Subject – Math AA(Higher Level)
Topic - Algebra
Year - May 2021 – Nov 2024
Paper -1
Questions

Question 1

[Maximum mark: 18]

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that
$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0.$ [4]

Question 2

[Maximum mark: 8]

(a) Show that $\log_9(\cos 2x+2)=\log_3\sqrt{\cos 2x+2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x)=\log_9(\cos 2x+2)$ for $0<x<\frac{\pi}{2}$. [5]

Question 3

[Maximum mark: 5]

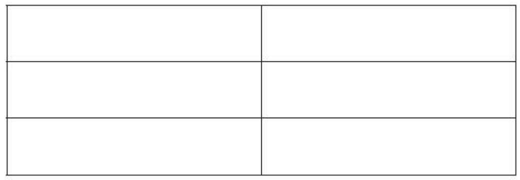
(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n\in\mathbb{Z}$. [2]

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

Question 4

[Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

Question 5

[Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where a, $b \in \mathbb{Z}$.

Find the possible values of a.

Question 6

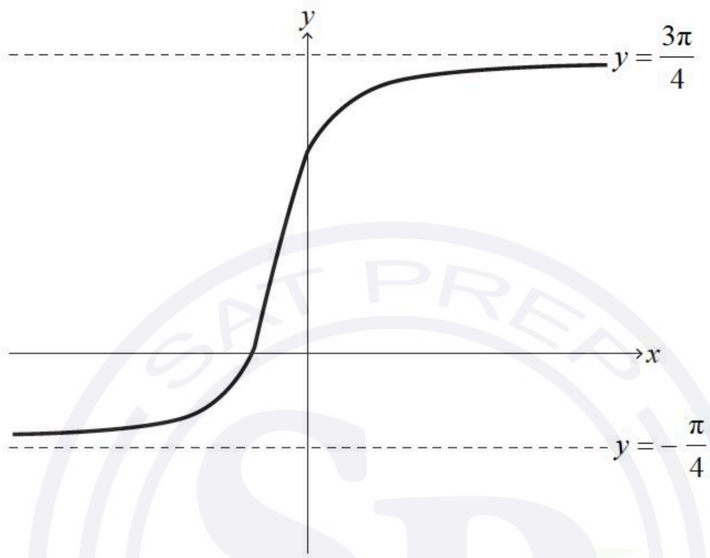
[Maximum mark: 5]

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, u_1, and the value of the common difference, d.

Question 7

[Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

Question 8

[Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

Question 9

[Maximum mark: 5]

In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k.

Question 10

[Maximum mark: 4]

Consider two consecutive positive integers, n and $n+1$.

Show that the difference of their squares is equal to the sum of the two integers.

Question 11

[Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

Question 12

[Maximum mark: 7]

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q}$, $a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of a and the value of b. [6]

(b) State the restriction which must be placed on x for this expansion to be valid. [1]

Question 13

[Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

Question 14

[Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

Question 15

[Maximum mark: 18]

Consider the series $\ln x + p\ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R}, x > 1$ and $p \in \mathbb{R}, p \neq 0$.

(a) Consider the case where the series is geometric.

(i) Show that $p = \pm\frac{1}{\sqrt{3}}$.

(ii) Hence or otherwise, show that the series is convergent.

(iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [6]

(b) Now consider the case where the series is arithmetic with common difference d.

(i) Show that $p = \frac{2}{3}$.

(ii) Write down d in the form $k\ln x$, where $k \in \mathbb{Q}$.

(iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n. [12]

Question 16

[Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = \left(1 - b^2\right) - 2b\mathrm{i}$, where $b \in \mathbb{R}, b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

(b) Hence, given that $\arg\left(z_1 z_2\right) = \frac{\pi}{4}$, find the value of b. [3]

Question 17

[Maximum mark: 6]

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

Question 18

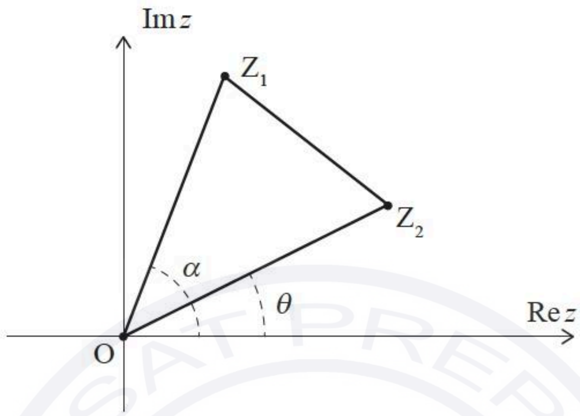
[Maximum mark: 5]

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

Question 19

[Maximum mark: 18]

In the following Argand diagram, the points Z_1, O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles α, θ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha, \theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2{}^* = r_1 r_2 e^{i(\alpha - \theta)}$ where $z_2{}^*$ is the complex conjugate of z_2. [2]

(b) Given that $\text{Re}(z_1 z_2{}^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2.

(ii) Hence show that $z_1{}^2 + z_2{}^2 = z_1 z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]

Question 20

[Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

Question 21

[Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of x. [5]

Question 22

[Maximum mark: 6]

Consider any three consecutive integers, $n-1$, n and $n+1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

Question 23

[Maximum mark: 5]

The n^{th} term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, u_1. [1]

(b) Given that the n^{th} term of this sequence is -33, find the value of n. [2]

(c) Find the common difference, d. [2]

Question 24

[Maximum mark: 18]

Let z_n be the complex number defined as $z_n = (n^2 + n + 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_0)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of n. [3]

Let $w_n = z_0 z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\dfrac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

Question 25

[Maximum mark: 16]

Consider a three-digit code abc, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

Question 26

[Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

(b) Find the value of p. [3]

Question 27

[Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a) (i) Find the sum of the first five terms.

(ii) Given that $S_6 = 60$, find u_6. [4]

(b) Find u_1. [2]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

(e) Given that $v_{99} < 0$, find v_5. [2]

Question 28

[Maximum mark: 6]

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$.

Question 29

[Maximum mark: 6]

Find the range of possible values of k such that $\mathrm{e}^{2x} + \ln k = 3\mathrm{e}^x$ has at least one real solution.

Question 30

[Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}\,\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

Question 31

[Maximum mark: 7]

Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

Question 32

[Maximum mark: 17]

(a) Find the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. [7]

Question 33

[Maximum mark: 5]

It is given that $z = 5 + q\text{i}$ satisfies the equation $z^2 + \text{i}z = -p + 25\text{i}$, where p, $q \in \mathbb{R}$.

Find the value of p and the value of q.

Question 34

[Maximum mark: 7]

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by 17 for all $n \in \mathbb{Z}^+$.

Question 35

[Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + \ldots + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k.

Question 36

[Maximum mark: 7]

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

(a) Find the value of p and the value of q. [5]

(b) Find the value of u_5. [2]

Question 37

[Maximum mark: 16]

Consider the arithmetic sequence a, p, $q \dots$, where a, p, $q \neq 0$.

(a) Show that $2p - q = a$. [2]

Consider the geometric sequence a, s, $t \dots$, where a, s, $t \neq 0$.

(b) Show that $s^2 = at$. [2]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Consider the case where $a = 9$, $s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(i) arithmetic sequence;

(ii) geometric sequence. [4]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9$, $u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e) (i) Find the common difference of the new sequence in terms of $\ln 3$.

(ii) Show that $\sum_{i=1}^{10} u_i = -90 - 25\ln 3$. [6]

Question 38

[Maximum mark: 7]

A teacher takes n students on a field trip. The students are assigned randomly into two groups.

For safety reasons there must be exactly three students in the first group and at least three students in the second group.

The teacher will randomly assign three students to the first group and the other students to the second group.

(a) Write down an expression for the number of ways that the students could be assigned. [1]

Two of the students ask the teacher not to work in the same group.

The teacher agrees and now finds that the number of ways to assign the students is halved.

(b) Determine the value of n. [6]

Question 39

[Maximum mark: 7]

A function $g(x)$ is defined by $g(x) = 2x^3 - 7x^2 + dx - e$, where $d, e \in \mathbb{R}$.

α, β and γ are the three roots of the equation $g(x) = 0$ where $\alpha, \beta, \gamma \in \mathbb{R}$.

(a) Write down the value of $\alpha + \beta + \gamma$. [1]

A function $h(z)$ is defined by $h(z) = 2z^5 - 11z^4 + rz^3 + sz^2 + tz - 20$, where $r, s, t \in \mathbb{R}$.

α, β and γ are also roots of the equation $h(z) = 0$.

It is given that $h(z) = 0$ is satisfied by the complex number $z = p + 3\text{i}$.

(b) Show that $p = 1$. [3]

It is now given that $h\left(\frac{1}{2}\right) = 0$, and $\alpha, \beta \in \mathbb{Z}^+$, $\alpha < \beta$ and $\gamma \in \mathbb{Q}$.

(c) (i) Find the value of the product $\alpha\beta$.

(ii) Write down the value of α and the value of β. [3]

Question 40

[Maximum mark: 5]

Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$.

Question 41

[Maximum mark: 20]

Consider $\phi = (a + b\mathrm{i})^3$, where $a, b \in \mathbb{R}$.

(a) In terms of a and b, find

(i) the real part of ϕ;

(ii) the imaginary part of ϕ. [3]

(b) Hence, or otherwise, show that $\left(1+\sqrt{3}\,\mathrm{i}\right)^3 = -8$. [2]

The roots of the equation $z^3 = -8$ are u, v and w, where $u = 1+\sqrt{3}\,\mathrm{i}$ and $v \in \mathbb{R}$.

(c) Write down v and w, giving your answers in Cartesian form. [2]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(d) Find the area of the triangle UVW. [3]

Each of the points U, V and W is rotated counter-clockwise (anticlockwise) about 0 through an angle of $\frac{\pi}{4}$ to form three new points U', V' and W'. These points represent the complex numbers u', v' and w' respectively.

(e) Find u', v' and w', giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \le \pi$. [4]

(f) Given that u', v' and w' are the solutions of $z^3 = c + d\mathrm{i}$, where $c, d \in \mathbb{R}$, find the value of c and the value of d. [3]

It is given that u, v, w, u', v' and w' are all solutions of $z^n = \alpha$ for some $\alpha \in \mathbb{C}$, where $n \in \mathbb{N}$.

(g) Find the smallest positive value of n. [3]

Question 42

[Maximum mark: 7]

Using mathematical induction and the definition ${}^{n}C_r = \frac{n!}{r!(n-r)!}$, prove that $\sum_{r=1}^{n} {}^{r}C_1 = {}^{n+1}C_2$ for all $n \in \mathbb{Z}^+$.

Question 43

[Maximum mark: 5]

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n - 1}{b}$, where $a, b \in \mathbb{Z}^+$. [1]

(b) Hence, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10\left(10^n - 1\right) - 9n}{81}$. [4]

Question 44

[Maximum mark: 5]

It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10}\left(\frac{1}{a}\right)$; [2]

(b) $\log_{1000} a$. [3]

Question 45

[Maximum mark: 20]

Consider the equation $z^4 = 16\mathrm{i}$, where $z \in \mathbb{C}$.

The equation has four roots z_1, z_2, z_3, z_4, where $z_i = r(\cos\theta_i + \mathrm{i}\sin\theta_i)$, $r > 0$ and $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$.

(a) Find z_1, z_2, z_3 and z_4. [6]

The roots z_1, z_2, z_3 and z_4 form a geometric sequence.

(b) Find the common ratio of the sequence, expressing your answer in Cartesian form. [3]

The roots z_1, z_2, z_3 and z_4 are represented by the points A, B, C and D respectively on an Argand diagram.

(c) Plot the points A, B, C and D on an Argand diagram. [3]

The equation $v^4 = a + b\mathrm{i}$, where $v \in \mathbb{C}$ and $a, b \in \mathbb{R}$ has roots z_1^*, z_2^*, z_3^* and z_4^*.

(d) Determine the value of a and the value of b. [3]

The midpoint of $[\mathrm{AB}]$ is A', the midpoint of $[\mathrm{BC}]$ is B', the midpoint of $[\mathrm{CD}]$ is C' and the midpoint of $[\mathrm{DA}]$ is D'.

Consider the equation $w^p = 2^q$, where $w \in \mathbb{C}$ and $p, q \in \mathbb{Z}^+$.

Four of the roots of $w^p = 2^q$ are represented by the points A', B', C' and D'.

(e) Find the least possible value of p and the corresponding value of q. [5]

Question 46

[Maximum mark: 6]

For a particular arithmetic sequence, $u_{10} = 16$ and $S_{25} = 100$.

Find the value of k such that $u_k = 0$.

Question 47

[Maximum mark: 4]

Prove that $(3n+2)^2 - (3n-2)^2$ is a multiple of 12 for all $n \in \mathbb{Z}^+$.