A-level

Topic :Integral Calculus

May 2013-May 2023

Questions

Question 1

(a) Show that $\int_2^4 4x \ln x \, dx = 56 \ln 2 - 12$. [5]

(b) Use the substitution $u = \sin 4x$ to find the exact value of $\int_0^{\frac{1}{24}\pi} \cos^3 4x \, dx$. [5]

Question 2

(i) By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3}) \tan \theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3 + x^2)}} \, dx,$$

expressing your answer as a single logarithm. [4]

Question 3

(i) Express $(\sqrt{3}) \cos x + \sin x$ in the form $R \cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{1}{((\sqrt{3}) \cos x + \sin x)^2} \, dx = \frac{1}{4}\sqrt{3}.$$ [4]

Question 4

Find the exact value of $\int_1^4 \frac{\ln x}{\sqrt{x}} \, dx$. [5]

Question 5

(i) Prove that $\cot \theta + \tan \theta \equiv 2 \operatorname{cosec} 2\theta$. [3]

(ii) Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta \, d\theta = \frac{1}{2} \ln 3$. [4]

Question 6

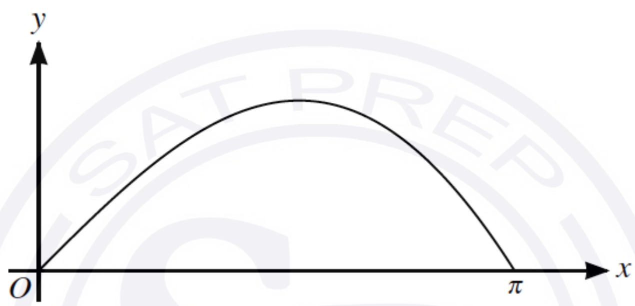
Use the substitution $u = 3x + 1$ to find $\int \frac{3x}{3x+1}\,dx$. [4]

Question 7

Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1+3\tan x)}}{\cos^2 x}\,dx.$$ [5]

Question 8



The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.

(i) Find $\frac{dy}{dx}$ and show that $4\frac{d^2y}{dx^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

(ii) Find the exact value of the area of the region enclosed by this part of the curve and the x-axis. [5]

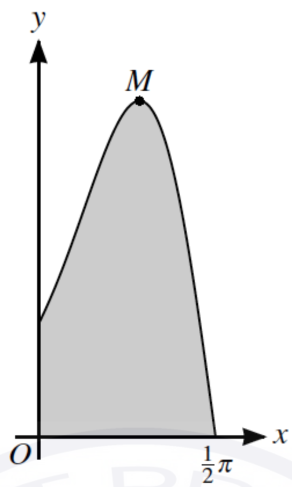
Question 9

Let $f(x) = \frac{6+6x}{(2-x)(2+x^2)}$.

(i) Express $f(x)$ in the form $\frac{A}{2-x} + \frac{Bx+C}{2+x^2}$. [4]

(ii) Show that $\int_{-1}^{1} f(x)\,dx = 3\ln 3$. [5]

Question 10



The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \sin x$, find the exact value of the area of the shaded region bounded by the curve and the axes. [5]

(ii) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

Question 11

(i) Use the trapezium rule with 3 intervals to estimate the value of

$$\int_{\frac{1}{6}\pi}^{\frac{2}{3}\pi} \operatorname{cosec} x \, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Using a sketch of the graph of $y = \operatorname{cosec} x$, explain whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part **(i)**. [2]

Question 12

It is given that $I = \int_0^{0.3} (1 + 3x^2)^{-2} \, dx$.

(i) Use the trapezium rule with 3 intervals to find an approximation to I, giving the answer correct to 3 decimal places. [3]

(ii) For small values of x, $(1 + 3x^2)^{-2} \approx 1 + ax^2 + bx^4$. Find the values of the constants a and b. Hence, by evaluating $\int_0^{0.3} (1 + ax^2 + bx^4) \, dx$, find a second approximation to I, giving the answer correct to 3 decimal places. [5]

Question 13

By first using the substitution $u = e^x$, show that

$$\int_0^{\ln 4} \frac{e^{2x}}{e^{2x} + 3e^x + 2} \, dx = \ln\left(\tfrac{8}{5}\right). \qquad [10]$$

Question 14

Use the trapezium rule with three intervals to find an approximation to

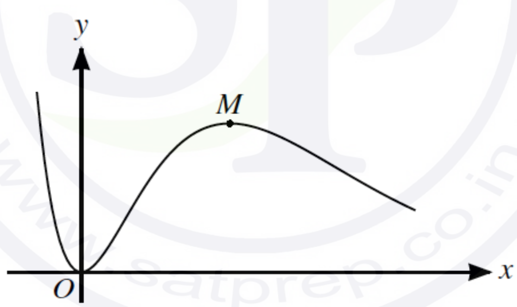
$$\int_0^3 |3^x - 10| \, dx. \qquad [4]$$

Question 15

(a) Find $\int (4 + \tan^2 2x) \, dx$. [3]

(b) Find the exact value of $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin(x + \frac{1}{6}\pi)}{\sin x} \, dx$. [5]

Question 16



The diagram shows the curve $y = x^2 e^{2-x}$ and its maximum point M.

(i) Show that the x-coordinate of M is 2. [3]

(ii) Find the exact value of $\displaystyle\int_0^2 x^2 e^{2-x} \, dx$. [6]

Question 17

Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{2}\pi} \ln(1 + \sin x) \, dx,$$

giving your answer correct to 2 decimal places. [3]

Question 18

Let $I = \int_0^1 \frac{\sqrt{x}}{2 - \sqrt{x}}\,dx$.

(i) Using the substitution $u = 2 - \sqrt{x}$, show that $I = \int_1^2 \frac{2(2-u)^2}{u}\,du$. [4]

(ii) Hence show that $I = 8\ln 2 - 5$. [4]

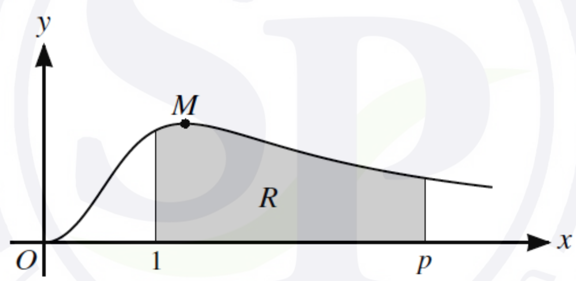
Question 19

Let $f(x) = \frac{11x + 7}{(2x-1)(x+2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\int_1^2 f(x)\,dx = \frac{1}{4} + \ln\left(\frac{9}{4}\right)$. [5]

Question 20



The diagram shows the curve $y = \frac{x^2}{1 + x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

Question 21

Use the substitution $u = 4 - 3\cos x$ to find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4 - 3\cos x)}}\,dx$. [8]

Question 22

(i) Show that $(x+1)$ is a factor of $4x^3 - x^2 - 11x - 6$. [2]

(ii) Find $\displaystyle\int \frac{4x^2 + 9x - 1}{4x^3 - x^2 - 11x - 6}\,dx$. [8]

Question 23

Let $I = \displaystyle\int_0^1 \frac{9}{(3+x^2)^2}\,dx$.

(i) Using the substitution $x = (\sqrt{3})\tan\theta$, show that $I = \sqrt{3}\displaystyle\int_0^{\frac{1}{6}\pi} \cos^2\theta\,d\theta$. [3]

(ii) Hence find the exact value of I. [4]

Question 24

Let $f(x) = \dfrac{3x^3 + 6x - 8}{x(x^2+2)}$.

(i) Express $f(x)$ in the form $A + \dfrac{B}{x} + \dfrac{Cx + D}{x^2 + 2}$. [5]

(ii) Show that $\displaystyle\int_1^2 f(x)\,dx = 3 - \ln 4$. [5]

Question 25

Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} xe^{-2x}\,dx$. [5]

Question 26

Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} x^2 \sin 2x\,dx$. [5]

Question 27

Let $f(x) = \dfrac{4x^2 + 7x + 4}{(2x+1)(x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\displaystyle\int_0^4 f(x)\,dx = 8 - \ln 3$. [5]

Question 28

Let $I = \int_0^1 \frac{x^5}{(1+x^2)^3}\,dx$.

(i) Using the substitution $u = 1 + x^2$, show that $I = \int_1^2 \frac{(u-1)^2}{2u^3}\,du$. [3]

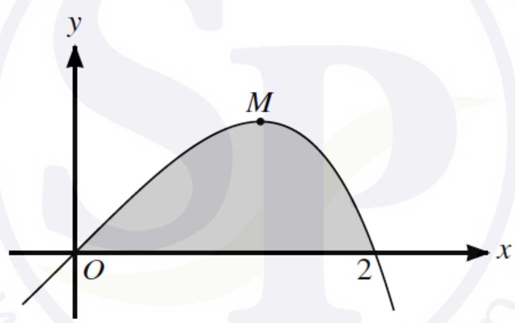
(ii) Hence find the exact value of I. [5]

Question 29

(i) Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. [4]

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$. [4]

Question 30



The diagram shows part of the curve $y = (2x - x^2)e^{\frac{1}{2}x}$ and its maximum point M.

(i) Find the exact x-coordinate of M. [4]

(ii) Find the exact value of the area of the shaded region bounded by the curve and the positive x-axis. [5]

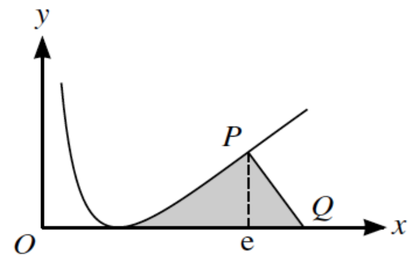
Question 31

Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})}\,dx$.

(i) Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u-1}{u+1}\,du$. [3]

(ii) Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

Question 32



The diagram shows the curve $y = (\ln x)^2$. The x-coordinate of the point P is equal to e, and the normal to the curve at P meets the x-axis at Q.

(i) Find the x-coordinate of Q. [4]

(ii) Show that $\int \ln x \, dx = x \ln x - x + c$, where c is a constant. [1]

(iii) Using integration by parts, or otherwise, find the exact value of the area of the shaded region between the curve, the x-axis and the normal PQ. [5]

Question 33

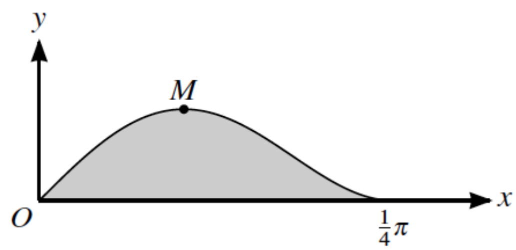
It is given that $x = \ln(1 - y) - \ln y$, where $0 < y < 1$.

(i) Show that $y = \dfrac{e^{-x}}{1 + e^{-x}}$. [2]

(ii) Hence show that $\int_0^1 y \, dx = \ln\left(\dfrac{2e}{e + 1}\right)$. [4]

(iii) Using integration by parts, or otherwise, find the exact value of the area of the shaded region between the curve, the x-axis and the normal PQ. [5]

Question 34



The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Find the x-coordinate of M. Give your answer correct to 2 decimal places. [6]

Question 35

(i) Prove that if $y = \dfrac{1}{\cos\theta}$ then $\dfrac{dy}{d\theta} = \sec\theta\tan\theta$. [2]

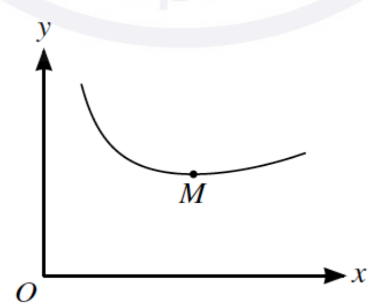
(ii) Prove the identity $\dfrac{1+\sin\theta}{1-\sin\theta} \equiv 2\sec^2\theta + 2\sec\theta\tan\theta - 1$. [3]

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{1+\sin\theta}{1-\sin\theta}\, d\theta$. [4]

Question 36

Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \theta \sin\tfrac{1}{2}\theta \, d\theta$. [4]

Question 37



The diagram shows a sketch of the curve $y = \dfrac{e^{\frac{1}{2}x}}{x}$ for $x > 0$,

Use the trapezium rule with two intervals to estimate the value of

$$\int_1^3 \frac{e^{\frac{1}{2}x}}{x}\,dx,$$

giving your answer correct to 2 decimal places. [3]

The estimate found in part **(ii)** is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with four intervals would be greater than E or less than E. [1]

Question 38

Let $f(x) = \dfrac{3x^2 - 4}{x^2(3x + 2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_1^2 f(x)\,dx = \ln\left(\frac{25}{8}\right) - 1$. [5]

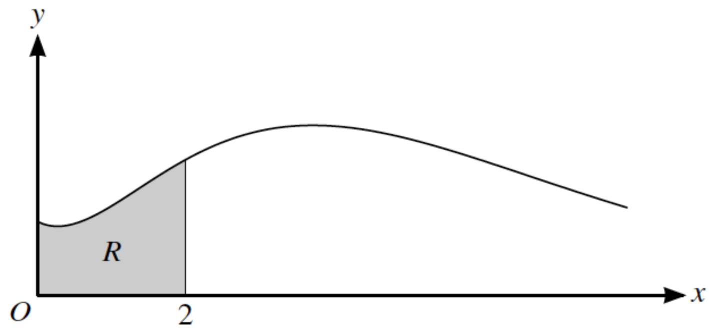
Question 39

Let $f(x) = \dfrac{4x^2 + 9x - 8}{(x + 2)(2x - 1)}$.

(i) Express $f(x)$ in the form $A + \dfrac{B}{x + 2} + \dfrac{C}{2x - 1}$. [4]

(ii) Hence show that $\int_1^4 f(x)\,dx = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$. [5]

Question 40

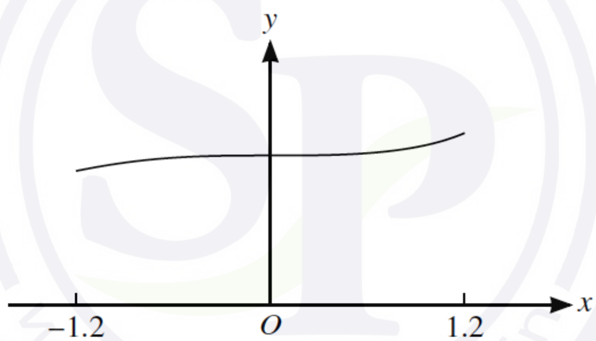


The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(i) Find the exact values of the x-coordinates of the stationary points of the curve. [4]

(ii) Show that the exact value of the area of R is $18 - \frac{42}{e}$. [5]

Question 41



The diagram shows a sketch of the curve $y = \frac{3}{\sqrt{(9 - x^3)}}$ for values of x from -1.2 to 1.2.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-1.2}^{1.2} \frac{3}{\sqrt{(9 - x^3)}} \, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

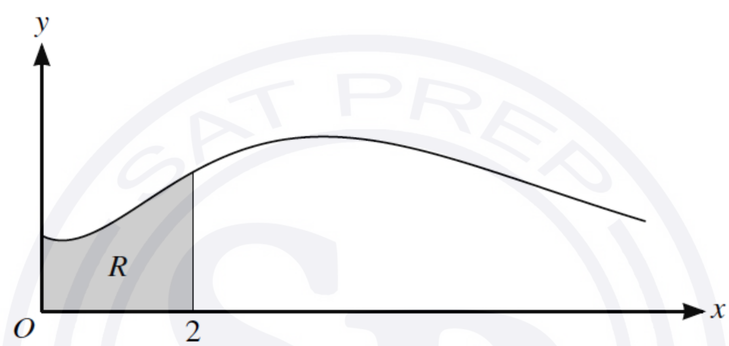
Question 42

Let $f(x) = \dfrac{4x^2 + 9x - 8}{(x+2)(2x-1)}$.

(i) Express $f(x)$ in the form $A + \dfrac{B}{x+2} + \dfrac{C}{2x-1}$. [4]

(ii) Hence show that $\int_1^4 f(x)\,dx = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$. [5]

Question 43



The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(i) Find the exact values of the x-coordinates of the stationary points of the curve. [4]

(ii) Show that the exact value of the area of R is $18 - \dfrac{42}{e}$. [5]

Question 44

Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 - \tan x)}\,dx,$$

giving your answer correct to 3 decimal places. [3]

Question 45

(i) Using the expansions of $\cos(3x + x)$ and $\cos(3x - x)$, show that

$$\tfrac{1}{2}(\cos 4x + \cos 2x) \equiv \cos 3x \cos x. \qquad [3]$$

(ii) Hence show that $\displaystyle\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x \cos x \, dx = \tfrac{3}{8}\sqrt{3}$. [3]

Question 46

Let $f(x) = \dfrac{5x^2 + x + 27}{(2x + 1)(x^2 + 9)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence find $\displaystyle\int_0^4 f(x)\, dx$, giving your answer in the form $\ln c$, where c is an integer. [5]

Question 47

Let $I = \displaystyle\int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1 - x}\right)}\, dx$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta \, d\theta$. [4]

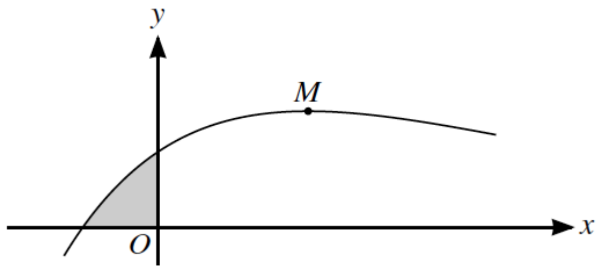
(ii) Hence find the exact value of I. [4]

Question 48

(i) Show that $\dfrac{2\sin x - \sin 2x}{1 - \cos 2x} \equiv \dfrac{\sin x}{1 + \cos x}$. [4]

(ii) Hence, showing all necessary working, find $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{2\sin x - \sin 2x}{1 - \cos 2x}\, dx$, giving your answer in the form $\ln k$. [4]

Question 49



The diagram shows the curve $y = (x + 1)e^{-\frac{1}{3}x}$ and its maximum point M.

(i) Find the x-coordinate of M. [4]

(ii) Find the area of the shaded region enclosed by the curve and the axes, giving your answer in terms of e. [5]

Question 50

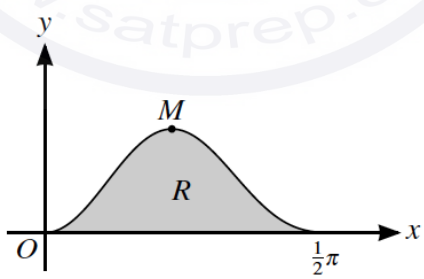
Showing all necessary working, find the value of $\int_0^{\frac{1}{6}\pi} x \cos 3x \, dx$, giving your answer in terms of π. [5]

Question 51

(i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and $\tan\alpha$. [3]

(ii) Hence, showing all necessary working, show that $\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2} \, d\theta = 5$. [5]

Question 52



The diagram shows the curve $y = 5\sin^2 x \cos^3 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M. The shaded region R is bounded by the curve and the x-axis.

(i) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [5]

(ii) Using the substitution $u = \sin x$ and showing all necessary working, find the exact area of R. [4]

Question 53

Let $f(x) = \dfrac{6x^2 + 8x + 9}{(2 - x)(3 + 2x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing all necessary working, show that $\int_{-1}^{0} f(x)\,dx = 1 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$. [5]

Question 54

(i) Find $\int \frac{\ln x}{x^3}\,dx$. [3]

(ii) Hence show that $\int_{1}^{2} \frac{\ln x}{x^3}\,dx = \frac{1}{16}(3 - \ln 4)$. [2]

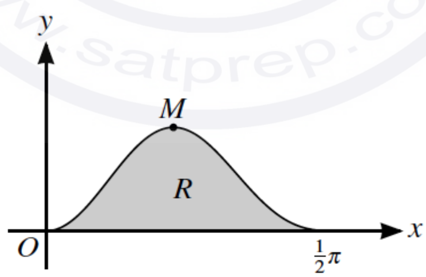
Question 55

A curve has equation $y = \dfrac{3\cos x}{2 + \sin x}$, for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Find the exact coordinates of the stationary point of the curve. [6]

(ii) The constant a is such that $\int_{0}^{a} \frac{3\cos x}{2 + \sin x}\,dx = 1$. Find the value of a, giving your answer correct to 3 significant figures. [4]

Question 56



The diagram shows the curve $y = 5\sin^2 x\cos^3 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M. The shaded region R is bounded by the curve and the x-axis.

(i) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [5]

(ii) Using the substitution $u = \sin x$ and showing all necessary working, find the exact area of R. [4]

Question 57

Let $f(x) = \dfrac{6x^2 + 8x + 9}{(2 - x)(3 + 2x)^2}$.

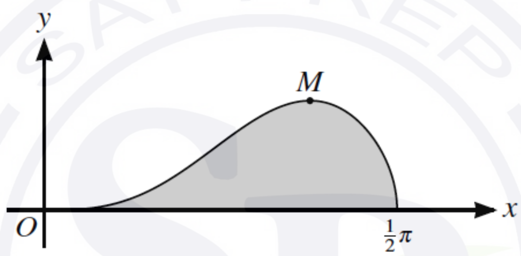
(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing all necessary working, show that $\int_{-1}^{0} f(x)\,dx = 1 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$. [5]

Question 58

Show that $\int_{1}^{4} x^{-\frac{3}{2}} \ln x \, dx = 2 - \ln 4$. [5]

Question 59



The diagram shows the curve $y = \sin^3 x \sqrt{(\cos x)}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Showing all your working, find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

Question 60

Use the trapezium rule with 3 intervals to estimate the value of

$$\int_{0}^{3} |2^x - 4| \, dx.$$ [3]

Question 61

(i) By first expanding $\sin(2x + x)$, show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$. [4]

(ii) Hence, showing all necessary working, find the exact value of $\int_{0}^{\frac{1}{3}\pi} \sin^3 x \, dx$. [4]

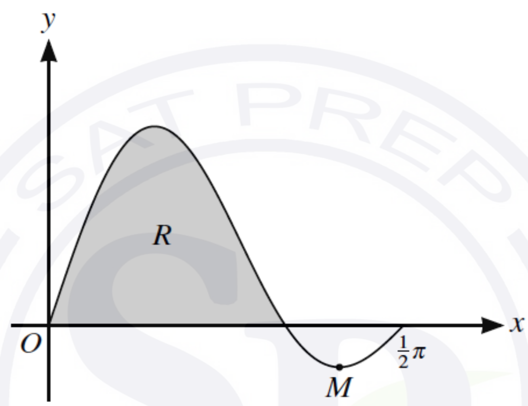
Question 62

Let $f(x) = \dfrac{10x + 9}{(2x + 1)(2x + 3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^1 f(x)\,dx = \frac{1}{2}\ln\frac{9}{5} + \frac{1}{5}$. [5]

Question 63



The diagram shows the curve $y = \sin 3x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and its minimum point M. The shaded region R is bounded by the curve and the x-axis.

(i) By expanding $\sin(3x + x)$ and $\sin(3x - x)$ show that

$$\sin 3x \cos x = \tfrac{1}{2}(\sin 4x + \sin 2x).$$ [3]

(ii) Using the result of part **(i)** and showing all necessary working, find the exact area of the region R. [4]

(iii) Using the result of part **(i)**, express $\dfrac{dy}{dx}$ in terms of $\cos 2x$ and hence find the x-coordinate of M, giving your answer correct to 2 decimal places. [5]

Question 64

Show that $\int_0^{\frac{1}{4}\pi} x^2 \cos 2x\,dx = \frac{1}{32}(\pi^2 - 8)$. [5]

Question 65

Let $f(\theta) = \dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta}$.

(i) Show that $f(\theta) = \tan\theta$. [3]

(ii) Hence show that $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} f(\theta)\,d\theta = \frac{1}{2}\ln\frac{3}{2}$. [4]

Question 66

(i) By differentiating $\dfrac{\cos x}{\sin x}$, show that if $y = \cot x$ then $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$. [2]

(ii) Show that $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} x\operatorname{cosec}^2 x\,dx = \frac{1}{4}(\pi + \ln 4)$. [6]

Question 67

(i) By first expanding $\cos(2x + x)$, show that $\cos 3x \equiv 4\cos^3 x - 3\cos x$. [4]

(ii) Hence solve the equation $\cos 3x + 3\cos x + 1 = 0$, for $0 \leqslant x \leqslant \pi$. [2]

(iii) Find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos^3 x\,dx$. [4]

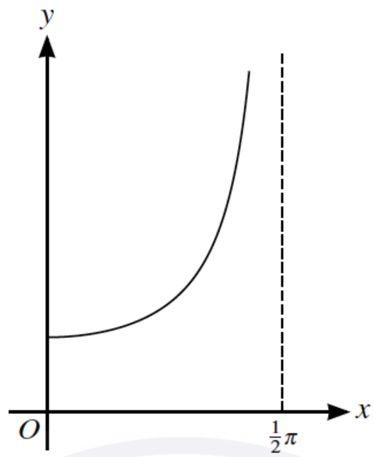
Question 68

Let $f(x) = \dfrac{2x^2 + x + 8}{(2x - 1)(x^2 + 2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing full working, find $\displaystyle\int_1^5 f(x)\,dx$, giving the answer in the form $\ln c$, where c is an integer. [5]

Question 69



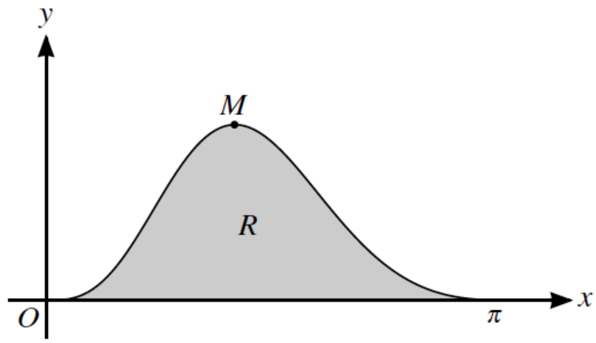
The diagram shows the graph of $y = \sec x$ for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Use the trapezium rule with 2 intervals to estimate the value of $\int_0^{1.2} \sec x \, dx$, giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part **(i)**. [1]

(iii) P is the point on the part of the curve $y = \sec x$ for $0 \leqslant x < \frac{1}{2}\pi$ at which the gradient is 2. By first differentiating $\frac{1}{\cos x}$, find the x-coordinate of P, giving your answer correct to 3 decimal places. [6]

Question 70



The diagram shows the graph of $y = e^{\cos x}\sin^3 x$ for $0 \leqslant x \leqslant \pi$, and its maximum point M. The shaded region R is bounded by the curve and the x-axis.

(i) Find the x-coordinate of M. Show all necessary working and give your answer correct to 2 decimal places. [5]

(ii) By first using the substitution $u = \cos x$, find the exact value of the area of R. [7]

Question 71

Find $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x\sec^2 x\,dx$. Give your answer in a simplified exact form. [7]

Question 72

(a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

(b) Using your answer to part **(a)**, find the exact value of $\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$. [5]

Question 73

Let $f(x) = \frac{\cos x}{1 + \sin x}$.

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$. [4]

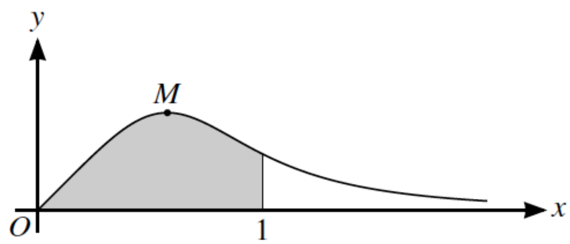
(b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

Question 74

Find the exact value of

$$\int_1^4 x^{\frac{3}{2}}\ln x\,dx.$$ [5]

Question 75



The diagram shows the curve $y = \dfrac{x}{1 + 3x^4}$, for $x \geqslant 0$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [4]

(b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [5]

Question 76

Find the exact value of $\displaystyle\int_0^1 (2 - x)e^{-2x}\,dx$. [5]

Question 77

Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

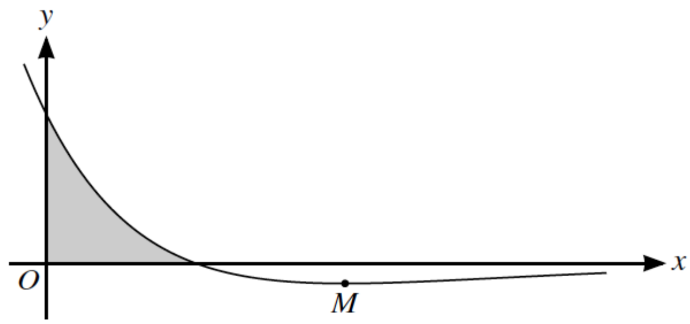
(a) Express $f(x)$ in partial fractions. [2]

(b) Using your answer to part **(a)**, show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

(c) Hence show that $\displaystyle\int_1^2 (f(x))^2\,dx = \tfrac{2}{5} + \tfrac{1}{2}\ln\left(\tfrac{5}{9}\right)$. [5]

Question 78



The diagram shows the curve $y = (2 - x)e^{-\frac{1}{2}x}$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

(b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e. [5]

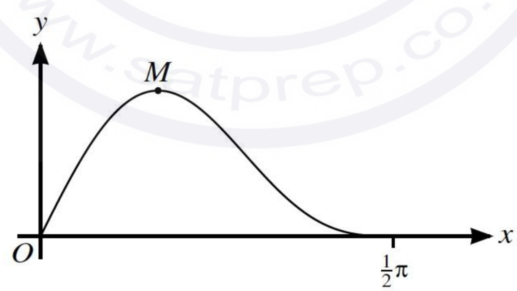
Question 79

Let $f(x) = \dfrac{x^2 + x + 6}{x^2(x + 2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

Question 80



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. [5]

(b) Find the exact x-coordinate of M. [6]

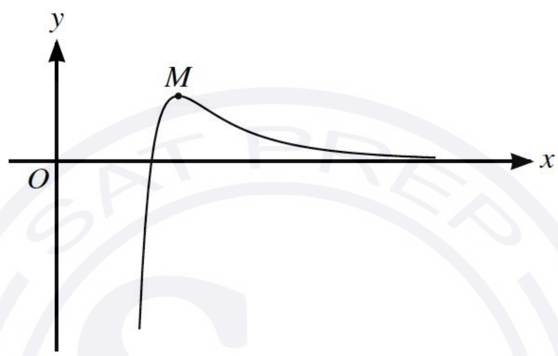
Question 81

Let $f(x) = \dfrac{5a}{(2x-a)(3a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence show that $\int_a^{2a} f(x)\,dx = \ln 6$. [4]

Question 82



The diagram shows the curve $y = \dfrac{\ln x}{x^4}$ and its maximum point M.

(a) Find the exact coordinates of M. [4]

(b) By using integration by parts, show that for all $a > 1$, $\int_1^a \dfrac{\ln x}{x^4}\,dx < \frac{1}{9}$. [6]

Question 83

Let $f(x) = \dfrac{15-6x}{(1+2x)(4-x)}$.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence find $\int_1^2 f(x)\,dx$, giving your answer in the form $\ln\left(\dfrac{a}{b}\right)$, where a and b are integers. [4]

Question 84

(a) Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$. [3]

(b) Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\,d\theta = \frac{1}{2}\ln 2$. [4]

Question 85

Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right) \mathrm{d}x$. [5]

Question 86

The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point. [5]

(b) Show that $\int_1^8 y \, \mathrm{d}x = 18 \ln 2 - 9$. [5]

Question 87

(a) Prove that $\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta} \equiv \tan^2 \theta$. [2]

(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \, \mathrm{d}\theta$. [4]

Question 88

Let $\mathrm{f}(x) = \dfrac{1}{(9 - x)\sqrt{x}}$.

(a) Find the x-coordinate of the stationary point of the curve with equation $y = \mathrm{f}(x)$. [4]

(b) Using the substitution $u = \sqrt{x}$, show that $\int_0^4 \mathrm{f}(x) \, \mathrm{d}x = \frac{1}{3} \ln 5$. [6]

Question 89

Find the exact value of $\displaystyle\int_{\frac{1}{3}\pi}^{\pi} x \sin \tfrac{1}{2}x \, \mathrm{d}x$. [5]

Question 90

(a) Using the expansions of $\sin(3x + 2x)$ and $\sin(3x - 2x)$, show that

$$\tfrac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x.$$ [3]

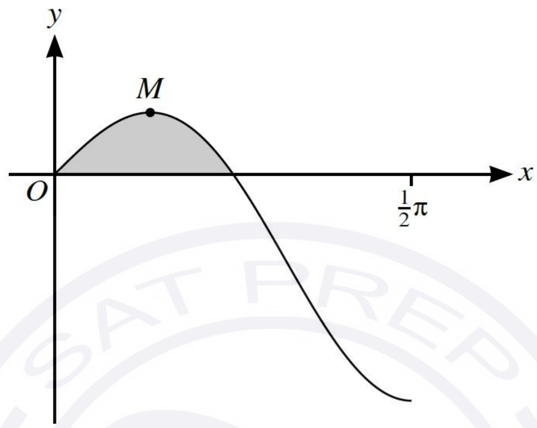
(b) Hence show that $\displaystyle\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x \, \mathrm{d}x = \tfrac{1}{5}(3 - \sqrt{2})$. [3]

Question 91

Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}}\,dx. \qquad [6]$$

Question 92



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 significant figures. [6]

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. [5]

Question 93

(a) Find the quotient and remainder when $8x^3 + 4x^2 + 2x + 7$ is divided by $4x^2 + 1$. [3]

(b) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{8x^3 + 4x^2 + 2x + 7}{4x^2 + 1}\,dx$. [5]

Question 94

Let $f(x) = \dfrac{x^2 + 9x}{(3x-1)(x^2+3)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find $\displaystyle\int_1^3 f(x)\,dx$, giving your answer in a simplified exact form. [5]

Question 95

Let $I = \int_0^3 \frac{27}{(9+x^2)^2}\,dx$.

(a) Using the substitution $x = 3\tan\theta$, show that $I = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta$. [4]

(b) Hence find the exact value of I. [4]

Question 96

Find the exact value of $\int_0^{\frac{1}{4}\pi} x\sec^2 x\,dx$. [5]

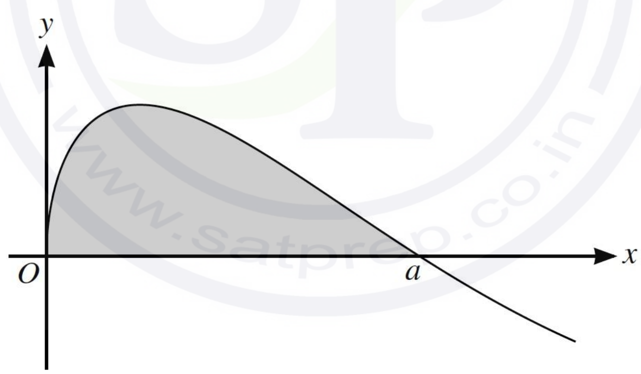
Question 97

Let $f(x) = \frac{5 - x + 6x^2}{(3-x)(1+3x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Find the exact value of $\int_0^1 f(x)\,dx$, simplifying your answer. [5]

Question 98



The diagram shows part of the curve $y = \sin\sqrt{x}$. This part of the curve intersects the x-axis at the point where $x = a$.

(a) State the exact value of a. [1]

(b) Using the substitution $u = \sqrt{x}$, find the exact area of the shaded region in the first quadrant bounded by this part of the curve and the x-axis. [7]

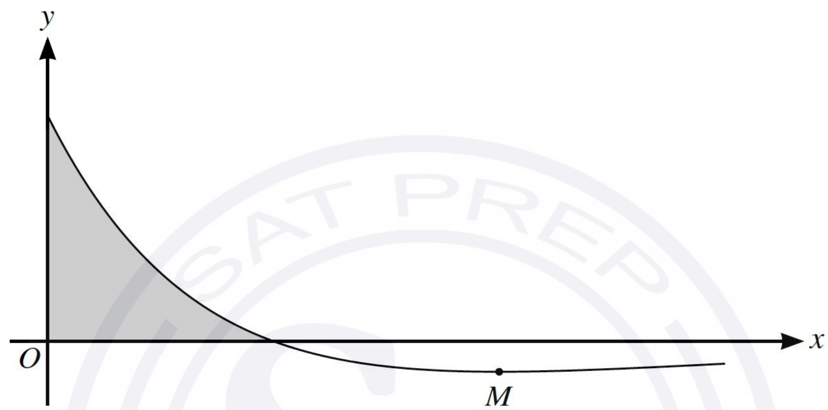
Question 99

Let $\mathrm{f}(x) = \dfrac{4 - x + x^2}{(1+x)(2+x^2)}$.

Express $\mathrm{f}(x)$ in partial fractions. [5]

(b) Find the exact value of $\int_0^4 \mathrm{f}(x)\,\mathrm{d}x$. Give your answer as a single logarithm. [5]

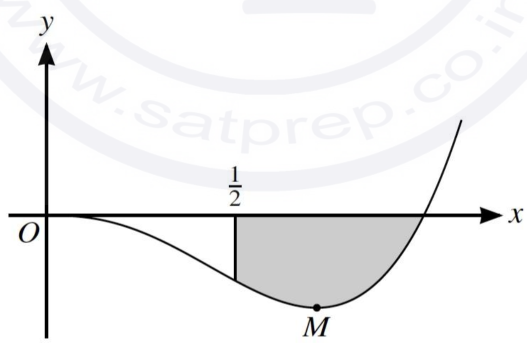
Question 100



The diagram shows part of the curve $y = (3 - x)\mathrm{e}^{-\frac{1}{3}x}$ for $x \geqslant 0$, and its minimum point M.

Find the area of the shaded region bounded by the curve and the axes, giving your answer in terms of e. [5]

Question 101



The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. [5]

Question 102

Let $f(x) = \dfrac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\displaystyle\int_0^2 f(x)\,dx = \ln 54 - \tfrac{1}{8}\pi$. [5]

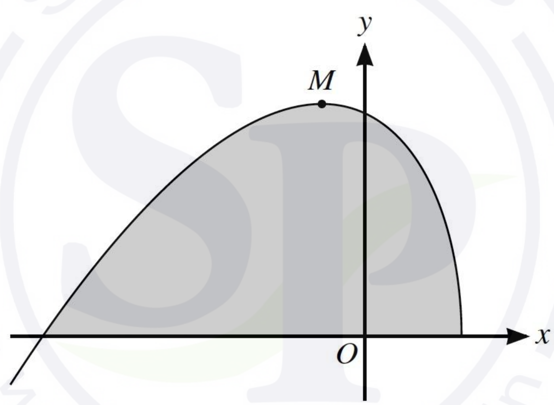
Question 103

(a) Use the substitution $u = \cos x$ to show that

$$\int_0^{\pi} \sin 2x\, e^{2\cos x}\,dx = \int_{-1}^{1} 2u e^{2u}\,du. \quad [4]$$

(b) Hence find the exact value of $\displaystyle\int_0^{\pi} \sin 2x\, e^{2\cos x}\,dx$. [4]

Question 104



The diagram shows the curve $y = (x + 5)\sqrt{3 - 2x}$ and its maximum point M.

Using the substitution $u = 3 - 2x$, find by integration the area of the shaded region bounded by the curve and the x-axis. Give your answer in the form $a\sqrt{13}$, where a is a rational number. [5]

Question 105

Let $f(x) = \dfrac{2x^2 + 17x - 17}{(1 + 2x)(2 - x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\displaystyle\int_0^1 f(x)\,dx = \tfrac{5}{2} - \ln 72$. [5]

Question 106

Let $f(x) = \dfrac{3 - 3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are integers. [5]