln 4+\arctan\sqrt{3}=-\ln 1+k$ ***(M1)***

$\Rightarrow k=\ln 2+\frac{\pi}{3}$ ***A1***

$\Rightarrow \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right|+\arctan\frac{y}{x}=-\ln\left|x\right|+\ln 2+\frac{\pi}{3}$

attempt to combine logarithms ***M1***

$\Rightarrow \frac{1}{2}\ln\left|\frac{y^2+x^2}{x^2}\right|+\frac{1}{2}\ln\left|x^2\right|=\ln 2+\frac{\pi}{3}-\arctan\frac{y}{x}$

$\Rightarrow \frac{1}{2}\ln\left|y^2+x^2\right|=\ln 2+\frac{\pi}{3}-\arctan\frac{y}{x}$ ***(A1)***

$\Rightarrow \sqrt{y^2+x^2}=\mathrm{e}^{\ln 2+\frac{\pi}{3}-\arctan\frac{y}{x}}$ ***(A1)***

$\Rightarrow \sqrt{y^2+x^2}=\mathrm{e}^{\ln 2}\times\mathrm{e}^{\frac{\pi}{3}-\arctan\frac{y}{x}}$ ***A1***

$\Rightarrow r=2\mathrm{e}^{\frac{\pi}{3}-\theta}$ ***AG***

[15 marks]