Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2019

Statistics and probability

Higher level

Paper 3

10 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

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(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

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AG Answer given in the question and so no marks are awarded.

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  Examples

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*Award **N** marks for **correct** answers where there is **no** working.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

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Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

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If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

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Students must always use correct mathematical notation, not calculator notation.
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13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) $H_0 : \rho = 0 \quad H_1 : \rho \neq 0$ ***A1***

Note: It must be ρ.

(ii) $p = 0.649$ ***A2***

Note: Accept anything that rounds to 0.65.

$0.649 > 0.05$ ***R1***

hence, we accept H_0 and conclude that Peter's claim is wrong ***A1***

Note: The ***A*** mark depends on the ***R*** mark and the answer must be given in context. Follow through the p-value in part (b).

[5 marks]

(b) a statement along along the lines of '(we have accepted that) the two variables are independent' or 'the two variables are weakly correlated' ***R1***

a statement along the lines of 'the use of the regression line is invalid' or 'it would give an inaccurate result' ***R1***

Note: Award the second ***R1*** only if the first ***R1*** is awarded.

Note: FT the conclusion in(a)(ii). If a candidate concludes that the claim is correct, mark as follows: (as we have accepted H_1) the 2 variables are dependent and 73 lies in the range of x values ***R1***, hence the use of the regression line is valid ***R1***.

[2 marks]

Total [7 marks]

2. (a) (i) attempt to find expected values *eg* $\mathrm{E}(T_1)$ ***(M1)***

$\mathrm{E}(T_1) = \frac{1}{3}\mathrm{E}(X_1 + X_2 + X_3) = \frac{1}{3}(\mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3))$

$= \mu$ ***A1***

$\mathrm{E}(T_2) = \frac{1}{3}\mathrm{E}(X_1 + 2X_2 + 3X_3) = \frac{1}{3}(\mathrm{E}(X_1) + 2\mathrm{E}(X_2) + 3\mathrm{E}(X_3))$

$= 2\mu$ ***A1***

$\mathrm{E}(T_3) = \frac{1}{3}\mathrm{E}(X_1 + 2X_2) = \frac{1}{3}(\mathrm{E}(X_1) + 2\mathrm{E}(X_2))$

$= \mu$ ***A1***

Note: Order does not matter.

$(2\mu \neq \mu)$ hence T_2 is biased, T_1 and T_3 are unbiased

(ii) use of variance of linear combinations **(M1)**

$\text{Var}(T_1) = \frac{1}{9}\left(\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3)\right)$

$= \frac{3}{9}\sigma^2 \left(= \frac{\sigma^2}{3}\right)$ **A1**

$\text{Var}(T_3) = \frac{1}{9}\left(\text{Var}(X_1) + 4\text{Var}(X_2)\right)$

$= \frac{5}{9}\sigma^2$ **A1**

$\frac{3}{9}\sigma^2 < \frac{5}{9}\sigma^2$ so T_1 is the more efficient estimator **R1A1**

Note: Award ***A1*** only if the ***R1*** is awarded.

Note: Follow through their variances and award ***R1*** for a comparison and ***A1*** if the ***M1*** was awarded.

[9 marks]

(b) (i) $\text{E}(\bar{Y}) = \text{E}(Y) = \frac{4}{p}$ **A1**

(ii) $\frac{\bar{Y}}{4}$ **A1**

[2 marks]

(c) (i) $\text{E}(W)\left(= 1 \times \frac{1}{2} + 2 \times \frac{1}{2}\right) = \frac{3}{2}$ **A1**

(ii) $\text{E}\left(\frac{1}{W}\right)\left(= 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{4}$ **(M1)A1**

(iii) the above example shows that in general $\text{E}\left(\frac{1}{T}\right) \neq \frac{1}{\text{E}(T)}$ (so that $E\left(\frac{4}{\bar{Y}}\right)$ may not equal $\frac{1}{E(\bar{Y}/4)} = p$). **R1**

Note: Do not award ***R1*** if the statement is given only in terms of W.

[4 marks]

Total [15 marks]

3. (a) for n (sufficiently) large the sample mean $\bar{X}$ approximately *A1*

$\sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ *A1*

Note: Award the first ***A1*** for n large and reference to the sample mean $\left(\bar{X}\right)$, the second ***A1*** is for normal and the two parameters.

Note: Award the second ***A1*** only if the first ***A1*** is awarded.

Note: Allow 'n tends to infinity' or '$n \geq 30$' in place of 'large'.

[2 marks]

(b) $[59.9, 60.5]$ *A1A1*

Note: Accept answers which round to the correct 3sf answers.

[2 marks]

(c) (i) under H_0, $\bar{X} \sim \mathrm{N}\left(60, \frac{4}{100}\right)$ *(A1)*

required to find k such that $P(\bar{X} > k) = 0.05$ *(M1)*

use of any valid method, *eg* GDC Inv(Normal) or $k = 60 + z\frac{\sigma}{\sqrt{n}}$ *(M1)*

hence critical region is $\bar{x} > 60.33$ *A1*

(ii) 0.05 *A1*

(iii) P(Type II error) = P(H_0 is accepted / H_0 is false) *(R1)*

Note: Accept Type II error means H_0 is accepted given H_0 is false.

$\Rightarrow \mathrm{P}(\bar{X} < 60.33) = 0.25$ when $\bar{X} \sim \mathrm{N}\left(\mu, \frac{4}{100}\right)$ *(M1)*

$$\Rightarrow \mathrm{P}\left(\frac{\bar{X} - \mu}{\frac{2}{10}} < \frac{60.33 - \mu}{\frac{2}{10}}\right) = 0.25$$ *(M1)*

$$\Rightarrow \mathrm{P}\left(Z < \frac{60.33 - \mu}{\frac{2}{10}}\right) = 0.25 \text{ where } Z \sim \mathrm{N}\left(0, 1^2\right)$$

$$\frac{60.33 - \mu}{\frac{2}{10}} = -0.6744...$$ *(A1)*

$$\mu = 60.33 + \frac{2}{10} \times 0.6744...$$

$$\mu = 60.5$$ *A1*

[10 marks]

Total [14 marks]

4. (a) $G_x'(t) = \dfrac{p^r r t^{r-1}(1-qt)^r - p^r t^r r(1-qt)^{r-1}(-q)}{(1-qt)^{2r}}$ ***M1A1***

use of $\mathrm{E}(X) = G_x'(1)$ ***M1***

$G_x'(1) = \dfrac{rp^r(1-q)^r - rp^r(1-q)^{r-1}(-q)}{(1-q)^{2r}}$ ***A1***

Note: Accept correct substitution of $t=1$ in any correct form of $G_x'(t)$.

$= \dfrac{rp^{2r} + rp^{2r-1} - rp^{2r}}{p^{2r}}$ ***A1***

Note: Accept any equivalent simplified expression which leads immediately to the final result $\dfrac{r}{p}$, for example $r\left(1+\dfrac{q}{p}\right)$.

$= \dfrac{r}{p}$

[5 marks]

(b) (i) $G_w(t) = G_x(t)G_y(t)$ ***(M1)***

$= \dfrac{p^r t^r}{(1-qt)^r} \times \dfrac{p^s t^s}{(1-qt)^s} = \dfrac{p^{r+s}t^{r+s}}{(1-qt)^{r+s}}$ ***A1***

(ii) $W \sim \mathrm{NB}(r+s,\ p)$ ***A1A1***

(iii) $\mathrm{P}(X=3 | W=7) = \dfrac{\mathrm{P}(X=3 \cap W=7)}{\mathrm{P}(W=7)}$ ***(M1)***

$= \dfrac{P(X=3) \times P(Y=4)}{P(W=7)}$ ***(A1)***

$= \dfrac{\binom{2}{1}p^2q\binom{3}{2}p^3q}{\binom{6}{4}p^5q^2}$ ***M1A1***

$= \dfrac{2}{5}$ ***A1***

[9 marks]

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1. (a) $\int_0^1 kx\mathrm{d}x + \int_1^2 kx^2\mathrm{d}x$ ***M1***

$= k\left[\frac{x^2}{2}\right]_0^1 + k\left[\frac{x^3}{3}\right]_1^2$ ***(A1)***

$= k\left[\frac{1}{2} + \left(\frac{8}{3} - \frac{1}{3}\right)\right]$ ***A1***

put expression equal to 1 ***M1***

$k = \frac{6}{17}$ ***AG***

[4 marks]

(b) $\int_0^x \frac{6}{17}t\mathrm{d}t = \frac{3x^2}{17}$ ***(A1)***

$F(x) = \frac{3x^2}{17}$, $0 \le x < 1$ ***A1***

$\int_1^x \frac{6}{17}t^2\mathrm{d}t = \frac{2x^3}{17} - \frac{2}{17}$ ***A1***

$F(x) = \frac{2x^3}{17} - \frac{2}{17} + F(1), 1 \le x \le 2$ ***M1***

$= \frac{2x^3}{17} + \frac{1}{17}$ ***A1***

$F(x) = 0$, $x < 0$ and $F(x) = 1$, $x > 2$ ***A1***

Note: Condone the use of x as the variable of integration.

Note: Accept the use of k in lines 1 and 3.

Note: Allow either weak or strong inequalities.

[6 marks]

(c) recognition that the median lies between 1 and 2 ***(M1)***

$F(m) = 0.5 \Rightarrow 0.5 = \frac{2}{17}m^3 + \frac{1}{17}$ ***(M1)***

$\Rightarrow m = 1.55$ ***A1***

Note: ***FT*** their $F(x)$ from (b) if possible.

[3 marks]

continued…

Question 1 continued

(d) $\mathrm{P}(-0.75 \le X - 1.55 \le 0.75)$

$= \mathrm{P}(0.8 \le X \le 2.3)$ ***(M1)***

$= F(2.3) - F(0.8)$ ***(M1)***

$1 - \frac{3}{17}(0.8036...)^2$

$= 0.886$ ***A1***

Note: Accept all answers that round to 0.89.

Note: ***FT*** their m from (c).

[3 marks]

Total [16 marks]

2.

Note: In question 2, accept answers that round correctly to 2 significant figures.

(a) $X \sim \mathrm{N}(150, 45^2)$

$\mathrm{P}(X > 180) = 0.252$ ***(M1)A1***

[2 marks]

(b) required to find $\mathrm{P}(X_1 + X_2 + X_3 < 540)$

let $S = X_1 + X_2 + X_3$

$\mathrm{E}(S) = 450$ ***(A1)***

$\mathrm{Var}(S) = 3\mathrm{Var}(X)$ ***(M1)***

$= 3 \times 45^2 \left(\Rightarrow \sigma = 45\sqrt{3}\right)$ $(= 6075)$ ***(A1)***

$\mathrm{P}(S < 540) = 0.876$ ***A1***

Note: In (b) and (c) condone incorrect notation, *eg*, $3X$ for $X_1 + X_2 + X_3$.

[4 marks]

(c) let $Y = (X_1 + X_2 + X_3 + X_4) - (X_5 + X_6 + X_7)$ ***(M1)***

$\mathrm{E}(Y) = \mathrm{E}(X) = 150$ ***(A1)***

$\mathrm{Var}(Y) = 4\mathrm{Var}(X) + 3\mathrm{Var}(X)\left(= 7\mathrm{Var}(X)\right)$ ***(M1)***

$= 14175$ ***(A1)***

required to find $\mathrm{P}(Y < 0)$ ***(M1)***

$= 0.104$ ***A1***

[6 marks]

Total [12 marks]

3. (a) $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{100}\right)$ ***A1***

Note: Accept n in place of 100.

[1 mark]

(b) $\hat{\mu} = \frac{\sum x}{n} = \frac{199.8}{100} = 1.998$ ***A1***

Note: Accept 2.00, 2.0 and 2.

[1 mark]

(c) ${s_{n-1}}^2 = \frac{n}{n-1}\left(\frac{\sum x^2}{n} - \bar{x}^2\right) = \frac{100}{99}\left(\frac{407.8}{100} - 1.998^2\right)$ ***(M1)***

$= 0.086864$

unbiased estimate for σ^2 is 0.0869 ***A1***

Note: Accept any answer which rounds to 0.087.

[2 marks]

(d) 90% confidence interval is $1.998 \pm 1.660\sqrt{\frac{0.0869}{100}}$ ***(M1)***

$= (1.95, 2.05)$ ***A1A1***

Note: ***FT*** their σ from (c).

Note: Condone the use of the z-value 1.645 since n is large.

Note: Accept any values that round to 1.95 and 2.05.

[3 marks]

(e) (i) p-value is 0.0377 ***A2***

Note: Award ***A1*** for the 2-tail value 0.0754.

Note: Award ***A2*** for 0.0377 and ***A1*** for any other value that rounds to 0.038.

Note: ***FT*** their estimated mean from (b), note that 2 gives $p = 0.032(0)$.

(ii) accept the null hypothesis ***A1***

Note: ***FT*** their p-value.

[3 marks]

Total [10 marks]

4. (a) **METHOD 1**

$\text{Cov}(X+c,Y)=\text{E}([X+c]Y)-\text{E}(X+c)\text{E}(Y)$ ***M1***

$=\text{E}(XY+cY)-\text{E}(X)\text{E}(Y)-c\text{E}(Y)$ ***A1***

$=\text{E}(XY)+\text{E}(cY)-\text{E}(X)\text{E}(Y)-c\text{E}(Y)$

$=\text{E}(XY)+c\text{E}(Y)-\text{E}(X)\text{E}(Y)-c\text{E}(Y)$ ***A1***

$=\text{Cov}(X,Y)$ ***AG***

METHOD 2

$\text{Cov}(X+c,Y)=\text{E}[(X+c-\text{E}(X+c))(Y-\text{E}(Y))]$ ***M1***

$=\text{E}[(X+c-\text{E}(X)-\text{E}(c))(Y-\text{E}(Y))]$ ***A1***

$=\text{E}[(X+c-\text{E}(X)-c)(Y-\text{E}(Y))]$ ***A1***

$=\text{Cov}(X,Y)$ ***AG***

[3 marks]

(b) **METHOD 1**

$\text{Cov}(X+Y,Z)=\text{E}([X+Y]Z)-\text{E}(X+Y)\text{E}(Z)$ ***M1***

$=\text{E}(XZ+YZ)-(\text{E}(X)+\text{E}(Y))\text{E}(Z)$ ***A1***

$=\text{E}(XZ)+\text{E}(YZ)-\text{E}(X)\text{E}(Z)-\text{E}(Y)\text{E}(Z)$ ***A1***

$=\text{Cov}(X,Z)+\text{Cov}(Y,Z)$ ***AG***

METHOD 2

$\text{Cov}(X+Y,Z)=\text{E}[(X+Y-\text{E}(X+Y))(Z-\text{E}(Z))]$ ***M1***

$=\text{E}[(X+Y-\text{E}(X)-\text{E}(Y))(Z-\text{E}(Z))]$

$=\text{E}[(X-\text{E}(X)+Y-\text{E}(Y))(Z-\text{E}(Z))]$ ***A1***

$=\text{E}[(X-\text{E}(X))(Z-\text{E}(Z))]+\text{E}[(Y-\text{E}(Y))(Z-\text{E}(Z))]$ ***A1***

$=\text{Cov}(X,Z)+\text{Cov}(Y,Z)$ ***AG***

[3 marks]

continued…

Question 4 continued

(c) $\text{Cov}\left(1+S, S+ST^2\right)$

$= \text{Cov}\left(S, S+ST^2\right)$ (from a) ***M1***

$= \text{Cov}\left(S, S\right) + \text{Cov}\left(S, ST^2\right)$ (from b) ***M1***

METHOD 1

$= \text{Var}\left(S\right) + \text{E}\left(S^2T^2\right) - \text{E}\left(S\right)\text{E}\left(ST^2\right)$ ***A1***

$= \text{Var}\left(S\right) + \text{E}\left(S^2\right)\text{E}\left(T^2\right) - \text{E}\left(S\right)\text{E}\left(ST^2\right)$ ***A1***

$= \text{Var}\left(S\right) + \text{Var}\left(S\right)\text{Var}\left(T\right) - \text{E}\left(S\right)\text{E}\left(ST^2\right)$ ***(A1)***

$= 1 + 1 - 0$

$= 2$ ***A1***

METHOD 2

$= \text{Var}(S) + \text{E}[(S - \text{E}(S))(ST^2 - \text{E}(ST^2))]$ ***A1***

$= \text{Var}(S) + \text{E}(S \times ST^2)$ ***A1***

$= \text{Var}(S) + \text{Var}(S)\text{Var}(T)$ ***(A1)***

$= 1 + 1 + 0$

$= 2$ ***A1***

[6 marks]

Total [12 marks]

Markscheme

November 2018

Statistics and probability

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2018**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a - b)$	Do not award the final ***A1***

3 ***N*** marks

Award ***N*** *marks for* ***correct*** *answers where there is* ***no*** *working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets*** *eg* ***(M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

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*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

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*Unless the question specifies otherwise, **accept** equivalent forms.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $\mathrm{E}(2X + 7Y) = 2\mathrm{E}(X) + 7\mathrm{E}(Y) = 6 + 28 = 34$ ***(M1)A1***

[2 marks]

(b) $\mathrm{Var}(X) = \mathrm{E}(X) = 3$ and $\mathrm{Var}(Y) = \mathrm{E}(Y) = 4$ ***(R1)***

$\mathrm{Var}(4X - 3Y) = 16\mathrm{Var}(X) + 9\mathrm{Var}(Y) = 48 + 36$ ***(M1)***

$= 84$ ***A1***

[3 marks]

(c) use of $\mathrm{E}\left(U^2\right) = \mathrm{Var}(U) + \left(\mathrm{E}(U)\right)^2$ ***(M1)***

$\mathrm{E}\left(X^2\right) = 3 + 3^2; \mathrm{E}\left(Y^2\right) = 4 + 4^2$ ***A1***

$\mathrm{E}\left(X^2 - Y^2\right) = \mathrm{E}\left(X^2\right) - \mathrm{E}\left(Y^2\right)$ ***(M1)***

$= -8$ ***A1***

[4 marks]

Total [9 marks]

2. (a) (i) $\bar{t} = \dfrac{\sum_{i=1}^{75} t_i}{75} = 28.866... = 28.9$ ***A1***

(ii) $s_{n-1}{}^2 = \dfrac{75}{74}\left(\dfrac{\sum_{i=1}^{75} t_i^2}{75} - \bar{t}^2\right) = 188.9009... = 189$ ***(M1)A1***

Note: Accept all answers that round to 28.9 and 189.

Note: Award ***M0*** if division by 75.

[3 marks]

(b) attempting to find a confidence interval. ***(M1)***

(i) 90% interval: $(26.2, 31.5)$ ***A1***

(ii) 95% interval: $(25.7, 32.0)$ ***A1***

Note: Accept any values which round to within 0.1 of the correct value.

Note: Award ***M1A1A0*** if only confidence limits are given in the form 28.9 ± 2.6.

[3 marks]

(c) 26 lies within the 95% interval but not within the 90% interval ***R1***

Note: Award ***R1*** for considering whether or not one or two of the intervals contain 26.

the belief is supported at the 5% level (accept 95%) ***A1***

the belief is not supported at the 10% level (accept 90%) ***A1***

Note: ***FT*** their intervals but award ***R1A1A0*** if both intervals give the same conclusion.

[3 marks]

Total [9 marks]

3.

Note: Accept all answers that round to the correct 2sf answer in (a), (b) and (c) but not in (d).

(a) (i) $X \sim N\left(550,\, 8^2\right)$ ***(M1)***

$P(X > 560) = 0.10564\ldots = 0.106$ ***A1***

(ii) $X_i \sim N\left(550,\, 8^2\right),\ i = 1,\ldots,11$

let $Y = \sum_{i=1}^{11} X_i$

$E(Y) = 11 \times 550\ (6050)$ ***A1***

$Var(Y) = 11 \times 8^2\ (704)$ ***(M1)A1***

$P(Y \le 6000) = 0.02975\ldots = 0.0298$ ***A1***

[6 marks]

(b) (i) t distribution with 7 degrees of freedom ***A1A1***

(ii) $p = 0.25779\ldots = 0.258$ ***A2***

(iii) $p > 0.05$ ***R1***

therefore we conclude that there is no evidence to reject H_0 ***A1***

Note: ***FT*** their p-value.

Note: Only award ***A1*** if ***R1*** awarded.

[6 marks]

(c) (i) $H_0 : \rho = 0,\ H_1 : \rho > 0$ ***A1***

Note: Do not accept r in place of ρ.

(ii) $r = 0.782$ ***A2***

(iii) $0.01095\ldots = 0.0110$ ***A1***

since $0.0110 < 0.05$ ***R1***

there is positive association between weight and length ***A1***

Note: ***FT*** their p-value.

Note: Only award ***A1*** if ***R1*** awarded.

Note: Conclusion must be in context.

[6 marks]

continued…

Question 3 continued

(d) regression line of y(weight) on x(length) is ***(M1)***
$y = 0.8267\ldots\, x + 255.96\ldots$ ***(A1)***
$x = 360$ gives $y = 554$ ***A1***

Note: Award ***M1A0A0*** for the wrong regression line, that is $y = 0.7393\ldots x - 51.62\ldots$.

[3 marks]

Total [21 marks]

4. (a) $G_X(t) = \sum_{x=1}^{\infty} pq^{x-1}t^x$ or equivalent ***M1A1***

Note: Condone omission of limits on summation.

$= \sum_{x=1}^{\infty} pt(qt)^{x-1}$

recognition of a geometric series ***M1***

$= \dfrac{pt}{1-qt}$ ***AG***

[3 marks]

(b) $G_X'(t) = \dfrac{p(1-qt)+pqt}{(1-qt)^2}$ ***M1A1***

$= \dfrac{p}{(1-qt)^2}$

$\mathrm{E}(X) = G_X'(1) = \dfrac{p}{(1-q)^2} = \dfrac{p}{p^2}$ ***M1A1***

$= \dfrac{1}{p}$ ***AG***

[4 marks]

(c) **METHOD 1**

$G_Y(t) = pt^5 + pqt^7 + pq^2t^9 + \ldots$ ***M1A1***
recognition of geometric series ***(M1)***

$= \dfrac{pt^5}{1-qt^2}$ ***A1***

continued…

Question 4 continued

METHOD 2

$G_Y(t) = E\left(t^Y\right) = E\left(t^{2X+3}\right)$ ***M1***

$= t^3 E\left(\left(t^2\right)^X\right)$ ***A1***

$= t^3 G_X\left(t^2\right)$ ***(M1)***

$= \dfrac{pt^5}{1-qt^2}$ ***A1***

[4 marks]

Total [11 marks]

Markscheme

May 2018

Statistics and probability

Higher level

Paper 3

10 pages

Instructions to Examiners

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13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

Note: In question 1, accept answers that round correctly to 2 significant figures.

1. (a) $\text{P}(4.75 < X < 4.85) = 0.197$ ***A1***

[1 mark]

(b) consider the random variable $X - 2Y$ ***(M1)***

$\text{E}(X - 2Y) = -0.6$ ***(A1)***

$\text{Var}(X - 2Y) = \text{Var}(X) + 4\text{Var}(Y)$ ***(M1)***

$= 0.13$ ***(A1)***

$X - 2Y \sim \text{N}(-0.6, 0.13)$

$\text{P}(X - 2Y > 0)$ ***(M1)***

$= 0.0480$ ***A1***

[6 marks]

(c) let $W = X_1 + X_2 + Y_1 + Y_2 + Y_3$ be the total weight

$\text{E}(W) = 17.7$ ***(A1)***

$\text{Var}(W) = 2\text{Var}(X) + 3\text{Var}(Y) = 0.1475$ ***(M1)(A1)***

$W \sim \text{N}(17.7, 0.1475)$

$\text{P}(W > 18) = 0.217$ ***A1***

[4 marks]

Total [11 marks]

2. (a) X is geometric (or negative binomial) ***A1***

[1 mark]

(b) $G(t) = \frac{1}{4}t + \frac{1}{4}\left(\frac{3}{4}\right)t^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 t^3 + \ldots$ ***M1A1***

recognition of GP $\left(u_1 = \frac{1}{4}t, r = \frac{3}{4}t\right)$ ***(M1)***

$= \dfrac{\frac{1}{4}t}{1 - \frac{3}{4}t}$ ***A1***

leading to $G(t) = \dfrac{t}{4 - 3t}$ ***AG***

[4 marks]

(c) attempt to use product or quotient rule ***M1***

$G'(t) = \dfrac{4}{(4 - 3t)^2}$ ***A1***

[2 marks]

continued…

Question 2 continued

(d) 4 ***A1***

Note: Award ***A1FT*** to a candidate that correctly calculates the value of $G'(1)$ from their $G'(t)$.

[1 mark]

Total [8 marks]

Note: In question 3, accept answers that round correctly to 2 significant figures.

3. (a) $H_0 : \mu = 9.5$; $H_1 : \mu \neq 9.5$ ***A1***

[1 mark]

(b) the critical values are $9.5 \pm 1.95996... \times \frac{0.4}{\sqrt{20}}$ ***(M1)(A1)***

i.e. $9.3247\ldots, 9.6753\ldots$

the critical region is $\bar{b} < 9.32$, $\bar{b} > 9.68$ ***A1A1***

Note: Award ***A1*** for correct inequalities, ***A1*** for correct values.

Note: Award ***M0*** if *t*-distribution used, note that $t(19)_{97.5} = 2.093\ldots$

[4 marks]

(c) $\bar{B} \sim \text{N}\left(9.8, \left(\frac{0.4}{\sqrt{20}}\right)^2\right)$ ***(A1)***

$\text{P}\left(9.3247\ldots < \bar{B} < 9.6753\ldots\right)$ ***(M1)***

$= 0.0816$ ***A1***

Note: FT the critical values from (b). Note that critical values of 9.32 and 9.68 give 0.0899.

[3 marks]

(d) **METHOD 1**

$X \sim \text{N}\left(10.8, \frac{1.2^2}{6}\right)$ ***(M1)(A1)***

$\text{P}(10.2 < X < 11.4) = 0.7793\ldots$ ***(A1)***

confidence level is 77.9% ***A1***

Note: Accept 78%.

METHOD 2

$11.4 - 10.2 = 2z \times \frac{1.2}{\sqrt{6}}$ ***(M1)***

$z = 1.224...$ ***(A1)***

$\text{P}(-1.224\ldots < Z < 1.224\ldots) = 0.7793\ldots$ ***(A1)***

confidence level is 77.9% ***A1***

Note: Accept 78%.

[4 marks]

Total [12 marks]

4. (a) $H_0 : \rho = 0$; $H_1 : \rho < 0$ ***A1***

[1 mark]

(b) (i) $t = -0.708\sqrt{\dfrac{11-2}{1-(-0.708)^2}}$ $(= -3.0075...)$ ***(M1)***

degrees of freedom $= 9$ ***(A1)***

$P(T < -3.0075...) = 0.00739$ ***A1***

Note: Accept any answer that rounds to 0.0074.

(ii) reject H_0 or equivalent statement ***R1***

Note: Apply follow through on the candidate's p-value.

[4 marks]

(c) (i) $\text{Cov}(U, V) = E\big((U - E(U))(V - E(V))\big)$

$= E\big(UV - E(U)V - E(V)U + E(U)E(V)\big)$ ***M1***

$= E(UV) - E\big(E(U)V\big) - E\big(E(V)U\big) + E\big(E(U)E(V)\big)$ ***(A1)***

$= E(UV) - E(U)E(V) - E(V)E(U) + E(U)E(V)$ ***A1***

$\text{Cov}(U, V) = E(UV) - E(U)E(V)$ ***AG***

(ii) $E(UV) = E(U)E(V)$ (independent random variables) ***R1***

$\Rightarrow \text{Cov}(U, V) = E(U)E(V) - E(U)E(V) = 0$ ***A1***

hence, $\rho = \dfrac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\text{Var}(V)}} = 0$ ***A1AG***

Note: Accept the statement that $\text{Cov}(U,V)$ is the numerator of the formula for ρ.

Note: Only award the first ***A1*** if the ***R1*** is awarded.

[6 marks]

Total [11 marks]

5. (a) $E(P) = E\left(\dfrac{X}{n}\right) = \dfrac{1}{n}E(X)$ ***M1***

$= \dfrac{1}{n}(np) = p$ ***A1***

so P is an unbiased estimator of p ***AG***

[2 marks]

continued…

Question 5 continued

(b) (i) $\mathrm{E}\left(nP(1-P)\right)=\mathrm{E}\left(n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right)$

$=\mathrm{E}(X)-\frac{1}{n}\mathrm{E}\left(X^2\right)$ ***M1A1***

use of $\mathrm{E}\left(X^2\right)=\mathrm{Var}(X)+\left(\mathrm{E}(X)\right)^2$ ***M1***

Note: Allow candidates to work with P rather than X for the above 3 marks.

$=np-\frac{1}{n}\left(np(1-p)+(np)^2\right)$ ***A1***

$=np-p(1-p)-np^2$

$=np(1-p)-p(1-p)$ ***A1***

Note: Award ***A1*** for the factor of $(1-p)$.

$=(n-1)\,p(1-p)$ ***AG***

(ii) an unbiased estimator is $\frac{n^2P(1-P)}{n-1}\left(=\frac{nU}{n-1}\right)$ ***A1***

[6 marks]

Total [8 marks]

Markscheme

November 2017

Statistics and probability

Higher level

Paper 3

Instructions to Examiners

Abbreviations

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(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

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3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

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*Award **N** marks for **correct** answers where there is **no** working.*

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- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

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*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

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*Unless the question specifies otherwise, **accept** equivalent forms.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

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If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

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A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $F(t)=\int_0^t\left(x-\frac{x^3}{4}\right)\mathrm{d}x\left(=\int_0^t\frac{x\left(4-x^2\right)}{4}\mathrm{d}x\right)$ ***M1***

$=\left[\frac{x^2}{2}-\frac{x^4}{16}\right]_0^t\left(=\left[\frac{x^2\left(8-x^2\right)}{16}\right]_0^t\right)\left(=\left[\frac{-(4-x^2)^2}{16}\right]_0^t\right)$ ***A1***

$=\frac{t^2}{2}-\frac{t^4}{16}\left(=\frac{t^2\left(8-t^2\right)}{16}\right)\left(=1-\frac{(4-t^2)^2}{16}\right)$ ***A1***

Note: Condone integration involving t only.

Note: Award ***M1A0A0*** for integration without limits *eg*, $\int\frac{t\left(4-t^2\right)}{4}\mathrm{d}t=\frac{t^2}{2}-\frac{t^4}{16}$ or equivalent.

Note: But allow integration $+\,C$ then showing $C=0$ or even integration without C if $F(0)=0$ or $F(2)=1$ is confirmed.

[3 marks]

(b) (i)

correct shape including correct concavity ***A1***
clearly indicating starts at origin and ends at (2,1) ***A1***

Note: Condone the absence of $(0,0)$.

Note: Accept 2 on the x-axis and 1 on the y-axis correctly placed.

(ii) attempt to solve $\frac{a^2}{2}-\frac{a^4}{16}=0.75$ (or equivalent) for a ***(M1)***

$a=1.41\left(=\sqrt{2}\right)$ ***A1***

Note: Accept any answer that rounds to 1.4.

[4 marks]

Total [7 marks]

2. (a) UE of μ is $8.01\ (= 8.0125)$ ***A1***
UE of σ^2 is 0.404 ***(M1)A1***

Note: Accept answers that round correctly to 2 sf.

Note: Condone incorrect notation, *ie*, μ instead of UE of μ and σ^2 instead of UE of σ^2.

Note: ***M0*** for squaring $0.594\ldots$ giving 0.354, ***M1A0*** for failing to square $0.635\ldots$

[3 marks]

(b) (i) attempting to use the t-test ***(M1)***
p-value is 0.0566 ***A2***

Note: Accept any answer that rounds correctly to 2 sf.

(ii) $0.0566 > 0.05$ ***R1***
we accept the null hypothesis (mean pumpkin weight is $7.5\,\text{kg}$) ***A1***

Note: Apply follow through on the candidate's p-value.

Note: Do not award ***A1*** if ***R1*** is not awarded.

[5 marks]

Total [8 marks]

3. (a) $\mathrm{E}(U) = \mathrm{E}\left(a\overline{X}_1 + (1-a)\overline{X}_2\right) = a\mathrm{E}\left(\overline{X}_1\right) + (1-a)\mathrm{E}\left(\overline{X}_2\right)$ ***(M1)***

$\mathrm{E}\left(\overline{X}_1\right) = \mu$ and $\mathrm{E}\left(\overline{X}_2\right) = \mu$

$E(U) = a\mu + (1-a)\mu$ (or equivalent) ***A1***

$= \mu$ ***A1***

hence U is an unbiased estimator of μ ***AG***

[3 marks]

(b) (i) $\mathrm{Var}(U) = \mathrm{Var}\left(a\overline{X}_1 + (1-a)\overline{X}_2\right)$

$= a^2\mathrm{Var}\left(\overline{X}_1\right) + (1-a)^2\mathrm{Var}\left(\overline{X}_2\right)$ ***M1***

stating that $\mathrm{Var}\left(\overline{X}_1\right) = \dfrac{\sigma^2}{n_1}$ and $\mathrm{Var}\left(\overline{X}_2\right) = \dfrac{\sigma^2}{n_2}$ ***A1***

$\Rightarrow \mathrm{Var}\,U = a^2\dfrac{\sigma^2}{n_1} + (1-a)^2\dfrac{\sigma^2}{n_2}$ ***AG***

Note: Line 3 or equivalent must be seen somewhere.

continued…

Question 3 continued

(ii) let $\text{Var}(U) = V$

EITHER

$$\frac{\mathrm{d}V}{\mathrm{d}a} = 2a\frac{\sigma^2}{n_1} - 2(1-a)\frac{\sigma^2}{n_2}$$ ***M1***

attempting to solve $\frac{\mathrm{d}V}{\mathrm{d}a} = 0$ for a ***R1***

Note: Award ***M1*** for obtaining a in terms of n_1, n_2 and σ.

OR

forming a quadratic in a

$$V = \left(\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}\right)a^2 - 2\frac{\sigma^2}{n_2}a + \frac{\sigma^2}{n_2}$$ ***M1***

attempting to find the axis of symmetry of V ***R1***

THEN

$$a = \frac{\frac{2\sigma^2}{n_2}}{2\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$ ***(A1)***

$$a = \frac{n_1}{n_1 + n_2}$$ ***A1***

(iii) substituting a into U ***(M1)***

$$U = \frac{n_1\overline{X}_1 + n_2\overline{X}_2}{n_1 + n_2}$$ ***A1***

Note: Do not ***FT*** an incorrect a for ***A1***, the ***M1*** may however be awarded.

this is an expression for the mean of the combined samples
OR this is a weighted mean of the two sample means ***R1***

[9 marks]

Total [12 marks]

4. (a) $H_0 : \rho = 0$; $H_1 : \rho \neq 0$ ***A1A1***

[2 marks]

(b) $\nu = 10$ ***(A1)***

$t_{0.005} = 3.16927\ldots$ ***(M1)(A1)***

we reject $H_0 : \rho = 0$ if $|t| > 3.16927\ldots$ ***(R1)***

attempting to solve $|r|\sqrt{\dfrac{10}{1-r^2}} > 3.16927\ldots$ for $|r|$ ***M1***

Note: Allow = instead of >.

(least value of $|r|$ is) 0.708 (3 sf) ***A1***

Note: Award ***A1M1A0R1M1A0*** to candidates who use a one-tailed test. Award ***A0M1A0R1M1A0*** to candidates who use an incorrect number of degrees of freedom or both a one-tailed test and incorrect degrees of freedom.

Note: Possible errors are
10 DF 1-tail, t = 2.763…, least value = 0.658
11 DF 2-tail, t = 3.105…, least value = 0.684
11 DF 1-tail, t = 2.718…, least value = 0.634.

[6 marks]

Total [8 marks]

5. (a) (i) $G_X'(t) = \lambda e^{\lambda(t-1)}$ ***A1***

$G_X''(t) = \lambda^2 e^{\lambda(t-1)}$ ***A1***

(ii) $\mathrm{Var}(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$ ***(M1)***

$G_X'(1) = \lambda$ and $G_X''(1) = \lambda^2$ ***(A1)***

$\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2$ ***A1***

$= \lambda$ ***AG***

[5 marks]

(b) $G_{X+Y}(t) = e^{\lambda(t-1)} \times e^{\mu(t-1)}$ ***M1***

Note: The ***M1*** is for knowing to multiply pgfs.

$= e^{(\lambda+\mu)(t-1)}$ ***A1***

which is the pgf for a Poisson distribution with mean $\lambda + \mu$ ***R1AG***

Note: Line 3 identifying the Poisson pgf must be seen.

[3 marks]

continued…

Question 5 continued

(c) (i) $P(X = x | X + Y = n) = \dfrac{P(X = x \cap Y = n - x)}{P(X + Y = n)}$ ***(M1)***

$= \left(\dfrac{e^{-\lambda}\lambda^{x}}{x!}\right)\left(\dfrac{e^{-\mu}\mu^{n-x}}{(n-x)!}\right)\left(\dfrac{n!}{e^{-(\lambda+\mu)}(\lambda+\mu)^{n}}\right)$ (or equivalent) ***M1A1***

$= \binom{n}{x}\dfrac{\lambda^{x}\mu^{n-x}}{(\lambda+\mu)^{n}}$ ***A1***

$= \binom{n}{x}\left(\dfrac{\lambda}{\lambda+\mu}\right)^{x}\left(\dfrac{\mu}{\lambda+\mu}\right)^{n-x}$ ***A1***

leading to $P(X = x | X + Y = n) = \binom{n}{x}\left(\dfrac{\lambda}{\lambda+\mu}\right)^{x}\left(1-\dfrac{\lambda}{\lambda+\mu}\right)^{n-x}$ ***AG***

(ii) $B\left(n, \dfrac{\lambda}{\lambda+\mu}\right)$ ***A1A1***

Note: Award ***A1*** for stating binomial and ***A1*** for stating correct parameters.

[7 marks]

Total [15 marks]

Markscheme

May 2017

Statistics and probability

Higher level

Paper 3

Instructions to Examiners

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1. (a) $H_0 : \mu = 7,\ H_1 : \mu < 7$ ***A1***

[1 mark]

(b) $\overline{x} = \dfrac{83.64}{12} = 6.97$ ***A1***

$s_{n-1}^2 = \dfrac{583.05}{11} - \dfrac{83.64^2}{132} = 0.0072$ ***(M1)A1***

[3 marks]

(c) (i) $t = \dfrac{6.97 - 7}{\sqrt{\dfrac{0.0072}{12}}} = -1.22(474\ldots)$ ***(M1)(A1)***

degrees of freedom $= 11$ ***(A1)***

p-value $= 0.123$ ***A1***

Note: Accept any answer that rounds correctly to 0.12.

(ii) because $p > 0.1$ ***R1***

the inspector's claim is not supported (at the $10\ \%$ level)
(or equivalent in context) ***A1***

Note: Only award the ***A1*** if the ***R1*** has been awarded

[6 marks]

Total [10 marks]

2. (a) (i) $P(0.25 \le X \le 0.75) = F(0.75) - F(0.25)$ ***(M1)***

$= 0.466$ ***A1***

Note: Accept any answer that rounds correctly to 0.466.

(ii) the median m satisfies $F(m) = 0.5$ ***(M1)***

$m = 0.685$ ***A1***

Note: Accept any answer that rounds correctly to 0.685.

[4 marks]

(b) (i) $f(x) = F'(x)$ ***(M1)***

$= \mathrm{e}^{x-1} + x\mathrm{e}^{x-1}$ ***A1***

$= (x+1)\mathrm{e}^{x-1}$ ***AG***

continued…

Question 2 continued

(ii) $\mu = \int_0^1 x(x+1)e^{x-1}dx$ ***(M1)***

$= 0.632 \quad \left(1 - \frac{1}{e}\right)$ ***A1***

Note: Accept any answer that rounds correctly to 0.632.

$\sigma^2 = \int_0^1 x^2(x+1)e^{x-1}dx - 0.632\ldots^2$ ***(M1)***

$= 0.0719 \quad \left(\frac{6}{e} - 2 - \frac{1}{e^2}\right)$ ***A1***

Note: Accept any answer that rounds correctly to 0.072.

[6 marks]

(c) (i) the central limit theorem states that the mean of a large sample from any distribution (with a finite variance) is approximately normally distributed ***A1***

(ii) $\bar{X}$ is approximately $N(0.632\ldots, 0.000719\ldots)$ ***(M1)(A1)***

$P\left(\bar{X} > 0.65\right) = 0.25$ (2 dps required) ***A1***

[4 marks]

Total [14 marks]

3. (a) $G(t) = \sum P(X = x)t^x$ ***(M1)***

$= p + pqt^2 + pq^2t^4 + \ldots$

(summing GP) $u_1 = p,\ r = qt^2$ ***A1***

$= \frac{p}{1 - qt^2}$ ***AG***

[2 marks]

(b) $G'(t) = -\frac{p}{\left(1 - qt^2\right)^2} \times -2qt$ ***M1A1***

$E(X) = G'(1)$ ***(M1)***

$= \frac{2pq}{(1-q)^2} \quad \left(= \frac{2q}{p}\right)$ ***A1***

[4 marks]

continued…

Question 3 continued

(c) **METHOD 1**

PGF of $Y = \sum P(Y = y)t^y$ **(*M1*)**

$= pt + pqt^5 + pq^2t^9 + \ldots$ ***A1***

$= \dfrac{pt}{1 - qt^4}$ ***A1***

METHOD 2

PGF of $Y = \mathrm{E}(t^Y)$ **(*M1*)**

$= \mathrm{E}\left(t^{2X+1}\right)$

$= \mathrm{E}\left((t^2)^X\right) \times \mathrm{E}(t)$ ***A1***

$= \dfrac{pt}{1 - qt^4}$ ***A1***

[3 marks]

Total [9 marks]

4. (a) $\mathrm{E}(U) = a\left(\mathrm{E}(X_1) + \mathrm{E}(X_2)\right) + b\left(\mathrm{E}(Y_1) + \mathrm{E}(Y_2) + \mathrm{E}(Y_3)\right)$ ***(M1)***

$= 2a\mu + 6b\mu$ ***A1***

(for an unbiased estimator,) $\mathrm{E}(U) = \mu$ ***R1***

giving $2a + 6b = 1$ ***AG***

Note: Condone omission of E on LHS.

[3 marks]

(b) $\mathrm{Var}(U) = a^2\left(\mathrm{Var}(X_1) + \mathrm{Var}(X_2)\right) + b^2\left(\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \mathrm{Var}(Y_3)\right)$ ***(M1)***

$= 4a^2\sigma^2 + 3b^2\sigma^2$ ***A1***

$= 4\left(\dfrac{1-6b}{2}\right)^2 \sigma^2 + 3b^2\sigma^2$ ***A1***

$= \left(39b^2 - 12b + 1\right)\sigma^2$ ***AG***

[3 marks]

(c) (i) the best unbiased estimator (of this form) will be found by minimising $\mathrm{Var}(U)$ ***(R1)***

For example, $\dfrac{\mathrm{d}}{\mathrm{d}b}\left(\mathrm{Var}(U)\right) = (78b - 12)\sigma^2$ **(*A1*)**

for a minimum, $b = \dfrac{12}{78}\left(= \dfrac{2}{13}\right)$ so that $a = \dfrac{3}{78}\left(= \dfrac{1}{26}\right)$ ***A1***

continued…

Question 4 continued

(ii) $\text{Var}\,U = \left(39\left(\frac{2}{13}\right)^2 - 12\left(\frac{2}{13}\right) + 1\right)\sigma^2$

$= \frac{\sigma^2}{13}$ $(0.0769\sigma^2)$ ***A1***

[4 marks]

Total [10 marks]

5. (a) $H_0 : \rho = 0; H_1 : \rho > 0$ ***A1***

Note: Do not accept *r* in place of ρ .

[1 mark]

(b) insufficient evidence to conclude that there is a (positive) association between marks in these two subjects (or equivalent statement in context) ***A1***

[1 mark]

(c) degrees of freedom $= 10$ ***(A1)***

required value of t = inverse $t(0.823)$ ***(M1)***

$= 0.972$ ***A1***

attempt to solve $t = r\sqrt{\frac{n-2}{1-r^2}}$ ***(M1)***

$r = 0.294$ ***A1***

Note: Accept any *r* value that rounds to 0.29.

Note: Follow through their *t* value to determine *r*.

[5 marks]

Total [7 marks]

N16/5/MATHL/HP3/ENG/TZ0/SP/M

Markscheme

November 2016

Statistics and probability

Higher level

Paper 3

9 pages

Instructions to Examiners

Abbreviations

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  Award ***A1*** for $\big(2\cos(5x-3)\big)5$, even if $10\cos(5x-3)$ is not seen.

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A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

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Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $r = 0.804$ ***A2***

Note: Accept any number that rounds to 0.80.

[2 marks]

(b) (i) t distribution with 8 degrees of freedom ***A1A1***

(ii) p-value $= 0.00254$ ***A2***

Notes: Accept any number that rounds to 0.0025. Award ***A1*** for 2-tail test giving an answer that rounds to 0.0051.

(iii) p-value < 0.01, so conclude that there is positive correlation ***R1A1***

Notes: Only award the ***A1*** if the ***R1*** is awarded. Do not accept just "reject H_0" or "accept H_1". The words "positive correlation" must be seen.

[6 marks]

(c) regression line of y (Exam 2 mark) on x (Exam 1 mark) is ***(M1)***
$y = 0.59407\ldots x + 21.387\ldots$ ***(A1)***
$x = 11$ gives $y = 28$ (to nearest integer) ***A1***

[3 marks]

(d) (i) applying the t test to the differences
t distribution with 9 degrees of freedom ***A1A1***

(ii) p-value$= 0.239$ ***A2***

Notes: Accept any number that rounds to 0.24. Award ***A1*** if subtraction done the wrong way round giving p-value $= 0.109$.

(iii) p-value > 0.05, so accept H_0 or $u_d = 6$ ***R1A1***

[6 marks]

Total [17 marks]

2. (a) (i) mean $= 119 \times 2 = 238$ ***A1***

(ii) variance $= 119 \times \frac{1}{9} = \frac{119}{9}$ $(=13.2)$ ***(M1)A1***

Note: If 120 is used instead of 119 award ***A0(M1)A0*** for part (a) and apply follow through for parts (b)-(d). (b) is unaffected and in (c) the interval becomes $(234, 246)$. In (d) the first 2 ***A1*** marks are for $0.3633\ldots$ and $0.0174\ldots$ so the final answer will round to 0.017.

[3 marks]

(b) justified by the Central Limit Theorem ***R1***
since n is large ***A1***

Note: Accept $n > 30$.

[2 marks]

continued…

Question 2 continued

(c) $X \sim N\left(238, \frac{119}{9}\right)$

$Z = \dfrac{X - 238}{\frac{\sqrt{119}}{3}} \sim N(0, 1)$ ***(M1)(A1)***

$P(Z < q) = 0.95 \Rightarrow q = 1.644\ldots$ ***(A1)***

so $P(-1.644\ldots < Z < 1.644\ldots) = 0.9$ ***(R1)***

$P\left(-1.644\ldots < \dfrac{X - 238}{\frac{\sqrt{119}}{3}} < 1.644\ldots\right) = 0.9$ ***(M1)***

interval is $232 < X < 244$ (3sf) $(A = 232, B = 244)$ ***A1A1***

Notes: Accept the use of inverse normal applied to the distribution of X. Alternative is to use the GDC to find a pretend Z confidence interval for a mean and then convert by multiplying by 119. Either A or B correct implies the five implied marks. Accept any numbers that round to these 3sf numbers.

[7 marks]

(d) under H_1, $X \sim N\left(238, \frac{119}{9}\right)$ ***(M1)***

$P(236 \le X \le 240) = 0.41769\ldots$ ***(A1)***

probability that all 4 values of X lie in this interval is

$(0.41769\ldots)^4 = 0.030439\ldots$ ***(M1)(A1)***

so probability of a Type II error is 0.0304 (3sf) ***A1***

Note: Accept any answer that rounds to 0.030.

[5 marks]

Total [17 marks]

3. (a) (i) $\text{NB}\left(2, \frac{1}{7}\right)$ ***A1A1A1***

Note: The final ***A1*** mark can be awarded for knowing that $p = \frac{1}{7}$ independent of the other two marks.

(ii) $\text{E}(X) = \dfrac{r}{p} = 14$ ***A1***

(iii) $\binom{4}{1}\left(\frac{6}{7}\right)^3\left(\frac{1}{7}\right)^2 = 0.0514$ ***(M1)A1***

Note: Accept any number that rounds to this 3sf number.

[6 marks]

continued…

Question 3 continued

(b) (i) $Y = Y_1 + Y_2$ (number up to1st + number up to 2nd) ***(M1)***

$Y_1 \sim Geo\left(\frac{1}{7}\right),\ Y_2 \sim Geo\left(\frac{3}{7}\right)$ ***(A1)***

Notes: The above ***(A1)*** is independent of the ***(M1)***. Could have NB $(1, p)$, instead of $Geo(p)$.

$\mathrm{E}(Y) = \frac{1}{\left(\frac{1}{7}\right)} + \frac{1}{\left(\frac{3}{7}\right)} = 7 + \frac{7}{3} = 9\frac{1}{3}\ (9.33)$ ***M1A1***

(ii) $Y = Y_1 + Y_2 = 5$ happens when ***(M1)***

$Y_1 = 1, Y_2 = 4$ or $Y_1 = 2, Y_2 = 3$ or $Y_1 = 3, Y_2 = 2$ or $Y_1 = 4, Y_2 = 1$ ***(A1)***

so probability is $\frac{1}{7}\frac{4}{7}\frac{4}{7}\frac{4}{7}\frac{3}{7} + \frac{6}{7}\frac{1}{7}\frac{4}{7}\frac{4}{7}\frac{3}{7} + \frac{6}{7}\frac{6}{7}\frac{1}{7}\frac{4}{7}\frac{3}{7} + \frac{6}{7}\frac{6}{7}\frac{6}{7}\frac{1}{7}\frac{3}{7}$ ***(M1)(A1)***

$= 0.0928\ \left(\frac{1560}{16807}\right)$ ***A1***

Note: Accept any answer that rounds to 0.093.

[9 marks]

Total [15 marks]

4. (a) $J(t) = G(t)H(t)$ ***A1***

[1 mark]

(b) (i) $J'(t) = G'(t)H(t) + G(t)H'(t)$ ***M1A1***

$J'(1) = G'(1)H(1) + \mathrm{G}(1)\mathrm{H}'(1)$ ***M1***

$J'(1) = G'(1) + \mathrm{H}'(1)$ ***A1***

so $E(Z) = E(X) + E(Y)$ ***AG***

(ii) $J''(t) = G''(t)H(t) + G'(t)H'(t) + G'(t)H'(t) + G(t)H''(t)$ ***M1A1***

$J''(1) = G''(1)H(1) + 2G'(1)H'(1) + G(1)H''(1)$

$= G''(1) + 2G'(1)H'(1) + H''(1)$ ***A1***

$\mathrm{Var}(Z) = J''(1) + J'(1) - (J'(1))^2$ ***M1***

$= G''(1) + 2G'(1)H'(1) + H''(1) + G'(1) + H'(1) - (G'(1) + H'(1))^2$ ***A1***

$= G''(1) + G'(1) - (G'(1))^2 + H''(1) + H'(1) - (H'(1))^2$ ***A1***

so $\mathrm{Var}(Z) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ ***AG***

Note: If addition is wrongly used instead of multiplication in (a) it is inappropriate to give ***FT*** apart from the second ***M*** marks in each part, as the working is too simple.

[10 marks]

Total [11 marks]

Markscheme

May 2016

Statistics and probability

Higher level

Paper 3

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1. (a) $z = 0.841\ldots$ ***(A1)***

$a = \mu + z\sigma$ ***(M1)***

$= 26.2$ ***A1***

[3 marks]

(b) let T denote the total time taken to complete 5 crosswords.

T is $\mathrm{N}(110, 125)$ ***(A1)(A1)***

Note: ***A1*** for the mean and ***A1*** for the variance.

$\mathrm{P}(T > 120) = 0.186$ ***A1***

[3 marks]

(c) consider the random variable $U = Y - 2X$ ***(M1)***

$\mathrm{E}(U) = -4$ ***A1***

$\mathrm{Var}(U) = \mathrm{Var}(Y) + 4\,\mathrm{Var}(X)$ ***(M1)***

$= 136$ ***A1***

$\mathrm{P}(Y > 2X) = \mathrm{P}(U > 0)$ ***(M1)***

$= 0.366$ ***A1***

[6 marks]

Total [12 marks]

2. (a) $H_0 : \rho = 0$; $H_1 : \rho \neq 0$ ***A1A1***

[2 marks]

(b) (i) $t = 0.486 \times \sqrt{\dfrac{10-2}{1-0.486^2}}$ ***(M1)***

$= 1.572\ldots$ ***(A1)***

degrees of freedom $= 8$ ***(A1)***

$\mathrm{P}(T > 1.5728\ldots)$ ***(M1)***

$= 0.0772$ ***(A1)***

p-value $= 0.154$ ***A1***

Note: Do not follow through for the final ***A1*** if their H_1 is one-sided.

(ii) accept H_0 or equivalent statement involving H_0 or H_1

(at the 5% significance level) ***R1***

Note: Follow through the candidate's p -value.

[7 marks]

continued…

Question 2 continued

(c) **EITHER**

because the above analysis suggests that X, Y are independent ***R1***

OR

the value of r suggests that X and Y are weakly correlated ***R1***

[1 mark]

Total [10 marks]

3. (a) $\mathrm{E}(U) = k\mathrm{E}(\bar{X}) = k\mathrm{E}(X)$ ***(M1)***

$= \dfrac{k\theta}{2}$ ***(A1)***

unbiased when $k = 2$ ***A1***

[3 marks]

(b) (i) for the data, $\Sigma x = 40.8$ ***(A1)***

$\Rightarrow \bar{x} = 5.1$ ***(A1)***

so that unbiased estimate for $\theta = 10.2$ ***A1***

(ii) this is impossible because of the sample value 10.3 ***R1***

[4 marks]

(c) (i) $\mathrm{Var}(U) = 4 \times \mathrm{Var}(\bar{X})$ ***(M1)***

$= 4 \times \dfrac{\theta^2}{24n}$ ***A1***

$= \dfrac{\theta^2}{6n}$ ***AG***

(ii) $\mathrm{E}(U^2) = \mathrm{Var}(U) + \left(\mathrm{E}(U)\right)^2$ ***M1***

$= \dfrac{\theta^2}{6n} + \theta^2$ ***A1***

$\mathrm{E}(U^2) \neq \theta^2$ ***R1***

so not unbiased ***AG***

(iii) $\mathrm{E}(U^2) = \dfrac{\theta^2}{6n}(1 + 6n)$ ***(A1)***

$\mathrm{E}\left(\left(\dfrac{6n}{1 + 6n}\right)U^2\right) = \theta^2$ ***(A1)***

therefore $\left(\left(\dfrac{6n}{1 + 6n}\right)U^2\right)$ is an unbiased estimator for θ^2 ***A1***

[8 marks]

Total [15 marks]

4. (a) $H_0: \mu = 2.2$; $H_1: \mu \neq 2.2$ ***A1A1***

[2 marks]

(b) (i) UE of mean $= \dfrac{42.0}{20} = 2.1$ ***A1***

UE of variance $= \dfrac{89.2}{19} - \dfrac{20 \times 2.1^2}{19} = 0.0526 \left(\dfrac{1}{19}\right)$ ***(M1)A1***

> **Note:** Award ***(M0)*** for division by 20 where there is no subsequent use of $\dfrac{20}{19}$.

(ii) $t = -1.95$ ***(A1)***

$\text{DF} = 19$ ***(A1)***

p-value $= 0.0662$ ***A1***

> **Note:** Allow follow through from (b)(i). In particular, 0.05 for the variance gives $t = -2$ and p-value 0.0600.

accept H_0, or equivalent statement involving H_0 or H_1, indicating that the mean weight is 2.2kg ***R1***

> **Note:** Follow through the candidate's p-value.

[7 marks]

(c) $[1.99, 2.21]$ ***A1A1***

> **Note:** Allow follow through from (b)(i). In particular, 0.05 for the variance gives $[2.00, 2.20]$.

[2 marks]

Total [11 marks]

5. (a) $\text{P}(Y = y) = \int_y^{y+1} \text{e}^{-x}\text{d}x$ ***M1A1***

$= \left[-\text{e}^{-x}\right]_y^{y+1}$ ***A1***

$= -\text{e}^{-(y+1)} + \text{e}^{-y}$ ***A1***

$= \text{e}^{-y}(1 - \text{e}^{-1})$ ***AG***

[4 marks]

continued…

Question 5 continued

(b) (i) attempt to use $G(t) = \sum P(Y = y)t^y$ ***(M1)***

$= \sum_{y=0}^{\infty} e^{-y}(1 - e^{-1})t^y$ ***A1***

Note: Accept a listing of terms without the use of Σ.

this is an infinite geometric series with first term $1 - e^{-1}$ and common ratio $e^{-1}t$ ***M1***

$G(t) = \dfrac{1 - e^{-1}}{1 - e^{-1}t}$ ***AG***

(ii) $E(Y) = G'(1)$ ***M1***

$G'(t) = \dfrac{1 - e^{-1}}{(1 - e^{-1}t)^2} \times e^{-1}$ ***(M1)(A1)***

$E(Y) = \dfrac{e^{-1}}{(1 - e^{-1})}$ ***(A1)***

$= 0.582$ ***A1***

Note: Allow the use of GDC to determine $G'(1)$.

[8 marks]

Total [12 marks]

Markscheme

November 2015

Statistics and probability

Higher level

Paper 3

Instructions to Examiners

Abbreviations

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3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

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$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

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A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

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13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) valid attempt to use $\bar{x} \pm z\frac{\sigma}{\sqrt{n}}$ ***(M1)***

$[182, 188]$ ***A1A1***

Note: Accept answers that round to the correct 3 sf.

[3 marks]

(b) $1.96 \times \frac{15.0}{\sqrt{n}} < 1.5$ ***M1A1***

$n > \left(\frac{15.0}{1.5} \times 1.96\right)^2$ ***(M1)***

Note: Award M1 for attempting to solve the inequality.

Note: Allow the use of =.

minimum value $n = 385$ ***A1***

[4 marks]

Total [7 marks]

2. (a) $r = -0.762$ ***(M1)A1***

Note: Accept answers that round to –0.76.

[2 marks]

(b) H_0: Moisture content and strength are independent or $\rho = 0$

H_1: Moisture content and strength are not independent or $\rho \neq 0$ ***A1***

EITHER

test statistic is –3.33 ***A1***

critical value is $(\pm)$ 2.306 ***A1***

since $-3.33 < -2.306$ or $3.33 > 2.306$, ***R1***

reject H_0 (or equivalent) ***A1***

OR

p-value is 0.0104 ***A2***

as $0.0104 < 0.05$, ***R1***

reject H_0 (or equivalent) ***A1***

Note: The ***R1*** and ***A1*** can be awarded as follow through from their test statistic or p-value.

[5 marks]

continued…

Question 2 continued

(c) $x =$ strength
$y =$ moisture content
$x = -0.629y + 28.1$ ***(M1)(A1)***
if $y = 9.5$ so $x = 22.1$ ***(M1)A1***

Note: Only accept answers that round to 22.1.

Note: Award ***M1A1M0A0*** for the other regression line $y = 30.1 - 0.924x$.

[4 marks]

Total [11 marks]

3. let X denote boys' height and Y denote girls' height
if BB, P(taller is boy) $= 1$ ***(A1)***
if GG, P(taller is boy) $= 0$ ***(A1)***

if BG or GB:
consider $X - Y$ ***(M1)***
$E(X - Y) = 178 - 169 = 9$ ***A1***
$\text{Var}(X - Y) = 5.2^2 + 5.4^2 \ (= 56.2)$ ***(M1)A1***
$\text{P}(X - Y > 0) = 0.885$ ***A1***
answer is $\frac{1}{4} \times 1 + \frac{1}{2} \times 0.885 = 0.693$ ***(M1)A1***

[9 marks]

4. (a) **METHOD 1**

$\mathrm{P}(U = u) = \frac{1}{4}\left(\frac{3}{4}\right)^{u-1}$ ***(M1)***

$F(u) = \mathrm{P}(U \le u) = \sum_{r=1}^{u} \frac{1}{4}\left(\frac{3}{4}\right)^{r-1}$ (or equivalent)

$= \dfrac{\frac{1}{4}\left(1 - \left(\frac{3}{4}\right)^{u}\right)}{1 - \frac{3}{4}}$ ***(M1)***

$= 1 - \left(\frac{3}{4}\right)^{u}$ ***A1***

METHOD 2

$\mathrm{P}(U \le u) = 1 - \mathrm{P}(U > u)$ ***(M1)***

$\mathrm{P}(U > u) =$ probability of u consecutive failures ***(M1)***

$\mathrm{P}(U \le u) = 1 - \left(\frac{3}{4}\right)^{u}$ ***A1***

[3 marks]

(b) $\mathrm{P}(U > 20) = 1 - \mathrm{P}(U \le 20)$ ***(M1)***

$= \left(\frac{3}{4}\right)^{20}$ $(= 0.00317)$ ***A1***

[2 marks]

(c) $G_U(t) = \sum_{r=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{r-1} t^r$ (or equivalent) ***M1A1***

$= \sum_{r=1}^{\infty} \frac{1}{3}\left(\frac{3}{4}t\right)^{r}$ ***(M1)***

$= \dfrac{\frac{1}{3}\left(\frac{3}{4}t\right)}{1 - \frac{3}{4}t} \left(= \dfrac{\frac{1}{4}t}{1 - \frac{3}{4}t}\right)$ ***A1***

$= \dfrac{t}{4 - 3t}$ ***AG***

[4 marks]

continued…

Question 4 continued

(d) (i) $E(U) = \frac{1}{\frac{1}{4}} = 4$ ***(A1)***

$E(U_1 + U_2 + U_3) = 4 + 4 + 4 = 12$ ***A1***

(ii) $\text{Var}(U) = \frac{\frac{3}{4}}{\left(\frac{1}{4}\right)^2} = 12$ ***A1***

$\text{Var}(U_1 + U_2 + U_3) = 12 + 12 + 12 = 36$ ***A1***

(iii) $G_V(t) = (G_U(t))^3$ ***(M1)***

$= \left(\frac{t}{4 - 3t}\right)^3$ ***A1***

[6 marks]

(e) $G_W{}'(t) = -3(4 - 3t)^{-4}(-3)\left(= \frac{9}{(4 - 3t)^4}\right)$ ***(M1)(A1)***

$E(W) = G_W{}'(1) = 9$ ***(M1)A1***

Note: Allow the use of the calculator to perform the differentiation.

[4 marks]

(f) **EITHER**

probability generating function of the constant 3 is t^3 ***A1***

OR

$G_{W+3}(t) = E\left(t^{W+3}\right) = E\left(t^W\right)E\left(t^3\right)$ ***A1***

THEN

$W + 3$ has generating function $G_{W+3} = \frac{1}{(4 - 3t)^3} \times t^3 = G_V(t)$ ***M1***

as the generating functions are the same $V = W + 3$ ***R1AG***

[3 marks]

Total [22 marks]

5. (a) let X denote the score on the die

(i) $$\mathrm{P}(X = x) = \begin{cases} \dfrac{1-p}{5} & , \quad x = 1, 2, 3, 4, 5 \\ p & , \quad x = 6 \end{cases}$$ ***(M1)***

$E(X_1) = (1+2+3+4+5)\dfrac{1-p}{5} + 6p$ ***M1***

$= 3 + 3p$ ***A1***

(ii) so an unbiased estimator for p would be $\dfrac{X_1 - 3}{3}$ ***A1***

[4 marks]

(b) (i) $E\left(k(X_1 - 3) + \left(\dfrac{1}{3} - k\right)(X_2 - 3)\right)$ ***M1***

$= kE\,(X_1 - 3) + \left(\dfrac{1}{3} - k\right)E(X_2 - 3)$ ***M1***

$= k\,(3p) + \left(\dfrac{1}{3} - k\right)(3p)$ ***A1***

any correct expression involving just k and p

$= p$ ***AG***

hence $k(X_1 - 3) + \left(\dfrac{1}{3} - k\right)(X_2 - 3)$ is an unbiased estimator of p

[3 marks]

(ii) $\mathrm{Var}\left(k(X_1 - 3) + \left(\dfrac{1}{3} - k\right)(X_2 - 3)\right)$ ***M1***

$= k^2\,\mathrm{Var}(X_1 - 3) + \left(\dfrac{1}{3} - k\right)^2 \mathrm{Var}(X_2 - 3)$ ***A1***

$= \left(k^2 + \left(\dfrac{1}{3} - k\right)^2\right)\sigma^2$ (where σ^2 denotes $\mathrm{Var}(X)$)

valid attempt to minimise the variance ***M1***

$k = \dfrac{1}{6}$ ***A1***

Note:	Accept an argument which states that the most efficient estimator is the one having equal coefficients of X_1 and X_2.

[4 marks]

Total [11 marks]

Markscheme

May 2015

Statistics and probability

Higher level

Paper 3

Instructions to Examiners

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1. (a) $\mathrm{P}(L \geq 4995) = 0.785$ ***(M1)A1***

Note: Accept any answer that rounds correctly to 0.79.
Award ***M1A0*** for 0.78.

Note: Award ***M1A0*** for any answer that rounds to 0.55 obtained by taking $\mathrm{SD} = 40$.

[2 marks]

(b) we are given that $L \sim \mathrm{N}(5000,\ 40)$ and $S \sim \mathrm{N}(1000,\ 25)$

consider $X = L - 5S$ (ignore $\pm$ 30) ***(M1)***

$\mathrm{E}(X) = 0$ (± 30 consistent with line above) ***A1***

$\mathrm{Var}(X) = \mathrm{Var}(L) + 25\,\mathrm{Var}(S) = 40 + 625 = 665$ ***(M1)A1***

require $\mathrm{P}(X \geq 30)$ (or $\mathrm{P}(X \geq 0)$ if -30 above) ***(M1)***

obtain 0.122 ***A1***

Note: Accept any answer that rounds correctly to 2 significant figures.

[6 marks]

(c) consider $Y = L - (S_1 + S_2 + S_3 + S_4 + S_5)$ (ignore ± 30) ***(M1)***

$\mathrm{E}(Y) = 0$ (± 30 consistent with line above) ***A1***

$\mathrm{Var}(Y) = 40 + 5 \times 25 = 165$ ***A1***

require $\mathrm{P}(Y \leq -30)$ $\left(\text{or } \mathrm{P}(Y \leq 0) \text{ if } +30 \text{ above}\right)$ ***(M1)***

obtain 0.00976 ***A1***

Note: Accept any answer that rounds correctly to 2 significant figures.

Note: Condone the notation $Y = L - 5S$ if the variance is correct.

[5 marks]

Total [13 marks]

2. (a) unbiased estimate of μ is $2.36(36\ldots)$ $(26/11)$ ***(M1)A1***

unbiased estimate of σ^2 is $33.65(45\ldots) = \left(5.801^2\right)$ $(1851/55)$ ***(M1)A1***

Note: Accept any answer that rounds correctly to 3 significant figures.

Note: Award ***M1A0*** for any unbiased estimate of σ^2 that rounds to 5.80.

[4 marks]

(b) (i) $\mathrm{H}_0 : \mu = 0;\ \mathrm{H}_1 : \mu > 0$ ***A1A1***

Note: Award ***A1A0*** if an inappropriate symbol is used for the mean, *eg*, r, $\bar{\mathrm{d}}$.

continued…

Question 2 continued

(ii) attempt to use t-test ***(M1)***
$t = 1.35$ ***(A1)***
$\text{DF} = 10$ ***(A1)***
p-value $= 0.103$ ***A1***

Note: Accept any answer that rounds correctly to 3 significant figures.

(iii) $0.103 > 0.05$ ***A1***
there is insufficient evidence at the 5% level to support the claim (that extra tuition improves examination marks)

OR

the claim (that extra tuition improves examination marks) is not supported at the 5% level (or equivalent statement) ***R1***

Note: Follow through the candidate's p-value.

Note: Do not award ***R1*** for Accept H_0 or Reject H_1.

[8 marks]

Total [12 marks]

3. (a) the (unbiased) estimate of μ is 9.793 ***(A1)***

the 99% CI is $9.793 \pm 2.576\frac{0.03}{\sqrt{6}}$ ***(M1)(A1)***

$= [9.761, 9.825]$ ***A1***

Note: Accept 9.762 and 9.824.

[4 marks]

(b) if this process is carried out a large number of times ***A1***
(approximately) 99% of the intervals will contain μ ***A1***

Note: Award ***A1A1*** for a consideration of any specific large value of times $(n \geq 100)$.

[2 marks]

continued…

Question 3 continued

(c) **METHOD 1**

If the interval is halved, 2.576 becomes 1.288 ***M1***
normal tail probability corresponding to $1.288 = 0.0988\ldots$ ***A1***
confidence level $= 80\%$ ***A1***

METHOD 2

half width $= 0.5 \times 0.063$ or 0.062 or $0.064 = 0.0315$ or 0.031 or 0.032 ***M1***

$\frac{2z \times 0.03}{\sqrt{6}} = 0.0315$ or 0.031 or 0.032

giving $z = 1.285\ldots$ or $1.265\ldots$ or $1.306\ldots$ ***A1***
confidence level $= 80\%$ or 79% or 81% ***A1***

Note: Follow through values from (a).

[3 marks]

Total [9 marks]

4. (a) (i) an estimator T is a formula (or statistic) that can be applied to the values in any sample, taken from X ***A1***
to estimate the value of μ ***A1***

(ii) an estimator is unbiased if $\mathrm{E}(T) = \mu$ ***A1***

[3 marks]

(b) (i) using linearity and the definition of an unbiased estimator ***M1***
$\mu = \alpha\mu + \beta\mu + (\alpha - \beta)\mu$ ***A1***
obtain $\alpha = \frac{1}{2}$ ***A1***

(ii) attempt to compute $\mathrm{Var}(U)$ using correct formula ***M1***

$\mathrm{Var}(U) = \frac{1}{4}\sigma^2 + \beta^2\sigma^2 + \left(\frac{1}{2} - \beta\right)^2 \sigma^2$ ***A1***

$\mathrm{Var}(U) = \sigma^2\left(2\beta^2 - \beta + \frac{1}{2}\right)$ ***AG***

(iii) attempt to minimise quadratic in β (or equivalent) ***(M1)***

$\beta = \frac{1}{4}$ ***A1***

(iv) $(U) = \frac{1}{2}X_1 + \frac{1}{4}X_2 + \frac{1}{4}X_3$ ***A1***

$\mathrm{Var}(U) = \frac{3}{8}\sigma^2$ ***A1***

continued…

Question 4 continued

(v) $\frac{1}{3}X_1+\frac{1}{3}X_2+\frac{1}{3}X_3$ ***A1***

$\text{Var}\left(\frac{1}{3}X_1+\frac{1}{3}X_2+\frac{1}{3}X_3\right)=\frac{3}{9}\sigma^2$ ***A1***

$<\text{Var}(U)$ ***R1***

> **Note:** Accept $\sum_{i=1}^{3}\lambda_i X_i$ if $\sum_{i=1}^{3}\lambda_i=1$ and $\sum_{i=1}^{3}\lambda_i^2<\frac{3}{8}$ and follow through to the variance if this is the case.

[12 marks]

Total [15 marks]

5. (a) $\text{P}(X=0)=1-p(=q)$; $\text{P}(X=1)=p$ ***(M1)(A1)***

$G_x(t)=\sum_r \text{P}(X=r)t^r$ (or writing out term by term) ***M1***

$=q+pt$ ***A1***

[4 marks]

(b) **METHOD 1**

PGF for $B(n,\ p)$ is $(q+pt)^n$ ***R1***

which is a polynomial of degree n ***R1***

METHOD 2

in n independent trials, it is not possible to obtain more than n successes (or equivalent, *eg,* $\text{P}(X>n)=0$) ***R1***

so $a_r=0$ for $r>n$ ***R1***

[2 marks]

continued…

Question 5 continued

(c) let $Y = X_1 + X_2$

$G_Y(t) = (q_1 + p_1 t)(q_2 + p_2 t)$ ***A1***

$G_Y(t)$ has degree two, so if Y is binomial then

$Y \sim \mathrm{B}(2,\ p)$ for some p ***R1***

$(q + pt)^2 = (q_1 + p_1 t)(q_2 + p_2 t)$ ***A1***

Note: The LHS could be seen as $q^2 + 2pqt + p^2 t^2$.

METHOD 1

by considering the roots of both sides, $\frac{q_1}{p_1} = \frac{q_2}{p_2}$ ***M1***

$\frac{1 - p_1}{p_1} = \frac{1 - p_2}{p_2}$ ***A1***

so $p_1 = p_2$ ***AG***

METHOD 2

equating coefficients,

$p_1 p_2 = p^2$, $q_1 q_2 = q^2$ or $(1 - p_1)(1 - p_2) = (1 - p)^2$ ***M1***

expanding,

$p_1 + p_2 = 2p$ so p_1, p_2 are the roots of $x^2 - 2px + p^2 = 0$ ***A1***

so $p_1 = p_2$ ***AG***

[5 marks]

Total [11 marks]

N14/5/MATHL/HP3/ENG/TZ0/SP/M

International Baccalaureate®
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MARKSCHEME

November 2014

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

11 pages

Instructions to Examiners

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$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

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A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

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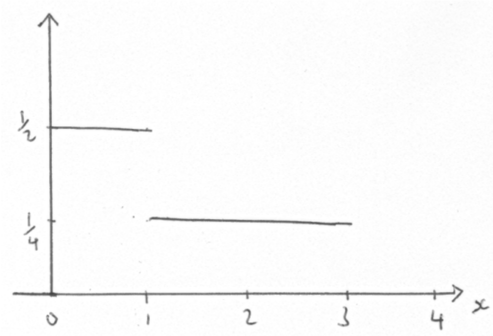
1 (a)



A1

Note: Ignore open / closed endpoints and vertical lines.

Note: Award ***A1*** for a correct graph with scales on both axes and a clear indication of the relevant values.

[1 mark]

(b)

$$F(x)=\begin{cases} 0 & x<0 \\ \dfrac{x}{2} & 0 \le x < 1 \\ \dfrac{x}{4}+\dfrac{1}{4} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

considering the areas in their sketch or using integration ***(M1)***

$F(x)=0,\ x<0,\ F(x)=1,\ x\ge 3$ ***A1***

$F(x)=\dfrac{x}{2},\ 0\le x<1$ ***A1***

$F(x)=\dfrac{x}{4}+\dfrac{1}{4},\ 1\le x<3$ ***A1A1***

Note: Accept < for ≤ in all places and also > for ≥ first ***A1***.

[5 marks]

(c) $Q_3 = 2, Q_1 = 0.5$ ***A1A1***

IQR is $2-0.5=1.5$ ***A1***

[3 marks]

Total [9 marks]

2. (a) **METHOD 1**

let X be the number of throws until Eric hits the target three times

$X \sim \text{NB}(3, 0.2)$ ***(M1)***

$\text{P}(X = 6) = \binom{5}{2} 0.8^3 \times 0.2^3$ ***(A1)***

$= 0.04096 \left(= \frac{128}{3125} \right) \text{(exact)}$ ***A1***

METHOD 2

let X be the number of hits in five throws

X is B(5, 0.2) ***(M1)***

$\text{P}(X = 2) = \binom{5}{2} 0.2^2 \times 0.8^3 \quad (0.2048)$ ***(A1)***

$\text{P(3rd hit on 6th throw)} = \binom{5}{2} 0.2^2 \times 0.8^3 \times 0.2 = 0.04096 \left(= \frac{128}{3125} \right) \text{(exact)}$ ***A1***

[3 marks]

(b) (i) expected number of throws $= \frac{3}{0.2} = 15$ ***(M1)A1***

(ii) profit $= (10 - 15) = -\$5$ or loss = \$5 ***A1***

[3 marks]

(c) **METHOD 1**

let Y be the number of times the target is hit in 8 throws

$Y \sim \text{B}(8, 0.2)$ ***(M1)***

$\text{P}(Y \le 2)$ ***(M1)***

$= 0.797$ ***A1***

METHOD 2

let the 3rd hit occur on the Yth throw

Y is NB(3, 0.2) ***(M1)***

$\text{P}(Y > 8) = 1 - \text{P}(Y \le 8)$ ***(M1)***

$= 0.797$ ***A1***

[3 marks]

Total [9 marks]

3. (a) **METHOD 1**

$\text{Cov}(X, Y) = \text{E}\left((X-\mu_X)(Y-\mu_Y)\right)$

$= \text{E}\left(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y\right)$ ***(M1)***

$= \text{E}(XY) - \mu_Y\text{E}(X) - \mu_X\text{E}(Y) + \mu_X\mu_Y$

$= \text{E}(XY) - \mu_X\mu_Y$ ***A1***

as X and Y are independent $\text{E}(XY) = \mu_X\mu_Y$ ***R1***

$\text{Cov}(X, Y) = 0$ ***AG***

METHOD 2

$\text{Cov}(X,Y) = \text{E}\left((X-\mu_x)(Y-\mu_y)\right)$

$= \text{E}(X-\mu_x)\text{E}(Y-\mu_y)$ ***(M1)***

since X,Y are independent ***R1***

$= (\mu_x-\mu_x)(\mu_y-\mu_y)$ ***A1***

$= 0$ ***AG***

[3 marks]

(b) $H_0 : \rho = 0 \quad H_1 : \rho < 0$ ***A1***

Note: The hypotheses must be expressed in terms of ρ.

test statistic $t_{test} = -0.35\sqrt{\dfrac{20-2}{1-(-0.35)^2}}$ ***(M1)(A1)***

$= -1.585\ldots$ ***(A1)***

degrees of freedom = 18 ***(A1)***

EITHER

p-value $= 0.0652$ ***A1***

this is greater than 0.05 ***M1***

OR

$t_{5\%}(18) = -1.73$ ***A1***

this is less than –1.59 ***M1***

THEN

hence accept H_0 or reject H_1 or equivalent or contextual equivalent ***R1***

Note: Allow follow through for the final ***R1*** mark.

[8 marks]

Total [11 marks]

4. (a) (i) $G'(t) = \lambda e^{\lambda(t-1)}$ ***A1***

$\mathrm{E}(X) = G'(1)$ ***M1***

$= \lambda$ ***AG***

(ii) $G''(t) = \lambda^2 e^{\lambda(t-1)}$ ***M1***

$\Rightarrow G''(1) = \lambda^2$ ***(A1)***

$\mathrm{Var}(X) = G''(1) + G'(1) - \left(G'(1)\right)^2$ ***(M1)***

$= \lambda^2 + \lambda - \lambda^2$ ***A1***

$= \lambda$ ***AG***

[6 marks]

(b) (i) $\mathrm{E}(S) = 2\lambda - \lambda = \lambda$ ***A1***

(ii) $\mathrm{Var}(S) = 4\lambda + \lambda = 5\lambda$ ***(A1)A1***

Note: First ***A1*** can be awarded for either 4λ or $+\lambda$.

[3 marks]

(c) (i) $\mathrm{E}(T) = \dfrac{\lambda}{2} + \dfrac{\lambda}{2} = \lambda$ (so T is an unbiased estimator) ***A1***

(ii) $\mathrm{Var}(T) = \dfrac{1}{4}\lambda + \dfrac{1}{4}\lambda = \dfrac{1}{2}\lambda$ ***A1***

this is less than $\mathrm{Var}(S)$, therefore T is the more efficient estimator ***R1AG***

Note: Follow through their variances from (b)(ii) and (c)(ii).

[3 marks]

(d) no, mean does not equal the variance ***R1***

[1 mark]

(e) $G_{X+Y}(t) = e^{\lambda(t-1)} \times e^{\lambda(t-1)} = e^{2\lambda(t-1)}$ ***M1A1***

which is the probability generating function for a Poisson with a mean of 2λ ***R1AG***

[3 marks]

(f) (i) $G_{X+Y}(1) = 1$ ***A1***

(ii) $G_{X+Y}(-1) = e^{-4\lambda}$ ***A1***

[2 marks]

continued …

Question 4 continued

(g) $G_{X+Y}(1) = p(0) + p(1) + p(2) + p(3)\ldots$
$G_{X+Y}(-1) = p(0) - p(1) + p(2) - p(3)\ldots$
so $2\mathrm{P}(\text{even}) = G_{X+Y}(1) + G_{X+Y}(-1)$ ***(M1)(A1)***
$\mathrm{P}(\text{even}) = \frac{1}{2}(1 + e^{-4\lambda})$ ***A1***

[3 marks]

Total [21 marks]

5. (a) $\bar{X} \sim N\left(5.2, \frac{1.2^2}{16}\right)$ ***(M1)***

critical value is $5.2 - 1.64485\ldots \times \frac{1.2}{4} = 4.70654\ldots$ ***(A1)***

critical region is $\left]-\infty, 4.71\right]$ ***A1***

Note: Allow follow through for the final ***A1*** from their critical value.

[3 marks]

Note: Follow through previous values in (b), (c) and (d).

(b) type II error probability = $\mathrm{P}(\bar{X} > 4.70654... \mid \bar{X}$ is $N\left(4.6, \frac{1.2^2}{16}\right)$ ***(M1)***

= 0.361 ***A1***

[2 marks]

(c) $0.9 \times 0.05 + 0.1 \times (1 - 0.361\ldots) = 0.108875997\ldots = 0.109$ ***M1A1***

Note: Award ***M1*** for a weighted average of probabilities with weights 0.1, 0.9.

[2 marks]

continued …

Question 5 continued

(d) attempt to use conditional probability formula ***M1***

$\frac{0.9 \times 0.05}{0.108875997\ldots}$ ***(A1)***

$= 0.41334\ldots = 0.413$ ***A1***

[3 marks]

Total [10 marks]

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MARKSCHEME

May 2014

MATHEMATICS
STATISTICS AND PROBABLITY

Higher Level

Paper 3

11 pages

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1. (a) (i) $P(X = 6) = 0.122$ ***(M1)A1***

(ii) $P(X = 6 \mid 5 \le X \le 8) = \dfrac{P(X = 6)}{P(5 \le X \le 8)} = \dfrac{0.122\ldots}{0.592\ldots - 0.0996\ldots}$ ***(M1)(A1)***

$= 0.248$ ***A1***

[5 marks]

(b) (i) $E(\bar{X}) = 8$ ***A1***

$\text{Var}(\bar{X}) = \dfrac{8}{n}$ ***A1***

(ii) $E(\bar{X}) \ne \text{Var}(\bar{X})$ (for $n > 1$) ***R1***

> **Note:** Only award the ***R1*** if the two expressions in (b)(i) are different.

[3 marks]

(c) (i) **EITHER**

$\bar{X} \sim N(8,\ 0.2)$ ***(M1)A1***

> **Note:** ***M1*** for normality, ***A1*** for parameters.

$P(7.1 < \bar{X} < 8.5) = 0.846$ ***A1***

OR

The expression is equivalent to
$P(283 \le \sum X \le 339)$ where $\sum X$ is Po(320) ***M1A1***
$= 0.840$ ***A1***

> **Note:** Accept 284, 340 instead of 283, 339
> Accept any answer that rounds correctly to 0.84 or 0.85.

continued…

Question 1 continued

(ii) **EITHER**

$k = 1.96\frac{\sigma}{\sqrt{n}}$ or $1.96\ \text{std}\left(\bar{X}\right)$ ***(M1)(A1)***

$k = 0.877$ or $1.96\ \sqrt{0.2}$ ***A1***

OR

The expression is equivalent to

$P(320 - 40k \le \sum X \le 320 + 40k) = 0.95$ ***(M1)***

$k = 0.875$ ***A2***

Note:	Accept any answer that rounds to 0.87 or 0.88. Award ***M1A0*** if modulus sign ignored and answer obtained rounds to 0.74 or 0.75

[6 marks]

Total [14 marks]

2. (a) $H_0: \rho = 0$ *A1*

$H_1: \rho > 0$ *A1*

[2 marks]

(b) 0.853 *A2*

Note: Accept any answer that rounds to 0.85.

[2 marks]

(c) $p\text{-value} = 0.00173$ (1-tailed) *A1*

Note: Accept any answer that rounds to 0.0017.
Accept any answer that rounds to 0.0035 obtained from 2-tailed test.

strong evidence to reject the hypothesis that there is no correlation between rainfall and yield or to accept the hypothesis that there is correlation between rainfall and yield *R1*

Note: Follow through the *p*-value for the conclusion.

[2 marks]

(d) $y = 1.78x + 40.5$ *A1A1*

Note: Accept numerical coefficients that round to 1.8 and 41.

[2 marks]

(e) $y = 1.77\ldots(19) + 40.5\ldots$ *M1*

74.3 *A1*

Note: Accept any answer that rounds to 74 or 75.

[2 marks]

(f) the gradient of the regression line *y* on *x* is 1.78 or equivalent *A1*

the regression line of *x* on *y* is $x = 0.409y - 12.2$ *(A1)*

the gradient of the regression line *x* on *y* is $\frac{1}{0.409}(= 2.44)$ *(M1)A1*

calculate $\arctan(2.44) - \arctan(1.78)$ *(M1)*

angle between regression lines is 7 degrees *A1*

Note: Accept any answer which rounds to ±7 degrees.

[6 marks]

Total [16 marks]

3. (a) $\mathrm{E}\left(\frac{X-b}{a}\right)=\frac{a\lambda+b-b}{a}$ ***M1A1***

$=\lambda$ ***A1***

(Therefore $\frac{X-b}{a}$ is an unbiased estimator for λ) ***AG***

[3 marks]

(b) (i) $f(y)\ge 0$ ***R1***

> **Note:** Only award ***R1*** if this statement is made explicitly.

recognition or showing that integral of f is 1 (seen anywhere) ***R1***

EITHER

$\int_{\lambda-3}^{\lambda}\frac{2}{9}(3+y-\lambda)\,\mathrm{d}y$ ***M1***

$=\frac{2}{9}\left[(3-\lambda)y+\frac{1}{2}y^2\right]_{\lambda-3}^{\lambda}$ ***A1***

$=\frac{2}{9}\left(\lambda(3-\lambda)+\frac{1}{2}\lambda^2-(3-\lambda)(\lambda-3)-\frac{1}{2}(\lambda-3)^2\right)$ or equivalent ***A1***

$=1$

OR

the graph of the probability density is a triangle with base length 3 and height $\frac{2}{3}$ ***M1A1***

its area is therefore $\frac{1}{2}\times 3\times\frac{2}{3}$ ***A1***

$=1$

(ii) $\mathrm{E}(Y)=\int_{\lambda-3}^{\lambda}\frac{2}{9}y(3+y-\lambda)\,\mathrm{d}y$ ***M1***

$=\frac{2}{9}\left[(3-\lambda)\frac{1}{2}y^2+\frac{1}{3}y^3\right]_{\lambda-3}^{\lambda}$ ***A1***

$=\frac{2}{9}\left((3-\lambda)\frac{1}{2}\left(\lambda^2-(\lambda-3)^2\right)+\frac{1}{3}\left(\lambda^3-(\lambda-3)^3\right)\right)$ ***M1***

$=\lambda-1$ ***A1A1***

> **Note:** Award 3 marks for noting that the mean is $\frac{2}{3}$rds the way along the base and then ***A1A1*** for $\lambda-1$

> **Note:** Award ***A1*** for λ and ***A1*** for -1.

continued…

Question 3 continued

(iii) unbiased estimator: $Y+1$ ***A1***

> **Note:** Accept $\bar{Y}+1$.
> Follow through their E(Y) if linear.

[11 marks]

Total [14 marks]

4. (a) use of $\mathrm{P}(X=n)=pq^{n-1}\ (q=1-p)$ ***(M1)***

$\mathrm{P}(X<4)=p+pq+pq^2\left(=1-q^3\right)\left(=1-(1-p)^3\right)(=3p-3p^2+p^3)$ ***A1***

[2 marks]

(b) $G_X(t)=\mathrm{P}(X=1)t+\mathrm{P}(X=2)t^2+\ldots$ ***(M1)***

$=pt+pqt^2+pq^2t^3+\ldots$ ***A1***

summing an infinite geometric series ***M1***

$=\dfrac{pt}{1-qt}$ ***AG***

[3 marks]

(c) (i) **EITHER**

$G_Y(t)=\mathrm{P}(Y=1)t+\mathrm{P}(Y=2)t^2+\ldots$ ***A1***

$=0\times t+\mathrm{P}(X=1)t^2+0\times t^3+\mathrm{P}(X=2)t^4+\ldots$ ***M1A1***

$=G_X\left(t^2\right)$ ***AG***

OR

$G_Y(t)=E(t^Y)=E(t^{2X})$ ***M1A1***

$=E\left((t^2)^X\right)$ ***A1***

$=G_X(t^2)$ ***AG***

continued…

Question 4 continued

(ii) $\mathrm{E}(Y) = G_Y'(1)$ ***A1***

EITHER

$= 2tG_X'\left(t^2\right)$ evaluated at $t=1$ ***M1A1***

$= 2\mathrm{E}(X)$ ***AG***

OR

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{pt^2}{(1-qt^2)}\right) = \frac{2pt(1-qt^2)+2pqt^3}{(1-qt^2)^2}$ evaluated at $t=1$ ***A1***

$= 2\times\frac{p(1-qt)+pqt}{(1-qt)^2}$ evaluated at $t=1$ (or $\frac{2}{p}$) ***A1***

$= 2\mathrm{E}(X)$ ***AG***

[6 marks]

(d) (i) $G_W(t) = t\,G_Y(t)$ (or equivalent) ***A2***

(ii) attempt to evaluate $G_W'(t)$ ***M1***

EITHER

obtain $1\times G_Y(t) + t\times G_Y{}'(t)$ ***A1***

substitute $t=1$ to obtain $1\times 1 + 1\times G_Y{}'(1)$ ***A1***

OR

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{pt^3}{(1-qt^2)}\right) = \frac{3pt^2(1-qt^2)+2pqt^4}{(1-qt^2)^2}$ ***A1***

substitute $t=1$ to obtain $1+\frac{2}{p}$ ***A1***

$= 1 + 2\mathrm{E}(X)$ ***AG***

[5 marks]

Total [16 marks]

N13/5/MATHL/HP3/ENG/TZ0/SP/M

MARKSCHEME

November 2013

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

12 pages

Instructions to Examiners

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R Marks awarded for clear **Reasoning**.

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AG Answer given in the question and so no marks are awarded.

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Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking November 2013**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
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All the marks will be added and recorded by Scoris.

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- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
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3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

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*Implied marks appear in **brackets, for example, (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

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*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

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- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a mis-read. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
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*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

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Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

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Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) $\bar{v} = \frac{1}{1000}(55 \times 5 + 65 \times 13 + \ldots + 145 \times 31)$ ***A1M1***

> **Note:** ***A1*** for mid-points, ***M1*** for use of the formula.

$= \frac{113210}{1000} = 113.21$ ***AG***

(ii) $s^2 = \frac{(55-113.21)^2 \times 5 + (65-113.21)^2 \times 13 + \ldots + (145-113.21)^2 \times 31}{999}$ ***(M1)***

$= \frac{362295.9}{999} = 362.6585\ldots = 363$ ***A1***

> **Note:** Award ***A1*** if answer rounds to 362 or 363.
>
> **Note:** Condone division by 1000.

[4 marks]

(b) $\bar{v} \pm \frac{t_{0.025} \times s}{\sqrt{n}}$ ***(M1)***

hence the confidence interval $I = [112.028, 114.392]$ ***A1***

> **Note:** Accept answers which round to 112 and 114.
>
> **Note:** Condone the use of $z_{0.025}$ for $t_{0.025}$ and σ for s.

[2 marks]

(c) less confidence implies narrower interval ***R2***

> **Note:** Accept equivalent statements or arguments having a meaningful diagram and/or relevant percentiles.

hence the confidence interval I at the 95% level contains the confidence interval J at the 90% level ***AG***

[2 marks]

Total [8 marks]

2. (a) let $W = X - 1.5Y$ ***(M1)***

$\text{E}(W) = 180 - 1.5 \times 150 (= -45)$ ***A1***

$\text{Var}(W) = 14^2 + 2.25 \times 12^2$ $(= 520 \text{ or } 22.8^2)$ ***(M1)A1***

$W \sim \text{N}(-45, 520)$ ***(M1)***

$\text{P}(W > 0) = 0.0242$ ***(M1)A1*** ***N4***

> **Note:** The penultimate ***(M1)*** is for recognising normality.

[7 marks]

(b) let $T = X_1 + X_2 + X_3 + X_4 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6$ (grams) denote the total weight ***(M1)***

$\text{E}(T) = 4 \times 180 + 6 \times 150$ $(= 1620)$ ***A1***

> **Note:** Condone correct expected value from $T = 4X + 6Y$.

$\text{Var}(T) = 4 \times 14^2 + 6 \times 12^2$ $(= 1648 \text{ or } 40.6^2)$ ***(M1)A1***

then $T \sim N(1620, 1648)$ ***(M1)***

$\text{P}(T > 1500) = 0.998$ ***(M1)A1*** ***N4***

> **Note:** Accept answers which round to 0.998.
>
> **Note:** The penultimate ***(M1)*** is for recognising normality.

[7 marks]

Total [14 marks]

3. (a) (i) H_0: all coins are fair (or the data are represented by $B(7, 0.5)$) ***A1***
H_1: not all coins are fair (or the data are not represented by $B(7, 0.5)$) ***A1***

(ii) $\chi^2_{cal} > \chi^2_{critical}$ (14.067) or p-value < 0.05 $\Rightarrow$ reject null hypothesis ***A1***

(iii) let T be the number of tails obtained, T is binomially distributed ***(M1)***
$T \sim B(7, 0.5)$ ***(A1)***

T	0	1	2	3	4	5	6	7
f_0	6	19	141	218	203	117	38	8
f_e	**5.859…**	**41.01…**	**123.0…**	**205.0…**	**205.0…**	**123.0…**	**41.01…**	**5.859…**

A2

Note: Allow tabular values which are correct to 3 significant figures.

Note: Award ***A1*** for 6 or 7 correct values.

$\chi^2_{calc} = 16.576\ldots$ ***A1***

Note: Accept answer which round to 16.6.

(iv) $\nu = 7$ ***(A1)***
since $16.576\ldots > 14.067$ or $p = 0.02(034\ldots) < 0.05$, H_0 is rejected ***R1***

Note: Follow through their χ^2_{calc} or p-value for the ***R1***.

[10 marks]

(b) reduce the significance level (or equivalent statement) ***R2***

[2 marks]

(c) (i) accepting H_0 (or failing to reject H_0) when it is false (or equivalent) ***A1***

(ii) increase the number of trials ***A1***

[2 marks]

Total [14 marks]

4. H_0: the training schedule does not help improve times (or $\mu = 0$) ***A1***
H_1: the training schedule does help improve times (or $\mu > 0$) ***A1***

Note: Subsequent marks can be awarded even if the hypotheses are not stated. (Assuming difference of times is normally distributed.)

let d = time before training – time after training ***(M1)***

Competitor	A	B	C	D	E
Time before training (in seconds)	75	74	60	69	69
Time after training (in seconds)	73	69	55	72	65
Difference *d*	2	5	5	–3	4

A1

EITHER

$n = 5, \sum d = 13, \sum d^2 = 79 \Rightarrow s_{n-1}^2 = \frac{1}{4}\left(79 - \frac{169}{5}\right) = 11.3$ ***(M1)***

(small sample) so use a one-sided t-test ***(M1)***

Note: The "one-sided" t-test may have been seen above when stating H_1.

$t = \frac{2.6}{\sqrt{\frac{11.3}{5}}} = 1.7\ldots$ ***A1***

$v = 4$, ***A1***
at the 1 % level the critical value is 3.7 ***A1***
since $3.7 > 1.7\ldots$
H_0 is accepted (insufficient evidence to reject H_0) ***R1***

Note: Follow through their t-value.

OR

(small sample) so use a one-sided t-test ***(M1)***
$p = 0.079\ldots$ ***A4***
since $0.079\ldots > 0.01$
H_0 is accepted (insufficient evidence to reject H_0) ***R1***

Note: Follow through their p-value.

Note: Accept d = time after training – time before training throughout.

[10 marks]

5. (a) $E(S) = 2E(X) + 3E(Y) = 6 + 6 = 12$ ***A1***

$\text{Var}(S) = 4\text{Var}(X) + 9\text{Var}(Y) = 12 + 18 = 30$ ***A1***

[2 marks]

(b) S does not have a Poisson distribution ***A1***

because $\text{Var}(S) \neq E(S)$ ***R1***

Note: Follow through their E(S) and Var(S) if different.

[2 marks]

(c) **EITHER**

$P(T = 3) = P((X, Y) = (3, 0)) + P((X, Y) = (2, 1)) +$

$+P((X, Y) = (1, 2)) + P((X, Y) = (0, 3))$ ***(M1)***

$= P(X = 3)P(Y = 0) + P(X = 2)P(Y = 1) +$

$+P(X = 1)P(Y = 2) + P(X = 0)P(Y = 3)$ ***(M1)***

$= \frac{125e^{-5}}{6} (= 0.140)$ ***A2***

Note: Accept answers which round to 0.14.

OR

T is $P_o(2 + 3) = P_o(5)$ ***(M1)(A1)***

$P(T = 3) = \frac{125e^{-5}}{6} (= 0.140)$ ***A2***

Note: Accept answers which round to 0.14.

[4 marks]

(d) $P(T = t) = P((X, Y) = (0, t)) + P((X, Y) = (1, t-1)) + \dots P((X, Y) = (t, 0))$ ***(M1)***

$= P(X = 0)P(Y = t) + P(X = 1)P(Y = t-1) + \dots + P(X = t)P(Y = 0)$ ***A1***

$= \sum_{r=0}^{t} P(X = r)P(Y = t - r)$ ***AG***

[2 marks]

(e) $\mathrm{P}(T=t) = \sum_{r=0}^{t} \mathrm{P}(X=r)\mathrm{P}(Y=t-r)$

$$= \sum_{r=0}^{t} \frac{e^{-3}3^r}{r!} \times \frac{e^{-2}2^{t-r}}{(t-r)!}$$ ***M1A1***

$$= \frac{e^{-5}}{t!} \sum_{r=0}^{t} \frac{t!}{r!(t-r)!} \times 3^r 2^{t-r}$$ ***M1***

$$= \frac{e^{-5}}{t!}(3+2)^t$$ ***A1***

$$\left(= \frac{e^{-5}5^t}{t!}\right)$$

hence T follows a Poisson distribution with mean 5 ***AG***

[4 marks]

Total [14 marks]

International Baccalaureate
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MARKSCHEME

May 2013

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

11 pages

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11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $\bar{x} = 14$ ***A1***

$s_{n-1}^2 = \frac{3977.57}{19} - \frac{280^2}{380}$ ***(M1)***

$= 3.03$ ***A1***

[3 marks]

Note: Accept any notation for these estimates including μ and σ^2.

Note: Award ***M0A0*** for division by 20.

(b) the 95% confidence limits are

$\bar{x} \pm t\sqrt{\frac{s_{n-1}{}^2}{n}}$ ***(M1)***

Note: Award ***M0*** for use of *z*.

ie, $14 \pm 2.093\sqrt{\frac{3.03}{20}}$ ***(A1)***

Note : ***FT*** their mean and variance from (a).

giving $[13.2, 14.8]$ ***A1***

Note : Accept any answers which round to 13.2 and 14.8.

[3 marks]

(c) Use of t-statistic $\left(= \frac{14-15}{\sqrt{\frac{3.03}{20}}} \right)$ ***(M1)***

Note : ***FT*** their mean and variance from (a).

Note : Award ***M0*** for use of *z*.

Note: Accept $\frac{15-14}{\sqrt{\frac{3.03}{20}}}$.

$= -2.569\ldots$ ***(A1)***

Note: Accept $2.569\ldots$

p-value $= 0.009392\ldots \times 2 = 0.0188$ ***A1***

Note: Accept any answer that rounds to 0.019.

Note : Award ***(M1)(A1)A0*** for any answer that rounds to 0.0094.

insufficient evidence to reject H_0 (or equivalent, *eg* accept H_0 or reject H_1) ***R1***

Note: ***FT*** on their *p*-value.

[4 marks]

Total [10 marks]

2. (a) H_0 : number of goals can be modelled by a Poisson distribution
H_1 : number of goals cannot be modelled by a Poisson distribution ***A1***

Note: Do not award ***A1*** if a value for the mean of the distribution is given.

[1 mark]

(b) (i) sample mean $= \dfrac{\sum fx}{\sum f}$ ***(M1)***

$= 2.3$ ***A1***

exp freq for *r* goals $= 60 \times \dfrac{e^{-2.3} \times 2.3^r}{r!} (r \leq 4)$ ***(M1)***

Number of goals	0	1	2	3	4	≥ 5
Expected Frequency	6.0155	13.8357	15.9111	12.1985	7.0141	5.0250

A3

Note: At this stage, accept tabular values correct to 3 significant figures.

Note: Award ***A2*** for 1 error, ***A1*** for 2 errors and ***A0*** for 3 or more errors.

(ii) $\chi^2_{calc} = \sum \dfrac{f_o^{\ 2}}{f_e} - N$ or $\sum \dfrac{(f_o - f_e)^2}{f_e} = 2.69$ ***(M1)A1***

Note: Do not ***FT*** from incorrect tabular values.

Note: Accept any answer that rounds to 2.7.

$\text{DF} = 4$ ***(A1)***

Note: ***FT*** the DF from the table, *ie* award the ***(A1)*** if the value given is 2 less than the number of cells.

p-value $= 0.612$ ***A1***

Note: Accept any answer that rounds to 0.61.

Note: Do not ***FT*** from incorrect tabular values.

(iii) the manager's belief is supported (at all reasonable significance levels) (or equivalent, *eg* accept H_0 or reject H_1) ***R1***

Note: Follow through their *p*-value or χ^2_{crit} .

[11 marks]

Total [12 marks]

3. (a) $H_0 : \mu = 1.2$; $H_1 : \mu < 1.2$ ***A1***

Note: Accept "H_0 : (30-day) mean $= 36$; H_1 : (30-day) mean < 36".

[1 mark]

(b) (i) let X denote the number of breakdowns in 30 days
then under H_0, $E(X) = 36$ ***(A1)***
sig level $= P(X \le 25 \mid \text{mean} = 36)$ ***(M1)(A1)***
$= 0.0345$ (3.45%) ***A1***

Note: Accept any answer that rounds to 0.035 (3.5%).

Note: Do not accept the use of a normal approximation.

(ii) under H_1, $E(X) = 22.5$ ***(A1)***
P (Type II error) $= P(X \ge 26 \mid \text{mean} = 22.5)$ ***(M1)(A1)***
$= 0.257$ ***A1***

Note: Accept any answer that rounds to 0.26.

Note: Do not accept the use of a normal approximation.

[8 marks]

Total [9 marks]

4. (a) (i) $F(x) = \int_1^x \frac{3u^2 + 2u}{10}\,\mathrm{d}u$ ***(M1)***

$= \left[\frac{u^3 + u^2}{10}\right]_1^x$ ***A1***

Note: Do not penalise missing or wrong limits at this stage. Accept the use of x in the integrand.

$= \frac{x^3 + x^2 - 2}{10}$ ***A1***

(ii) the median m satisfies the equation $F(m) = \frac{1}{2}$ so ***(M1)***

$m^3 + m^2 - 7 = 0$ ***(A1)***

Note: Do not ***FT*** from an incorrect $F(x)$.

$m = 1.63$ ***A1***

Note: Accept any answer that rounds to 1.6.

[6 marks]

continued …

Question 4 continued

(b) (i) the mean of a large sample from any distribution is approximately normal ***A1***

Note: This is the minimum acceptable explanation.

(ii) we require the mean μ and variance σ^2 of X

$\mu = \int_1^2 \left(\frac{3x^3 + 2x^2}{10}\right) \mathrm{d}x$ ***(M1)***

$= \frac{191}{120}$ (1.591666…) ***A1***

$\sigma^2 = \int_1^2 \left(\frac{3x^4 + 2x^3}{10}\right) \mathrm{d}x - \mu^2$ ***(M1)***

$= 0.07659722\ldots$ ***A1***

the central limit theorem states that

$\bar{X} \approx N\left(\mu, \frac{\sigma^2}{n}\right)$, *i.e.* $N(1.591666\ldots, 0.0005106481\ldots)$ ***M1A1***

$\mathrm{P}(\bar{X} > 1.6) = 0.356$ ***A1***

Note: Accept any answer that rounds to 0.36.

[8 marks]

Total [14 marks]

5. (a) (i) the number of hits, $X \sim \mathrm{B}(8,\ 0.4)$ ***(A1)***

$\mathrm{P}(X=4)=\binom{8}{4}\times 0.4^4\times 0.6^4$ ***(M1)***

$=0.232$ ***A1***

Note: Accept any answer that rounds to 0.23.

(ii) let the 4th hit occur on the Yth shot so that $Y \sim \mathrm{NB}(4,\ 0.4)$ ***(A1)***

$\mathrm{P}(Y=8)=\binom{7}{3}\times 0.4^4\times 0.6^4$ ***(M1)***

$=0.116$ ***A1***

Note: Accept any answer that rounds to 0.12.

[6 marks]

(b) (i) $X \sim \mathrm{NB}(10,\ 0.4)$ ***(M1)***

$\mathrm{E}(X)=\frac{10}{0.4}=25$ ***A1***

(ii) let P_x denote $\mathrm{P}(X=x)$

$\mathrm{P}_x=\binom{x-1}{9}\times 0.4^{10}\times 0.6^{x-10}$ ***A1***

$\frac{\mathrm{P}_x}{\mathrm{P}_{x-1}}=\frac{\binom{x-1}{9}\times 0.4^{10}\times 0.6^{x-10}}{\binom{x-2}{9}\times 0.4^{10}\times 0.6^{x-11}}$ ***M1A1***

$=\frac{(x-1)!}{9!(x-10)!}\times\frac{9!(x-11)!\times 0.6}{(x-2)!}$ ***A1***

Note: Award ***A1*** for correct evaluation of combinatorial terms.

$=\frac{3(x-1)}{5(x-10)}$ ***AG***

(iii) $\mathrm{P}_x > \mathrm{P}_{x-1}$ as long as

$3x-3>5x-50$ ***(M1)***

i.e. $x<23.5$ ***(A1)***

the most likely value is 23 ***A1***

Note: Allow solutions based on creating a table of values of P_x.

[9 marks]

Total [15 marks]

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MARKSCHEME

November 2012

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

11 pages

Instructions to Examiners

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*Award **N** marks for **correct** answers where there is **no** working.*

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6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a mis-read. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

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7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

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12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

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13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) A has the hypergeometric distribution $\left(\text{Hyp}(5, 4, 10)\right)$ ***A1***

[1 mark]

(b) $\text{P}(A=3)=\dfrac{\binom{4}{3}\times\binom{6}{2}}{\binom{10}{5}}=\dfrac{4\times 15}{252}=\dfrac{5}{21}\quad (=0.238)$ ***(M1)A1***

[2 marks]

(c) $\text{P}(A=5)=0$ (since there are only 4 chocolate biscuits) ***A1***

[1 mark]

(d) B has the binomial distribution $\left(\text{B}\left(5, \dfrac{4}{10}\right)\right)$ ***A1***

[1 mark]

(e) $\text{P}(B=3)=\left(\binom{5}{3}\left(\dfrac{4}{10}\right)^3\left(\dfrac{6}{10}\right)^2=\right)\dfrac{144}{625}\quad (=0.2304)$ ***(M1)A1***

Note: Accept 0.230.

[2 marks]

(f) $\text{P}(B=5)=\left(\left(\dfrac{4}{10}\right)^5=\right)\dfrac{32}{3125}\quad (=0.01024)$ ***(M1)A1***

Note: Accept 0.0102.

[2 marks]

(g) $E(D)=E(B)-E(A)$ ***(M1)***

$=5\left(\dfrac{4}{10}\right)-5\left(\dfrac{4}{10}\right)=0$ ***A1***

[2 marks]

(h) $\text{Var}(D)=\text{Var}(B-A)=1^2\,\text{Var}(B)+(-1)^2\,\text{Var}(A)=\text{Var}(B)+\text{Var}(A)$ ***M1***

since B and A are independent ***R1***

$=5\left(\dfrac{4}{10}\right)\left(\dfrac{6}{10}\right)\left(\dfrac{10-5}{10-1}\right)+5\left(\dfrac{4}{10}\right)\left(\dfrac{6}{10}\right)$ ***(A1)(A1)***

$=\dfrac{28}{15}\quad (=1.87)$ ***A1***

[5 marks]

Total [16 marks]

2. (a) (i) $2\mu, 2\sigma^2$ ***A1A1***

(ii) $3\mu, 9\sigma^2$ ***A1A1***

(iii) $\mu, 3\sigma^2$ ***A1A1***

(iv) $\mu, \dfrac{\sigma^2}{n}$ ***A1A1***

> **Note**: If candidate clearly and correctly gives the standard deviations rather than the variances, give ***A1*** for 2 or 3 standard deviations and ***A1A1*** for 4 standard deviations.

[8 marks]

(b) $\mathrm{Var}(X_1) = \mathrm{E}(X_1^2) - (\mathrm{E}(X_1))^2$ ***(M1)***

$\sigma^2 = \mathrm{E}(X_1^2) - \mu^2$ ***(A1)***

$\mathrm{E}(X_1^2) = \sigma^2 + \mu^2$ ***A1***

[3 marks]

Total [11 marks]

3. (a) (i) $\mathrm{H}_0: \mu = 3, \mathrm{H}_1: \mu < 3$

1 tailed *z* test as σ^2 is known

under H_0, $X \sim \mathrm{N}\left(3, \frac{1}{4}\right)$ so $\overline{X} \sim \mathrm{N}\left(3, \frac{\frac{1}{4}}{36}\right) = \mathrm{N}\left(3, \frac{1}{144}\right)$ ***(M1)***

$z = \dfrac{\overline{x} - 3}{\frac{1}{12}}$ is $\mathrm{N}(0, 1)$ ***(A1)***

$\mathrm{P}(z < -1.64485\ldots) = 0.05$ ***(A1)***

so inequality is given by $\dfrac{\overline{x} - 3}{\frac{1}{12}} < -1.64485\ldots$ giving $\overline{x} < 2.8629\ldots$ ***M1***

$\overline{x} < 2.863$ (4sf) ***A1***

> **Note:** Candidates can get directly to the answer from $\mathrm{N}\left(3, \frac{1}{144}\right)$ they do not have to go via *z* is $\mathrm{N}(0, 1)$. However they must give some explanation of what they have done; they cannot just write the answer down.

(ii) a Type I error is accepting H_1 when H_0 is true ***A1***

(iii) a Type II error is accepting H_0 when H_1 is true ***A1***

continued …

Question 3 continued

(iv) 0.05 ***A1***

Note: Accept anything that rounds to 0.050 if they do the conditional calculation.

(v) $\bar{X} \sim \mathrm{N}\left(2.75, \frac{1}{144}\right)$ ***(M1)***

$\mathrm{P}(\bar{x} > 2.8629\ldots) = 0.0877$ (3sf) ***(M1)A1***

Note: Accept any answer between 0.0875 and 0.0877 inclusive.

Note: Accept anything that rounded is between 0.087and 0.089 if there is evidence that the candidate has used tables.

[11 marks]

(b) (i) *t*-test ***A1***

(ii) $\mathrm{H}_0: \mu = 3,\ \mathrm{H}_1: \mu < 3$

1 tailed *t* test as σ^2 is unknown

$t = \dfrac{\bar{y} - 3}{\frac{1}{12}}$ has the *t*-distribution with $\nu = 35$ ***(M1)***

the *p*-value is $0.0509\ldots$ ***A2***

this is > 0.05 ***R1***

so we accept that the mean wave height is 3 ***R1***

Note: Allow "Accept H_0" provided H_0 has been stated.

Note: Accept ***FT*** on the *p*-value for the ***R1***s.

(iii) $2.719 < \mu < 3.001$ (4 sf) ***A1A1***

Note: $2.860 \pm 1.6896\ldots \times \dfrac{\frac{1}{2}}{6}$ would gain ***M1***.

Note: Award ***A1A0*** if answer are only given to 3sf.

[8 marks]

Total [19 marks]

4. (a) $X \sim \text{Geo}\left(\frac{1}{6}\right)$ or $\text{NB}\left(1, \frac{1}{6}\right)$ ***A1***

[1 mark]

(b) $\text{E}(X) = 6$ ***A1***

[1 mark]

(c) H_0 : the probability that the dice lands with a "six" uppermost is $\frac{1}{6}$

H_1 : the probability is not $\frac{1}{6}$

under H_0, the expected values are given by the following table

Value of *X*	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Frequency	36	30	25	20.833…	17.361…	14.468…	12.056…

Value of *X*	**8**	**9**	**10**	**≥11**
Frequency	10.047…	8.372…	6.977…	34.885…

A3

Note: Award ***A2*** for one error, ***A1*** for two errors and ***A0*** for more than two errors.

Note: Accept answers that agree with the above to 1dp.

$\nu = 11 - 1 = 10$ ***A1***

(applying a χ^2 goodness of fit test)

EITHER

$p = 0.935\ldots$ ***A3***

Note: Accept answers within a tolerance of ± 0.004.

$0.935\ldots > 0.10$ so we accept H_0 ***R1***

OR

$\chi^2_{calc} = 4.248\ldots$ ***A2***

Note: Accept answers within a tolerance of ± 0.02.

$\chi^2_{crit} = 15.987\ldots$ ***A1***

$4.248 < 15.987$ so we accept H_0 ***R1***

Note: Incorrect combination of cells (grouping 10 or more) leading to $p = 0.926\ldots$ or $\chi^2_{calc} = 3.77\ldots$ and $\chi^2_{crit} = 14.686\ldots$ or incorrect combination of cells (grouping 9 and 10) leading to $p = 0.900\ldots$ or $\chi^2_{calc} = 4.17\ldots$ and $\chi^2_{crit} = 14.686\ldots$ both with $\nu = 9$ would gain ***A1*** (rather than the full ***A3***) ***A1*** and then either ***A3R1*** or ***A2A1R1*** respectively. Use the same tolerances as in the main markscheme.

Note: Allow follow through on their *p*-value or χ^2_{calc} value.

[8 marks]

continued …

Question 4 continued

(d) Y is $\mathrm{NB}\left(2, \frac{1}{6}\right)$ ***A1***

[1 mark]

(e) $\mathrm{P}(Y = y) = \frac{1}{36}$ gives $y = 2$ ***A1***

(as all other probabilities would have a factor of 5 in the numerator)

[1 mark]

(f) $\mathrm{P}(Y \le 6) = \left(\frac{1}{6}\right)^2 + 2\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2 + 3\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^2 + 5\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^2$ ***(M1)***

$= 0.263$ ***A1***

[2 marks]

Total [14 marks]

M12/5/MATHL/HP3/ENG/TZ0/SP/M

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MARKSCHEME

May 2012

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

12 pages

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- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) unbiased estimate of the mean: 795 (grams) *A1*
unbiased estimate of the variance: $108(\text{grams}^2)$ *(M1)A1*

[3 marks]

(b) null hypothesis $H_0: \mu = 800$ *A1*
alternative hypothesis $H_1: \mu < 800$ *A1*
using 1-tailed *t*-test *(M1)*

EITHER

$p = 0.0812\ldots$ *A3*

OR

with 9 degrees of freedom *(A1)*

$t_{calc} = \frac{\sqrt{10}(795-800)}{\sqrt{108}} = -1.521$ *A1*

$t_{crit} = -1.383$ *A1*

Note: Accept 2sf intermediate results.

THEN

so the baker's claim is rejected *R1*

Note: Accept "reject H_0" provided H_0 has been correctly stated.

Note: ***FT*** for the final ***R1***.

[7 marks]

(c) proportion rejected from sample $p_s = \frac{5}{40} = 0.125$ *(A1)*
using formula for confidence interval at 95 % level: *(M1)*

$p_s \pm 1.96\sqrt{\frac{p_s(1-p_s)}{n}}$

$= 0.125 \pm 0.102 = [0.0225, 0.227]$ *A1A1*

[4 marks]

Total [14 marks]

2. (a) $\mathrm{P}(X \le n) = \sum_{i=1}^{n} \mathrm{P}(X = i) = \sum_{i=1}^{n} pq^{i-1}$ ***M1A1***

$= p\frac{1-q^n}{1-q}$ ***A1***

$= 1-(1-p)^n$ ***AG***

[3 marks]

(b) $(1-p)^m - (1-p)^n$ ***A1***

[1 mark]

(c) attempt to solve $0.8-(0.8)^n > 0.5$ ***M1***

obtain $n = 6$ ***A1***

[2 marks]

Total [6 marks]

3. (a) mean $= 2.06$ ***A1***
variance $= 1.94$ ***A1***

[2 marks]

(b) a Poisson distribution has the property that its mean and variance are the same ***R1***

[1 mark]

(c) H_0 : the data can be modelled by a Poisson distribution
H_1 : the data cannot be modelled by a Poisson distribution ***A1***

Note: If a parameter is stated, award ***A0***.

METHOD 1

use the estimated mean to find expected values

number of injuries	0	1	2	3	4	5	6 or more
observed number of weeks	6	14	15	9	(5) 8	(2)	(1)
expected number of weeks	6.64	13.67	14.06	9.65	(4.96) 7.98	(2.04)	(0.98)

full table ***A3***

Note: Award ***A2*** if 5 or 6 correct expected values, ***A1*** if 4 correct values, ***A0*** otherwise.

Note: Allow ***FT*** on an $n = 6$ value in the final column.

the last three columns should be combined ***M1***
$\chi^2_{calc} = 0.176$ ***(M1)A1***
degrees of freedom $= 7 - 1 - 1 - 2 = 3$ ***A1***

EITHER

$\chi^2_{5\%}(3) = 7.81 > 0.176$ ***A1***

OR

p-value $= 0.981 > 0.05$ ***A1***

THEN

conclude that the data can be modelled by a Poisson distribution ***R1***

continued …

Question 3 continued

METHOD 2

use mean $= 2$ to find expected values

number of injuries	0	1	2	3	4	5	6 or more
observed number of weeks	6	14	15	9	(5) 8	(2)	(1)
expected number of weeks	7.037	14.07	14.07	9.38	(4.69) 7.43	(1.88)	(0.86)

full table ***A3***

Note: Award ***A2*** if 5 or 6 correct expected values, ***A1*** if 4 correct values, ***A0*** otherwise.

.

Note: Allow ***FT*** on an $n = 6$ value in the final column.

the last three columns should be combined ***M1***

$\chi^2_{\text{calc}} = 0.272$ ***(M1)A1***

degrees of freedom $= 7 - 1 - 1 - 2 = 3$ ***A1***

EITHER

$\chi^2_{5\%}(3) = 7.81 > 0..272$ ***A1***

OR

p-value $= 0.965 > 0.05$ ***A1***

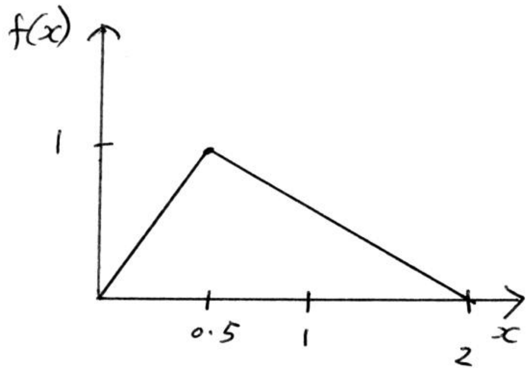
THEN

conclude that the data can be modelled by a Poisson distribution ***R1***

[10 marks]

Total [13 marks]

4. (a) piecewise linear graph



correct shape **A1**
with vertices $(0, 0)$, $(0.5, 1)$ and $(2, 0)$ **A1**
LQ: $x = 0.5$, because the area of the triangle is 0.25 **R1**

[3 marks]

(b) (i) $\mathrm{E}(X) = \int_0^{0.5} x \times 2x\,\mathrm{d}x + \int_{0.5}^{2} x \times \left(\frac{4}{3} - \frac{2}{3}x\right)\mathrm{d}x = \frac{5}{6}$ (=0.833...) ***(M1)A1***

(ii) $\mathrm{E}(X^2) = \int_0^{0.5} x^2 \times 2x\,\mathrm{d}x + \int_{0.5}^{2} x^2 \times \left(\frac{4}{3} - \frac{2}{3}x\right)\mathrm{d}x = \frac{7}{8}$ (=0.875) ***(M1)A1***

[4 marks]

(c) (i) $\mathrm{E}(Y - 2X) = 2\mathrm{E}(X) - 2\mathrm{E}(X) = 0$ **A1**

(ii) $\mathrm{Var}(X) = \left(\mathrm{E}(X^2) - \mathrm{E}(X)^2\right) = \frac{13}{72}$ **A1**

$Y \sim X_1 + X_2 \Rightarrow \mathrm{Var}(Y) = 2\mathrm{Var}(X)$ ***(M1)***

$\mathrm{Var}(Y - 2X) = 2\mathrm{Var}(X) + 4\mathrm{Var}(X) = \frac{13}{12}$ ***M1A1***

[5 marks]

continued …

Question 4 continued

(d) (i) attempt to use $cf(x) = \int f(u)\,\mathrm{d}u$ ***M1***

obtain $cf(x) = \begin{cases} x^2, & 0 \le x \le 0.5, \\ \frac{4x}{3} - \frac{1}{3}x^2 - \frac{1}{3}, & 0.5 \le x \le 2, \end{cases}$ ***A1*** ***A2***

(ii) attempt to solve $cf(x) = 0.5$ ***M1***

$\frac{4x}{3} - \frac{1}{3}x^2 - \frac{1}{3} = 0.5$ ***(A1)***

obtain 0.775 ***A1***

Note:	Accept attempts in the form of an integral with upper limit the unknown median.

Note:	Accept exact answer $2 - \sqrt{1.5}$.

[7 marks]

Total [19 marks]

5. (a) $\frac{m^{k-1}e^{-m}}{(k-1)!} = \frac{m^{k+1}e^{-m}}{(k+1)!}$ ***M1***

$\Rightarrow 1 = \frac{m^2}{(k+1)k}$ ***A1***

Note: Award ***A1*** for any correct intermediate step.

$\Rightarrow m^2 = (k+1)k$ ***AG***

[2 marks]

(b) $\frac{\mathrm{P}(X=k)}{\mathrm{P}(X=k-1)} = \frac{e^{-m} \times \frac{m^k}{k!}}{e^{-m} \times \frac{m^{k-1}}{(k-1)!}}$ ***M1***

$= \frac{m}{k}$ ***A1***

$= \frac{\sqrt{k(k+1)}}{k}$ ***M1***

$= \sqrt{\frac{k+1}{k}} > 1$ ***R1***

so $\mathrm{P}(X=k) > \mathrm{P}(X=k-1)$ ***R1***

similarly $\mathrm{P}(X=k) > \mathrm{P}(X=k+1)$ ***R1***

hence k is the mode ***AG***

[6 marks]

Total [8 marks]

N11/5/MATHL/HP3/ENG/TZ0/SP/M

International Baccalaureate®
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MARKSCHEME

November 2011

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

12 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

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Unless the question specifies otherwise, ***accept*** *equivalent forms.*

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Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty ***(AP)*** *no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award ***A0****. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

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12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) let S be the weight of tea in a random *Supermug* tea bag

$S \sim \mathrm{N}(4.2,\ 0.15^2)$

$\mathrm{P}(S > 3.9) = 0.977$ ***(M1)A1***

[2 marks]

(b) let M be the weight of tea in a random *Megamug* tea bag

$M \sim \mathrm{N}(5.6,\ 0.17^2)$

$\mathrm{P}(M > 5.4) = 0.880\ldots$ ***(A1)***

$\mathrm{P}(M < 5.4) = 1 - 0.880\ldots = 0.119\ldots$ ***(A1)***

required probability $= 2 \times 0.880\ldots \times 0.119\ldots = 0.211$ ***M1A1***

[4 marks]

(c) $\mathrm{P}(S_1 + S_2 + S_3 + S_4 + S_5 < 20.5)$

let $S_1 + S_2 + S_3 + S_4 + S_5 = A$ ***(M1)***

$\mathrm{E}(A) = 5\mathrm{E}(S)$

$= 21$ ***A1***

$\mathrm{Var}(A) = 5\mathrm{Var}(S)$

$= 0.1125$ ***A1***

$A \sim \mathrm{N}(21,\ 0.1125)$

$\mathrm{P}(A < 20.5) = 0.0680$ ***A1***

[4 marks]

(d) $\mathrm{P}\big(S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 - (M_1 + M_2 + M_3 + M_4 + M_5) > 0\big)$

let $S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 - (M_1 + M_2 + M_3 + M_4 + M_5) = B$ ***(M1)***

$\mathrm{E}(B) = 7\mathrm{E}(S) - 5\mathrm{E}(M)$

$= 1.4$ ***A1***

Note: Above ***A1*** is independent of first ***M1***.

$\mathrm{Var}(B) = 7\mathrm{Var}(S) + 5\mathrm{Var}(M)$ ***(M1)***

$= 0.302$ ***A1***

$\mathrm{P}(B > 0) = 0.995$ ***A1***

[5 marks]

Total [15 marks]

2. (a) $H_0 : p = 0.75$, $H_1 : p > 0.75$ ***A1***

one-tailed test

$\bar{X} \sim N\left(0.75, \frac{0.75 \times 0.25}{200}\right)$ ***(M1)***

$\bar{X} \sim N(0.75, 9.375 \times 10^{-4})$

EITHER

p-value $= 0.0512$ ***A1***

accept null hypothesis because p-value > 0.05 ***R1***

OR

$z = \frac{0.8 - 0.75}{\sqrt{9.375 \times 10^{-4}}} = 1.63$ ***A1***

accept null hypothesis because $z < 1.64$ ***R1***

Note:	Accept alternative solutions using binomial distribution, giving p-value of 0.0578.

Note:	Allow follow through on final reasoning mark.

[4 marks]

(b) confidence interval $= 0.8 \pm 1.96\sqrt{\frac{\frac{160}{200} \times \frac{40}{200}}{200}}$ ***(M1)***

$= (0.745, 0.855)$ ***A1A1***

[3 marks]

Total [7 marks]

3. (a) exponential distribution with mean $\frac{1}{\lambda}$ ***A1***

[1 mark]

(b) $\int \lambda e^{-\lambda t} dt = -e^{-\lambda t}$ (+c) ***A1***

$\Rightarrow F(x) = \left[-e^{-\lambda t}\right]_0^x$ ***(M1)***

$= 1 - e^{-\lambda x}$ $(x \geq 0)$ ***A1***

[3 marks]

(c) $1 - F\left(\frac{2}{\lambda}\right)$ ***M1***

$= e^{-2}$ $(= 0.135)$ ***A1***

[2 marks]

(d) $F(m) = \frac{1}{2}$ ***(M1)***

$\Rightarrow e^{-\lambda m} = \frac{1}{2}$ ***A1***

$\Rightarrow -\lambda m = \ln\frac{1}{2}$

$\Rightarrow m = \frac{1}{\lambda}\ln 2$ ***A1***

[3 marks]

(e) $F\left(\frac{1}{\lambda}\right) - F\left(\frac{\ln 2}{\lambda}\right)$ ***M1***

$= \frac{1}{2} - e^{-1}$ $(= 0.132)$ ***A1***

[2 marks]

Total [11 marks]

4. (a) $\mathrm{H}_0\colon X \sim \mathrm{B}\left(5, \frac{1}{2}\right)$, $\mathrm{H}_1\colon X$ does not follow $\mathrm{B}\left(5, \frac{1}{2}\right)$ ***A1***

[1 mark]

(b) $\mathrm{P}(X=0)=0.03125$
$\mathrm{P}(X=1)=0.15625$
$\mathrm{P}(X=2)=0.3125$
$\mathrm{P}(X=3)=0.3125$
$\mathrm{P}(X=4)=0.15625$
$\mathrm{P}(X=5)=0.03125$ ***(A3)***

> **Note:** Award ***A2*** for one error or premature rounding, ***A1*** for two errors, and ***A0*** otherwise.

X	O	E
0	2	3.125
1	15	15.625
2	s	31.25
3	$69-s$	31.25
4	12	15.625
5	2	3.125

(M1)
(A1)

> **Note:** Award method mark for any attempt to multiply the probability by 100.

combine classes: ***M1***

X	O	E
0 or 1	17	18.75
2	s	31.25
3	$69-s$	31.25
4 or 5	14	18.75

$\chi^2_{calc} = 0.16\dot{3} + 31.25 - 2s + 0.032s^2 + 45.602 - 2.416s + 0.032s^2 + 1.20\dot{3}$ ***M1***

$= \frac{8}{125}s^2 - \frac{552}{125}s + \frac{29332}{375} = 0.064s^2 - 4.42s + 78.2$ ***A1***

> **Note:** Award ***M1A0*** if candidates do not combine classes, obtaining $\chi^2_{calc} = 0.064s^2 - 4.42s + 78.5$.

[8 marks]

continued …

Question 4 continued

(c) $\nu = n - 1 = 4 - 1 = 3$ ***(A1)***
critical value $= 6.25$ ***A1***
solving: $0.064s^2 - 4.42s + 78.2 < 6.25$ ***M1A1***

Note: Accept use of $=$ in above line.

$\Rightarrow 26.3 < s < 42.8$
$\Rightarrow 27 \le s \le 42$ ***A1A1A1***

Note: Award ***A1*** for each correct endpoint and ***A1*** for correct inequalities. Only penalize one mark if end points are not integers but otherwise correct.

Note: If candidates do not combine classes in part (b) award full ***FT*** marks for the solution below:

$\nu = n - 1 = 6 - 1 = 5$
critical value $= 9.24$
solving: $0.064s^2 - 4.42s + 78.5 < 9.24$

Note: Accept use of $=$ in above line.

$\Rightarrow 24.0 < s < 45.0$

$\Rightarrow 25 \le s \le 45$ (accept 24 and 44)

Note: Award ***A1*** for each correct endpoint and ***A1*** for correct inequalities. Only penalize one mark if endpoints are not integers but otherwise correct.

[7 marks]

Total [16 marks]

5. (a) (i) $X = 2U \Rightarrow X \le \frac{3}{2}$

$X = 4U \Rightarrow X > 3$

X is only defined when $X \le \frac{3}{2}, X > 3$ ***M1A1***

hence X cannot take values such that $\frac{3}{2} < X \le 3$ ***AG***

(ii) **EITHER**

pdf is given by $f(u) = 1$ ***(M1)***

$\text{P}\left(0 < X \le \frac{3}{2}\right) = \int_0^{\frac{3}{4}} 1\,\text{d}u$ ***(A1)***

$= [u]_0^{\frac{3}{4}} = \frac{3}{4}$ ***A1***

OR

pdf is given by $f(x) = \frac{1}{2}$ ***(M1)***

$\text{P}\left(0 < X \le \frac{3}{2}\right) = \int_0^{\frac{3}{2}} \frac{1}{2}\text{d}x$ ***(A1)***

$= \left[\frac{x}{2}\right]_0^{\frac{3}{2}} = \frac{3}{4}$ ***A1***

(iii) $\text{P}(3 < X \le 4) = 1 - \frac{3}{4} = \frac{1}{4}$ ***A1***

[6 marks]

(b) **EITHER**

$\int_0^{\frac{Q_1}{2}} 1\,\text{d}u = \frac{1}{4}$ ***(M1)(A1)***

$\Rightarrow [u]_0^{\frac{Q_1}{2}} = \frac{1}{4}$

$\Rightarrow \frac{Q_1}{2} = \frac{1}{4}$

$\Rightarrow Q_1 = \frac{1}{2}$ ***A1***

OR

$\int_0^{Q_1} \frac{1}{2}\text{d}x = \frac{1}{4}$ ***(M1)(A1)***

$\Rightarrow \left[\frac{x}{2}\right]_0^{Q_1} = \frac{1}{4}$

$\Rightarrow \frac{Q_1}{2} = \frac{1}{4}$

$\Rightarrow Q_1 = \frac{1}{2}$ ***A1***

[3 marks]

continued …

Question 5 continued

(c) **EITHER**

$$\mathrm{E}(X)=\int_0^{\frac{3}{4}} 2u\,\mathrm{d}u+\int_{\frac{3}{4}}^{1} 4u\,\mathrm{d}u$$ ***M1***

$$=\left[u^2\right]_0^{\frac{3}{4}}+\left[2u^2\right]_{\frac{3}{4}}^{1}$$

$$=\frac{9}{16}+2-\frac{9}{8}=\frac{23}{16}\ (=1.44)$$ ***A1***

OR

$$\mathrm{E}(X)=\int_0^{\frac{3}{2}}\frac{x}{2}\mathrm{d}x+\int_3^4\frac{x}{4}\mathrm{d}x$$ ***M1***

$$=\left[\frac{x^2}{4}\right]_0^{\frac{3}{2}}+\left[\frac{x^2}{8}\right]_3^4$$

$$=\frac{9}{16}+\frac{16}{8}-\frac{9}{8}=\frac{23}{16}\ (=1.44)$$ ***A1***

[2 marks]

Total [11 marks]