Subject – Math AA(Higher Level)
Topic - Algebra
Year - May 2021 – Nov 2024
Paper -1
Answers

Question 1

(a) attempt to find modulus **(M1)**

$r = 2\sqrt{3}\left(= \sqrt{12}\right)$ **A1**

attempt to find argument in the correct quadrant **(M1)**

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ **A1**

$= \frac{5\pi}{6}$ **A1**

$-3 + \sqrt{3}\mathrm{i} = \sqrt{12}\mathrm{e}^{\frac{5\pi \mathrm{i}}{6}}\left(= 2\sqrt{3}\mathrm{e}^{\frac{5\pi \mathrm{i}}{6}}\right)$

[5 marks]

(b) attempt to find a root using de Moivre's theorem **M1**

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi \mathrm{i}}{18}}$ **A1**

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument **M1**

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi \mathrm{i}}{18}}$ **A1**

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi \mathrm{i}}{18}}$ **A1**

Note: Ignore labels for u, v and w at this stage.

[5 marks]

Question 2

(a) attempting to use the change of base rule **M1**

$\log_9(\cos 2x+2)=\dfrac{\log_3(\cos 2x+2)}{\log_3 9}$ **A1**

$=\dfrac{1}{2}\log_3(\cos 2x+2)$ **A1**

$=\log_3\sqrt{\cos 2x+2}$ **AG**

[3 marks]

(b) $\log_3(2\sin x)=\log_3\sqrt{\cos 2x+2}$

$2\sin x=\sqrt{\cos 2x+2}$ **M1**

$4\sin^2 x=\cos 2x+2$ (or equivalent) **A1**

use of $\cos 2x=1-2\sin^2 x$ ***(M1)***

$6\sin^2 x=3$

$\sin x=(\pm)\dfrac{1}{\sqrt{2}}$ **A1**

$x=\dfrac{\pi}{4}$ **A1**

Note: Award ***A0*** if solutions other than $x=\dfrac{\pi}{4}$ are included.

[5 marks]

Total [8 marks]

(c) ***METHOD 1***

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

$\text{Area} = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$.

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

METHOD 2

$\text{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

$\text{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using $\text{Area} = \frac{1}{2} \times \text{UV} \times \text{VW} \times \sin\alpha$ for example ***M1***

$\text{Area} = \frac{1}{2}\left(\sqrt{3} \times 12^{\frac{1}{6}}\right)\left(\sqrt{3} \times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

(d) $u + v + w = 0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + i\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + i\sin\frac{17\pi}{18}\right) = 0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{7\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$ ***AG***

[4 marks]

Total [18 marks]

Question 3

(a) attempting to expand the LHS **(M1)**

$\text{LHS} = \left(4n^2 - 4n + 1\right) + \left(4n^2 + 4n + 1\right)$ **A1**

$= 8n^2 + 2\left(= \text{RHS}\right)$ **AG**

[2 marks]

(b) ***METHOD 1***

recognition that $2n-1$ and $2n+1$ represent two consecutive odd integers (for all odd integers n) **R1**

$8n^2 + 2 = 2\left(4n^2 + 1\right)$ **A1**

valid reason *eg* divisible by 2 (2 is a factor) **R1**

so the sum of the squares of any two consecutive odd integers is even **AG**

[3 marks]

METHOD 2

recognition, *eg* that n and $n+2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) **R1**

$n^2 + \left(n+2\right)^2 = 2\left(n^2 + 2n + 2\right)$ **A1**

valid reason *eg* divisible by 2 (2 is a factor) **R1**

so the sum of the squares of any two consecutive odd integers is even **AG**

[3 marks]

Total [5 marks]

Question 4

(a) **METHOD 1**

B has one less pen to select *(M1)*

EITHER

A and B can be placed in 6×5 ways *(A1)*

C, D, E have 6 choices each *(A1)*

OR

A (or B), C, D, E have 6 choices each *(A1)*

B (or A) has only 5 choices *(A1)*

THEN

$5\times6^4\ (=6480)$ ***A1***

METHOD 2

total number of ways $=6^5$ *(A1)*

number of ways with Amber and Brownie together $=6^4$ *(A1)*

attempt to subtract (may be seen in words) *(M1)*

6^5-6^4

$=5\times6^4\ (=6480)$ ***A1***

[4 marks]

(b) **METHOD 1**

total number of ways $=6!(=720)$ *(A1)*

number of ways with Amber and Brownie sharing a boundary

$=2\times7\times4!(=336)$ *(A1)*

attempt to subtract (may be seen in words) *(M1)*

$720-336=384$ ***A1***

METHOD 2

case 1: number of ways of placing A in corner pen

$3\times4\times3\times2\times1$

Four corners total no of ways is $4\times(3\times4\times3\times2\times1)=12\times4!(=288)$ *(A1)*

case 2: number of ways of placing A in the middle pen

$2\times4\times3\times2\times1$

two middle pens so $2\times(2\times4\times3\times2\times1)=4\times4!(=96)$ *(A1)*

attempt to add (may be seen in words) *(M1)*

total no of ways $=288+96$

$=16\times4!(=384)$ ***A1***

[4 marks]

Total [8 marks]

Question 5

METHOD 1

other two roots are $a-b\text{i}$ and $b-a\text{i}$	***A1***
sum of roots $=-4$ and product of roots $=400$	***A1***
attempt to set sum of four roots equal to -4 or 4 OR attempt to set product of four roots equal to 400	***M1***
$a+b\text{i}+a-b\text{i}+b+a\text{i}+b-a\text{i}=-4$	
$2a+2b=-4(\Rightarrow a+b=-2)$	***A1***
$(a+b\text{i})(a-b\text{i})\ (b+a\text{i})(b-a\text{i})=400$	
$(a^2+b^2)^2=400$	***A1***
$a^2+b^2=20$	
attempt to solve simultaneous equations	***(M1)***
$a=2$ or $a=-4$	***A1A1***

[8 marks]

METHOD 2

other two roots are $a-b\text{i}$ and $b-a\text{i}$	***A1***
$(z-(a+b\text{i}))(z-(a-b\text{i}))(z-(b+a\text{i}))(z-(b-a\text{i}))(=0)$	***A1***
$((z-a)^2+b^2)((z-b)^2+a^2)(=0)$	
$(z^2-2az+a^2+b^2)(z^2-2bz+b^2+a^2)(=0)$	***A1***
Attempt to equate coefficient of z^3 and constant with the given quartic equation	***M1***
$-2a-2b=4$ and $(a^2+b^2)^2=400$	***A1***
attempt to solve simultaneous equations	***(M1)***
$a=2$ or $a=-4$	***A1A1***

[8 marks]

Question 6

METHOD 1 (finding u_1 first, from S₈)

$4(u_1+8)=8$ **(A1)**

$u_1=-6$ **A1**

$u_1+7d=8$ OR $4(2u_1+7d)=8$ (may be seen with their value of u_1) **(A1)**

attempt to substitute their u_1 **(M1)**

$d=2$ **A1**

METHOD 2 (solving simultaneously)

$u_1+7d=8$ **(A1)**

$4(u_1+8)=8$ OR $4(2u_1+7d)=8$ OR $u_1=-3d$ **(A1)**

attempt to solve linear or simultaneous equations **(M1)**

$u_1=-6, d=2$ **A1A1**

[5 marks]

Question 7

(a) **EITHER**

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$) ***A1***

(may be seen anywhere)

[3 marks]

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha+\beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$ ***AG***

[4 marks]

(c) **METHOD 1**

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

[3 marks]

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$\tan\left(\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right)\right)$

$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$ ***A1***

$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$ ***A1***

$= \arctan 1$ ***AG***

[3 marks]

METHOD 3

$\tan\left(\arctan\left(2x+1\right)\right)=\tan\left(\arctan\left(\frac{x}{x+1}\right)+\frac{\pi}{4}\right)$

$\tan\frac{\pi}{4}=1$ (or equivalent) ***A1***

LHS = $2x+1$ ***A1***

RHS = $\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}}\left(=2x+1\right)$ ***A1***

[3 marks]

(d) let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n = 1$, $\sum_{r=1}^{1} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{1}{2}\right) = \mathrm{RHS}$ and so $\mathrm{P}(1)$ is true **_R1_**

assume $\mathrm{P}(k)$ is true, ie. $\sum_{r=1}^{k} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{k}{k+1}\right)$ $\left(k \in \mathbb{Z}^+\right)$ **_M1_**

Note: Award ***M0*** for statements such as "let $n = k$".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1} \arctan\left(\frac{1}{2r^2}\right) = \sum_{r=1}^{k} \arctan\left(\frac{1}{2r^2}\right) + \arctan\left(\frac{1}{2(k+1)^2}\right)$$ **_(M1)_**

$$= \arctan\left(\frac{k}{k+1}\right) + \arctan\left(\frac{1}{2(k+1)^2}\right)$$ **_A1_**

$$= \arctan\left(\frac{\frac{k}{k+1} + \frac{1}{2(k+1)^2}}{1 - \left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$$ **_M1_**

$$= \arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{2(k+1)^3 - k}\right)$$ **_A1_**

Note: Award ***A1*** for correct numerator, with $(k+1)$ factored. Denominator does not need to be simplified

$$= \arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{2k^3+6k^2+5k+2}\right)$$ **_A1_**

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$= \arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{(k+2)\left(2k^2+2k+1\right)}\right) = \arctan\left(\frac{k+1}{k+2}\right)$$ **_A1_**

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so
$\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ **R1**

Note: Award the final ***R1*** mark provided at least four of the previous marks have been awarded. **Note:** To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

[9 marks]
Total [19 marks]

Question 8

$\alpha+\beta+\alpha+\beta=k$ ***(A1)***

$\alpha+\beta=\frac{k}{2}$

$\alpha\beta(\alpha+\beta)=-3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right)=-3k\ \left(-\frac{k^3}{8}=-3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8}+3k=0$ (or equivalent) for k ***(M1)***

$k=2\sqrt{6}\ \left(=\sqrt{24}\right)(k>0)$ ***A1***

Note: Award ***A0*** for $k=\pm 2\sqrt{6}\ \left(\pm\sqrt{24}\right)$.

[5 marks]

Question 9

EITHER

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0 + {}^7C_1x^6k^1 + {}^7C_2x^5k^2 + \ldots$ (or ${}^7C_0k^7x^0 + {}^7C_1k^5x^1 + {}^7C_2k^5x^2 + \ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_rx^rk^{7-r}$ (or ${}^7C_rk^rx^{7-r}$) ***(M1)***

$r=2$ (or $r=5$) ***(A1)***

THEN

${}^7C_2=21$ (or ${}^7C_5=21$) (seen anywhere) ***(A1)***

$21x^5k^2=63x^5$ $(21k^2=63,\ k^2=3)$ ***A1***

$k=\pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k=\sqrt{3}$.

[5 marks]

Question 10

attempt to subtract squares of integers **(M1)**

$(n+1)^2 - n^2$

EITHER

correct order of subtraction and correct expansion of $(n+1)^2$, seen anywhere ***A1A1***

$= n^2 + 2n + 1 - n^2 \left(= 2n+1\right)$

OR

correct order of subtraction and correct factorization of difference of squares ***A1A1***

$= (n+1-n)(n+1+n)\left(= 2n+1\right)$

THEN

$= n + n + 1$ =RHS ***A1***

Note: Do not award final ***A1*** unless all previous working is correct.

which is the sum of n and $n+1$ ***AG***

Note: If expansion and order of subtraction are correct, award full marks for candidates who find the sum of the integers as $2n+1$ and then show that the difference of the squares (subtracted in the correct order) is $2n+1$.

[4 marks]

Question 11

(a) (i) $\left(1+e^{i\frac{\pi}{6}}-1\right)^3$

$=\left(e^{i\frac{\pi}{6}}\right)^3$ **A1**

$=e^{i\frac{\pi}{2}}$ **A1**

$=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$=i$ **AG**

Note: Candidates who solve the equation correctly can be awarded the above two marks. The working for part (i) may be seen in part (ii).

(ii) $(z-1)^3=e^{i\left(\frac{\pi}{2}+2\pi k\right)}$ ***(M1)***

$z-1=e^{i\left(\frac{\pi}{6}+\frac{4\pi k}{6}\right)}$ ***(M1)***

$(k=1)\Rightarrow \omega_2=1+e^{i\frac{5\pi}{6}}$ **A1**

$(k=2)\Rightarrow \omega_3=1+e^{i\frac{9\pi}{6}}$ **A1**

[6 marks]

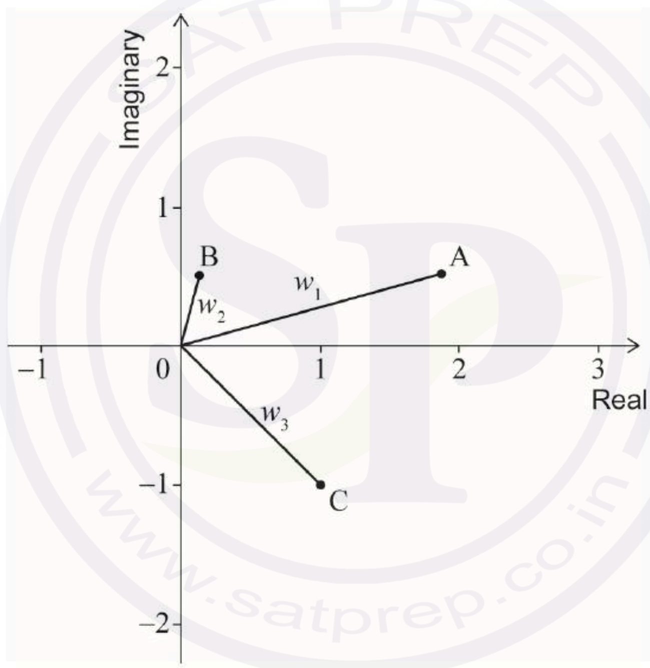
(b) **EITHER**

attempt to express $e^{i\frac{\pi}{6}}$, $e^{i\frac{5\pi}{6}}$, $e^{i\frac{9\pi}{6}}$ in Cartesian form and translate 1 unit in the positive direction of the real axis ***(M1)***

OR

attempt to express w_1, w_2 and w_3 in Cartesian form ***(M1)***

THEN



Note: To award ***A*** marks, it is not necessary to see A,B or C, the w_i , or the solid lines

A1A1A1

[4 marks]

(c) valid attempt to find $\omega_1 - \omega_3$ (or $\omega_3 - \omega_1$) ***M1***

$$\omega_1 - \omega_3 = \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - (1 - \mathrm{i}) = \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \text{ OR } \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{2}$$

valid attempt to find $\left|\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}\right|$ ***M1***

$$= \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$\mathrm{AC} = \sqrt{3}$ ***A1***

[3 marks]

(d) **METHOD 1**

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(\frac{z-1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$\frac{\alpha - 1}{\alpha} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$$\alpha - 1 = \alpha \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$$

$$\alpha - \alpha \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = 1$$

$$\alpha\left(1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) = 1$$

$\alpha = \frac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

METHOD 2

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(1-\frac{1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$1-\frac{1}{z} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \frac{1}{\alpha}$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

(e) **METHOD 1**

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

$$=\frac{2}{2-\sqrt{3}-i}$$ ***A1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{4-2\sqrt{3}+2i}{\left(2-\sqrt{3}\right)^2+1}$$ ***A1***

$$=\frac{4-2\sqrt{3}+2i}{8-4\sqrt{3}}$$ ***A1***

$$=\frac{1}{2}+\frac{1}{4-2\sqrt{3}}i$$

$$\Rightarrow \text{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

METHOD 2

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{1}{\left(1-\cos\frac{\pi}{6}\right)-i\sin\frac{\pi}{6}}\times\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)^2+\sin^2\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{1-2\cos\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}}$$

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

$$=\frac{1}{2}+\frac{i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

METHOD 3

attempt to multiply through by $-\frac{e^{-i\frac{\pi}{12}}}{e^{-i\frac{\pi}{12}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=-\frac{e^{-i\frac{\pi}{12}}}{e^{i\frac{\pi}{12}}-e^{-i\frac{\pi}{12}}}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=-\frac{\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)}{\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}-\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)}$$ ***A1***

$$=-\frac{\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}}{2i\sin\frac{\pi}{12}}$$ ***A1***

$$=\frac{1}{2}-\frac{1}{2i}\cot\frac{\pi}{12}\ \left(=\frac{1}{2}+\frac{1}{2}i\cot\frac{\pi}{12}\right)$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

[6 marks]

Question 12

(a) attempt to expand binomial with negative fractional power **(M1)**

$$\frac{1}{\sqrt{1+ax}}=(1+ax)^{-\frac{1}{2}}=1-\frac{ax}{2}+\frac{3a^2x^2}{8}+\ldots$$ **A1**

$$\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1-\frac{x}{2}-\frac{x^2}{8}+\ldots$$ **A1**

$$\frac{1}{\sqrt{1+ax}}-\sqrt{1-x}=\frac{(1-a)}{2}x+\left(\frac{3a^2+1}{8}\right)x^2+\ldots$$

attempt to equate coefficients of x or x^2 **(M1)**

$$x:\quad \frac{1-a}{2}=4b;\ x^2:\frac{3a^2+1}{8}=b$$

attempt to solve simultaneously **(M1)**

$$a=-\frac{1}{3},b=\frac{1}{6}$$ **A1**

[6 marks]

(b) $|x|<1$ **A1**

[1 mark]

Total [7 marks]

Question 13

(a) attempt to use discriminant $b^2 - 4ac(>0)$ **M1**

$(2p)^2 - 4(3p)(1-p)(>0)$

$16p^2 - 12p(>0)$ ***(A1)***

$p(4p-3)(>0)$

attempt to find critical values $\left(p=0,\ p=\frac{3}{4}\right)$ **M1**

recognition that discriminant >0 ***(M1)***

$p<0$ or $p>\frac{3}{4}$ **A1**

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) $p=4 \Rightarrow 12x^2+8x-3=0$

valid attempt to use $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (or equivalent) **M1**

$x=\frac{-8\pm\sqrt{208}}{24}$

$x=\frac{-2\pm\sqrt{13}}{6}$

$a=-2$ **A1**

[2 marks]

Total [7 marks]

Question 14

attempt to use change the base **(M1)**

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3\left(4x^3\right)$$

attempt to use the power rule **(M1)**

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3\left(4x^3\right)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ **(M1)**

$$\log_3 \sqrt{x} = \log_3\left(4\sqrt{2}x^3\right)$$

Note: The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen.

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

$$x^5 = \frac{1}{32}$$ **(A1)**

$$x = \frac{1}{2}$$ **A1**

[5 marks]

Question 15

(a) (i) **EITHER**

attempt to use a ratio from consecutive terms ***M1***

$$\frac{p\ln x}{\ln x} = \frac{\frac{1}{3}\ln x}{p\ln x} \quad \text{OR} \quad \frac{1}{3}\ln x = (\ln x)r^2 \quad \text{OR} \quad p\ln x = \ln x\left(\frac{1}{3p}\right)$$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in geometric sequence. Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm\frac{1}{\sqrt{3}}$ ***A1***

$p = \pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

(ii) **EITHER**

since, $|p| = \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < 1$ ***R1***

OR

since, $|p| = \frac{1}{\sqrt{3}}$ and $-1 < p < 1$ ***R1***

THEN

$\Rightarrow$ the geometric series converges. ***AG***

> **Note:** Accept r instead of p.
>
> Award ***R0*** if both values of p not considered.

(iii) $\frac{\ln x}{1 - \frac{1}{\sqrt{3}}} \left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x = 3 - \sqrt{3} + \sqrt{3} - 1$ $(\Rightarrow \ln x = 2)$ ***A1***

$x = e^2$ ***A1***

[6 marks]

(b) (i) **METHOD 1**

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p\ln x - \ln x = \frac{1}{3}\ln x - p\ln x$ OR $\frac{1}{3}\ln x = \ln x + 2(p\ln x - \ln x)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.
Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$.

$2p\ln x = \frac{4}{3}\ln x \quad \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1 + u_3}{2}$ ***M1***

$p\ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ ***A1***

$2p\ln x = \frac{4}{3}\ln x \quad \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 3

attempt to find difference using u_3 **M1**

$\frac{1}{3}\ln x = \ln x + 2d \quad \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p\ln x - \ln x = -\frac{1}{3}\ln x$ **A1**

$p\ln x = \frac{2}{3}\ln x$ **A1**

$p = \frac{2}{3}$ **AG**

(ii) $d = -\frac{1}{3}\ln x$ **A1**

(iii) **METHOD 1**

$$S_n = \frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right]$$

attempt to substitute into S_n and equate to $\ln\left(\frac{1}{x^3}\right)$ ***(M1)***

$$\frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right] = \ln\left(\frac{1}{x^3}\right)$$

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2\ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n\ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right)\ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ (or equivalent)

Note: Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ ***(M1)***

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2) = 0$

$n = 9$ ***A1***

METHOD 2

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

listing the first 7 terms of the sequence ***(A1)***

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + \ldots$

recognizing first 7 terms sum to 0 ***M1***

8th term is $-\frac{4}{3}\ln x$ ***(A1)***

9th term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8th and 9th terms $= -3\ln x$ ***(A1)***

$n = 9$ ***A1***

[12 marks]

Total [18 marks]

Question 16

(a) $z_1 z_2 = (1+b\mathrm{i})\left((1-b^2)-(2b)\mathrm{i}\right)$

$= (1-b^2-2\mathrm{i}^2 b^2)+\mathrm{i}(-2b+b-b^3)$ ***M1***

$= (1+b^2)+\mathrm{i}(-b-b^3)$ ***A1A1***

Note: Award ***A1*** for $1+b^2$ and ***A1*** for $-b\mathrm{i}-b^3\mathrm{i}$.

[3 marks]

(b) $\arg(z_1 z_2) = \arctan\left(\frac{-b-b^3}{1+b^2}\right) = \frac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \frac{\pi}{4}$ (since $1+b^2 \neq 0, \text{for } b \in \mathbb{R}$) ***A1***

OR

$-b-b^3 = 1+b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

Total [6 marks]

Question 17

Assume that a and b are both odd. ***M1***

Note: Award ***M0*** for statements such as "let a and b be both odd".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m+1$ and $b = 2n+1$ ***A1***

$a^2+b^2 \equiv (2m+1)^2+(2n+1)^2$

$= 4m^2+4m+1+4n^2+4n+1$ ***A1***

$= 4\left(m^2+m+n^2+n\right)+2$ ***(A1)***

($4\left(m^2+m+n^2+n\right)$ is always divisible by 4) but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2+b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

Note: Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

Question 18

EITHER

attempt to obtain the general term of the expansion

$T_{r+1} = {}^nC_r\left(8x^3\right)^{n-r}\left(-\frac{1}{2x}\right)^r$ OR $T_{r+1} = {}^nC_{n-r}\left(8x^3\right)^r\left(-\frac{1}{2x}\right)^{n-r}$ ***(M1)***

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent) ***(M1)***

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - (n - r) = 0$ (or equivalent) ***A1***

r is a multiple of 3 $(r = 3,6,9,...)$ or one correct value of n (seen anywhere) ***(A1)***

$n = 4k, k \in \mathbb{Z}^+$ ***A1***

Note: Accept n is a (positive) multiple of 4 or $n = 4,8,12,...$

Do not accept $n = 4,8,12$

Note: Award full marks for a correct answer using trial and error approach showing $n = 4,8,12,...$ and for recognizing that this pattern continues.

[5 marks]

Question 19

(a) $z_2^* = r_2 e^{-i\theta}$ **(A1)**

$z_1 z_2^* = r_1 e^{i\alpha} r_2 e^{-i\theta}$ **A1**

$z_1 z_2^* = r_1 r_2 e^{i(\alpha-\theta)}$ **AG**

Note: Accept working in modulus-argument form

[2 marks]

(b) $\mathrm{Re}\left(z_1 z_2^*\right) = r_1 r_2 \cos(\alpha-\theta) \quad (=0)$ **A1**

$\alpha - \theta = \arccos 0 \quad (r_1, r_2 > 0)$

$\alpha - \theta = \frac{\pi}{2}$ (as $0 < \alpha - \theta < \pi$) **A1**

so Z_1OZ_2 is a right-angled triangle **AG**

[2 marks]

(c) (i) **EITHER**

$\frac{z_1}{z_2}\left(= \frac{r_1}{r_2} e^{i(\alpha-\theta)}\right) = e^{i\frac{\pi}{3}}$ (since $r_1 = r_2$) **(M1)**

OR

$z_1 = r_2 e^{i\left(\theta + \frac{\pi}{3}\right)} \quad \left(= r_2 e^{i\theta} e^{i\frac{\pi}{3}}\right)$ **(M1)**

THEN

$z_1 = z_2 e^{i\frac{\pi}{3}}$ **A1**

(ii) substitutes $z_1 = z_2 e^{i\frac{\pi}{3}}$ into $z_1^2 + z_2^2$ ***M1***

$z_1^2 + z_2^2 = z_2^2 e^{i\frac{2\pi}{3}} + z_2^2 \left(= z_2^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$ ***A1***

EITHER

$e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$ ***A1***

OR

$z_2^2 \left(e^{i\frac{2\pi}{3}} + 1 \right) = z_2^2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i + 1 \right)$

$= z_2^2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$ ***A1***

THEN

$z_1^2 + z_2^2 = z_2^2 e^{i\frac{\pi}{3}}$

$= z_2 \left(z_2 e^{i\frac{\pi}{3}} \right)$ and $z_2 e^{i\frac{\pi}{3}} = z_1$ ***A1***

so $z_1^2 + z_2^2 = z_1 z_2$ ***AG***

(d) **METHOD 1**

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$a^2 = z_1^{\ 2} + z_2^{\ 2} + 2z_1 z_2$ ***A1***

$a^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

METHOD 2

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$\left(z_1 + z_2\right)^2 = z_1^{\ 2} + z_2^{\ 2} + 2z_1 z_2$ ***A1***

$\left(z_1 + z_2\right)^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ and $z_1 + z_2 = -a$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

[5 marks]

(e) $a^2 - 3 \times 12 = 0$

$a = \pm 6 \ \left(\Rightarrow z^2 \pm 6z + 12 = 0\right)$ ***A1***

for $a = -6$:

$z_1 = 3 + \sqrt{3}\mathrm{i}, z_2 = 3 - \sqrt{3}\mathrm{i}$ and $\alpha - \theta = -\frac{5\pi}{3}$ which does not satisfy $0 < \alpha - \theta < \pi$ ***R1***

for $a = 6$:

$z_1 = -3 - \sqrt{3}\mathrm{i}, z_2 = -3 + \sqrt{3}\mathrm{i}$ and $\alpha - \theta = \frac{\pi}{3}$ ***A1***

so (for $0 < \alpha - \theta < \pi$), only one equilateral triangle can be formed from point O and the two roots of this equation ***AG***

[3 marks]

Total [18 marks]

Question 20

METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.

Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

Note: Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

Question 21

(a) **EITHER**

recognises the required term (or coefficient) in the expansion *(M1)*

$bx^5 = {}^7C_2 x^5 1^2$ OR $b = {}^7C_2$ OR ${}^7C_5$

$b = \frac{7!}{2!5!} \left(= \frac{7!}{2!(7-2)!} \right)$

correct working ***A1***

$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$ OR $\frac{7\times6}{2!}$ OR $\frac{42}{2}$

OR

lists terms from row 7 of Pascal's triangle *(M1)*

$1, 7, 21, \ldots$ ***A1***

THEN

$b = 21$ ***AG***

[2 marks]

(b) $a = 7$ *(A1)*

correct equation ***A1***

$21x^5 = \frac{ax^6 + 35x^4}{2}$ OR $21x^5 = \frac{7x^6 + 35x^4}{2}$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve **their** quadratic *(M1)*

$(x-1)(x-5) = 0$ OR $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$

$x = 1, x = 5$ ***A1***

Note: Award final ***A0*** for obtaining $x = 0, x = 1, x = 5$.

[5 marks]

Total [7 marks]

Question 22

(a) $(n-1)+n+(n+1)$ **(A1)**

$=3n$ **A1**

which is always divisible by 3 **AG**

[2 marks]

(b) $(n-1)^2+n^2+(n+1)^2$ (= $n^2-2n+1+n^2+n^2+2n+1$) **A1**

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept n^2-1 or n^2+1) **(M1)**

$=3n^2+2$ **A1**

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 **R1**

$3n^2$ is divisible by 3 and so $3n^2+2$ is never divisible by 3

OR the first term is divisible by 3, the second is not

OR $3\left(n^2+\frac{2}{3}\right)$ OR $\frac{3n^2+2}{3}=n^2+\frac{2}{3}$

hence the sum of the squares is never divisible by 3 **AG**

[4 marks]

Total [6 marks]

Question 23

(a) $u_1 = 12$ **A1**

[1 mark]

(b) $15 - 3n = -33$ ***(A1)***

$n = 16$ **A1**

[2 marks]

(c) valid approach to find d ***(M1)***

$u_2 - u_1 = 9 - 12$ OR recognize gradient is -3 OR attempts to solve $-33 = 12 + 15d$

$d = -3$ **A1**

[2 marks]

Total [5 marks]

Question 24

(a) (i) $z_0 = 1 + \mathrm{i}$ ***(A1)***

$\arg(z_0) = \arctan(1) = \frac{\pi}{4} = 45°$ ***A1***

Note: Accept any of these three forms, including an answer marked on an Argand diagram.

(ii) $\arg(z_n) = \arctan\left(\frac{1}{n^2+n+1}\right)$ ***A1***

[3 marks]

(b) (i) attempt to use the compound angle formula for tan ***M1***

$$\tan(\arctan(a)+\arctan(b)) = \frac{\tan(\arctan(a))+\tan(\arctan(b))}{1-\tan(\arctan(a))\tan(\arctan(b))}$$

$= \frac{a+b}{1-ab}$ ***A1***

$\Rightarrow \arctan(a)+\arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ ***AG***

(ii) **METHOD 1**

$\arg(w_1) = \arg(z_0 z_1) = \arg(z_0) + \arg(z_1)$ ***M1***

$= \arctan(1) + \arctan\left(\frac{1}{3}\right)$ ***(A1)***

$= \arctan\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$ ***A1***

$= \arctan(2)$ ***AG***

(c) let $n=0$

$$\text{LHS} = \arg(w_0) = \arg(z_0) = \arctan(1)\left(=\frac{\pi}{4}\right)$$

$\text{RHS} = \arctan(1)\left(=\dfrac{\pi}{4}\right)$ so LHS = RHS **R1**

> **Note:** Award ***R0*** for not starting at $n=0$, for example by referring to the result in (b) (ii) for $n=1$. Award subsequent marks.

assume true for $n=k$, (so $\arg(w_k) = \arctan(k+1)$) **M1**

> **Note:** Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

$\arg(w_{k+1})$

$= \arg(w_k z_{k+1})\left(= \arg(w_k) + \arg(z_{k+1})\right)$ ***(M1)***

$= \arctan(k+1) + \arctan\left(\dfrac{1}{(k+1)^2 + (k+1) + 1}\right)$ ***A1***

$= \arctan\left(\dfrac{(k+1) + \left(\dfrac{1}{(k+1)^2+(k+1)+1}\right)}{1-(k+1)\left(\dfrac{1}{(k+1)^2+(k+1)+1}\right)}\right)$ ***M1***

$= \arctan\left(\dfrac{(k+1) + \left(\dfrac{1}{k^2+3k+3}\right)}{1-(k+1)\left(\dfrac{1}{k^2+3k+3}\right)}\right)$ ***(A1)***

$= \arctan\left(\dfrac{(k+1)(k^2+3k+3)+1}{(k^2+3k+3)-(k+1)}\right)$

$$= \arctan\left(\frac{k^3 + 4k^2 + 6k + 4}{k^2 + 2k + 2}\right)$$ ***A1***

$$= \arctan\left(\frac{(k+2)(k^2 + 2k + 2)}{k^2 + 2k + 2}\right)$$ ***A1***

$$= \arctan(k+2)\left(= \arctan((k+1)+1)\right)$$ ***A1***

since true for $n = 0$, and true for $n = k+1$ if true for $n = k$, the statement is true for all $n \in \mathbb{N}$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** four of the previous marks must have been awarded.

[10 marks]

Total [18 marks]

Question 25

(a) (i) 5^3 **(A1)**

$=125$ **A1**

(ii) ${}^5P_3 = 5\times4\times3$ **(A1)**

$=60$ **A1**

[4 marks]

(b) (i) **METHOD 1**

$x^2+3x+2=(x+1)(x+2)$ **(A1)**

correct use of factor theorem for at least one of their factors **(M1)**

$P(-1)=0$ or $P(-2)=0$

attempt to find two equations in a, b and c **(M1)**

$(-1)^3+a(-1)^2+b(-1)+c=0(\Rightarrow -1+a-b+c=0)$

$(-2)^3+a(-2)^2+b(-2)+c=0$

$-8+4a-2b+c=0$ and $-1+a-b+c=0$ **A1**

attempt to combine their two equations in $-8+4a-2b+c=0$ to eliminate c **(M1)**

$b=3a-7$ **A1**

METHOD 2

$P(x) = x^3 + ax^2 + bx + c = (x^2 + 3x + 2)(x + d)$ ***(M1)***

$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d$ ***(A1)***

attempt to compare coefficients of x^2 and x ***(M1)***

$a = 3 + d$ and $b = 2 + 3d$ ***A1***

attempt to eliminate d ***(M1)***

$\Rightarrow b = 3a - 7$ ***A1***

(ii) **METHOD 1**

$a = 1, 2, 5$ lead to invalid values for b ***R1***

$a = 3, b = 2 \Rightarrow c = 0$ so not possible ***R1***

so $a = 4, b = 5, c = 2$ is the only solution ***AG***

METHOD 2

$c = 2a - 6$ ***R1***

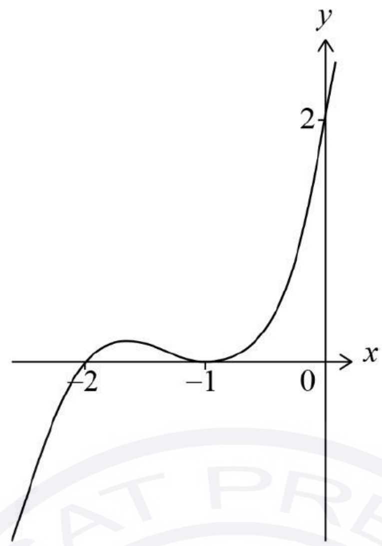
correctly argues $a = 4$ is the only possibility ***R1***

so $a = 4, b = 5, c = 2$ is the only solution ***AG***

(iii) $x^3 + 4x^2 + 5x + 2 = (x^2 + 3x + 2)(x + 1)$

$= (x + 2)(x + 1)(x + 1)$ ***A1***

(iv)



positive cubic shape with y-intercept at $(0, 2)$ **A1**

x-intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ **A1**

local minimum point at $(-1, 0)$ **A1**

ote: Accept answers from an approach based on calculus.

[12 marks]

Total [16 marks]

Question 26

(a) product of roots $=80$ **(A1)**

$3-\mathrm{i}$ is a root **(A1)**

attempt to set up an equation involving the product of their four roots and ± 80 **(M1)**

$(3+\mathrm{i})(3-\mathrm{i})\alpha^3=80 \Rightarrow 10\alpha^3=80$

$\alpha=2$ **A1**

[4 marks]

(b) **METHOD 1**

sum of roots $=-p$ **(A1)**

$-p=3+\mathrm{i}+3-\mathrm{i}+2+4$ **(M1)**

Note: Accept $p=3+\mathrm{i}+3-\mathrm{i}+2+4$ for ***(M1)***

$p=-12$ **A1**

METHOD 2

$(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))(z-2)(z-4)$ **(M1)**

$((z-3)-i)((z-3)+i)(z-2)(z-4)$ **(A1)**

$(z^2-6z+10)(z^2-6z+8)=z^4-12z^3+\ldots$

$p=-12$ **A1**

[3 marks]

Total [7 marks]

Question 27

(a) (i) recognition that $n = 5$ *(M1)*

$S_5 = 45$ ***A1***

(ii) **METHOD 1**

recognition that $S_5 + u_6 = S_6$ *(M1)*

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ *(M1)*

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ *(M1)*

$u_6 = 15$ ***A1***

[4 marks]

(b) recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 *(M1)*

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}(u_1 + 15)$

$u_1 = 5$ ***A1***

[2 marks]

(c) **EITHER**

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}(5 + u_n)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n-1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

(d) recognition that $v_2 r^2 = v_4$ OR $(v_3)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3} \quad \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

Total [14 marks]

Question 28

METHOD 1

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

Note: Award ***A1A1*** for $P(3\mathrm{i})=0$ and $P(-3\mathrm{i})=0$ seen in their working. Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\mathrm{i})=0$ or $P(-3\mathrm{i})=0$ ***(M1)***

$$4m-3m\mathrm{i}+\frac{36}{m}(3\mathrm{i})^2-(3\mathrm{i})^3=0$$

$$4m-3m\mathrm{i}-\frac{36}{m}(-9)+27\mathrm{i}=0$$

attempt to equate the real or imaginary parts ***(M1)***

$27-3m=0$ OR $9\times\frac{36}{m}=4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

Note: Accept sum of three roots set to $-\frac{36}{m}$. Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$$3\mathrm{i}-3\mathrm{i}+r=\frac{36}{m}\left(\Rightarrow r=\frac{36}{m}\right)$$

substitute their r into product of roots ***(M1)***

$(3\mathrm{i})(-3\mathrm{i})\left(\frac{36}{m}\right)=4m$ OR $(z^2+9)\left(\frac{36}{m}-z\right)$

$9\times\frac{36}{m}=4m$ OR $\frac{4m}{9}=\frac{36}{m}$

OR

attempt to equate product of three roots to $4m$ ***(M1)***

Note: Accept product of three roots set to $-4m$. Award ***M0*** for stating product of roots is $\pm 4m$.

$(3\text{i})(-3\text{i})\times r = 4m\left(\Rightarrow r = \frac{4m}{9}\right)$

substitute their r into sum of roots ***(M1)***

$3\text{i}-3\text{i}+\frac{4m}{9}=\frac{36}{m}$ OR $(z^2+9)\left(\frac{4m}{9}-z\right)$

$\frac{4m}{9}=\frac{36}{m}$

THEN

$m=9$ ***(A1)***

third root is 4 ***A1***

[6 marks]

METHOD 2

3i (is a root) ***A1***

(other complex root is) -3i ***A1***

recognition that the other factor is $(z+3\text{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\text{i})(z+3\text{i})(r-z)$ OR $(z-3\text{i})(z+3\text{i})=z^2+9 \Rightarrow P(z)=(z^2+9)\left(\frac{4m}{9}-z\right)$

Note: Accept any attempt at long division of $P(z)$ by z^2+9. Award ***M0*** for stating other factor is $(z+3\text{i})$ or obtaining z^2+9 with no further working.

attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

$m=9$ ***(A1)***

third root is 4 ***A1***

Note: Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\text{i})(z+3\text{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

Question 29

recognition of quadratic in e^x **(M1)**

$\left(e^x\right)^2 - 3e^x + \ln k\,(=0)$ OR $A^2 - 3A + \ln k\,(=0)$

recognizing discriminant ≥ 0 (seen anywhere) **(M1)**

$(-3)^2 - 4(1)(\ln k)$ OR $9 - 4\ln k$ **(A1)**

$\ln k \le \frac{9}{4}$ **(A1)**

$e^{9/4}$ (seen anywhere) **A1**

$0 < k \le e^{9/4}$ **A1**

[6 marks]

Question 30

(a) **METHOD 1**

$|u| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}\ \left(= \sqrt{1+3}\right)$ **A1**

$= 2$ **AG**

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}\left(\sqrt{3}\right)$ OR $\arg u = \pi + \tan^{-1}\left(-\sqrt{3}\right)$ **M1**

$= \pi - \frac{\pi}{3}$ **A1**

Note: Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ and so $u = 2e^{i\frac{2\pi}{3}}$ **AG**

[3 marks]

METHOD 2

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}\left(\sqrt{3}\right)$ OR $\arg u = \pi + \tan^{-1}\left(-\sqrt{3}\right)$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

: Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ ***AG***

$r\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = -1 + \sqrt{3}\mathrm{i}$

$r = \frac{-1}{\cos\frac{2\pi}{3}} = \frac{-1}{-\frac{1}{2}}$ OR $r = \frac{\sqrt{3}}{\sin\frac{2\pi}{3}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}}$ ***A1***

$= 2$ and so $u = 2e^{\mathrm{i}\frac{2\pi}{3}}$ ***AG***

[3 marks]

(b) (i) $u^n \in \mathbb{R} \Rightarrow \frac{2n\pi}{3} = k\pi \ \ (k \in \mathbb{Z})$ ***(M1)(A1)***

Note: Award ***M1*** for noting that $\sin\frac{2n\pi}{3} = 0$ from $u^n = 2^n\left(\cos\frac{2n\pi}{3} + \mathrm{i}\sin\frac{2n\pi}{3}\right)$.

Award ***(A1)*** for a multiple of 3 considered.

$n = 3$ ***A1***

(ii) substitutes their value (must be a multiple of 3) for n into u^n ***(M1)***

$u^3 = 2^3\cos 2\pi$

$= 8$ ***A1***

[5 marks]

(c) (i) $-1-\sqrt{3}\mathrm{i}$ is a root (by the conjugate root theorem) ***A1***

Note: Accept $2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$.

let $z = c$ be the real root

EITHER

uses sum of roots (equated to ± 5) ***(M1)***

$\left(\left(-1+\sqrt{3}\mathrm{i}\right)+\left(-1-\sqrt{3}\mathrm{i}\right)+c\right)=-5$ ***(A1)***

$-2+c=-5$ ***(A1)***

OR

uses product of roots (equated to ± 12) ***(M1)***

$\left(-1+\sqrt{3}\mathrm{i}\right)\left(-1-\sqrt{3}\mathrm{i}\right)c=-12$ ***(A1)***

$4c=-12$ ***(A1)***

OR

$\left(z-\left(-1+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(-1-\sqrt{3}\mathrm{i}\right)\right)=z^2+2z+4$ ***(A1)***

compares coefficients *eg* ***(M1)***

$(z-c)\left(z^2+2z+4\right)=z^3+5z^2+10z+12$

$-4c=12$ ***(A1)***

THEN

$c=-3$ (and so $z=-3$ is a root) ***A1***

(ii) **METHOD 1**

compares $z^3+5z^2+10z+12=0$ and $1+5w+10w^2+12w^3=0$

$z=\frac{1}{w}\Rightarrow w=\frac{1}{z}$ ***A2***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\mathrm{i}}\left(=\frac{-1\pm\sqrt{3}\mathrm{i}}{4}\right)$ ***A1A1***

METHOD 2

attempts to factorize into a product of a linear factor and a quadratic factor ***(M1)***

$1+5w+10w^2+12w^3=(3w+1)(4w^2+2w+1)$ ***A1***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\mathrm{i}}\left(=\frac{-1\pm\sqrt{3}\mathrm{i}}{4}\right)$ ***A1A1***

[9 marks]

(d) $(a+b\mathrm{i})^2=2(a-b\mathrm{i})$ ***A1***

attempts to expand and equate real and imaginary parts: ***M1***

$a^2-b^2+2ab\mathrm{i}=2a-2b\mathrm{i}$

$a^2-b^2=2a$ and $2ab=-2b$

attempts to find the value of a or b ***M1***

$2b(a+1)=0$

$b=0\Rightarrow a^2=2a\Rightarrow a=2$ (real root) ***A1***

$a=-1\Rightarrow 1-b^2=-2\Rightarrow b=\pm\sqrt{3}$ (complex roots $-1\pm\sqrt{3}\mathrm{i}$) ***A1***

[5 marks]

Total [22 marks]

Question 31

let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers, $n \geq 1$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = \frac{1}{2}$ and $\mathrm{RHS} = \frac{1}{2}$ and so $\mathrm{P}(1)$ is true **_R1_**

assume $\mathrm{P}(k)$ is true *ie*, $\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$ **_M1_**

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = \sum_{r=1}^{k} \frac{r}{(r+1)!} + \frac{k+1}{((k+1)+1)!}$$ **_(M1)_**

$$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$ **_A1_**

$$= 1 - \frac{(k+2)-(k+1)}{(k+2)!}$$ **_A1_**

$$= 1 - \frac{1}{(k+2)!} \left(= 1 - \frac{1}{((k+1)+1)!} \right)$$ **_A1_**

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true (for all integers, $n \geq 1$) **_R1_**

Note: To obtain the final ***R1***, any four of the previous marks must have been awarded.

[7 marks]

Question 32

(a) attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos^5\theta + {}^5C_1\mathrm{i}\cos^4\theta\sin\theta + {}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta + {}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$$ ***A1***

$$= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) + \mathrm{i}\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$$ ***A1A1***

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ ***A1***

substitute $\cos^2\theta = 1 - \sin^2\theta$ ***(M1)***

$\sin 5\theta = 5\left(1 - \sin^2\theta\right)^2\sin\theta - 10\sin^3\theta\left(1 - \sin^2\theta\right) + \sin^5\theta$ ***A1***

$\sin 5\theta = 5\left(1 - 2\sin^2\theta + \sin^4\theta\right)\sin\theta - 10\sin^3\theta\left(1 - \sin^2\theta\right) + \sin^5\theta$ ***A1***

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***AG***

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

Question 33

METHOD 1

attempt to substitute solution into given equation **(M1)**

$(5+q\text{i})^2 + \text{i}(5+q\text{i}) = -p + 25\text{i}$

$25 - q^2 + 10q\text{i} - q + 5\text{i} + p - 25\text{i} = 0$ OR $25 - q^2 + 10q\text{i} - q + 5\text{i} = -p + 25\text{i}$ **A1**

$25 - q^2 + p - q + (10q - 20)\text{i} = 0$

attempt to equate real or imaginary parts: **(M1)**

$10q - 20 = 0$ OR $25 - q^2 + p - q = 0$

$q = 2$, $p = -19$ **A1A1**

METHOD 2

$z^2 + \text{i}z + p - 25\text{i} = 0$

sum of roots $= -\text{i}$, product of roots $= p - 25\text{i}$ **M1**

one root is $(5+q\text{i})$ so other root is $(-5 - q\text{i} - \text{i})$ **A1**

product $(5+q\text{i})(-5 - q\text{i} - \text{i}) = -25 - 5q\text{i} - 5\text{i} - 5q\text{i} + q^2 + q = p - 25\text{i}$

equating real and imaginary parts for product of roots **(M1)**

Im: $-25 = -10q - 5$ Re: $p = -25 + q^2 + q$

$q = 2$, $p = -19$ **A1A1**

[5 marks]

Question 34

base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ **A1**

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

Note: The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=(5^2)5^{2k}-(2^3)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8(5^{2k}-2^{3k})$ OR $25(5^{2k}-2^{3k})+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17(5^{2k}+8s)$ OR $17(25s+2^{3k})$ which is divisible by 17 ***A1***

OR

$(5^{2k}-2^{3k})\times 5^2=5^{2k+2}-25\times 2^{3k}=17s\times 25$ ***M1***

$=5^{2k+2}-8\times 2^{3k}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times 2^{3k}=17s\times 25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times 25+17\times 2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17(25s+2^{3k})$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if 4 of the previous 6 marks have been awarded **Note:** 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

Question 35

attempt to apply binomial expansion **(M1)**

$(1+kx)^n = 1 + {}^nC_1 kx + {}^nC_2 k^2x^2 + \ldots$ OR ${}^nC_1 k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ **(A1)**

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ **(A1)**

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve **(M1)**

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ **A1**

$\Rightarrow k = \frac{3}{2}$ **A1**

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

Question 36

(a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3, \ q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1, \ d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =)65 - 40$ OR $(u_5 =)1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

Question 37

(a) attempt to find a difference *(M1)*

$d = p - a, 2d = q - a, d = q - p$ OR $p = a + d, q = a + 2d, q = p + d$

correct equation ***A1***

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ ***AG***

[2 marks]

(b) attempt to find a ratio *(M1)*

$r = \frac{s}{a}, r^2 = \frac{t}{a}, \quad r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation ***A1***

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ ***AG***

[2 marks]

(c) **EITHER**

$2p-1=s^2$ (or equivalent) ***A1***

$\left(s^2>0\right) \Rightarrow 2p-1>0$ OR $s=\sqrt{2p-1} \Rightarrow 2p-1>0$ OR $p=\dfrac{s^2+1}{2}$ (and $s^2>0$) ***R1***

OR

$2p-1=a$ and $s^2=a$ ***A1***

($s^2>0$, so) $a>0 \Rightarrow 2p-1>0$ OR $p=\dfrac{a+1}{2}$ and $a>0$ ***R1***

$\Rightarrow p>\dfrac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

(d) (i) $9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

(ii) $9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[4 marks]

(e) (i) attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR $d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2\ln 3$ OR $\ln 1 = 0$ OR $\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

(ii) **METHOD 1**

attempt to substitute first term and their common difference into S_{10} ***(M1)***

$\frac{10}{2}\left(2(9+\ln 9)+9(-4-\ln 3)\right)$ OR $\frac{10}{2}\left(2(9+2\ln 3)+9(-4-\ln 3)\right)$ (or equivalent) ***A1***

$=5(-18-5\ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

METHOD 2

$u_{10} = 9+\ln 9+9(-4-\ln 3)(=-27+\ln 9-9\ln 3)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}\left(2(9+\ln 9)+9(-4-\ln 3)\right)$ OR $\frac{10}{2}(9+\ln 9-27+\ln 9-9\ln 3)$ OR

$\frac{10}{2}\left(2(9+2\ln 3)+9(-4-\ln 3)\right)$ OR $\frac{10}{2}(9+\ln 9-27-7\ln 3)$ (or equivalent) ***A1***

$=5(-18-5\ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

[6 marks]

Total [16 marks]

Question 38

(a) ${}^{n}C_{3}$ **A1**

[1 mark]

(b) **EITHER**

finding the number of ways to assign the students with the two students apart

number of ways to assign two students ${}^{2}C_{1}$ (seen anywhere) ***(A1)***

number of ways to assign others ${}^{n-2}C_{2}$ to have one group of 3 (seen anywhere) ***(A1)***

number of ways = ${}^{2}C_{1}\times{}^{n-2}C_{2}$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}{}^{n}C_{3}={}^{2}C_{1}\times{}^{n-2}C_{2}$ OR ${}^{n}C_{3}=2\times{}^{2}C_{1}\times{}^{n-2}C_{2}$

valid attempt to eliminate all factorials from their equation ***(M1)***

$\frac{n(n-1)(n-2)}{3\times 2}=2\times 2\times\frac{(n-2)(n-3)}{2}$ or equivalent with no factorials

$n(n-1)=12(n-3)$

OR

finding the number of ways to assign the students with the two students together

number of ways to assign two students and one other to the first group ${}^{n-2}C_1$
(seen anywhere) ***(A1)***

number of ways to assign three other students the first group ${}^{n-2}C_3$ (seen anywhere) ***(A1)***

number of ways $= {}^{n-2}C_1 + {}^{n-2}C_3$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}\,{}^{n}C_3 = {}^{n-2}C_1 + {}^{n-2}C_3$ OR ${}^{n}C_3 = 2\left({}^{n-2}C_1 + {}^{n-2}C_3\right)$

valid attempt to eliminate all factorials from their equation ***(M1)***

$$\frac{n(n-1)(n-2)}{3\times 2} = 2\times(n-2) + 2\times\frac{(n-2)(n-3)(n-4)}{3\times 2}$$

$$n(n-1) = 12 + 2(n-3)(n-4)$$

THEN

$n^2 - 13n + 36 = 0$ ***A1***

$(n-9)(n-4) = 0$

$n = 9$ ***A1***

Note: Do not award the final ***A1*** if additional values of n are given.

[6 marks]

Total [7 marks]

Question 39

(a) $\alpha+\beta+\gamma=\frac{7}{2}$ **A1**

[1 mark]

(b) $p-3\text{i}$ is also a root (seen anywhere) **A1**

recognition of 5 roots and attempt to sum these roots ***(M1)***

$p+3\text{i}+p-3\text{i}+\frac{7}{2}$

$p+3\text{i}+p-3\text{i}+\frac{7}{2}=\frac{11}{2}$ **A1**

$p=1$ **AG**

[3 marks]

(c) (i) attempt to find product of 5 roots and equate to ± 10 ***(M1)***

$(1+3\text{i})(1-3\text{i})\frac{1}{2}\alpha\beta=10$

$\alpha\beta=2$ **A1**

(ii) $\alpha=1$ and $\beta=2$ **A1**

[3 marks]

Total [7 marks]

Question 40

recognising a quadratic in 3^x **(M1)**

$$3\times\left(3^x\right)^2+5\times3^x-2=0$$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) **(M1)**

$\left(3\times3^x-1\right)\left(3^x+2\right)=0$ OR $3^x=\dfrac{-5\pm\sqrt{25+24}}{6}$ (or equivalent) **(A1)**

$3^x=\dfrac{1}{3}$ (or $3^x=-2$) **(A1)**

$x=-1$ ***A1***

Note: Award the final ***A1*** if candidate's answer includes $x=-1$ and $x=\log_3(-2)$. Award ***A0*** if other incorrect answers are given.

[5 marks]

Question 41

(a) attempt to expand the brackets or attempt to find modulus and argument of ϕ ***(M1)***

$(a+b\mathrm{i})^3 = a^3 + 3a^2b\mathrm{i} + 3a(b\mathrm{i})^2 + (b\mathrm{i})^3$ OR $\left(\sqrt{a^2+b^2}\right)^3 \operatorname{cis}\left(3\arctan\left(\frac{b}{a}\right)\right)$

(i) real part is $a^3 - 3ab^2$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

(ii) imaginary part is $3a^2b - b^3$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \sin\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

Note: Award ***(M1)A1A0*** for $\left(a^3-3ab^2\right)+\left(3a^2b-b^3\right)\mathrm{i}$ OR
$\left(a^2+b^2\right)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right) + \left(a^2+b^2\right)^{\frac{3}{2}} \mathrm{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

For (ii) condone $\left(3a^2b-b^3\right)\mathrm{i}$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \mathrm{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

[3 marks]

(b) attempt to substitute $a=1$ and $b=\sqrt{3}$ into their real or imaginary part found in (a) OR to expand the brackets OR to use polar form ***M1***

$(1-9)+\left(3\sqrt{3}-3\sqrt{3}\right)\mathrm{i}$ OR $\left(2\sqrt{3}\mathrm{i}-2\right)\left(1+\sqrt{3}\mathrm{i}\right) = 2\sqrt{3}\mathrm{i}-2-6-2\sqrt{3}\mathrm{i}$ OR

$\left(2\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\right)^3 = 8\mathrm{e}^{\mathrm{i}\pi}$ OR $\left(2\operatorname{cis}\left(60^\circ\right)\right)^3 = 8\operatorname{cis}\left(180^\circ\right)$ ***A1***

$= -8$ ***AG***

[2 marks]

(c) $v=-2,\ w=1-\sqrt{3}\mathrm{i}$ ***A1A1***

Note: Award ***A1A0*** for $v=1-\sqrt{3}\mathrm{i},\ w=-2$ or if the labels v and w are not clearly specified or missing. Candidates may be awarded full ***FT*** marks for subsequent parts.

[2 marks]

(d) **METHOD 1**

triangle UVW has height $h = 3$ and base $b = 2\sqrt{3}$ ***(A1)***

attempt to find area of triangle with their height and base ***(M1)***

area = $\frac{1}{2} \times 2\sqrt{3} \times 3$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 2

triangle UVW has sides of length $\left(\sqrt{3^2 + \left(\sqrt{3}\right)^2} = \right)\sqrt{12}$ ***(A1)***

attempt to find area of equilateral triangle with their side length ***(M1)***

area = $\frac{1}{2}\left(\sqrt{12}\right)^2 \sin\frac{\pi}{3}$ OR $\frac{1}{2}\sqrt{12}(3)$ OR $\left(\sqrt{12}\right)^2 \times \frac{\sqrt{3}}{4}$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 3

triangle UVO has sides of length $\left(\sqrt{1^2 + \left(\sqrt{3}\right)^2} = \right)2$ ***(A1)***

attempt to find area of three isosceles triangles with their side length and angle $\frac{2\pi}{3}$ ***(M1)***

area $= 3\left(\frac{1}{2}(2)^2 \sin\frac{2\pi}{3}\right)$

$= 3\sqrt{3}$ (square units) ***A1***

[3 marks]

(e) attempt to express u, v or w in the form $re^{i\theta}$ and multiply by $e^{\frac{\pi}{4}i}$ ***(M1)***

$\left(u' = 2e^{\frac{\pi}{3}i}e^{\frac{\pi}{4}i} = \right)2e^{\frac{7\pi}{12}i}$ ***A1***

$\left(v' = 2e^{\pi i}e^{\frac{\pi}{4}i} = 2e^{\frac{5\pi}{4}i} = \right)2e^{-\frac{3\pi}{4}i}$ ***A1***

$\left(w' = 2e^{-\frac{\pi}{3}i}e^{\frac{\pi}{4}i} = \right)2e^{-\frac{\pi}{12}i}$ ***A1***

Note: These ***A1*** marks should be awarded independently and in any order.

[4 marks]

(f) **EITHER**

attempt to find one of $(u')^3, (v')^3$ or $(w')^3$ ***(M1)***

$(u')^3 = \left(2e^{\frac{7\pi}{12}i}\right)^3 = 8e^{\frac{7\pi}{4}i} = 8e^{-\frac{\pi}{4}i}$ OR $(v')^3 = \left(2e^{-\frac{3\pi}{4}i}\right)^3 = 8e^{-\frac{\pi}{4}i}$ OR

$(w')^3 = \left(2e^{-\frac{\pi}{12}i}\right)^3 = 8e^{-\frac{\pi}{4}i}$ ***(A1)***

OR

attempt to find product of their three roots u', v' and w' ***(M1)***

$u' \times v' \times w' = (c + d\,i)$

$2e^{\frac{7\pi}{12}i} \times 2e^{-\frac{3\pi}{4}i} \times 2e^{-\frac{\pi}{12}i} = 8e^{-\frac{\pi}{4}i}$ OR $2e^{\frac{7\pi}{12}i} \times 2e^{\frac{5\pi}{4}i} \times 2e^{-\frac{\pi}{12}i} = 8e^{-\frac{\pi}{4}i}$ (or equivalent) ***(A1)***

OR

attempt to find $\left(ze^{\frac{\pi}{4}i}\right)^3$ for any z such that $z^3 = -8$ OR to rotate -8 by $\frac{3\pi}{4}$ ***(M1)***

$\left(\left(ze^{\frac{\pi}{4}i}\right)^3 = z^3 e^{\frac{3\pi}{4}i} = \right) -8e^{\frac{3\pi}{4}i}$ OR $8e^{-\frac{\pi}{4}i}$ OR $8\text{cis}(-45^\circ)$ ***(A1)***

THEN

$8e^{-\frac{\pi}{4}i} = 8\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{8}{\sqrt{2}}(1-i)$

$= 4\sqrt{2} - 4\sqrt{2}i$ ***A1***

Note: Accept $c = \frac{8}{\sqrt{2}}, d = -\frac{8}{\sqrt{2}}$.

[3 marks]

(g) **METHOD 1**

attempt to write the arguments of u, v, w, u', v' and w' over a common denominator OR to write the arguments in degrees ***(M1)***

$\frac{-9\pi}{12}, \frac{-4\pi}{12}, \frac{-\pi}{12}, \frac{4\pi}{12}, \frac{7\pi}{12}, \frac{12\pi}{12}$ OR $-135°, -60°, -15°, 60°, 105°, 180°$

THEN

arguments of u, v, w, u', v' and w' differ by $\frac{3\pi}{12}$ and $\frac{5\pi}{12}$ OR $45°$ and $75°$

so arguments of polygon vertices differ by $\frac{\pi}{12}$ or $15°$ ***(A1)***

$n = 24$ ***A1***

METHOD 2

Let $z = r\,\text{cis}\,\theta \Rightarrow z^n = r^n \text{cis}(n\theta) = r^n \text{cis}(n\theta)$, where θ is the argument of u, v, w, u', v' and w'.

recognition to find $n\theta$ where $n = 6, 12, 18, \ldots$ and $\theta = -\frac{3\pi}{4}, -\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \pi$ ***(M1)***

when $n = 6 \Rightarrow (n\theta =) -\frac{9\pi}{2}, -2\pi, -\frac{\pi}{2}, 2\pi, \frac{7\pi}{2}, 6\pi$

when $n = 12 \Rightarrow (n\theta =) -9\pi, -4\pi, -\pi, 4\pi, 7\pi, 12\pi$ (which is not a multiple of 2π) ***(A1)***

$n = 24$ ***A1***

[3 marks]

Total [20 marks]

Question 42

Base case $n=1$: LHS $= {}^1C_1 = 1$ and RHS $= {}^2C_2 = 1$, so true for $n=1$ ***R1***

Note: Award ***R0*** if the value of ${}^1C_1$ and ${}^2C_2$ are not evaluated.
Subsequent marks can still be awarded.

assume true for $n=k$ ie $\sum_{r=1}^{k} {}^rC_1 = {}^{k+1}C_2$ for some $k \in \mathbb{Z}^+$ ***M1***

Note: The assumption of truth must be clear.
Award ***M0*** for statements such as "let $n=k$" or "$n=k$ is true".
Subsequent marks can still be awarded.

consider $n=k+1$

LHS $= \sum_{r=1}^{k+1} {}^rC_1$

$= \sum_{r=1}^{k} {}^rC_1 + {}^{k+1}C_1$ ***(M1)***

$= {}^{k+1}C_2 + {}^{k+1}C_1$ OR $\frac{(k+1)!}{2(k-1)!} + \frac{(k+1)!}{k!}$ ***A1***

EITHER

attempt to cancel factorials and use a common denominator ***M1***

$= \frac{(k+1)k + 2(k+1)}{2} \left(= \frac{(k+2)(k+1)}{2} \right)$

OR

attempt to use a common denominator ***M1***

$= \frac{k(k+1)!}{2k!} + \frac{2(k+1)!}{2k!} \left(= \frac{(k+2)(k+1)!}{2k!} \right)$

THEN

$= \frac{(k+2)!}{2!k!} \left(= \frac{(k+2)!}{2!(k+2-2)!} \right)$ ***A1***

$= {}^{k+2}C_2$

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if 4 of the previous 6 marks have been awarded.

Total *[7 marks]*

Question 43

(a) $S_n = \dfrac{10^n - 1}{9}$ **A1**

$(a = 10, b = 9)$

[1 mark]

(b) **METHOD 1**

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \dfrac{10-1}{9} + \dfrac{10^2-1}{9} + \ldots + \dfrac{10^n-1}{9}$ ***(A1)***

$= \dfrac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\dfrac{9\left(10-1+10^2-1+10^3-1+\ldots+10^n-1\right)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10 + 10^2 + 10^3 + \ldots + 10^n = \dfrac{10\left(10^n-1\right)}{10-1}$ and $-1-1-1\ldots = -n$ ***A1***

$= \dfrac{\dfrac{10\left(10^n-1\right)}{10-1} - n}{9}$ OR $\dfrac{9\left(\dfrac{10\left(10^n-1\right)}{10-1}\right) - 9n}{81}$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$= \dfrac{10\left(10^n-1\right) - 9n}{81}$ ***AG***

METHOD 2

attempt to create sum using sigma notation with S_n **M1**

$$\sum_{i=1}^{n} \frac{10^i - 1}{9} \quad \left(= \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1 \right) \right)$$

$$\sum_{i=1}^{n} 10^i = \frac{10\left(10^n - 1\right)}{9}$$ **A1**

$$\sum_{i=1}^{n} 1 = n$$ **A1**

$$= \frac{1}{9}\left(\frac{10\left(10^n - 1\right)}{9} - n \right) \text{ OR } \frac{1}{9}\left(\frac{10\left(10^n - 1\right) - 9n}{9} \right)$$ **A1**

$$= \frac{10\left(10^n - 1\right) - 9n}{81}$$ **AG**

METHOD 3

let $\mathrm{P}(n)$ be the proposition that $S_1 + S_2 + S_3 + \ldots + S_n = \dfrac{10\left(10^n - 1\right) - 9n}{81}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = S_1 = \dfrac{10^1 - 1}{9} = 1$ and $\mathrm{RHS} = \dfrac{10\left(10^1 - 1\right) - 9(1)}{81} = 1$ and so $\mathrm{P}(1)$ is true **_R1_**

assume $\mathrm{P}(k)$ is true *i.e.* $S_1 + S_2 + S_3 + \ldots + S_k = \dfrac{10\left(10^k - 1\right) - 9k}{81}$ **_M1_**

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks a this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$$S_1 + S_2 + S_3 + \ldots + S_{k+1} = \frac{10\left(10^k - 1\right) - 9k}{81} + \frac{10^{k+1} - 1}{9}$$

$$= \frac{10^{k+1} - 10 - 9k + 9\left(10^{k+1}\right) - 9}{81}$$ **_A1_**

$$= \frac{10\left(10^{k+1} - 1\right) - 9(k+1)}{81}$$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true **_R1_**

(for all integers $n \geq 1$)

Note: To obtain the final ***R1***, the first ***R1*** and ***A1*** must have been awarded.

[4 marks]

Total [5 marks]

Question 44

(a) $\log_{10}1-\log_{10}a$ OR $\log_{10}a^{-1}=-\log_{10}a$ OR $\log_{10}10^{-\frac{1}{3}}$ OR $10^x=\frac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$=-\frac{1}{3}$ ***A1***

[2 marks]

(b) $\frac{\log_{10}a}{\log_{10}1000}$ OR $\frac{1}{3}\log_{1000}10$ OR $\log_{1000}\sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}}=1000^x\left(=\left(10^3\right)^x\right)$ ***(A1)***

$\frac{\log_{10}a}{3}$ OR $\frac{1}{3}\log_{1000}1000^{\frac{1}{3}}$ OR $\log_{1000}1000^{\frac{1}{9}}$ OR $3x=\frac{1}{3}$ ***(A1)***

$=\frac{1}{9}$ ***A1***

[3 marks]

Total [5 marks]

Question 45

(a) $|16\mathrm{i}| = 16$ and $\arg(16\mathrm{i}) = \frac{\pi}{2}$ ***(A1)***

attempt to use De Moivre's Theorem ***(M1)***

$z_1 = 2\left(\cos\left(\frac{\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)$ ***A1***

attempts to find other solutions using $z = 2\left(\cos\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{8} + \frac{\pi k}{2}\right)\right)$ or equivalent ***(M1)***

$z_2 = 2\left(\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)$ (or any other root) ***A1***

$z_3 = 2\left(\cos\left(\frac{9\pi}{8}\right) + \mathrm{i}\sin\left(\frac{9\pi}{8}\right)\right)$ and $z_4 = 2\left(\cos\left(\frac{13\pi}{8}\right) + \mathrm{i}\sin\left(\frac{13\pi}{8}\right)\right)$ ***A1***

Note: Award a maximum of ***(A1)(M1)A1(M1)A1A0*** for more than four roots or any roots outside the range.

Note: Allow use of r-cis form throughout.

[6 marks]

(b) attempt to evaluate a ratio with their roots eg $\frac{z_2}{z_1}$ ***(M1)***

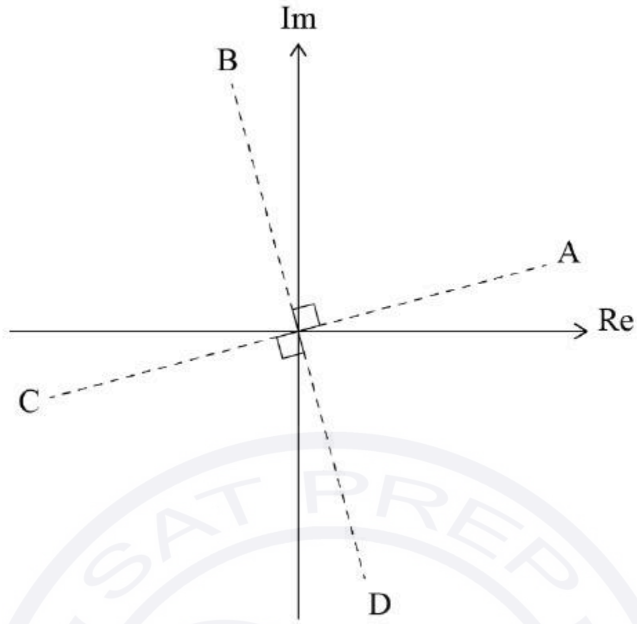
$$\frac{z_2}{z_1} = \frac{2\left(\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)}{2\left(\cos\left(\frac{\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)}$$ or equivalent

$= \left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right)$ ***(A1)***

$= \mathrm{i}$ ***A1***

[3 marks]

(c)



point A in approximately correct place in first quadrant **A1**

points A, B, C and D approximately the same distance from the origin **A1**

approximate angular separation of $\frac{\pi}{2}$ **A1**

Note: Dotted lines not required.

[3 marks]

(d) **EITHER**

$\left(z_i^{*}\right)^4 = \left(z_i^{4}\right)^*$ ***(A1)***

$= (16\text{i})^*$ ***(A1)***

OR

$z_1^* = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$ ***(A1)***

$\left(z_1^*\right)^4 = 2^4\left(\cos\left(-\frac{4\pi}{8}\right) + \text{i}\sin\left(-\frac{4\pi}{8}\right)\right)$ ***(A1)***

OR

$z_1 z_2 z_3 z_4 = -16\text{i}$ ***(A1)***

$\left(z_1 z_2 z_3 z_4\right)^* = (-16\text{i})^*$

$z_1^* z_2^* z_3^* z_4^* = 16\text{i}$ ***(A1)***

THEN

$\left(z^4 =\right) -16\text{i}$ ***A1***

$(a = 0,\ b = -16)$

[3 marks]

(e) $\arg w_1 \left(= \frac{\pi}{8} + \frac{\pi}{4}\right) = \frac{3\pi}{8}$ ***A1***

$\text{AB} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow \text{AA}' = \sqrt{2}$

$\Rightarrow \text{OA}' = |w_1| = \sqrt{2}$ ***A1***

$\therefore w_1 = \sqrt{2}\,\text{cis}\frac{3\pi}{8}$

considers when $\arg\left(w_1^{\,p}\right) \in \mathbb{Z}^+$ (multiple of 2π) ***(M1)***

$\Rightarrow p = 16$ ***A1***

$\Rightarrow q = 8$ ***A1***

[5 marks]

Total [20 marks]

Question 46

attempt to use $u_n = u_1 + (n-1)d$ or $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ or $S_n = \frac{n}{2}[u_1 + u_n]$ to

set up at least one equation in u_1 and d ***(M1)***

$16 = u_1 + 9d$ and $100 = \frac{25}{2}\left[2u_1 + 24d\right]$ ***(A1)***

attempt to solve their two linear equations in u_1 and d simultaneously (must eliminate one variable) ***(M1)***

$d = -4 (\Rightarrow u_1 = 52)$ **A1**

attempt to solve $u_k = 0$ with their d (or with their d and u_1) ***(M1)***

$\Rightarrow k = 14$ **A1**

[6 marks]

Question 47

METHOD 1

attempt to expand $(3n+2)^2-(3n-2)^2$ ***M1***

---

ote: Award ***M0*** for invalid attempts such as $(3n+2)^2=9n^2+4$.

$=9n^2+12n+4-(9n^2-12n+4)$ or equivalent ***A1***

$=24n$ OR $12n+12n$ ***A1***

$=12(2n)$ OR $\frac{24}{12}=2$ OR $\frac{24n}{2}=12n$ OR $12n+12n=12(n+n)$ (or equivalent) ***R1***

so is a multiple of 12 ***AG***

---

ote: Do not award the ***R1*** unless both ***A*** marks have been awarded.

METHOD 2

use of $a^2-b^2=(a+b)(a-b)$ where $a=3n+2, b=3n-2$ ***M1***

$=(3n+2+3n-2)(3n+2-3n+2)$

$=6n\times 4$ ***A1***

$=24n$ ***A1***

$=12(2n)$ OR $\frac{24n}{12}=2n$ OR $\frac{24n}{2}=12n$ OR $12n+12n=12(n+n)$ (or equivalent) ***R1***

so is a multiple of 12 ***AG***

METHOD 3

base case $n=1$: $\left(3(1)+2\right)^2-\left(3(1)-2\right)^2=25-1=24$

so true for $n=1$ ***A1***

assume true for $n=k$ i.e. $(3k+2)^2-(3k-2)^2$ is a multiple of 12 ***M1***

consider $n=k+1$:

$\left(3(k+1)+2\right)^2-\left(3(k+1)-2\right)^2$

$\left((3k+2)+3\right)^2-\left((3k-2)+3\right)^2$

$(3k+2)^2+6(3k+2)+9-\left((3k-2)^2+6(3k-2)+9\right)$

$(3k+2)^2-(3k-2)^2+24$

using the assumption $(3k+2)^2-(3k-2)^2=12M$

$12M+24$

$12(M+2)$

which is a multiple of 12, hence true for $n=k+1$ ***A1***

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

[4 marks]