

**A-level**  
**Topic : Numerical Equation and Solution**  
**May 2013-May 2025**  
**Questions**

Question 1

$$V = \frac{1}{k}(80 - 80e^{-kt}).$$

It is observed that  $V = 500$  when  $t = 15$ , so that  $k$  satisfies the equation

$$k = \frac{4 - 4e^{-15k}}{25}.$$

Use an iterative formula, based on this equation, to find the value of  $k$  correct to 2 significant figures. Use an initial value of  $k = 0.1$  and show the result of each iteration to 4 significant figures. [3]

Question 2

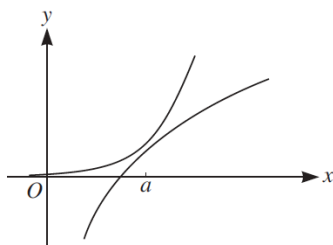
The sequence of values given by the iterative formula

$$x_{n+1} = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)},$$

with initial value  $x_1 = 3.5$ , converges to  $\alpha$ .

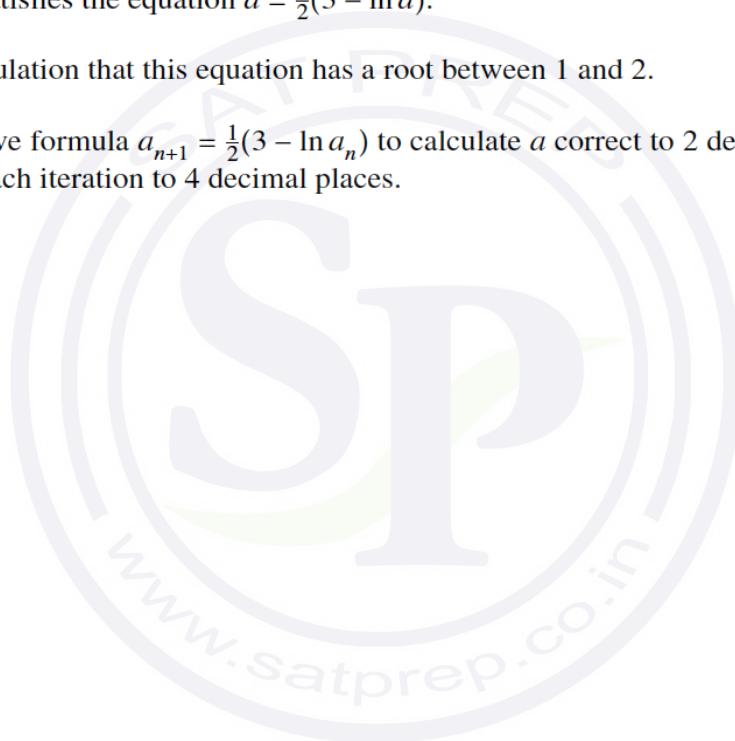
- (i) Use this formula to calculate  $\alpha$  correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]
- (ii) State an equation satisfied by  $\alpha$  and hence find the exact value of  $\alpha$ . [2]

Question 3

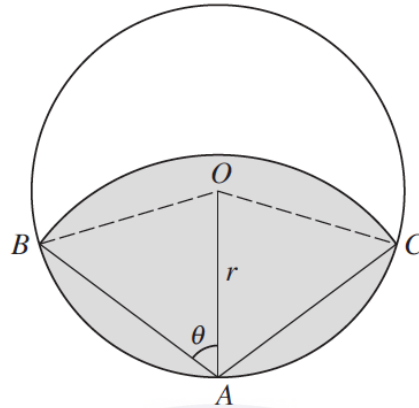


The diagram shows the curves  $y = e^{2x-3}$  and  $y = 2 \ln x$ . When  $x = a$  the tangents to the curves are parallel.

- (i) Show that  $a$  satisfies the equation  $a = \frac{1}{2}(3 - \ln a)$ . [3]
- (ii) Verify by calculation that this equation has a root between 1 and 2. [2]
- (iii) Use the iterative formula  $a_{n+1} = \frac{1}{2}(3 - \ln a_n)$  to calculate  $a$  correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]



Question 4



In the diagram,  $A$  is a point on the circumference of a circle with centre  $O$  and radius  $r$ . A circular arc with centre  $A$  meets the circumference at  $B$  and  $C$ . The angle  $OAB$  is  $\theta$  radians. The shaded region is bounded by the circumference of the circle and the arc with centre  $A$  joining  $B$  and  $C$ . The area of the shaded region is equal to half the area of the circle.

(i) Show that  $\cos 2\theta = \frac{2 \sin 2\theta - \pi}{4\theta}$ . [5]

(ii) Use the iterative formula

$$\theta_{n+1} = \frac{1}{2} \cos^{-1} \left( \frac{2 \sin 2\theta_n - \pi}{4\theta_n} \right),$$

with initial value  $\theta_1 = 1$ , to determine  $\theta$  correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

Question 5

It is given that  $\int_0^p 4xe^{-\frac{1}{2}x} dx = 9$ , where  $p$  is a positive constant.

(i) Show that  $p = 2 \ln \left( \frac{8p + 16}{7} \right)$ . [5]

(ii) Use an iterative process based on the equation in part (i) to find the value of  $p$  correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures. [3]

Question 6

- (i) By sketching each of the graphs  $y = \operatorname{cosec} x$  and  $y = x(\pi - x)$  for  $0 < x < \pi$ , show that the equation

$$\operatorname{cosec} x = x(\pi - x)$$

has exactly two real roots in the interval  $0 < x < \pi$ . [3]

- (ii) Show that the equation  $\operatorname{cosec} x = x(\pi - x)$  can be written in the form  $x = \frac{1 + x^2 \sin x}{\pi \sin x}$ . [2]

- (iii) The two real roots of the equation  $\operatorname{cosec} x = x(\pi - x)$  in the interval  $0 < x < \pi$  are denoted by  $\alpha$  and  $\beta$ , where  $\alpha < \beta$ .

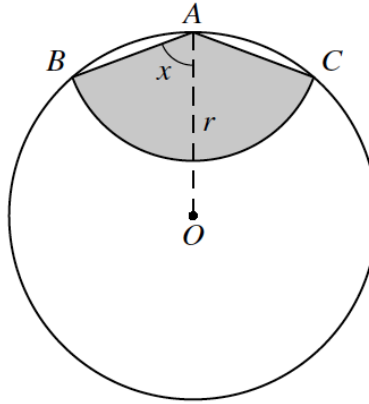
- (a) Use the iterative formula

$$x_{n+1} = \frac{1 + x_n^2 \sin x_n}{\pi \sin x_n}$$

to find  $\alpha$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

- (b) Deduce the value of  $\beta$  correct to 2 decimal places. [1]

Question 7



In the diagram,  $A$  is a point on the circumference of a circle with centre  $O$  and radius  $r$ . A circular arc with centre  $A$  meets the circumference at  $B$  and  $C$ . The angle  $OAB$  is equal to  $x$  radians. The shaded region is bounded by  $AB$ ,  $AC$  and the circular arc with centre  $A$  joining  $B$  and  $C$ . The perimeter of the shaded region is equal to half the circumference of the circle.

(i) Show that  $x = \cos^{-1}\left(\frac{\pi}{4 + 4x}\right)$ . [3]

(ii) Verify by calculation that  $x$  lies between 1 and 1.5. [2]

(iii) Use the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{\pi}{4 + 4x_n}\right)$$

to determine the value of  $x$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 8

The equation  $x = \frac{10}{e^{2x} - 1}$  has one positive real root, denoted by  $\alpha$ .

(i) Show that  $\alpha$  lies between  $x = 1$  and  $x = 2$ . [2]

(ii) Show that if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{2} \ln\left(1 + \frac{10}{x_n}\right)$$

converges, then it converges to  $\alpha$ . [2]

(iii) Use this iterative formula to determine  $\alpha$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 9

It is given that  $\int_1^a \ln(2x) dx = 1$ , where  $a > 1$ .

(i) Show that  $a = \frac{1}{2} \exp\left(1 + \frac{\ln 2}{a}\right)$ , where  $\exp(x)$  denotes  $e^x$ . [6]

(ii) Use the iterative formula

$$a_{n+1} = \frac{1}{2} \exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 10

(i) Sketch the curve  $y = \ln(x + 1)$  and hence, by sketching a second curve, show that the equation

$$x^3 + \ln(x + 1) = 40$$

has exactly one real root. State the equation of the second curve. [3]

(ii) Verify by calculation that the root lies between 3 and 4. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sqrt[3]{(40 - \ln(x_n + 1))},$$

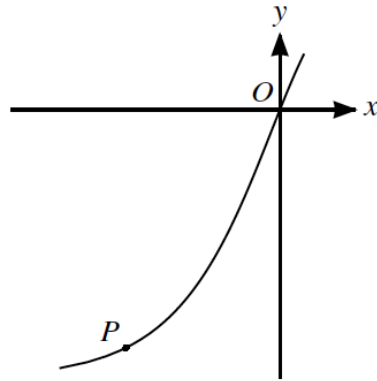
with a suitable starting value, to find the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Deduce the root of the equation

$$(e^y - 1)^3 + y = 40,$$

giving the answer correct to 2 decimal places. [2]

Question 11

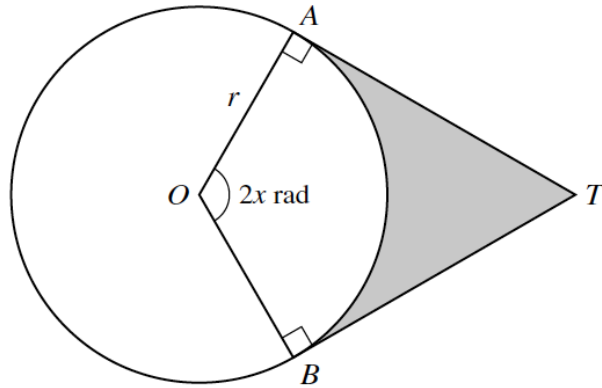


The diagram shows part of the curve with parametric equations

$$x = 2 \ln(t + 2), \quad y = t^3 + 2t + 3.$$

- (i) Find the gradient of the curve at the origin. [5]
- (ii) At the point  $P$  on the curve, the value of the parameter is  $p$ . It is given that the gradient of the curve at  $P$  is  $\frac{1}{2}$ .
- (a) Show that  $p = \frac{1}{3p^2 + 2} - 2$ . [1]
- (b) By first using an iterative formula based on the equation in part (a), determine the coordinates of the point  $P$ . Give the result of each iteration to 5 decimal places and each coordinate of  $P$  correct to 2 decimal places. [4]

Question 12



The diagram shows a circle with centre  $O$  and radius  $r$ . The tangents to the circle at the points  $A$  and  $B$  meet at  $T$ , and the angle  $AOB$  is  $2x$  radians. The shaded region is bounded by the tangents  $AT$  and  $BT$ , and by the minor arc  $AB$ . The perimeter of the shaded region is equal to the circumference of the circle.

- (i) Show that  $x$  satisfies the equation

$$\tan x = \pi - x. \quad [3]$$

- (ii) This equation has one root in the interval  $0 < x < \frac{1}{2}\pi$ . Verify by calculation that this root lies between 1 and 1.3. [2]

- (iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi - x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 13

It is given that  $\int_0^a x \cos x \, dx = 0.5$ , where  $0 < a < \frac{1}{2}\pi$ .

- (i) Show that  $a$  satisfies the equation  $\sin a = \frac{1.5 - \cos a}{a}$ . [4]

- (ii) Verify by calculation that  $a$  is greater than 1. [2]

- (iii) Use the iterative formula

$$a_{n+1} = \sin^{-1} \left( \frac{1.5 - \cos a_n}{a_n} \right)$$

to determine the value of  $a$  correct to 4 decimal places, giving the result of each iteration to 6 decimal places. [3]

Question 14

The equation  $x^3 - x^2 - 6 = 0$  has one real root, denoted by  $\alpha$ .

(i) Find by calculation the pair of consecutive integers between which  $\alpha$  lies. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to  $\alpha$ . [2]

(iii) Use this iterative formula to determine  $\alpha$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 15

A curve has parametric equations

$$x = t^2 + 3t + 1, \quad y = t^4 + 1.$$

The point  $P$  on the curve has parameter  $p$ . It is given that the gradient of the curve at  $P$  is 4.

(i) Show that  $p = \sqrt[3]{2p + 3}$ . [3]

(ii) Verify by calculation that the value of  $p$  lies between 1.8 and 2.0. [2]

(iii) Use an iterative formula based on the equation in part (i) to find the value of  $p$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 16

The equation  $x^5 - 3x^3 + x^2 - 4 = 0$  has one positive root.

(i) Verify by calculation that this root lies between 1 and 2. [2]

(ii) Show that the equation can be rearranged in the form

$$x = \sqrt[3]{\left(3x + \frac{4}{x^2} - 1\right)}. \quad [1]$$

(iii) Use an iterative formula based on this rearrangement to determine the positive root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 17

- (i) By sketching a suitable pair of graphs, show that the equation

$$5e^{-x} = \sqrt{x}$$

has one root.

[2]

- (ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2} \ln\left(\frac{25}{x_n}\right)$$

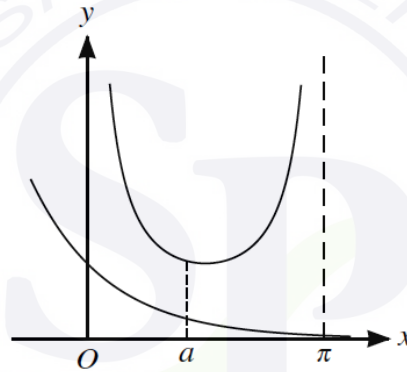
converges, then it converges to the root of the equation in part (i).

[2]

- (iii) Use this iterative formula, with initial value  $x_1 = 1$ , to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[3]

Question 18



The diagram shows the curve  $y = \operatorname{cosec} x$  for  $0 < x < \pi$  and part of the curve  $y = e^{-x}$ . When  $x = a$ , the tangents to the curves are parallel.

- (i) By differentiating  $\frac{1}{\sin x}$ , show that if  $y = \operatorname{cosec} x$  then  $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$ .

[3]

- (ii) By equating the gradients of the curves at  $x = a$ , show that

$$a = \tan^{-1}\left(\frac{e^a}{\sin a}\right).$$

[2]

- (iii) Verify by calculation that  $a$  lies between 1 and 1.5.

[2]

- (iv) Use an iterative formula based on the equation in part (ii) to determine  $a$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

[3]

Question 19

The curve with equation  $y = x^2 \cos \frac{1}{2}x$  has a stationary point at  $x = p$  in the interval  $0 < x < \pi$ .

- (i) Show that  $p$  satisfies the equation  $\tan \frac{1}{2}p = \frac{4}{p}$ . [3]
- (ii) Verify by calculation that  $p$  lies between 2 and 2.5. [2]
- (iii) Use the iterative formula  $p_{n+1} = 2 \tan^{-1}\left(\frac{4}{p_n}\right)$  to determine the value of  $p$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 20

- (i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} \frac{1}{2}x = \frac{1}{3}x + 1$$

has one root in the interval  $0 < x \leq \pi$ . [2]

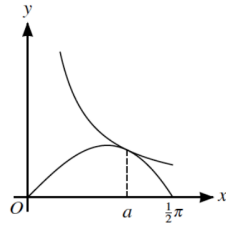
- (ii) Show by calculation that this root lies between 1.4 and 1.6. [2]
- (iii) Show that, if a sequence of values in the interval  $0 < x \leq \pi$  given by the iterative formula

$$x_{n+1} = 2 \sin^{-1}\left(\frac{3}{x_n + 3}\right)$$

converges, then it converges to the root of the equation in part (i). [2]

- (iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 21



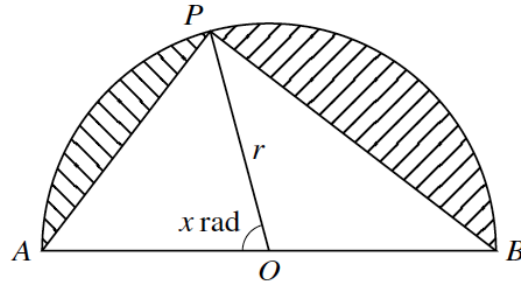
The diagram shows the curves  $y = x \cos x$  and  $y = \frac{k}{x}$ , where  $k$  is a constant, for  $0 < x \leq \frac{1}{2}\pi$ . The curves touch at the point where  $x = a$ .

- (i) Show that  $a$  satisfies the equation  $\tan a = \frac{2}{a}$ . [5]
- (ii) Use the iterative formula  $a_{n+1} = \tan^{-1}\left(\frac{2}{a_n}\right)$  to determine  $a$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]
- (iii) Hence find the value of  $k$  correct to 2 decimal places. [2]

Question 22

- (i) By sketching suitable graphs, show that the equation  $e^{-\frac{1}{2}x} = 4 - x^2$  has one positive root and one negative root. [2]
- (ii) Verify by calculation that the negative root lies between  $-1$  and  $-1.5$ . [2]
- (iii) Use the iterative formula  $x_{n+1} = -\sqrt{4 - e^{-\frac{1}{2}x_n}}$  to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

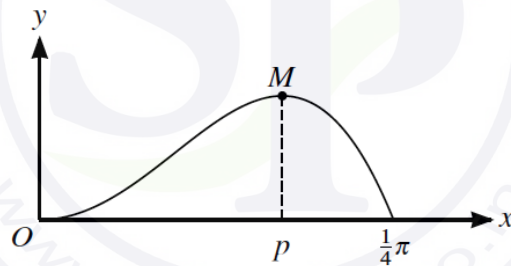
Question 23



The diagram shows a semicircle with centre  $O$ , radius  $r$  and diameter  $AB$ . The point  $P$  on its circumference is such that the area of the minor segment on  $AP$  is equal to half the area of the minor segment on  $BP$ . The angle  $AOP$  is  $x$  radians.

- (i) Show that  $x$  satisfies the equation  $x = \frac{1}{3}(\pi + \sin x)$ . [3]
- (ii) Verify by calculation that  $x$  lies between 1 and 1.5. [2]
- (iii) Use an iterative formula based on the equation in part (i) to determine  $x$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 24



The diagram shows the curve  $y = x^2 \cos 2x$  for  $0 \leq x \leq \frac{1}{4}\pi$ . The curve has a maximum point at  $M$  where  $x = p$ .

- (i) Show that  $p$  satisfies the equation  $p = \frac{1}{2} \tan^{-1} \left( \frac{1}{p} \right)$ . [3]
- (ii) Use the iterative formula  $p_{n+1} = \frac{1}{2} \tan^{-1} \left( \frac{1}{p_n} \right)$  to determine the value of  $p$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]
- (iii) Find, showing all necessary working, the exact area of the region bounded by the curve and the  $x$ -axis. [5]

### Question 25

The equation  $\cot x = 1 - x$  has one root in the interval  $0 < x < \pi$ , denoted by  $\alpha$ .

- (i) Show by calculation that  $\alpha$  is greater than 2.5. [2]
- (ii) Show that, if a sequence of values in the interval  $0 < x < \pi$  given by the iterative formula  $x_{n+1} = \pi + \tan^{-1}\left(\frac{1}{1-x_n}\right)$  converges, then it converges to  $\alpha$ . [2]
- (iii) Use this iterative formula to determine  $\alpha$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

### Question 26

The equation  $x^3 = 3x + 7$  has one real root, denoted by  $\alpha$ .

- (i) Show by calculation that  $\alpha$  lies between 2 and 3. [2]

Two iterative formulae,  $A$  and  $B$ , derived from this equation are as follows:

$$x_{n+1} = (3x_n + 7)^{\frac{1}{3}}, \quad (A)$$

$$x_{n+1} = \frac{x_n^3 - 7}{3}. \quad (B)$$

Each formula is used with initial value  $x_1 = 2.5$ .

- (ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate  $\alpha$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [4]

### Question 27

It is given that  $\int_1^a x^{\frac{1}{2}} \ln x \, dx = 2$ , where  $a > 1$ .

- (i) Show that  $a^{\frac{3}{2}} = \frac{7 + 2a^{\frac{3}{2}}}{3 \ln a}$ . [5]
- (ii) Show by calculation that  $a$  lies between 2 and 4. [2]
- (iii) Use the iterative formula

$$a_{n+1} = \left( \frac{7 + 2a_n^{\frac{3}{2}}}{3 \ln a_n} \right)^{\frac{2}{3}}$$

to determine  $a$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 28

(i) By sketching suitable graphs, show that the equation  $e^{2x} = 6 + e^{-x}$  has exactly one real root. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1. [2]

(iii) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{3} \ln(1 + 6e^{x_n})$$

converges, then it converges to the root of the equation in part (i). [2]

Question 29

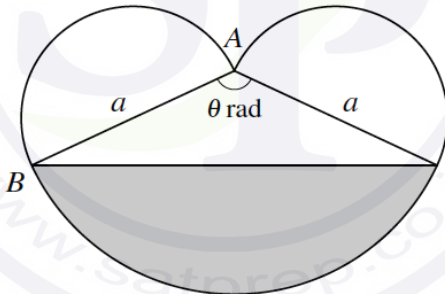
The positive constant  $a$  is such that  $\int_0^a x e^{-\frac{1}{2}x} dx = 2$ .

(i) Show that  $a$  satisfies the equation  $a = 2 \ln(a + 2)$ . [5]

(ii) Verify by calculation that  $a$  lies between 3 and 3.5. [2]

(iii) Use an iteration based on the equation in part (i) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 30



The diagram shows a triangle  $ABC$  in which  $AB = AC = a$  and angle  $BAC = \theta$  radians. Semicircles are drawn outside the triangle with  $AB$  and  $AC$  as diameters. A circular arc with centre  $A$  joins  $B$  and  $C$ . The area of the shaded segment is equal to the sum of the areas of the semicircles.

(i) Show that  $\theta = \frac{1}{2}\pi + \sin \theta$ . [3]

(ii) Verify by calculation that  $\theta$  lies between 2.2 and 2.4. [2]

(iii) Use an iterative formula based on the equation in part (i) to determine  $\theta$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 31

The curve with equation  $y = \frac{\ln x}{3+x}$  has a stationary point at  $x = p$ .

- (i) Show that  $p$  satisfies the equation  $\ln x = 1 + \frac{3}{x}$ . [3]
- (ii) By sketching suitable graphs, show that the equation in part (i) has only one root. [2]
- (iii) It is given that the equation in part (i) can be written in the form  $x = \frac{3+x}{\ln x}$ . Use an iterative formula based on this rearrangement to determine the value of  $p$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 32

- (i) By sketching a suitable pair of graphs, show that the equation  $x^3 = 3 - x$  has exactly one real root. [2]
- (ii) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

- converges, then it converges to the root of the equation in part (i). [2]
- (iii) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 33

The equation of a curve is  $y = x \ln(8 - x)$ . The gradient of the curve is equal to 1 at only one point, when  $x = a$ .

- (i) Show that  $a$  satisfies the equation  $x = 8 - \frac{8}{\ln(8 - x)}$ . [3]
- (ii) Verify by calculation that  $a$  lies between 2.9 and 3.1. [2]
- (iii) Use an iterative formula based on the equation in part (i) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 34

(i) By sketching a suitable pair of graphs, show that the equation  $x^3 = 3 - x$  has exactly one real root. [2]

(ii) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation in part (i). [2]

(iii) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 35

The sequence of values given by the iterative formula

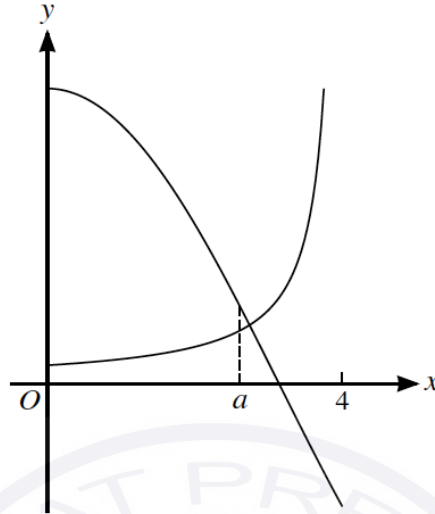
$$x_{n+1} = \frac{2x_n^6 + 12x_n}{3x_n^5 + 8},$$

with initial value  $x_1 = 2$ , converges to  $\alpha$ .

(i) Use the formula to calculate  $\alpha$  correct to 4 decimal places. Give the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by  $\alpha$  and hence find the exact value of  $\alpha$ . [2]

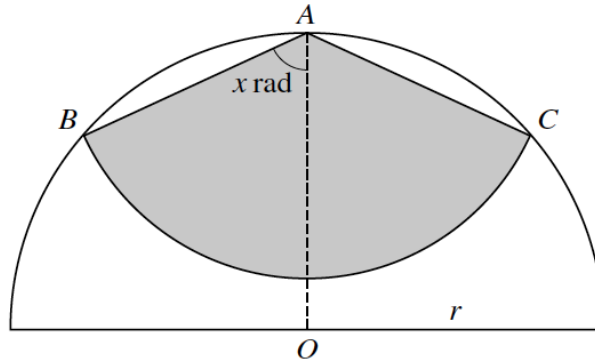
Question 36



The diagram shows the curves  $y = 4 \cos \frac{1}{2}x$  and  $y = \frac{1}{4-x}$ , for  $0 \leq x < 4$ . When  $x = a$ , the tangents to the curves are perpendicular.

- (i) Show that  $a = 4 - \sqrt{(2 \sin \frac{1}{2}a)}$ . [4]
- (ii) Verify by calculation that  $a$  lies between 2 and 3. [2]
- (iii) Use an iterative formula based on the equation in part (i) to determine  $a$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 37



In the diagram,  $A$  is the mid-point of the semicircle with centre  $O$  and radius  $r$ . A circular arc with centre  $A$  meets the semicircle at  $B$  and  $C$ . The angle  $OAB$  is equal to  $x$  radians. The area of the shaded region bounded by  $AB$ ,  $AC$  and the arc with centre  $A$  is equal to half the area of the semicircle.

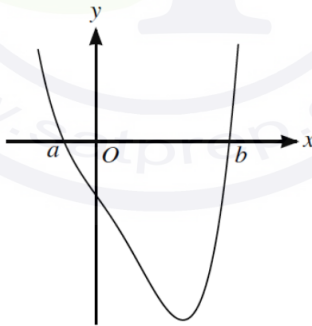
(i) Use triangle  $OAB$  to show that  $AB = 2r \cos x$ . [1]

(ii) Hence show that  $x = \cos^{-1} \sqrt{\left(\frac{\pi}{16x}\right)}$ . [2]

(iii) Verify by calculation that  $x$  lies between 1 and 1.5. [2]

(iv) Use an iterative formula based on the equation in part (ii) to determine  $x$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 38



The diagram shows the curve  $y = x^4 - 2x^3 - 7x - 6$ . The curve intersects the  $x$ -axis at the points  $(a, 0)$  and  $(b, 0)$ , where  $a < b$ . It is given that  $b$  is an integer.

(i) Find the value of  $b$ . [1]

(ii) Hence show that  $a$  satisfies the equation  $a = -\frac{1}{3}(2 + a^2 + a^3)$ . [4]

(iii) Use an iterative formula based on the equation in part (ii) to determine  $a$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 39

The curve with equation  $y = e^{-2x} \ln(x - 1)$  has a stationary point when  $x = p$ .

- (i) Show that  $p$  satisfies the equation  $x = 1 + \exp\left(\frac{1}{2(x-1)}\right)$ , where  $\exp(x)$  denotes  $e^x$ . [3]
- (ii) Verify by calculation that  $p$  lies between 2.2 and 2.6. [2]
- (iii) Use an iterative formula based on the equation in part (i) to determine  $p$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 40

It is given that  $\int_0^a x \cos \frac{1}{3}x \, dx = 3$ , where the constant  $a$  is such that  $0 < a < \frac{3}{2}\pi$ .

- (i) Show that  $a$  satisfies the equation
- $$a = \frac{4 - 3 \cos \frac{1}{3}a}{\sin \frac{1}{3}a}. \quad [5]$$
- (ii) Verify by calculation that  $a$  lies between 2.5 and 3. [2]
- (iii) Use an iterative formula based on the equation in part (i) to calculate  $a$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

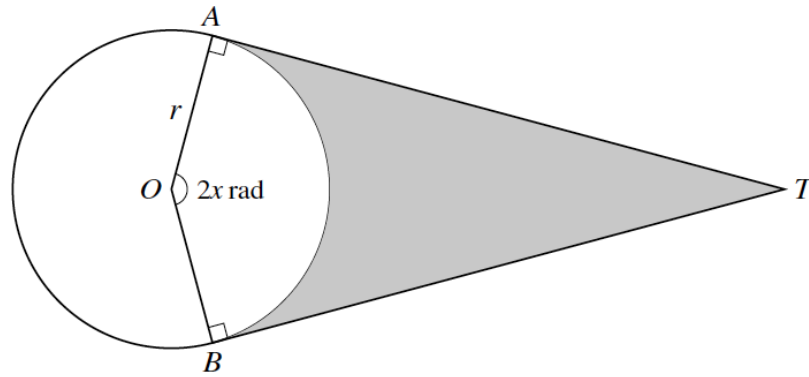
Question 41

- (i) By sketching a suitable pair of graphs, show that the equation  $\ln(x + 2) = 4e^{-x}$  has exactly one real root. [2]
- (ii) Show by calculation that this root lies between  $x = 1$  and  $x = 1.5$ . [2]
- (iii) Use the iterative formula  $x_{n+1} = \ln\left(\frac{4}{\ln(x_n + 2)}\right)$  to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 42

- (a) By sketching a suitable pair of graphs, show that the equation  $\sec x = 2 - \frac{1}{2}x$  has exactly one root in the interval  $0 \leq x < \frac{1}{2}\pi$ . [2]
- (b) Verify by calculation that this root lies between 0.8 and 1. [2]
- (c) Use the iterative formula  $x_{n+1} = \cos^{-1}\left(\frac{2}{4 - x_n}\right)$  to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 43



The diagram shows a circle with centre  $O$  and radius  $r$ . The tangents to the circle at the points  $A$  and  $B$  meet at  $T$ , and angle  $AOB$  is  $2x$  radians. The shaded region is bounded by the tangents  $AT$  and  $BT$ , and by the minor arc  $AB$ . The area of the shaded region is equal to the area of the circle.

(a) Show that  $x$  satisfies the equation  $\tan x = \pi + x$ . [3]

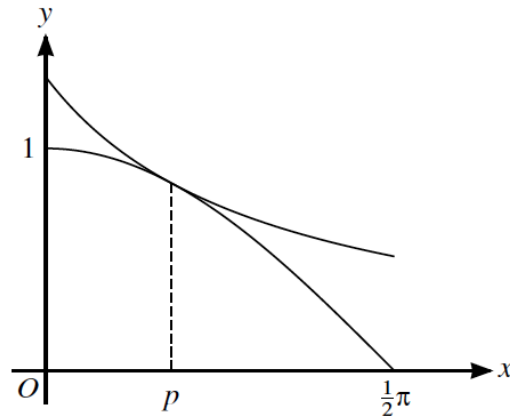
(b) This equation has one root in the interval  $0 < x < \frac{1}{2}\pi$ . Verify by calculation that this root lies between 1 and 1.4. [2]

(c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 44



The diagram shows the curves  $y = \cos x$  and  $y = \frac{k}{1+x}$ , where  $k$  is a constant, for  $0 \leq x \leq \frac{1}{2}\pi$ . The curves touch at the point where  $x = p$ .

(a) Show that  $p$  satisfies the equation  $\tan p = \frac{1}{1+p}$ . [5]

(b) Use the iterative formula  $p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$  to determine the value of  $p$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(c) Hence find the value of  $k$  correct to 2 decimal places. [2]

Question 45

(a) By sketching a suitable pair of graphs, show that the equation  $x^5 = 2 + x$  has exactly one real root. [2]

(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part (a). [2]

(c) Use the iterative formula with initial value  $x_1 = 1.5$  to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 46

- (a) By sketching a suitable pair of graphs, show that the equation  $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$  has exactly two roots in the interval  $0 < x < \pi$ . [2]

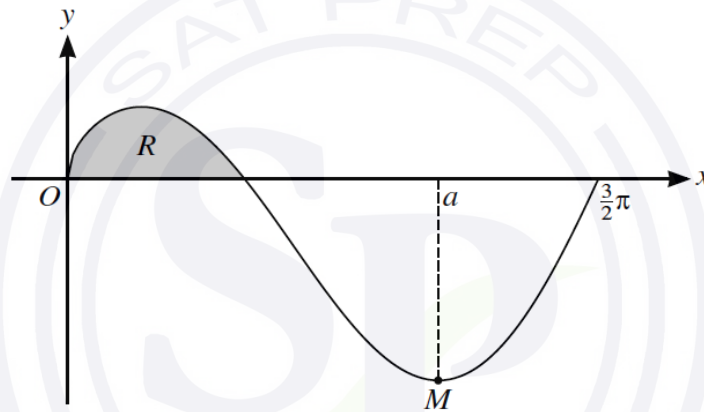
- (b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1} \left( \frac{1}{e^{-\frac{1}{2}x_n} + 1} \right),$$

with initial value  $x_1 = 2$ , converges to one of these roots.

Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 47



The diagram shows the curve  $y = \sqrt{x} \cos x$ , for  $0 \leq x \leq \frac{3}{2}\pi$ , and its minimum point  $M$ , where  $x = a$ . The shaded region between the curve and the  $x$ -axis is denoted by  $R$ .

- (a) Show that  $a$  satisfies the equation  $\tan a = \frac{1}{2a}$ . [3]

- (b) The sequence of values given by the iterative formula  $a_{n+1} = \pi + \tan^{-1} \left( \frac{1}{2a_n} \right)$ , with initial value  $x_1 = 3$ , converges to  $a$ .

Use this formula to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

- (c) Find the volume of the solid obtained when the region  $R$  is rotated completely about the  $x$ -axis. Give your answer in terms of  $\pi$ . [6]

Question 48

- (a) By sketching a suitable pair of graphs, show that the equation  $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$  has exactly two roots in the interval  $0 < x < \pi$ . [2]

- (b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right),$$

with initial value  $x_1 = 2$ , converges to one of these roots.

Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 49

Let  $f(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$ , for  $x > 0$ .

- (a) The equation  $x = f(x)$  has one root, denoted by  $a$ .

Verify by calculation that  $a$  lies between 1 and 1.5. [2]

- (b) Use an iterative formula based on the equation in part (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

- (c) Find  $f'(x)$ . Hence find the exact value of  $x$  for which  $f'(x) = -8$ . [6]

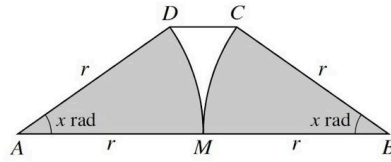
Question 50

- (a) By sketching a suitable pair of graphs, show that the equation  $\cot \frac{1}{2}x = 1 + e^{-x}$  has exactly one root in the interval  $0 < x \leq \pi$ . [2]

- (b) Verify by calculation that this root lies between 1 and 1.5. [2]

- (c) Use the iterative formula  $x_{n+1} = 2 \tan^{-1}\left(\frac{1}{1 + e^{-x_n}}\right)$  to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 51



The diagram shows a trapezium  $ABCD$  in which  $AD = BC = r$  and  $AB = 2r$ . The acute angles  $BAD$  and  $ABC$  are both equal to  $x$  radians. Circular arcs of radius  $r$  with centres  $A$  and  $B$  meet at  $M$ , the midpoint of  $AB$ .

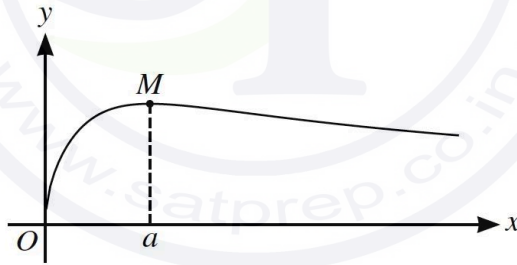
- (a) Given that the sum of the areas of the shaded sectors is 90% of the area of the trapezium, show that  $x$  satisfies the equation  $x = 0.9(2 - \cos x) \sin x$ . [3]
- (b) Verify by calculation that  $x$  lies between 0.5 and 0.7. [2]
- (c) Show that if a sequence of values in the interval  $0 < x < \frac{1}{2}\pi$  given by the iterative formula

$$x_{n+1} = \cos^{-1} \left( 2 - \frac{x_n}{0.9 \sin x_n} \right)$$

converges, then it converges to the root of the equation in part (a). [2]

- (d) Use this iterative formula to determine  $x$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 52



The diagram shows the curve  $y = \frac{\tan^{-1} x}{\sqrt{x}}$  and its maximum point  $M$  where  $x = a$ .

- (a) Show that  $a$  satisfies the equation

$$a = \tan \left( \frac{2a}{1 + a^2} \right). \quad [4]$$

- (b) Verify by calculation that  $a$  lies between 1.3 and 1.5. [2]
- (c) Use an iterative formula based on the equation in part (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 53

A large plantation of area  $20 \text{ km}^2$  is becoming infected with a plant disease. At time  $t$  years the area infected is  $x \text{ km}^2$  and the rate of increase of  $x$  is proportional to the ratio of the area infected to the area not yet infected.

When  $t = 0$ ,  $x = 1$  and  $\frac{dx}{dt} = 1$ .

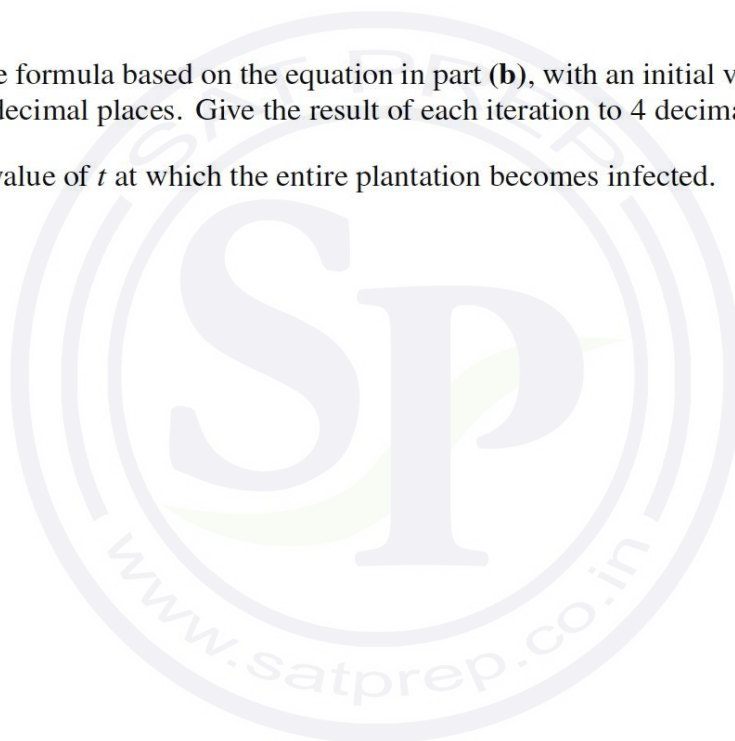
(a) Show that  $x$  and  $t$  satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}. \quad [2]$$

(b) Solve the differential equation and show that when  $t = 1$  the value of  $x$  satisfies the equation  $x = e^{0.9+0.05x}$ . [5]

(c) Use an iterative formula based on the equation in part (b), with an initial value of 2, to determine  $x$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(d) Calculate the value of  $t$  at which the entire plantation becomes infected. [1]

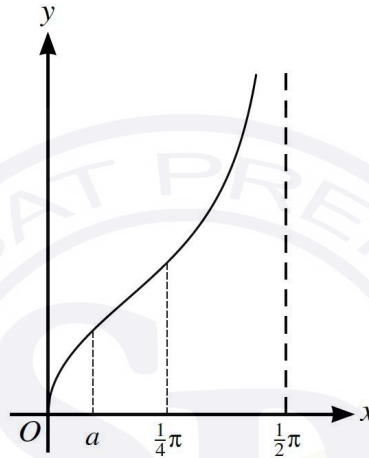


Question 54

The equation of a curve is  $y = \sqrt{\tan x}$ , for  $0 \leq x < \frac{1}{2}\pi$ .

- (a) Express  $\frac{dy}{dx}$  in terms of  $\tan x$ , and verify that  $\frac{dy}{dx} = 1$  when  $x = \frac{1}{4}\pi$ . [4]

The value of  $\frac{dy}{dx}$  is also 1 at another point on the curve where  $x = a$ , as shown in the diagram.



- (b) Show that  $t^3 + t^2 + 3t - 1 = 0$ , where  $t = \tan a$ . [4]
- (c) Use the iterative formula

$$a_{n+1} = \tan^{-1} \left( \frac{1}{3}(1 - \tan^2 a_n - \tan^3 a_n) \right)$$

to determine  $a$  correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

Question 55

The constant  $a$  is such that  $\int_1^a \frac{\ln x}{\sqrt{x}} dx = 6$ .

(a) Show that  $a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$ . [5]

[ $\exp(x)$  is an alternative notation for  $e^x$ .]

(b) Verify by calculation that  $a$  lies between 9 and 11. [2]

(c) Use an iterative formula based on the equation in part (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 56

(a) By sketching a suitable pair of graphs, show that the equation  $4 - x^2 = \sec \frac{1}{2}x$  has exactly one root in the interval  $0 \leq x < \pi$ . [2]

(b) Verify by calculation that this root lies between 1 and 2. [2]

(c) Use the iterative formula  $x_{n+1} = \sqrt{4 - \sec \frac{1}{2}x_n}$  to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 57

The constant  $a$  is such that  $\int_1^a x^2 \ln x dx = 4$ .

(a) Show that  $a = \left(\frac{35}{3 \ln a - 1}\right)^{\frac{1}{3}}$ . [5]

(b) Verify by calculation that  $a$  lies between 2.4 and 2.8. [2]

(c) Use an iterative formula based on the equation in part (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

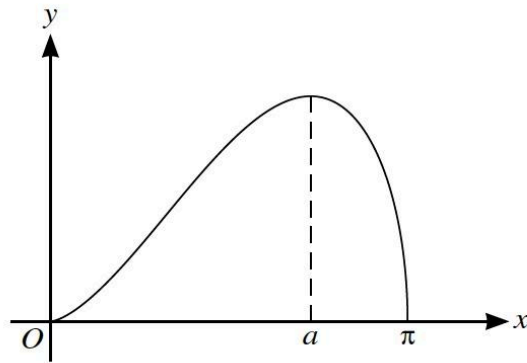
Question 58

(a) By sketching a suitable pair of graphs, show that the equation  $\ln x = 3x - x^2$  has one real root. [2]

(b) Verify by calculation that the root lies between 2 and 2.8. [2]

(c) Use the iterative formula  $x_{n+1} = \sqrt{3x_n - \ln x_n}$  to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 59



The curve  $y = x\sqrt{\sin x}$  has one stationary point in the interval  $0 < x < \pi$ , where  $x = a$  (see diagram).

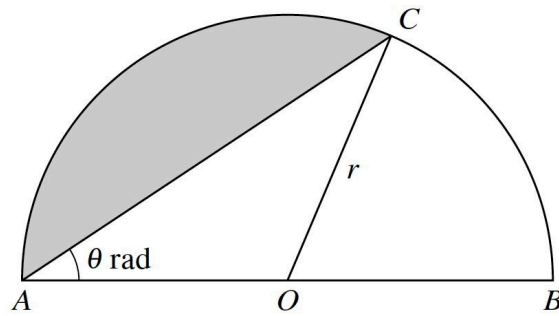
- (a) Show that  $\tan a = -\frac{1}{2}a$ . [4]
- (b) Verify by calculation that  $a$  lies between 2 and 2.5. [2]
- (c) Show that if a sequence of values in the interval  $0 < x < \pi$  given by the iterative formula  $x_{n+1} = \pi - \tan^{-1}\left(\frac{1}{2}x_n\right)$  converges, then it converges to  $a$ , the root of the equation in part (a). [2]
- (d) Use the iterative formula given in part (c) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 60

The curve with equation  $y = \frac{x^3}{e^x - 1}$  has a stationary point at  $x = p$ , where  $p > 0$ .

- (a) Show that  $p = 3(1 - e^{-p})$ . [3]
- (b) Verify by calculation that  $p$  lies between 2.5 and 3. [2]
- (c) Use an iterative formula based on the equation in part (a) to determine  $p$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 61



The diagram shows a semicircle with diameter  $AB$ , centre  $O$  and radius  $r$ . The shaded region is the minor segment on the chord  $AC$  and its area is one third of the area of the semicircle. The angle  $CAB$  is  $\theta$  radians.

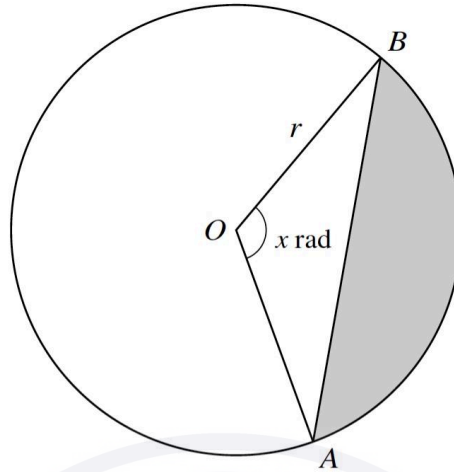
- (a) Show that  $\theta = \frac{1}{3}(\pi - 1.5 \sin 2\theta)$ . [4]
- (b) Verify by calculation that  $0.5 < \theta < 0.7$ . [2]
- (c) Use an iterative formula based on the equation in part (a) to determine  $\theta$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 62

The equation of a curve is  $y = \frac{x}{\cos^2 x}$ , for  $0 \leq x < \frac{1}{2}\pi$ . At the point where  $x = a$ , the tangent to the curve has gradient equal to 12.

- (a) Show that  $a = \cos^{-1} \left( \sqrt[3]{\frac{\cos a + 2a \sin a}{12}} \right)$ . [3]
- (b) Verify by calculation that  $a$  lies between 0.9 and 1. [2]
- (c) Use an iterative formula based on the equation in part (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

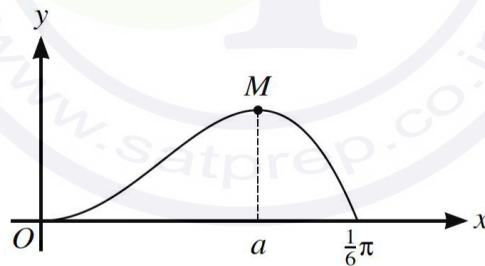
Question 63



The diagram shows a circle with centre  $O$  and radius  $r$ . The angle of the **minor** sector  $AOB$  of the circle is  $x$  radians. The area of the **major** sector of the circle is 3 times the area of the shaded region.

- (a) Show that  $x = \frac{3}{4} \sin x + \frac{1}{2}\pi$ . [4]
- (b) Show by calculation that the root of the equation in (a) lies between 2 and 2.5. [2]
- (c) Use an iterative formula based on the equation in (a) to calculate this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 64



The diagram shows the part of the curve  $y = x^2 \cos 3x$  for  $0 \leq x \leq \frac{1}{6}\pi$ , and its maximum point  $M$ , where  $x = a$ .

- (a) Show that  $a$  satisfies the equation  $a = \frac{1}{3} \tan^{-1}\left(\frac{2}{3a}\right)$ . [3]
- (b) Use an iterative formula based on the equation in (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 65

The equation  $\cot \frac{1}{2}x = 3x$  has one root in the interval  $0 < x < \pi$ , denoted by  $\alpha$ .

(a) Show by calculation that  $\alpha$  lies between 0.5 and 1. [2]

(b) Show that, if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{3} \left( x_n + 4 \tan^{-1} \left( \frac{1}{3x_n} \right) \right)$$

converges, then it converges to  $\alpha$ . [2]

(c) Use this iterative formula to calculate  $\alpha$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 66

The constant  $a$  is such that  $\int_0^a x e^{-2x} dx = \frac{1}{8}$ .

(a) Show that  $a = \frac{1}{2} \ln(4a + 2)$ . [5]

(b) Verify by calculation that  $a$  lies between 0.5 and 1. [2]

(c) Use an iterative formula based on the equation in (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 67

(a) By sketching a suitable pair of graphs, show that the equation

$$\cot x = 2 - \cos x$$

has one root in the interval  $0 < x \leq \frac{1}{2}\pi$ . [2]

(b) Show by calculation that this root lies between 0.6 and 0.8. [2]

(c) Use the iterative formula  $x_{n+1} = \tan^{-1} \left( \frac{1}{2 - \cos x_n} \right)$  to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 68

- (a) By sketching a suitable pair of graphs, show that the equation

$$\sqrt{x} = e^x - 3$$

has only one root. [2]

- (b) Show by calculation that this root lies between 1 and 2. [2]

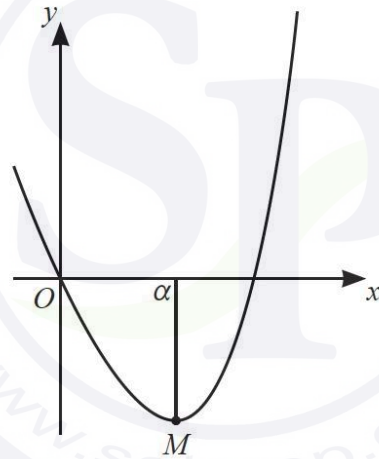
- (c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \ln(3 + \sqrt{x_n})$$

converges, then it converges to the root of the equation in (a). [1]

- (d) Use the iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 69



The diagram shows the curve  $y = xe^{2x} - 5x$  and its minimum point  $M$ , where  $x = \alpha$ .

- (a) Show that  $\alpha$  satisfies the equation  $\alpha = \frac{1}{2} \ln\left(\frac{5}{1+2\alpha}\right)$ . [3]

- (b) Verify by calculation that  $\alpha$  lies between 0.4 and 0.5. [2]

- (c) Use an iterative formula based on the equation in part (a) to determine  $\alpha$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 70

- (a) It is given that the equation  $e^{2x} = 5 + \cos 3x$  has only one root.

Show by calculation that this root lies in the interval  $0.7 < x < 0.8$ . [2]

- (b) Show that if a sequence of values in the interval  $0.7 < x < 0.8$  given by the iterative formula

$$x_{n+1} = \frac{1}{2} \ln(5 + \cos 3x_n)$$

converges then it converges to the root of the equation in part (a). [1]

- (c) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 71

- (a) By sketching a suitable pair of graphs, show that the equation  $\operatorname{cosec} \frac{1}{2}x = e^x - 3$  has exactly one root, denoted by  $\alpha$ , in the interval  $0 < x < \pi$ . [2]

- (b) Verify by calculation that  $\alpha$  lies between 1 and 2. [2]

- (c) Show that if a sequence of values in the interval  $0 < x < \pi$  given by the iterative formula

$$x_{n+1} = \ln(\operatorname{cosec} \frac{1}{2}x_n + 3)$$

converges, then it converges to  $\alpha$ . [1]

- (d) Use this iterative formula with an initial value of 1.4 to determine  $\alpha$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

- (e) State the minimum number of calculated iterations needed with this initial value to determine  $\alpha$  correct to 2 decimal places. [1]

Question 72

Let  $f(x) = 2x^3 - 5x^2 + 4$ .

- (a) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\frac{4}{5-2x_n}}$$

converges, then it converges to a root of the equation  $f(x) = 0$ . [2]

- (b) The equation has a root close to 1.2 .

Use the iterative formula from part (a) and an initial value of 1.2 to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 73

- (a) By sketching a suitable pair of graphs, show that the equation  $\cot 2x = \sec x$  has exactly one root in the interval  $0 < x < \frac{1}{2}\pi$ . [2]

- (b) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{1}{2} \tan^{-1}(\cos x_n)$$

converges, then it converges to the root in part (a). [1]

Question 74

- (a) By sketching a suitable pair of graphs, show that the equation  $2 + e^{-0.2x} = \ln(1+x)$  has only one root. [2]

- (b) Show by calculation that this root lies between 7 and 9. [2]

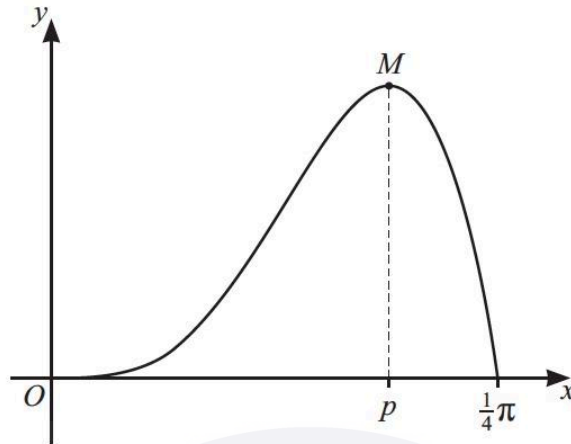
- (c) Use the iterative formula

$$x_{n+1} = \exp(2 + e^{-0.2x_n}) - 1$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[ $\exp(x)$  is an alternative notation for  $e^x$ .] [3]

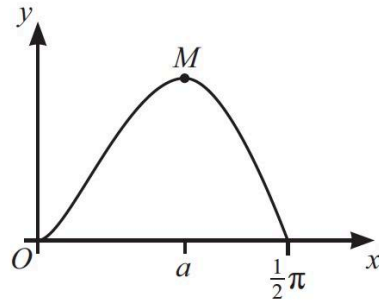
Question 75



The diagram shows the curve  $y = x^3 \cos 2x$  for  $0 \leq x \leq \frac{1}{4}\pi$ . The curve has a maximum point at  $M$ , where  $x = p$ .

- (a) Show that  $p$  satisfies the equation  $p = \frac{1}{2} \tan^{-1} \left( \frac{3}{2p} \right)$ . [3]
- (b) Show by calculation that  $0.5 < p < 0.7$ . [2]
- (c) Use an iterative formula based on the equation in part (a) to calculate  $p$  correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Question 76



The diagram shows the curve  $y = \sqrt{x} \sin 2x$  for  $0 \leq x \leq \frac{1}{2}\pi$ . The curve has a maximum point at  $M$ , where  $x = a$ .

(a) Show that  $\tan 2a = -4a$  [4]

(b) Show by calculation that  $0.9 < a < 0.95$ . [2]

(c) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}(\pi - \tan^{-1}(4x_n))$$

converges, then it converges to  $a$ . [2]

(d) Use the iterative formula in part (c) to calculate  $a$  correct to 4 decimal places. Give the result of each iteration to 6 decimal places. [3]

Question 77

(a) By sketching a suitable pair of graphs, show that the equation

$$|x - 2| = 2 \sin \frac{1}{2}x$$

has only one root in the interval  $0 < x < \pi$ . [2]

(b) Show by calculation that this root lies between 1 and 1.5. [2]

(c) Use the iterative formula  $x_{n+1} = 2 - 2 \sin \frac{1}{2}x_n$  with an initial value of 1.03 to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Question 78

The constant  $a$  is such that  $\int_1^a 6x \ln x \, dx = 4$ .

- (a) Show that  $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$ , where  $\exp(x)$  denotes  $e^x$ . [5]
- (b) Verify by calculation that  $a$  lies between 2 and 2.1. [2]
- (c) Use an iterative formula based on the equation in part (a) to determine  $a$  correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

